Exact Quench Dynamics from Thermal Pure Quantum States

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(Dated: October 24, 2025)

We present an exact solution for the real-time dynamics following a quench from a thermal pure quantum (TPQ) state in an integrable system. Although equilibration is expected in this setting because the TPQ state already encodes the thermal expectation values of all conserved quantities, the approach to equilibrium shows highly nontrivial coherent dynamics. In the spin-1/2 XX chain, local observables become stationary at their thermal values after dephasing, while the entanglement entropy exhibits a characteristic double-plateau structure. We obtain this behavior exactly using three complementary approaches: two-dimensional (2D) conformal field theory (CFT) on the Klein bottle, an exact numerical evolution based on the matrix Riccati equation, and an asymptotically exact quasiparticle picture. These results demonstrate that the non-monotonic entanglement evolution is a macroscopic manifestation of the coherent dephasing of anomalous pairing correlations in the initial TPQ state.

A central pillar of modern statistical mechanics is the eigenstate thermalization hypothesis (ETH), which provides a framework for understanding how isolated, chaotic quantum systems approach equilibrium [1–4]. The ETH asserts that individual energy eigenstates of chaotic systems are already locally thermal and reflect canonical typicality, meaning that a generic pure state from a microcanonical energy shell is locally indistinguishable from a thermal ensemble [5]. Consequently, the real-time evolution from a state that is already thermal-like, such as a thermal pure quantum (TPQ) state [6], is expected to display little or no dynamical change in local observables or entanglement entropy. In contrast, integrable systems, endowed with an extensive number of conservation laws, do not follow ETH and retain detailed memory of their initial conditions, relaxing instead to nontrivial stationary states described by a generalized Gibbs ensemble (GGE) [7, 8].

This raises a fundamental question: how does an integrable system evolve if it is prepared in a TPQ state? Does the dynamics retain the imprint of integrability, or reflect the thermal nature of the initial state? In this Letter, we provide an exact solution to this question and uncover a third dynamical paradigm, which is neither the trivial evolution of a chaotic system nor the standard non-thermal relaxation to a GGE. We consider a special TPQ state $|\Psi_\beta\rangle$ in the integrable spin-1/2 XX chain, with both the preparation and time evolution governed by the same Hamiltonian H.The TPQ state is constructed by imaginary-time evolving a nonlocal, maximally entangled crosscap state $|\mathcal{C}\rangle$ [9–17] as $|\Psi_\beta\rangle \propto e^{-\beta H/4}|\mathcal{C}\rangle$ [18–20].

The resulting dynamics reveal a distinctive, highly coherent route to equilibrium. Local observables reach to the thermal value after dephasing, yet the entanglement entropy displays a nonmonotonic evolution with a universal "double-plateau" structure. We establish this behavior from three complementary and exact perspectives. First, using 2D conformal field theory (CFT), we derive an analytical formula for the time-dependent entanglement entropy in terms of elliptic theta functions by evaluating vertex-operator correlators on the *Klein bottle* geometry. Second, we obtain an exact numerical benchmark by solving the governing *matrix Riccati equation* for the TPQ covariance matrix and evolving it in real time. Third, we show that the entire profile is quantitatively captured by an asymptotically exact quasiparticle picture [21–25], where the ballistic propagation of antipodally entangled quasiparticle pairs governs the coherent dephasing underlying the entropy double plateau. The perfect agreement among these approaches demonstrates that the phenomenon is exact and universal.

While it has been established that isolated integrable systems generally relax to non-thermal GGEs [26, 27], exceptions of true Gibbs-like equilibration are extremely rare and typically restricted to CFT or infinite-temperature cases [28, 29]. Our results provide an explicit lattice realization of a pure-state evolution that attains a canonical Gibbs ensemble, showing that equilibrium in integrable systems can emerge through a coherent, structured dephasing process rather than through chaos. This defines a new, exactly solvable pathway to quantum equilibration.

Thermal Pure Quantum State — In 2D CFT, a crosscap state $|\mathcal{C}\rangle$ is a boundary state [30] embodying a non-orientable spacetime, defined by the constraint $(L_n - (-1)^n \bar{L}_{-n})|\mathcal{C}\rangle = 0$ on the Virasoro generators [9]. While abstract, its physical essence is captured on a lattice of length L as an entangled antipodal pair (EAP) state [18]. For a spin-1/2 chain, this is

$$|\mathcal{C}\rangle = \bigotimes_{i=1}^{L/2} \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_i |\uparrow\rangle_{i+L/2} + |\downarrow\rangle_i |\downarrow\rangle_{i+L/2} \right) . \quad (1)$$

This construction embeds maximal, non-local entanglement into the system, profoundly distinguishing it from

short-range correlated ground states of local Hamiltonians.

A defining feature of $|\mathcal{C}\rangle$ [Eq. (1)] is its entanglement profile. The reduced density matrix ρ_A of any contiguous subsystem A of length $l \leq L/2$ is maximally mixed, $\rho_A = \frac{I_A}{2^l}$. This results in a volume-law entanglement entropy, $S_A(0) = l \log 2$, with a full "Page curve" profile $S_A(0) = \min(l, L-l) \log 2$. This property establishes the crosscap state as an infinite-temperature ($\beta = 0$) TPQ state, as it perfectly mimics a thermal Gibbs state for any local operator.

We generalize this initial condition to a finite temperature by defining a generic TPQ state $|\Psi_\beta\rangle$ via imaginary time evolution:

$$|\Psi_\beta\rangle \equiv \frac{e^{-\frac{\beta}{4}H}|\mathcal{C}\rangle}{\sqrt{\langle\mathcal{C}|\, e^{-\frac{\beta}{2}H}\,|\mathcal{C}\rangle}}\,. \tag{2}$$

This state is a deterministic, structured pure state which is, by construction, locally indistinguishable from a canonical Gibbs ensemble at inverse temperature β [19]. The quench dynamics from $|\Psi_\beta\rangle$ thus explore the evolution from a state of "fake" thermal equilibrium, allowing us to investigate how its globally stored quantum information is scrambled and redistributed over time.

The CFT Approach — The physical system under investigation is described at low energies by the $c = 1$ free compact boson CFT, which is dual to the massless Dirac fermion. We now outline the analytical calculation of the time-dependent entanglement entropy following the quench from the state $|\Psi_\beta\rangle$. The system is prepared on a Euclidean cylinder of circumference 2π and length $\beta/2$ with crosscap boundary conditions, defining the TPQ state at $t = 0$. This state then evolves in real time t under the massless Hamiltonian H. We introduce the complex coordinates $(y, \bar{y})$ for this spacetime: $y = \tau - i\sigma, \quad \bar{y} = \tau + i\sigma, \quad (0 \leq \tau \leq \frac{\beta}{2}, 0 \leq \sigma \leq 2\pi)$.

To compute the von Neumann entropy of a subsystem A, $S_A(t)$, we employ the replica trick, requiring the analytic continuation of the moments $\mathrm{Tr}[\rho_A(t)^n]$ to $n \to 1$, where $\rho_A(t) = \mathrm{Tr}_{\bar{A}}(e^{-iHt}|\Psi_\beta\rangle\langle\Psi_\beta|e^{iHt})$. The quantity $\mathrm{Tr}[\rho_A(t)^n]$ is computed as a path integral on an n-sheeted Riemann surface representing the replicated spacetime. This problem can be mapped to a system of n free Dirac fields on a single cylinder, where the replicas are coupled by twisted boundary conditions at the subsystem's endpoints [31].

In the bosonized language, these fermionic replica twists are implemented by the insertion of specific twist fields, $\mathcal{T}_k$ and $\bar{\mathcal{T}}_k$, where k labels the replica mode after a Fourier transform in the replica index. For the free boson at the self-dual radius ($R = 1$), these twist fields are primary vertex operators $V_{(\pm k,\pm k)}(y, \bar{y}) =: e^{\pm i\frac{k}{n}(\varphi(y)+\bar{\varphi}(\bar{y}))} :$. The crucial insight is that the geometry of the initial state preparation (the crosscap) effectively transforms the problem of computing the replicated partition function to that of computing a two-point function of these vertex operators on a Klein bottle $\mathcal{K}$. The total replicated partition function is then the product over all replica modes:

$$\mathrm{Tr}[\rho_A(t)^n] = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} Z_k = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle \mathcal{T}_k(y_1)\bar{\mathcal{T}}_k(y_2)\rangle_{\mathcal{K}}\,. \tag{3}$$

Here, y_1 and y_2 are the complex coordinates of the subsystem's endpoints in the spacetime describing the quench, given by $(y_1, \bar{y}_1) = (\frac{\beta}{4} + it - i\sigma_1, \frac{\beta}{4} + it + i\sigma_1)$ and $(y_2, \bar{y}_2) = (\frac{\beta}{4} + it - i\sigma_2, \frac{\beta}{4} + it + i\sigma_2)$. We denote $\sigma = \sigma_1 - \sigma_2$ being the length of the subsystem.

In Eq. (3), we need to compute the normalized two-point function of vertex operators on a Klein bottle, which is a highly nontrivial task, although similar calculations have been carried out for the conformal boundary state in Ref. [32]. For a subsystem of length σ at time t after the quench from the TPQ state prepared with parameter β, the entanglement entropy is given by [33]

$$S_A(t,\sigma) = \frac{1}{6}\log\frac{\eta(\frac{i\beta}{2\pi})^{-6}\left|\theta_1(\frac{\sigma}{2\pi}|\frac{i\beta}{2\pi})\theta_2(\frac{\beta+4it}{4\pi i}|\frac{i\beta}{2\pi})\right|^2}{\left|\theta_2(\frac{\beta+4it+2i\sigma}{4\pi i}|\frac{i\beta}{2\pi})\theta_2(\frac{\beta+4it-2i\sigma}{4\pi i}|\frac{i\beta}{2\pi})\right|}\,. \tag{4}$$

This expression involving the Dedekind eta function $\eta(\tau)$ and Jacobi theta functions $\theta_{1,2}(z|\tau)$, provides a complete, exact prediction for the entanglement dynamics, revealing the double-plateau structure and its dependence on β. Throughout this work, we adopt the conventions for the theta and eta functions as given in Ref. [34].

The Numerical Benchmark — To validate our analytical CFT predictions, we perform exact numerical simulations on the lattice realization of the theory, namely the spin-1/2 XX chain at half filling. The Hamiltonian for a system of $L = 2N$ sites with periodic boundary conditions is

$$H = -J\sum_{j=1}^{2N}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y)\,. \tag{5}$$

After a Jordan-Wigner transformation, $\sigma_j^+ = \left(\prod_{k=1}^{j-1}(-\sigma_k^z)\right)c_j^\dagger$, $\sigma_j^- = \left(\prod_{k=1}^{j-1}(-\sigma_k^z)\right)c_j$, $\sigma_j^z = 2c_j^\dagger c_j - 1$. The model is described by a Hamiltonian of non-interacting fermions, $H = -2J\sum_{j=1}^{2N}(c_j^\dagger c_{j+1} + \text{h.c.})$, with an anti-periodic boundary condition. The single-particle energy dispersion is given by $E(k) = -4J\cos(k)$. The ground state is formed by filling all negative energy states, which corresponds to half-filling ($N_f = N$). The low-energy excitations occur near the two Fermi points, $k_F = \pm\pi/2$, where the dispersion becomes linear: $E(k_F \pm q) \approx \pm(4J)q$. This linear dispersion is the defining characteristic of the massless (1+1)D Dirac fermion, with an emergent "speed of light" (Fermi velocity) of $v_F = 4J$ (in units where the lattice spacing $a = 1$).

This mapping allows us to leverage the powerful formalism of Gaussian states, as the entire quench protocol can be described exactly through the evolution of a fermionic covariance matrix Γ, which is defined in terms of Majorana fermion operators. For a system with $2N$ sites, we have $4N$ Majorana operators. The operators for site j are $\gamma_{2j-1} = c_j + c_j^\dagger$, $\gamma_{2j} = i(c_j^\dagger - c_j)$, and the elements of fermionic covariance matrix Γ is defined as $\Gamma_{mn} = \frac{i}{2}\langle[\gamma_m, \gamma_n]\rangle = i\langle\gamma_m\gamma_n\rangle - i\delta_{mn}$.

To validate our analytical CFT predictions, we perform exact numerical simulations on the spin-1/2 XX chain. The initial TPQ state $|\Psi_\beta\rangle$ is prepared by evolving the crosscap state $|\mathcal{C}\rangle$ in imaginary time, as described in Eq. (2). For the numerical implementation, we employ a fermionic representation of the system via the Jordan-Wigner transformation. It is important to note a subtle but important technical point regarding the lattice crosscap state itself. As was recently pointed out in Ref. [15], the Jordan-Wigner transformation of the strict spin crosscap state of Eq. (1) results in a superposition of two distinct fermionic Gaussian states. For simplicity and to maintain consistency with the Gaussian state formalism used in our analytical and quasiparticle pictures, our numerical simulation, following the approach in Ref. [16], targets the evolution from a Gaussian version of the crosscap state $|\mathcal{C}\rangle = \bigotimes_{j=1}^{L/2} \frac{1}{\sqrt{2}}(1 + c_j^\dagger c_{j+L/2}^\dagger)|0\rangle$. We expect that this simplification does not affect the qualitative features of the entanglement dynamics reported here, as the essential non-local entanglement structure is captured by either Gaussian component, though a quantitative investigation of the full superposition remains an interesting direction for future work.

The initial crosscap state $|\mathcal{C}\rangle$ is a fermionic Gaussian state, and its properties are fully determined by its two-point correlation matrices. The normal correlator is $C_{ij}(0) \equiv \langle\mathcal{C}|\, c_i^\dagger c_j \,|\mathcal{C}\rangle = \frac{1}{2}\delta_{ij}$, and the anomalous correlator is $F_{ij}(0) \equiv \langle\mathcal{C}|\, c_i c_j \,|\mathcal{C}\rangle = \frac{1}{2}(\delta_{j,i+N} - \delta_{i,j+N})$. It is more convenient to combine these into the $4N \times 4N$ real, anti-symmetric fermionic covariance matrix, $\Gamma_\mathcal{C}$. For the crosscap state, $\Gamma_\mathcal{C}$ is zero except for 2×2 blocks connecting antipodal sites i and $i + N$, which take the form $\Gamma_{i,i+N} = \sigma_x$.

The imaginary time evolution operator $e^{-\beta H/4}$ is a Gaussian operator, meaning the initial TPQ state $|\Psi_\beta\rangle$ is also a Gaussian state. Its covariance matrix, Γ_β, can be found by solving the governing *matrix Riccati equation* [35–37] for the imaginary time evolution [38, 39].

$$\frac{d\Gamma(\tau)}{d\tau} = -\mathcal{H} - \Gamma(\tau)\mathcal{H}\Gamma(\tau)\,. \tag{6}$$

The equation above hold if the Hamiltonian is quadratic: $H = \sum_{i,j=1}^{2N} h_{ij} c_i^\dagger c_j = \frac{i}{4}\sum_{ij} \mathcal{H}_{ij}\gamma_i\gamma_j$. By applying a trick, we can linearize this equation and solve it exactly as

$$\Gamma(\tau) = (\cos(\mathcal{H}\tau)\Gamma_0 + \sin(\mathcal{H}\tau))(\cos(\mathcal{H}\tau) - \Gamma_0 \sin(\mathcal{H}\tau))^{-1}\,, \tag{7}$$

where $\mathcal{H}$ is the $4N \times 4N$ matrix representation of the single-particle Hamiltonian in the Majorana basis. For the XX spin chain, $\mathcal{H} = \mathbf{h} \otimes (i\sigma_y)$. The covariance matrix for the TPQ state is the given by taking $\tau = \frac{\beta}{4}$ in Eq. (7), *i.e.* $\Gamma_\beta = \Gamma(\tau = \frac{\beta}{4})$ and set $\Gamma_0 = \Gamma_\mathcal{C}$.

Having obtained the exact covariance matrix Γ_β for the initial state, the real-time evolution for the quench is given by the linear equation:

$$\Gamma(t) = e^{\mathcal{H}t}\Gamma_\beta e^{-\mathcal{H}t}\,. \tag{8}$$

The entanglement entropy of a subsystem A of length l at any time t can then be computed directly from the eigenvalues $\pm i\nu_j$ of its restricted covariance matrix $\Gamma_A(t)$, which is the $2l \times 2l$ sub-block of $\Gamma(t)$ corresponding to the sites in A. The von Neumann entropy is a sum of binary entropies:

$$S_A(t) = -\sum_{j=1}^{l} \left[\frac{1+\nu_j}{2}\ln\frac{1+\nu_j}{2} + \frac{1-\nu_j}{2}\ln\frac{1-\nu_j}{2}\right]. \tag{9}$$

This procedure provides an exact, numerically efficient method to simulate the dynamics for large systems, serving as a rigorous benchmark for our analytical CFT results.

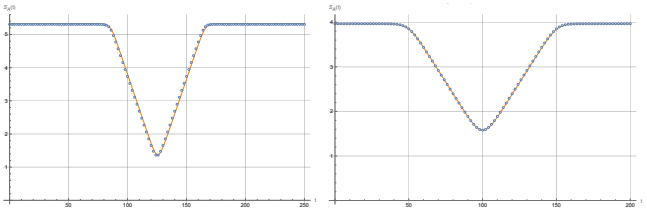

FIG. 1. Comparison between the exact numerical results and the CFT prediction for the entanglement entropy $S_A(t)$. Solid lines: numerical data; circles: CFT prediction. Left: $L = 500, l = 80, \beta = 20, v_F = 1$. Right: $L = 400, l = 100, \beta = 40, v_F = 1$.

As shown in Fig. 1, the CFT prediction (symbols) is in excellent quantitative agreement with the exact lattice calculation (solid line), accurately capturing both the double-plateau structure and the characteristic time scales after the rescaling $\beta \to 2\pi v_F\beta/L, \sigma \to 2\pi l/L, t \to 2\pi v_F t/L$ in Eq. (4).

The Quasiparticle Picture — The exact entanglement dynamics can be quantitatively understood through the quasiparticle picture [21–25]. In the standard quench paradigm from a low-entanglement state, the quench acts as a source of locally entangled quasiparticle pairs that propagate ballistically, causing entanglement to grow. The TPQ state, however, is a high-energy, extensively

entangled state whose dynamics require a fundamentally different interpretation. Here, the dominant process is not the creation of entanglement, but the transport and eventual destruction of the pre-existing, non-local entanglement.

The initial state $|\Psi_\beta\rangle$ can be viewed as a sea of quasiparticle pairs with opposite momenta $(k, -k)$. Crucially, the entanglement is non-local: a quasiparticle with momentum k at site x is entangled not with its partner at x, but with the particle of momentum $-k$ at the antipodal site $x + L/2$. After the quench at $t = 0$, these pairs propagate ballistically. The entanglement entropy of a subsystem A decreases when both members of an entangled pair, which were previously shared between A and its complement $\bar{A}$, become fully contained within A. This process leads to the following prediction for the entanglement dynamics:

$$S_A(t) = S_A(0) - 2\int_{-\pi}^{\pi} \frac{dk}{2\pi} s(k) \max\left(0, l - |v(k)\tau_k - L/2|\right) , \tag{10}$$

which is a straightforward generalization of the formula proposed in Ref. [16]. Let's explain this formula term-by-term. The first term, $S_A(0) = l\int_{-\pi}^{\pi} \frac{dk}{2\pi} s(k)$, is the initial entanglement entropy of the subsystem of length l. The second term is a subtractive correction representing the entropy removed by the quasiparticles. The factor of 2 accounts for the two quasiparticles in a pair. The counting function, $\max(0, \dots)$, is the heart of the picture: it counts the number of antipodal pairs with momentum k that are both inside the subsystem A at time t. The term $|v(k)\tau_k - L/2|$ is the effective separation of the two members of an antipodal pair, which starts at $L/2$ and decreases as they travel towards each other. Here, $v(k)$ is the velocity of a quasiparticle with momentum k, For the XX chain at half filling, $v(k) = 4J\sin(k)$ and $\tau_k = t \pmod{L/|v(k)|}$ is a periodic time variable that accounts for revivals in the finite system.

The function $s(k)$ is the effective entanglement density carried by each quasiparticle with momentum k. As proved in the appendix, the initial state is a BCS-like state in the thermodynamic limit

$$|\Psi_\beta\rangle = \prod_{k>0} \left(u_k(\beta) + v_k(\beta) c_k^\dagger c_{-k}^\dagger\right) |0\rangle \tag{11}$$

with Bogoliubov coefficients

$$|u_k(\beta)|^2 = \frac{1}{1 + e^{-\beta E(k)}}, \qquad |v_k(\beta)|^2 = \frac{e^{-\beta E(k)}}{1 + e^{-\beta E(k)}} . \tag{12}$$

This allows for a direct calculation of the entanglement entropy per mode, which is given by the binary entropy of the mode occupation:

$$s(k) = -n_k \ln n_k - (1 - n_k)\ln(1 - n_k) , \tag{13}$$

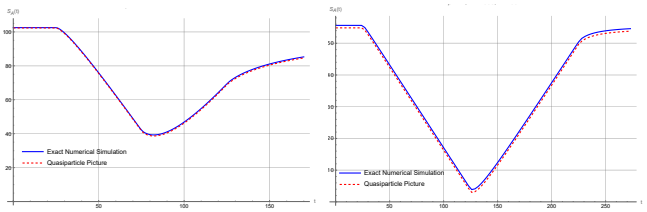

FIG. 2. Comparison between the exact numerical results and the quasiparticle prediction for the entanglement entropy $S_A(t)$. Solid lines: numerical data; dashed line: quasi-particle prediction. Left: $L = 600, l = 200, \beta = 1, J = 0.5$. Right: $L = 1000, l = 400, \beta = 4, J = 0.5$.

where the occupation n_k is given by the Fermi-Dirac distribution at the inverse temperature β of the initial state:

$$n_k \equiv \langle\Psi_\beta| c_k^\dagger c_k |\Psi_\beta\rangle = \frac{1}{e^{\beta E(k)} + 1} . \tag{14}$$

The formula (10) provides a complete quantitative picture that accurately reproduces our exact analytical and numerical results. The initial plateau is the time before any antipodal pairs can fully enter the subsystem. The subsequent decrease is governed by the rate at which pairs enter, weighted by their entanglement density $s(k)$. The rise and second plateau are due to the same pairs exiting the subsystem as they traverse outside the subsystem. This quasiparticle picture thus provides a powerful physical interpretation for the exact thermalization observed in the system. Figure 2 shows a comparison between the exact numerical results and the quasi-particle prediction for the entanglement entropy.

Mechanism of Coherent Equilibration — We now elucidate the microscopic mechanism underlying the observed equilibration. The process is not driven by chaos but by the dephasing of the initial state's quantum coherences, which can be described exactly in our model. After dephasing, the stationary state coincides with a GGE that, for this particular initial condition, reduces identically to the canonical Gibbs ensemble.

A defining feature of the TPQ state $|\Psi_\beta\rangle$ is its nonzero anomalous pairing amplitude,

$$\phi_k(\beta) = \langle c_k c_{-k}\rangle = \frac{-i\sin(kN)}{2\cosh[E(k)\beta/2]} , \tag{15}$$

which signals quantum coherence between sectors of different fermion number. In the thermal Gibbs ensemble, such coherence vanishes identically. To understand how this coherence disappears, we analyze the BCS representation of the TPQ state in the thermodynamic limit,

$$|\Psi_\beta(t)\rangle = \prod_{k>0} \left[u_k(\beta) + v_k(\beta) e^{-2iE(k)t} c_k^\dagger c_{-k}^\dagger\right] |0\rangle . \tag{16}$$

The occupations $n_k = \langle c_k^\dagger c_k\rangle$ remain constant, while the pairing amplitude oscillates as $\phi_k(t, \beta) =$

$\phi_k(0,\beta)e^{-i2E(k)t}$. Hence, observables sensitive to this coherence, such as $\langle c_j c_{j+1}\rangle$, display explicit temporal oscillations,

$$\langle c_j c_{j+1}\rangle_t = \frac{1}{L}\sum_k e^{-ik(2j+1)}\phi_k(0)e^{-2iE(k)t}, \tag{17}$$

showing that the initial state is not stationary.

At long times, these oscillations dephase, and the time average satisfies $\overline{\phi_k(t)} = 0$ for all $E(k) \neq 0$. Consequently, $\overline{\langle c_j c_{j+1}\rangle_t} = 0$, exactly matching the thermal expectation value $\mathrm{Tr}(\rho_{\mathrm{th}} c_j c_{j+1}) = 0$. In contrast, number-conserving observables such as $\langle c_j^\dagger c_{j+1}\rangle$ remain stationary and coincide with their thermal values at all times. Thus, all local observables approach the canonical values once the coherence encoded in ϕ_k has dephased. The entanglement entropy provides the macroscopic witness of this microscopic process. The nonmonotonic "double-plateau" profile directly tracks the coherent loss and reorganization of pairing correlations.

After dephasing, the remaining stationary state is fully characterized by the conserved occupations n_k. The corresponding GGE density matrix reads

$$\rho_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}}\exp\left[-\sum_k \lambda_k \hat{n}_k\right], \tag{18}$$

with Lagrange multipliers fixed by $\mathrm{Tr}(\rho_{\mathrm{GGE}}\hat{n}_k) = n_k = (e^{\beta E(k)}+1)^{-1}$, which yields $\lambda_k = \beta E(k)$. Substituting this into ρ_{GGE} gives

$$\rho_{\mathrm{st}} = \frac{e^{-\beta H}}{Z}. \tag{19}$$

Therefore, equilibration proceeds in two coherent stages: (i) the unitary dynamics dephase the nonconserved pairing coherences, and (ii) the remaining conserved populations define a GGE that, for this special state, coincides exactly with the Gibbs ensemble. This dephasing-based mechanism provides an explicit, solvable realization of thermalization without chaos.

Conclusion and outlooks — In this work, we study the entanglement evolution following a quench from a TPQ state and uncover a distinctive, coherent pathway to equilibrium. We provide a complete theoretical account of this process and show that the stationary state coincides with the canonical Gibbs ensemble. This apparent thermal behavior arises because the special symmetries of the initial TPQ state enforce all higher conserved charges that would otherwise lead to a GGE to vanish identically. The resulting nontrivial dynamics are the macroscopic signature of an underlying microscopic mechanism. The initial state possesses nonzero anomalous pairing correlations, $\langle c_k c_{-k}\rangle \neq 0$, representing quantum coherence forbidden in the final thermal state. The observed "double plateau" is the real-time fingerprint of the coherent dephasing of these pairing correlations, a process driven by the ballistic propagation of stable quasiparticles. This highlights a central finding of our work that a structured, nonmonotonic entanglement evolution can emerge as the macroscopic manifestation of microscopic quantum coherence during equilibration. Conceptually, this connects our results to other observations of intermediate-time coherent dynamics in the XX chain, such as the emergence and disappearance of short-range coherence in quenches leading to an infinite-temperature state [40]. While the initial states and final ensembles differ, both reveal that the route to equilibrium in integrable systems can be remarkably rich and structured.

Our results pave the way for exploring nonequilibrium dynamics from highly structured initial states and for extending the CFT quench framework to non-orientable manifolds. The CFT analysis and exact Gaussian numerics apply directly to mutual information and entanglement negativity, while the quasiparticle picture can be adapted with minor modifications. Exploring interacting integrable models in this context would be both interesting and important for future work. This perspective can be generalized to richer scenarios, including inhomogeneous quenches such as the Möbius or sine-square deformation (SSD) [41–46]. The experimental exploration of crosscap-type initial states and the associated entanglement dynamics would be a fascinating direction for future studies, potentially realizable in quantum-simulation platforms such as ultracold atoms or superconducting qubits.

Acknowledgments — We thank Pasquale Calabrese and Konstantinos Chalas for very helpful discussions. This work was supported by the National Natural Science Foundation of China (Grant Nos.12005081, and 12465014), and Program of China Scholarship Council (Grant No. 202308360098).

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Supplemental Material for "Exact Quench Dynamics from Thermal Pure Quantum States"

CONTENTS

The CFT approach to the quench dynamics from TPQ states

In this section, we give the full derivation of formula (4) in the main text.

Crosscap state in free compact boson CFT

We work in the bosonized description of the free massless Dirac fermion, setting $\alpha' = 2$ in string theory conventions. We consider a free scalar field $\phi(y, \bar{y}) = \varphi(y) + \bar{\varphi}(\bar{y})$ on the cylinder with circumference 2π. The field is compactified on a circle of radius R, satisfying

$$\phi(\sigma + 2\pi, t) = \phi(\sigma, t) + 2\pi R w \,, \tag{S1}$$

where $w \in \mathbb{Z}$ is the winding number. We can expand holomorphic part and anti-holomorphic part of the field as

$$\varphi(y) = \varphi_0 - i\alpha_0 y + i \sum_{m\neq 0} \frac{\alpha_m}{m} e^{-my}, \quad \bar{\varphi}(\bar{y}) = \bar{\varphi}_0 - i\bar{\alpha}_0 \bar{y} + i \sum_{m\neq 0} \frac{\bar{\alpha}_m}{m} e^{-m\bar{y}} \,, \tag{S2}$$

where we have defined

$$\alpha_0 = \frac{n}{R} + \frac{wR}{2}, \qquad \bar{\alpha}_0 = \frac{n}{R} - \frac{wR}{2} \tag{S3}$$

with commutation relations:

$$[\varphi_0, \alpha_0] = i \,, \quad [\bar{\varphi}_0, \bar{\alpha}_0] = i \,, \quad [\alpha_m, \alpha_n] = m\delta_{n+m,0} \,, \quad [\bar{\alpha}_m, \bar{\alpha}_n] = m\delta_{n+m,0} \,. \tag{S4}$$

The rest of the commutators vanish.

The Hamiltonian is $H = L_0 + \bar{L}_0 - \frac{1}{12}$, with

$$L_0 = \frac{1}{2}\big(\frac{n}{R} + \frac{wR}{2}\big)^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \,, \qquad \bar{L}_0 = \frac{1}{2}\big(\frac{n}{R} - \frac{wR}{2}\big)^2 + \sum_{n=1}^{\infty} \bar{\alpha}_{-n}\bar{\alpha}_n \,. \tag{S5}$$

The vacuum state is labeled by the winding number w and the momentum integer n and is denoted by $|n,w\rangle$.

The crosscap state satisfy

$$(\alpha_n - (-1)^n \bar{\alpha}_{-n})\,|\mathcal{C}\rangle = 0\,. \tag{S6}$$

This constraint can be solved as

$$|\mathcal{C}\rangle = \mathcal{N} e^{\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \alpha_{-n}\bar{\alpha}_{-n}} \sum_{n=-\infty}^{\infty} |n,0\rangle\ , \tag{S7}$$

where $\mathcal{N}$ is a normalization factor.

Although the subsequent calculations largely parallel those in [32], subtle differences arise at certain points, ultimately resulting in a qualitatively distinct behavior of the entanglement entropy.

The two-point functions

The unnormalized two-point function of the vertex operators $V_{k,\bar{k}} =: e^{ik\varphi + i\bar{k}\bar{\varphi}} :$ on the cylinder is

$$\langle\mathcal{C}|\, e^{-\beta H/2} V_{k,\bar{k}}(y_1,\bar{y}_1) V_{-k,-\bar{k}}(y_2,\bar{y}_2)\,|\mathcal{C}\rangle\ . \tag{S8}$$

Using the Baker-Campbell-Hausdorff (BCH) formula, $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$, the zero-mode part of the correlator above evaluates to

$$\sum_{n=-\infty}^{\infty} e^{-\frac{n^2\beta}{2R^2}} e^{\frac{n}{R}\left(k(y_1-y_2)+\bar{k}(\bar{y}_1-\bar{y}_2)\right)} e^{\frac{k^2}{2}(y_2-y_1)+\frac{\bar{k}^2}{2}(\bar{y}_2-\bar{y}_1)}\,. \tag{S9}$$

To deal with the massive modes, let us first focus on a single mode, say the n-th massive mode. Notice that $[\alpha_n,\alpha_{-n}] = n$, which means if we define $\hat{\alpha} \equiv \frac{e^{\frac{\pi i n}{2}}\alpha_n}{\sqrt{n}}$ and $\hat{\alpha}^\dagger \equiv \frac{e^{-\frac{\pi i n}{2}}\alpha_{-n}}{\sqrt{n}}$, then $\hat{\alpha}$ and $\hat{\alpha}^\dagger$ are the standard harmonic creation and anihation operators, satisfy $[\hat{\alpha},\hat{\alpha}^\dagger] = 1$. Similarly, for the anti-holomorphic part, we define $\hat{\beta} \equiv \frac{e^{\frac{\pi i n}{2}}\bar{\alpha}_n}{\sqrt{n}}$ and $\hat{\beta}^\dagger \equiv \frac{e^{-\frac{\pi i n}{2}}\bar{\alpha}_{-n}}{\sqrt{n}}$, which also satisfy $[\hat{\beta},\hat{\beta}^\dagger] = 1$. These oscillators are contained in

$$\begin{aligned} ik\varphi(y_i):\ & -\frac{k e^{-\frac{\pi i n}{2}}}{\sqrt{n}} e^{-n y_i}\hat{\alpha} + \frac{k e^{\frac{\pi i n}{2}}}{\sqrt{n}} e^{n y_i}\hat{\alpha}^\dagger, \qquad \text{with } n>0 \\ i\bar{k}\bar{\varphi}(y_i):\ & -\frac{\bar{k} e^{-\frac{\pi i n}{2}}}{\sqrt{n}} e^{-n \bar{y}_i}\hat{\beta} + \frac{\bar{k} e^{\frac{\pi i n}{2}}}{\sqrt{n}} e^{n \bar{y}_i}\hat{\beta}^\dagger, \qquad \text{with } n>0\,. \end{aligned} \tag{S10}$$

Then $V_{k,\bar{k}}(y_1,\bar{y}_1)V_{-k,-\bar{k}}(y_2,\bar{y}_2)$ contain $e^{a\hat{\alpha}+\bar{a}\hat{\beta}} e^{b\hat{\alpha}^\dagger+\bar{b}\hat{\beta}^\dagger}$, where

$$\begin{aligned} a &= -\frac{e^{-\frac{\pi i n}{2}}k}{\sqrt{n}}\left(e^{-n y_1} - e^{-n y_2}\right), \quad \bar{a} = -\frac{e^{-\frac{\pi i n}{2}}\bar{k}}{\sqrt{n}}\left(e^{-n \bar{y}_1} - e^{-n \bar{y}_2}\right), \\ b &= \frac{e^{\frac{\pi i n}{2}}k}{\sqrt{n}}\left(e^{n y_1} - e^{n y_2}\right), \quad \bar{b} = \frac{e^{\frac{\pi i n}{2}}\bar{k}}{\sqrt{n}}\left(e^{n \bar{y}_1} - e^{n \bar{y}_2}\right)\,. \end{aligned} \tag{S11}$$

In the definition of the crosscap state [Eq. (S7)], the factor is $e^{\hat{\alpha}^\dagger\hat{\beta}^\dagger}$. They also appear in the left most of Eq. (S8): $\langle 0|\, e^{\hat{\alpha}\hat{\beta}} e^{-\frac{n\beta}{2}(\hat{\alpha}^\dagger\hat{\alpha}+\hat{\beta}^\dagger\hat{\beta})} = \langle 0|\, e^{z\hat{\alpha}\hat{\beta}}$, with $z = e^{-n\beta}$, where we have used the BCH formula $e^X e^Y = \exp(e^{ad_X}Y)e^X$. Therefore, we need to compute $\langle 0|\, e^{\hat{\alpha}\hat{\beta}z} e^{a\hat{\alpha}+\bar{\alpha}\hat{\beta}} e^{b\hat{\alpha}^\dagger+\bar{b}\hat{\beta}^\dagger} e^{\hat{\alpha}^\dagger\hat{\beta}^\dagger}\,|0\rangle$, and the finial results is

$$\langle 0|\, e^{\hat{\alpha}\hat{\beta}z} e^{a\hat{\alpha}+\bar{\alpha}\hat{\beta}} e^{b\hat{\alpha}^\dagger+\bar{b}\hat{\beta}^\dagger} e^{\hat{\alpha}^\dagger\hat{\beta}^\dagger}\,|0\rangle = \frac{1}{1-z} e^{\frac{ab+\bar{a}\bar{b}+a\bar{a}+zb\bar{b}}{1-z}}\,. \tag{S12}$$

This identity can be proved in several ways, and we present one proof at the end of this section.

We should sum over all the massive modes as $\prod_{n=1}^{\infty}\frac{1}{1-z}\prod_{m=0}^{\infty}\exp(z^m(ab+\bar{a}\bar{b}+a\bar{a}+z b\bar{b}))$. Here we have used the identity $\frac{1}{1-z}=\sum_{m=0}^{\infty}z^m$. The constant term $-\frac{1}{12}$ in the Hamiltonian and the term $\prod_{n=1}^{\infty}\frac{1}{1-z}$ is direct related to Dedekind η-function

$$e^{\frac{1}{12}\frac{\beta}{2}}\prod_{n=1}^{\infty}\frac{1}{1-z}=\frac{1}{\eta\left(\frac{i\beta}{2\pi}\right)}\,. \tag{S13}$$

After regularizing the divergent $m=0$ term contribution, the contribution from the $z^m ab$ term in the exponent gives the following result

$$\prod_{n=1}^{\infty}\prod_{m=0}^{\infty}\exp(z^m ab)=\prod_{m=0}^{\infty}\exp\left(\sum_{n=1}^{\infty}\frac{2k^2e^{-mn\beta}}{n}(\cosh(ny_2-ny_1)-1)\right)=\left(\frac{\eta\left(\frac{i\beta}{2\pi}\right)^3}{\theta_1\left(\frac{y_2-y_1}{2\pi i}|\frac{i\beta}{2\pi}\right)}\right)^{k^2}\,, \tag{S14}$$

where we have used the formula $\sum_{n=1}^{\infty}\frac{x^n}{n}=-\log(1-x)$ and the definition of eta and theta functions [34]. The other terms can be computed in a similar way:

$$\prod_{n=1}^{\infty}\prod_{m=0}^{\infty}\exp(z^m(a\bar{a}+z b\bar{b}))=\left(\frac{\theta_2\left(\frac{y_1+\bar{y}_1}{2\pi i}|\frac{i\beta}{2\pi}\right)\theta_2\left(\frac{y_2+\bar{y}_2}{2\pi i}|\frac{i\beta}{2\pi}\right)}{\theta_2\left(\frac{y_1+\bar{y}_2}{2\pi i}|\frac{i\beta}{2\pi}\right)\theta_2\left(\frac{y_2+\bar{y}_1}{2\pi i}|\frac{i\beta}{2\pi}\right)}\right)^{-k\bar{k}}\,. \tag{S15}$$

Combining all these terms, we get

$$\begin{aligned}
&\langle\mathcal{C}|e^{-\frac{\beta}{2}H}V_{(k,\bar{k})}(y_1,\bar{y}_1)V_{(-k,-\bar{k})}(y_2,\bar{y}_2)|\mathcal{C}\rangle\\
&=\mathcal{N}^2\left[\sum_{n=-\infty}^{\infty}e^{-\frac{n^2\beta}{2R^2}}e^{\frac{n}{R}\left(k(y_1-y_2)+\bar{k}(\bar{y}_1-\bar{y}_2)\right)}e^{\frac{k^2}{2}(y_2-y_1)+\frac{\bar{k}^2}{2}(\bar{y}_2-\bar{y}_1)}\right]\cdot\frac{1}{\eta\left(\frac{i\beta}{2\pi}\right)}\\
&\left(\frac{\eta\left(\frac{i\beta}{2\pi}\right)^3}{\theta_1\left(\frac{y_2-y_1}{2\pi i}|\frac{i\beta}{2\pi}\right)}\right)^{k^2}\left(\frac{\eta\left(\frac{i\beta}{2\pi}\right)^3}{\theta_1\left(\frac{\bar{y}_2-\bar{y}_1}{2\pi i}|\frac{i\beta}{2\pi}\right)}\right)^{\bar{k}^2}\cdot\left(\frac{\theta_2\left(\frac{y_1+\bar{y}_1}{2\pi i}|\frac{i\beta}{2\pi}\right)\theta_2\left(\frac{y_2+\bar{y}_2}{2\pi i}|\frac{i\beta}{2\pi}\right)}{\theta_2\left(\frac{y_1+\bar{y}_2}{2\pi i}|\frac{i\beta}{2\pi}\right)\theta_2\left(\frac{y_2+\bar{y}_1}{2\pi i}|\frac{i\beta}{2\pi}\right)}\right)^{-k\bar{k}}\,.
\end{aligned} \tag{S16}$$

After substituting the explicit values of the coordinates $(y_1,\bar{y}_1)=(\frac{\beta}{4}+it-i\sigma_1,\frac{\beta}{4}+it+i\sigma_1)$ and $(y_2,\bar{y}_2)=(\frac{\beta}{4}+it-i\sigma_2,\frac{\beta}{4}+it+i\sigma_2)$ and taking into account that twist fields at $R=1$ having $k=\bar{k}$, the second line of the equation above simplifies as

$$\frac{\sum_{n=-\infty}^{\infty}e^{-\frac{n^2\beta}{2}}}{\eta\left(\frac{i\beta}{2\pi}\right)}=\frac{\theta(0|\frac{i\beta}{2\pi})}{\eta\left(\frac{i\beta}{2\pi}\right)}\,. \tag{S17}$$

This is exactly equals to $\langle\mathcal{C}|\,e^{-\frac{\beta}{2}H}\,|\mathcal{C}\rangle$, which can be interpreted as the partition function on the Klein bottle.

The entanglement entropy

To calculate the von Neumann entanglement entropy of a subsystem A, $S_A(t)$, we employ the replica trick. This requires first computing $\mathrm{Tr}(\rho_A(t)^N)$ and then performing an analytic continuation in the replica index $N\to 1$:

$$S_A(t)=-\frac{\partial}{\partial N}\log\left(\mathrm{Tr}[\rho_A(t)^N]\right)\Big|_{N=1}\,. \tag{S18}$$

The quantity $\mathrm{Tr}[\rho_A(t)^N]$ is computed as a partition function on an N-sheeted Riemann surface. This problem can be mapped to a system of N free Dirac fermion fields, $\{\psi^{(a)}\}_{a=1}^{N}$, on a single cylinder, where the interaction between replicas is encoded in twisted boundary conditions at the endpoints of the subsystem A. For example, at endpoint y_1, we have

$$\psi^{(a)}(e^{2\pi i}y_1)=\psi^{(a+1)}(y_1)\,. \tag{S19}$$

This system of coupled fields can be diagonalized by a discrete Fourier transform in the replica index a, resulting in N decoupled fermion theories, each with a diagonal phase twist at the boundaries. For the k-th mode, the boundary condition becomes

$$\psi_k(e^{2\pi i}y_1) = e^{2\pi ik/N}\psi_k(y_1)\,. \tag{S20}$$

A (1+1)D free massless Dirac fermion, with left- and right-moving components $\psi, \bar{\psi}$, is equivalent to a free scalar boson ϕ compactified on a circle of radius $R = 1$. The fermionic fields are represented as exponentials of the bosonic field:

$$\psi(y) = e^{i\varphi(y)}, \quad \bar{\psi}(\bar{y}) = e^{i\bar{\varphi}(\bar{y})}\,, \tag{S21}$$

where $\phi(y, \bar{y}) = \varphi(y) + \bar{\varphi}(\bar{y})$.

A twisted boundary condition for a fermion, such as $\psi_k(e^{2\pi i}y_1) = e^{2\pi ik/N}\psi_k(y_1)$, is equivalent to the insertion of a specific local operator, a twist field, at that point. In the bosonized theory, the twist fields that implement the fermionic replica twists are vertex operators. For the k-th decoupled theory, the boundary condition is created by inserting a pair of twisted-sector vertex operators, $\mathcal{T}_k$ and $\bar{\mathcal{T}}_k$, at the endpoints of the interval, y_1 and y_2. These vertex operators are primary fields of the bosonic CFT. For the Dirac fermion, they are

$$\mathcal{T}_k(y, \bar{y}) = V_{(\frac{k}{N}, \frac{k}{N})}(y, \bar{y}) =: e^{i\frac{k}{N}(\varphi(y)+\bar{\varphi}(\bar{y}))} : \tag{S22}$$

$$\bar{\mathcal{T}}_k(y, \bar{y}) = V_{(-\frac{k}{N}, -\frac{k}{N})}(y, \bar{y}) =: e^{-i\frac{k}{N}(\varphi(y)+\bar{\varphi}(\bar{y}))} :, \tag{S23}$$

where $V_{(k,\bar{k})}$ is the general vertex operator with left and right charges $(k, \bar{k})$.

With this mapping, the problem of calculating the partition function for the k-th replicated fermion is transformed into calculating a two-point function of bosonic vertex operators on a Klein bottle

$$Z_k = \langle \mathcal{T}_k(y_1, \bar{y}_1)\bar{\mathcal{T}}_k(y_2, \bar{y}_2)\rangle_{\mathcal{K}}\,. \tag{S24}$$

The total replicated partition function is then the product over all modes k

$$\mathrm{Tr}[\rho_A(t)^N] = \prod_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} Z_k = \prod_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \frac{\langle \mathcal{C}|\, e^{-\frac{\beta}{2}H}\mathcal{T}_k(y_1, \bar{y}_1)\bar{\mathcal{T}}_k(y_2, \bar{y}_2)\,|\mathcal{C}\rangle}{\langle \mathcal{C}|\, e^{-\frac{\beta}{2}H}\,|\mathcal{C}\rangle}\,. \tag{S25}$$

The crucial ingredient for this calculation is the explicit form of the normalized two-point function on the Klein bottle. This has already been done in the previous section (cf. Eq. (S16) and Eq. (S17)), and the result is

$$\langle \mathcal{T}_k(y_1, \bar{y}_1)\bar{\mathcal{T}}_k(y_2, \bar{y}_2)\rangle_{\mathcal{K}} = \left(\frac{\eta(\frac{i\beta}{2\pi})^6|\theta_2(\frac{\beta+4it}{4\pi i} + \frac{\sigma}{2\pi}|\frac{i\beta}{2\pi})||\theta_2(\frac{\beta+4it}{4\pi i} - \frac{\sigma}{2\pi}|\frac{i\beta}{2\pi})|}{|\theta_1(\frac{\sigma}{2\pi}|\frac{i\beta}{2\pi})|^2|\theta_2(\frac{\beta+4it}{4\pi i}|\frac{i\beta}{2\pi})|^2}\right)^{\frac{k^2}{N^2}}\,. \tag{S26}$$

From Eq. (S26) and using Eq. (S18), it's straightforward to obtain the entanglement entropy as

$$S_A(t, \sigma) = \frac{1}{6}\log\frac{|\theta_1(\frac{\sigma}{2\pi}|\frac{i\beta}{2\pi})|^2|\theta_2(\frac{\beta+4it}{4\pi i}|\frac{i\beta}{2\pi})|^2}{\eta(\frac{i\beta}{2\pi})^6|\theta_2(\frac{\beta+4it}{4\pi i} + \frac{\sigma}{2\pi}|\frac{i\beta}{2\pi})||\theta_2(\frac{\beta+4it}{4\pi i} - \frac{\sigma}{2\pi}|\frac{i\beta}{2\pi})|}\,. \tag{S27}$$

Proof of the identity S12

Since the identity (S12) is extremely crucial in our derivation of the twist operator correlator on the Klein bottle, here we provide its proof. We want to prove the following identity for bosonic creation and annihilation operators satisfying $[\hat{\alpha}, \hat{\alpha}^\dagger] = 1$ and $[\hat{\beta}, \hat{\beta}^\dagger] = 1$:

$$I \equiv \langle 0|e^{\hat{\alpha}\hat{\beta}z}e^{a\hat{\alpha}+\bar{a}\hat{\beta}}e^{b\hat{\alpha}^\dagger+\bar{b}\hat{\beta}^\dagger}e^{\hat{\alpha}^\dagger\hat{\beta}^\dagger}|0\rangle = \frac{1}{1-z}\exp\left[\frac{ab + \bar{a}\bar{b} + a\bar{a} + zb\bar{b}}{1-z}\right]\,. \tag{S28}$$

We can use the two-mode *coherent state* $|\alpha, \beta\rangle$ satisfying $\hat{\alpha}\,|\alpha, \beta\rangle = \alpha\,|\alpha, \beta\rangle\,, \hat{\beta}\,|\alpha, \beta\rangle = \beta\,|\alpha, \beta\rangle$ to transform the left hand side of the identity above to an integral by inserting a completeness relation for two-mode coherent states,

$1 = \int \frac{d^2\alpha\, d^2\beta}{\pi^2} |\alpha,\beta\rangle\langle\alpha,\beta|$, into the middle of the operator string. This converts the operator expectation value into a Gaussian integral over c-number variables α and β.

We insert the completeness relation to split the expression into two distinct matrix elements

$$I = \int \frac{d^2\alpha\, d^2\beta}{\pi^2} \langle 0|e^{\hat{\alpha}\hat{\beta}z} e^{a\hat{\alpha}+\bar{a}\hat{\beta}}|\alpha,\beta\rangle\langle\alpha,\beta|e^{b\hat{\alpha}^\dagger+\bar{b}\hat{\beta}^\dagger} e^{\hat{\alpha}^\dagger\hat{\beta}^\dagger}|0\rangle\,. \tag{S29}$$

We now evaluate the two matrix elements in the integrand separately. The first one is

$$M_L = \langle 0|e^{\hat{\alpha}\hat{\beta}z+a\hat{\alpha}+\bar{a}\hat{\beta}}|\alpha,\beta\rangle = e^{z\alpha\beta+a\alpha+\bar{a}\beta}\langle 0|\alpha,\beta\rangle\,. \tag{S30}$$

Using the standard overlap formula of coherent state $\langle 0|\alpha,\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}$, we get

$$M_L = e^{z\alpha\beta+a\alpha+\bar{a}\beta} e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\,. \tag{S31}$$

Similarly, for the second matrix element, we have

$$M_R = \langle\alpha,\beta|e^{b\hat{\alpha}^\dagger+\bar{b}\hat{\beta}^\dagger+\hat{\alpha}^\dagger\hat{\beta}^\dagger}|0\rangle = e^{b\alpha^*+\bar{b}\beta^*+\alpha^*\beta^*}\langle\alpha,\beta|0\rangle\,. \tag{S32}$$

Using the overlap formula $\langle\alpha,\beta|0\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}$, we obtain

$$M_R = e^{b\alpha^*+\bar{b}\beta^*+\alpha^*\beta^*} e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\,. \tag{S33}$$

We now substitute M_L and M_R back into the integral for I, obtaining

$$I = \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} \exp\left[-|\alpha|^2 - |\beta|^2 + z\alpha\beta + \alpha^*\beta^* + a\alpha + \bar{a}\beta + b\alpha^* + \bar{b}\beta^*\right]\,. \tag{S34}$$

We can solve this by iterated integration, first over α and then over β.

Integrating over α To integrate over α, we collect all terms in the exponent involving α, α^* and complete the square as

$$-\alpha^*\alpha + (a+z\beta)\alpha + (b+\beta^*)\alpha^* = -(\alpha - (b+\beta^*))(\alpha^* - (a+z\beta)) + (a+z\beta)(b+\beta^*)\,. \tag{S35}$$

Since $\int \frac{d^2\alpha}{\pi} e^{-(\alpha-J)(\alpha^*-K)} = 1$, integrating over α leaves the term $\exp\left[(a+z\beta)(b+\beta^*)\right]$.

Integrating over β We combine the result from the α-integration with the remaining β-dependent terms. The exponent for the β integral is

$$(a+z\beta)(b+\beta^*) - |\beta|^2 + \bar{a}\beta + \bar{b}\beta^* = ab - (1-z)|\beta|^2 + (\bar{a}+zb)\beta + (a+\bar{b})\beta^*\,. \tag{S36}$$

The remaining integral is

$$I = e^{ab}\int \frac{d^2\beta}{\pi} \exp\left[-(1-z)|\beta|^2 + (\bar{a}+zb)\beta + (a+\bar{b})\beta^*\right]\,. \tag{S37}$$

Using the Gaussian integral formula $\int \frac{d^2\zeta}{\pi} e^{-A|\zeta|^2+J\zeta+K\zeta^*} = \frac{1}{A}e^{JK/A}$, with $A = 1-z$, $J = \bar{a}+zb$, and $K = a+\bar{b}$, we obtain

$$I = e^{ab}\cdot\frac{1}{1-z}\exp\left[\frac{(\bar{a}+zb)(a+\bar{b})}{1-z}\right] = \frac{1}{1-z}\exp\left[\frac{ab+\bar{a}\bar{b}+a\bar{a}+zb\bar{b}}{1-z}\right]\,. \tag{S38}$$

This result perfectly matches the right-hand side of the identity we set out to prove.

Imaginary-time v.s. real-time evolution of the fermionic covariance matrix

In this section we give a complete derivation of the fermionic covariance matrix under imaginary-time and real-time evolution.

The imaginary-time evolution

Consider an arbitrary Gaussian state $\rho(0)$ and evolve it according to

$$\rho(\tau) = \frac{e^{-H\tau}\rho(0)e^{-H\tau}}{\mathrm{Tr}(e^{-2H\tau}\rho(0))}\,. \tag{S39}$$

From the equation above, we can obtain the equation of motion for $\rho(\tau)$ for any Hamiltonian H as

$$\frac{d\rho}{d\tau} = -\{H,\rho(\tau)\} + 2\rho(\tau)\mathrm{Tr}(\rho(\tau)H)\,, \tag{S40}$$

where $\{A,B\} = AB + BA$ is the anti-commutator.

Then the evolution equation for the covariance matrix Γ is

$$\frac{d\Gamma_{mn}(\tau)}{d\tau} = -\frac{i}{2}\mathrm{Tr}\left(\rho(\tau)\{[\gamma_m,\gamma_n],H\}\right) + 2\Gamma_{mn}(\tau)\mathrm{Tr}(\rho(\tau)H)\,. \tag{S41}$$

We need to evaluate the expectation value of the operator $\{[\gamma_m,\gamma_n],H\}$. Let's expand it as

$$\{[\gamma_m,\gamma_n],H\} = (\gamma_m\gamma_n - \gamma_n\gamma_m)H + H(\gamma_m\gamma_n - \gamma_n\gamma_m)\,. \tag{S42}$$

We can calculate the expectation value of each of the four terms (e.g., $\langle\gamma_m\gamma_n H\rangle$) separately. Let's focus on the first term and substitute the Hamiltonian $H = \frac{i}{4}\sum_{k,l}\mathcal{H}_{kl}\gamma_k\gamma_l$, obtaining

$$\langle\gamma_m\gamma_n H\rangle = \frac{i}{4}\sum_{k,l}\mathcal{H}_{kl}\langle\gamma_m\gamma_n\gamma_k\gamma_l\rangle\,. \tag{S43}$$

This quantity is the expectation value of four Majorana operators. For any Gaussian state, it can be reduced to two-point correlators $M_{ab} \equiv \langle\gamma_a\gamma_b\rangle$ by Wick's theorem. The two-point functions are directly related to the covariance matrix as

$$\langle\gamma_a\gamma_b\rangle = -i\Gamma_{ab} + \delta_{ab}\,, \tag{S44}$$

or equivalently, $M = -i\Gamma + I$. Now, we apply Wick's theorem to $\langle\gamma_m\gamma_n\gamma_k\gamma_l\rangle$

$$\langle\gamma_m\gamma_n\gamma_k\gamma\rangle = \langle\gamma_m\gamma_n\rangle\langle\gamma_k\gamma_l\rangle - \langle\gamma_m\gamma_k\rangle\langle\gamma_n\gamma_l\rangle + \langle\gamma_m\gamma_l\rangle\langle\gamma_n\gamma_k\rangle\,. \tag{S45}$$

Let's substitute this into the expression for $\langle\gamma_m\gamma_n H\rangle$, we get

$$\frac{i}{4}\sum_{k,l}\mathcal{H}_{kl}\left(\langle\gamma_m\gamma_n\rangle\langle\gamma_k\gamma_l\rangle - \langle\gamma_m\gamma_k\rangle\langle\gamma_n\gamma_l\rangle + \langle\gamma_m\gamma_l\rangle\langle\gamma_n\gamma_k\rangle\right)\,. \tag{S46}$$

This looks complicated, but we can simplify by using the properties of the matrices. Since the matrix $\mathcal{H}$ is anti-symmetric. The first term is proportion to $M_{mn}\mathrm{Tr}(\mathcal{H}^T M) = -M_{mn}\mathrm{Tr}(\mathcal{H}M)$. The second and third term are all proportion to $-(M\mathcal{H}M^T)_{mn}$. Thus, we have

$$\begin{aligned}\langle\gamma_m\gamma_n H\rangle &= \frac{i}{4}(-M_{mn}\mathrm{Tr}(\mathcal{H}M) - 2(M\mathcal{H}M^T)_{mn})\\ &= -\frac{i}{4}(M_{mn}\mathrm{Tr}(\mathcal{H}M) + 2(\Gamma\mathcal{H}\Gamma)_{mn} - 2i(\Gamma\mathcal{H})_{mn} + 2i(\mathcal{H}\Gamma)_{mn} + 2\mathcal{H}_{mn})\,,\end{aligned} \tag{S47}$$

Similarly, we have

$$\begin{aligned}\langle H\gamma_m\gamma_n\rangle &= \frac{i}{4}(-M_{mn}\mathrm{Tr}(\mathcal{H}M) - 2(M^T\mathcal{H}M)_{mn})\\ &= -\frac{i}{4}(M_{mn}\mathrm{Tr}(\mathcal{H}M) + 2(\Gamma\mathcal{H}\Gamma)_{mn} + 2i(\Gamma\mathcal{H})_{mn} - 2i(\mathcal{H}\Gamma)_{mn} + 2\mathcal{H}_{mn})\,.\end{aligned} \tag{S48}$$

For $m \neq n$, $M_{mn} = -i\Gamma_{mn}, M_{mn} = -M_{nm}$, and $-\frac{i}{4}\mathrm{Tr}(\mathcal{H}M) = \mathrm{Tr}(\rho(\tau)H)$. The other two terms from Eq. (S42) are obtained from Eq. (S47) and Eq. (S48) by exchange m and n. Combining all the contributions from these four terms, we arrive at the result

$$\langle\{[\gamma_m,\gamma_n],H\}\rangle = -2i(\mathcal{H}_{mn} + (\Gamma\mathcal{H}\Gamma)_{mn}) - 4i\Gamma_{mn}\mathrm{Tr}(\rho(\tau)H)\,. \tag{S49}$$

Then substitute the result above into Eq. (S41), we obtain the following imaginary time equation of motion

$$\frac{d\Gamma(\tau)}{d\tau} = -\mathcal{H} - \Gamma(\tau)\mathcal{H}\Gamma(\tau)\,. \tag{S50}$$

Let's make a simple consistent check of this equation. We should verify that our derived equation of motion (S50) is consistent with the purity constraint. That is, if the condition $\Gamma^2 = -I$ holds at $\tau = 0$, it must hold for all $\tau > 0$. To prove this, we show that the time derivative of Γ^2 is zero if Γ satisfies the equation of motion. We compute the derivative of Γ^2 using the chain rule

$$\frac{d}{d\tau}(\Gamma^2) = \frac{d\Gamma}{d\tau}\Gamma + \Gamma\frac{d\Gamma}{d\tau}\,. \tag{S51}$$

Next, we substitute the expression for $\frac{d\Gamma}{d\tau}$ from the Riccati equation (S50)

$$\begin{aligned}\frac{d}{d\tau}(\Gamma^2) &= -\mathcal{H}\Gamma - \Gamma\mathcal{H}\Gamma^2 - \Gamma\mathcal{H} - \Gamma^2\mathcal{H}\Gamma \\ &= -\mathcal{H}\Gamma - \Gamma\mathcal{H}(-I) - \Gamma\mathcal{H} - (-I)\mathcal{H}\Gamma = 0\,,\end{aligned} \tag{S52}$$

where we have used the purity condition, $\Gamma^2 = -I$, which holds at time $\tau = 0$. Since the time derivative of Γ^2 is identically zero, the property $\Gamma^2 = -I$ is a constant of motion.

The real-time evolution

The real-time evolution of an arbitrary density matrix is given by

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}, \tag{S53}$$

which is equivalent to

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)]\,. \tag{S54}$$

We want to derive the formula that governs the time evolution of the fermionic covariance matrix $\Gamma(t)$ for a system evolving under a quadratic Hamiltonian H. The state of the system is $|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$, where $|\Psi_0\rangle$ is a pure Gaussian state (like the TPQ state) with a known initial covariance matrix $\Gamma(0)$.

The most direct and unambiguous way to derive the evolution of an expectation value is through the Heisenberg picture, where the quantum state remains fixed at $|\Psi_0\rangle$, and the operators evolve in time. The evolution of any operator $\mathcal{O}(t)$ in the Heisenberg picture is given by

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}\,. \tag{S55}$$

Its equation of motion is the Heisenberg equation (setting $\hbar = 1$)

$$i\frac{d\mathcal{O}(t)}{dt} = [\mathcal{O}(t), H]\,. \tag{S56}$$

We apply this to the vector of Majorana fermion operators $\vec{\gamma}(t)$

$$i\frac{d\vec{\gamma}(t)}{dt} = [\vec{\gamma}(t), H]\,. \tag{S57}$$

This is the bridge that connects the many-body Hamiltonian H to the single-particle matrix $\mathcal{H}$. We need to compute the commutator $[H, \gamma_m]$. The Hamiltonian is $H = \frac{i}{4}\sum_{k,l}\mathcal{H}_{kl}\gamma_k\gamma_l$.

$$[H, \gamma_m] = \left[\frac{i}{4}\sum_{k,l}\mathcal{H}_{kl}\gamma_k\gamma_l, \gamma_m\right] = \frac{i}{4}\sum_{k,l}\mathcal{H}_{kl}[\gamma_k\gamma_l, \gamma_m]\,. \tag{S58}$$

Using the identity [AB, C] = A{B,C} − {A,C}B and the Majorana anti-commutator $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$, we get

$$[\gamma_k\gamma_l, \gamma_m] = 2\delta_{lm}\gamma_k - 2\delta_{km}\gamma_l\,. \tag{S59}$$

Substituting this back into the sum, we obtain

$$[H, \gamma_m] = \frac{i}{4}\sum_{k,l}\mathcal{H}_{kl}(2\delta_{lm}\gamma_k - 2\delta_{km}\gamma_l) = \frac{i}{2}\left(\sum_k \mathcal{H}_{km}\gamma_k - \sum_l \mathcal{H}_{ml}\gamma_l\right)\,. \tag{S60}$$

Using the anti-symmetry of the Majorana Hamiltonian matrix, $\mathcal{H}_{km} = -\mathcal{H}_{mk}$, we have

$$[H, \gamma_m] = \frac{i}{2}\left(-\sum_k \mathcal{H}_{mk}\gamma_k - \sum_l \mathcal{H}_{ml}\gamma_l\right) = -i\sum_k \mathcal{H}_{mk}\gamma_k\,. \tag{S61}$$

In matrix notation, the commutator is simply

$$[H, \vec{\gamma}] = -i\mathcal{H}\vec{\gamma}\,. \tag{S62}$$

Now substituting this commutator back into the Heisenberg equation of motion, we have

$$i\frac{d\vec{\gamma}(t)}{dt} = [\vec{\gamma}(t), H]\,. \tag{S63}$$

Since the commutator has the same form for the time-evolved operators, we have

$$i\frac{d\vec{\gamma}(t)}{dt} = i\mathcal{H}\vec{\gamma}(t)\,. \tag{S64}$$

The solution is given by the matrix exponential as

$$\vec{\gamma}(t) = e^{\mathcal{H}t}\vec{\gamma}(0)\,. \tag{S65}$$

Finally, we compute the covariance matrix at time t. Its definition involves the expectation value of the time-evolved operators in the initial state $|\Psi_0\rangle$.

$$\Gamma_{mn}(t) = i\langle\Psi_0|[\gamma_m(t), \gamma_n(t)]|\Psi_0\rangle\,. \tag{S66}$$

In matrix form, the equation above reads

$$\Gamma(t) = i\langle[\vec{\gamma}(t), \vec{\gamma}(t)^T]\rangle_0\,. \tag{S67}$$

Now, substitute the solution of $\vec{\gamma}(t)$ to obtain

$$\Gamma(t) = i\langle[e^{\mathcal{H}t}\vec{\gamma}(0), (e^{\mathcal{H}t}\vec{\gamma}(0))^T]\rangle_0\,. \tag{S68}$$

The matrix exponential $e^{-\mathcal{H}t}$ is a matrix of constant coefficients, so we can pull it out of the expectation value and the commutator

$$\Gamma(t) = e^{\mathcal{H}t}\left(i\langle[\vec{\gamma}(0), \vec{\gamma}(0)^T]\rangle_0\right)(e^{\mathcal{H}t})^T = e^{\mathcal{H}t}\Gamma(0)(e^{\mathcal{H}t})^T\,. \tag{S69}$$

Since $\mathcal{H}$ is a real, anti-symmetric matrix, the matrix exponential $U(t) = e^{-\mathcal{H}t}$ is a real, orthogonal matrix. An orthogonal matrix satisfies the property that its transpose is its inverse: $U^T = U^{-1}$.

$$(e^{-\mathcal{H}t})^T = e^{-\mathcal{H}t}\,. \tag{S70}$$

Substituting this back into our expression for $\Gamma(t)$, we finally have

$$\Gamma(t) = e^{\mathcal{H}t}\Gamma(0)e^{-\mathcal{H}t}\,. \tag{S71}$$

The solution of general matrix Riccati equations

In this section, I will outline the standard, powerful method for solving the most general matrix Riccati equation. A standard form of the matrix Riccati differential equation for an $n \times n$ matrix $X(t)$ is

$$\frac{dX(t)}{dt} = A + BX(t) + X(t)C + X(t)DX(t)\,, \tag{S72}$$

where A, B, C, D are all $n \times n$ constant matrices.

Our imaginary time evolution equation for the covariance matrix reads

$$\frac{d\Gamma(\tau)}{d\tau} = -\mathcal{H} - \Gamma(\tau)\mathcal{H}\Gamma(\tau)\,, \tag{S73}$$

which can be obtained from the general Riccati equation (S72) by setting $A = -\mathcal{H}$, $B = C = \mathbf{0}$ (the zero matrix), and $D = -\mathcal{H}$. The strategy is to find a solution of the form $X(t) = P(t)Q(t)^{-1}$, where the block matrix $\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix}$ evolves according to a simple linear differential equation.

We propose that the solution to the Riccati equation can be written as the ratio of two matrices

$$X(t) = P(t)Q(t)^{-1}\,, \tag{S74}$$

where $P(t)$ is $n \times n$ and $Q(t)$ is an invertible $n \times n$ matrix. We now differentiate this expression using the product rule and the rule for the derivative of a matrix inverse ($\frac{d}{dt}(A^{-1}) = -A^{-1}\frac{dA}{dt}A^{-1}$):

$$\frac{dX}{dt} = \frac{d}{dt}(PQ^{-1}) = P'Q^{-1} + P(Q^{-1})' = P'Q^{-1} - PQ^{-1}Q'Q^{-1}\,. \tag{S75}$$

Now, substituting the ansatz $X = PQ^{-1}$ and its derivative into the original Riccati equation, we obtain

$$P'Q^{-1} - PQ^{-1}Q'Q^{-1} = A + B(PQ^{-1}) + (PQ^{-1})C + (PQ^{-1})D(PQ^{-1})\,. \tag{S76}$$

Multiplying the entire equation from the right by $Q(t)$, we get

$$P' - PQ^{-1}Q' = AQ + BP + PQ^{-1}CQ + PQ^{-1}DP\,. \tag{S77}$$

Let's group terms for P' and Q' and rearrange the equation as

$$P' = (BP + AQ) + PQ^{-1}(Q' + CQ + DP)\,. \tag{S78}$$

This equation is satisfied if we require the block vector $\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix}$ evolves as

$$\frac{d}{dt}\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} = \begin{pmatrix} B & A \\ -D & -C \end{pmatrix}\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix}\,. \tag{S79}$$

This is now a linear system of differential equations. Let's write it out explicitly

$$P' = BP + AQ, \qquad Q' = -DP - CQ\,. \tag{S80}$$

The solution to the linear system is given by the matrix exponential

$$\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} = \exp\left(t\begin{pmatrix} B & A \\ -D & -C \end{pmatrix}\right)\begin{pmatrix} P(0) \\ Q(0) \end{pmatrix}\,. \tag{S81}$$

Let's call the $2n \times 2n$ matrix $M = \begin{pmatrix} B & A \\ -D & -C \end{pmatrix}$, then

$$\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} = e^{tM}\begin{pmatrix} P(0) \\ Q(0) \end{pmatrix}\,. \tag{S82}$$

Let's partition the matrix exponential e^{tM} into four $n \times n$ blocks

$$e^{tM} = \begin{pmatrix} E_{11}(t) & E_{12}(t) \\ E_{21}(t) & E_{22}(t) \end{pmatrix}. \tag{S83}$$

The initial condition of the original equation (S72) is $X(0) = X_0$. We choose the initial conditions for the linear system (S79) to satisfy this, for instance, by setting $P(0) = X_0$ and $Q(0) = I$ (the identity matrix). Then, the solution of P and Q is

$$\begin{aligned} P(t) &= E_{11}(t)P(0) + E_{12}(t)Q(0) = E_{11}(t)X_0 + E_{12}(t)\,, \\ Q(t) &= E_{21}(t)P(0) + E_{22}(t)Q(0) = E_{21}(t)X_0 + E_{22}(t)\,. \end{aligned} \tag{S84}$$

The final solution to the original Riccati equation is given by

$$P(t) = \left(E_{11}(t)X_0 + E_{12}(t)\right)\left(E_{21}(t)X_0 + E_{22}(t)\right)^{-1}. \tag{S85}$$

For our problem, the linear system corresponds to Eq. (S73) is

$$\frac{d}{d\tau}\begin{pmatrix} P(\tau) \\ Q(\tau) \end{pmatrix} = \begin{pmatrix} 0 & -\mathcal{H} \\ \mathcal{H} & 0 \end{pmatrix}\begin{pmatrix} P(\tau) \\ Q(\tau) \end{pmatrix}. \tag{S86}$$

The solution to this linear system is given by the matrix exponential

$$\begin{pmatrix} P(\tau) \\ Q(\tau) \end{pmatrix} = \exp\left(\tau\begin{pmatrix} 0 & -\mathcal{H} \\ \mathcal{H} & 0 \end{pmatrix}\right)\begin{pmatrix} P(0) \\ Q(0) \end{pmatrix}. \tag{S87}$$

The exponential of this block-matrix can be calculated easily. Let $M = \begin{pmatrix} 0 & -\mathcal{H} \\ \mathcal{H} & 0 \end{pmatrix}$, we have

$$e^{\tau M} = \begin{pmatrix} \cos(\mathcal{H}\tau) & -\sin(\mathcal{H}\tau) \\ \sin(\mathcal{H}\tau) & \cos(\mathcal{H}\tau) \end{pmatrix}. \tag{S88}$$

So the evolved matrices are

$$\begin{aligned} P(\tau) &= \cos(\mathcal{H}\tau)P(0) - \sin(\mathcal{H}\tau)Q(0)\,, \\ Q(\tau) &= \sin(\mathcal{H}\tau)P(0) + \cos(\mathcal{H}\tau)Q(0)\,. \end{aligned} \tag{S89}$$

Now we specify the initial conditions by imposing $\Gamma_0 = P(0)Q(0)^{-1}$. Choosing $P(0) = \Gamma_0$ and $Q(0) = I$, we then obtain

$$\begin{aligned} P(\tau) &= \cos(\mathcal{H}\tau)\Gamma_0 - \sin(\mathcal{H}\tau)\,, \\ Q(\tau) &= \sin(\mathcal{H}\tau)\Gamma_0 + \cos(\mathcal{H}\tau)\,. \end{aligned} \tag{S90}$$

The final solution is obtained from $\Gamma(\tau) = X(\tau)Y(\tau)^{-1}$ as

$$\Gamma(\tau) = \left(\cos(\mathcal{H}\tau)\Gamma_0 - \sin(\mathcal{H}\tau)\right)\left(\sin(\mathcal{H}\tau)\Gamma_0 + \cos(\mathcal{H}\tau)\right)^{-1}. \tag{S91}$$

Properties of the crosscap states and TPQ states

In this section, we will thoroughly discuss various aspects of the crosscap state and TPQ state. We first observed that in the crosscap state, a pair structure in momentum space appears in the thermodynamic limit. Then we construct a parent Hamiltonian of the crosscap state, which means the ground state of this Hamiltonian is coincide with the crosscap state. In the third part, we consider the thermodynamic limit, which simplify the Hamiltonian a lot and a BCS-type wave function of the crosscap state is obtained. Finally, we derived the Fermi-Dirac distribution can be reproduced in the TPQ state.

The pair structure in the thermodynamic limit

In this section, we will prove that the crosscap state, defined on a finite chain of size $2N$, becomes a simple BCS state of independent $(k,-k)$ pairs in the thermodynamic limit ($N\to\infty$). The momentum distribution of the crosscap state is obtain from the correlator $C_{ij}(0)=\frac{1}{2}\delta_{ij}$ as

$$\langle\mathcal{C}|c_k^\dagger c_q|\mathcal{C}\rangle=\frac{1}{2N}\sum_{i,j}e^{-iki}e^{iqj}\langle\mathcal{C}|c_i^\dagger c_j|\mathcal{C}\rangle=\frac{1}{2N}\sum_j e^{i(q-k)j}\left(\frac{1}{2}\right)=\frac{1}{2}\delta_{k,q}\,. \tag{S92}$$

Thus, the normal correlation matrix is exactly diagonal for any finite system size N, which implies that it is also diagonal in the limit $N\to\infty$. The equation (S92) also implies that the occupation of the crosscap state is uniform $n(k,0)=1/2$ for all k, which is the property of an infinite temperature system.

Then we need to derive the anomalous correlator in the momentum space. The rigorous form of the real space anomalous correlator for crosscap state is

$$F_{ij}=\frac{1}{2}\sum_{l=1}^{N}\left(\delta_{i,l}\delta_{j,l+N}-\delta_{j,l}\delta_{i,l+N}\right). \tag{S93}$$

Then we compute

$$\begin{aligned}\langle c_k c_q\rangle&=\frac{1}{2N}\sum_{i,j=1}^{2N}e^{-iki}e^{-iqj}\left[\frac{1}{2}\sum_{l=1}^{N}\left(\delta_{i,l}\delta_{j,l+N}-\delta_{j,l}\delta_{i,l+N}\right)\right]\\&=\frac{1}{4N}\sum_{l=1}^{N}\left(e^{-ikl}e^{-iq(l+N)}-e^{-ik(l+N)}e^{-iql}\right)\\&=\frac{1}{4N}\left(e^{-iqN}-e^{ikN}\right)\sum_{l=1}^{N}e^{-i(k+q)l}\,.\end{aligned} \tag{S94}$$

The momenta are quantized on a chain of size $2N$ with anti-periodic boundary conditions

$$k=\frac{\pi}{N}\left(m+\frac{1}{2}\right)\quad\text{and}\quad q=\frac{\pi}{N}\left(p+\frac{1}{2}\right), \tag{S95}$$

where m,p are integers in the range $[-N,N-1]$.

The core of the proof lies in the exact evaluation and analysis of the sum over half the chain. Let $\theta=k+q$. The sum in Eq. (S94) is a standard geometric series,

$$S_N(\theta)=\sum_{l=1}^{N}e^{-i\theta l}=e^{-i\theta}\frac{1-e^{-iN\theta}}{1-e^{-i\theta}}\,. \tag{S96}$$

This expression is exact for any $\theta\neq 2\pi n$. It is more illuminating to rewrite this in terms of sine functions using the identity $1-e^{-i\phi}=e^{-i\phi/2}(2i\sin(\phi/2))$:

$$S_N(\theta)=e^{-i\theta}\frac{e^{-iN\theta/2}(2i\sin(N\theta/2))}{e^{-i\theta/2}(2i\sin(\theta/2))}=e^{-i(N+1)\theta/2}\frac{\sin(N\theta/2)}{\sin(\theta/2)}\,. \tag{S97}$$

This is the well-known *Dirichlet kernel*. Our exact finite-N correlator is therefore

$$\langle c_k c_q\rangle=\frac{1}{4N}\left(e^{-iqN}-e^{ikN}\right)\left(e^{-i(N+1)(k+q)/2}\frac{\sin(N(k+q)/2)}{\sin((k+q)/2)}\right). \tag{S98}$$

We now analyze this exact expression as $N\to\infty$. Let's consider the anti-diagonal case ($q=-k$) first. In this case, the angle $\theta=k-k=0$. The Dirichlet kernel form is indeterminate (0/0), so we must return to the original sum

$$S_N(0)=\sum_{l=1}^{N}e^{0}=N\,. \tag{S99}$$

Now, we substitute this result back into the full expression for the correlator

$$\langle c_k c_{-k}\rangle = \frac{1}{4N}\left(e^{ikN} - e^{-ikN}\right) N = \frac{i}{2}\sin(kN)\,. \tag{S100}$$

Substituting the quantized momentum $k = \frac{\pi}{N}(m + 1/2)$, we get

$$\langle c_k c_{-k}\rangle = \frac{i}{2}\sin\left(\pi\left(m + \frac{1}{2}\right)\right) = \frac{i}{2}(-1)^m\,, \tag{S101}$$

which is of order $O(1)$ and does not vanish as $N \to \infty$. The correlation between a mode k and its partner $-k$ persists and is finite in the thermodynamic limit.

Now consider the off-diagonal case ($q \neq -k$). In this case, the angle $\theta = k + q \neq 0$. We can use the Dirichlet kernel form. Let's analyze the magnitude of the full expression,

$$|\langle c_k c_q\rangle| = \left|\frac{1}{4N}\right| \cdot \left|e^{-iqN} - e^{ikN}\right| \cdot \left|e^{-i(N+1)\theta/2}\right| \cdot \left|\frac{\sin(N\theta/2)}{\sin(\theta/2)}\right| \leq \frac{1}{4N} \cdot 2 \cdot 1 \cdot \frac{1}{|\sin((k+q)/2)|}\,. \tag{S102}$$

For any fixed k and q (with $q \neq -k$), the denominator $|\sin((k+q)/2)|$ is a finite number, and the entire expression is bounded by a constant times $1/N$. Therefore, in the thermodynamic limit, we have

$$\lim_{N\to\infty} |\langle c_k c_q\rangle| = 0 \quad (\text{for } q \neq -k)\,. \tag{S103}$$

The correlation between a mode k and any other mode $q \neq -k$ vanishes in the thermodynamic limit.

In summary, we have formally proven that the momentum-space correlation matrices of the crosscap state have the following asymptotic behavior. For normal correlator, we have

$$\lim_{N\to\infty} \langle c_k^\dagger c_q\rangle = \frac{1}{2}\delta_{k,q}\,, \tag{S104}$$

and for the anomalous correlator, we have

$$\lim_{N\to\infty} \langle c_k c_q\rangle = \left(\frac{i}{2}\sin(kN)\right)\delta_{k,-q}\,. \tag{S105}$$

The correlation matrices become exactly block-diagonal in the thermodynamic limit, which implies that the crosscap state asymptotically becomes a simple BCS state of independent $(k, -k)$ pairs.

The parent Hamiltonian of the crosscap states

Although the analysis in the previous section is enough to determine the final form of the crosscap state as an BCS state in the thermodynamic limit. In this section, we will try another method, which regard the crosscap state as the ground state of an special Hamiltonian– the parent Hamiltonian of the crosscap state. We find the method is inspiring and interesting in its own right.

We begin with the fundamental definition of the crosscap state in the real-space basis of fermion operators $c_j^\dagger$ on a chain of $2N$ sites:

$$|\mathcal{C}\rangle = \frac{1}{2^{\frac{N}{2}}}\left(\prod_{j=1}^{N}(1 + c_j^\dagger c_{j+N}^\dagger)\right)|0\rangle\,. \tag{S106}$$

where $|0\rangle = \otimes_{i=1}^{N}|0\rangle_i$ is the global vacuum. This state is built from maximally entangled pairs between antipodal sites.

It's easy to check that the crosscap state are annihilated by the operators $a_j = c_j - c_{j+N}^\dagger$:

$$\begin{aligned}&(c_j - c_{j+N}^\dagger)(|0\rangle + c_j^\dagger c_{j+N}^\dagger|0\rangle)\\ &= 0 - c_{j+N}^\dagger|0\rangle + c_j c_j^\dagger c_{j+N}^\dagger|0\rangle - c_{j+N}^\dagger c_j^\dagger c_{j+N}^\dagger|0\rangle = 0\,.\end{aligned} \tag{S107}$$

The condition $a_j|\mathcal{C}\rangle = 0$ for all $j = 1, \ldots, N$ uniquely defines the state. Let's construct a Hamiltonian whose ground state is exactly being the crosscap state. The parent Hamiltonian can be constructed as

$$H_{\text{parent}} = \sum_{j=1}^{N} a_j^\dagger a_j = \sum_{j=1}^{N} (c_j^\dagger - c_{j+N})(c_j - c_{j+N}^\dagger)\,. \tag{S108}$$

Dropping the constant term, the parent Hamiltonian becomes

$$H_{\text{parent}} = \sum_{j=1}^{N} (c_j^\dagger c_j - c_{j+N}^\dagger c_{j+N} - c_j^\dagger c_{j+N}^\dagger - c_{j+N} c_j)\,. \tag{S109}$$

We now rewrite the Hamiltonian above in the momentum basis. For a chain of size $2N$ with anti-periodic boundary conditions, the Fourier transform is:

$$c_j = \frac{1}{\sqrt{2N}} \sum_k e^{ikj} c_k \quad \text{where} \quad k = \frac{\pi}{N}(m + 1/2), \quad m \in \{-N, \ldots, N-1\}\,. \tag{S110}$$

Let's first consider the number operator part $H_{\text{number}} = \sum_{j=1}^{N} (c_j^\dagger c_j - c_{j+N}^\dagger c_{j+N})$, which transforms to

$$\begin{aligned} H_{\text{number}} &= \sum_{j=1}^{N} \left(\frac{1}{2N} \sum_{k,q} e^{-ikj} e^{iqj} c_k^\dagger c_q - \frac{1}{2N} \sum_{k,q} e^{-ik(j+N)} e^{iq(j+N)} c_k^\dagger c_q \right) \\ &= \frac{1}{2N} \sum_{k,q} c_k^\dagger c_q \left((1 - e^{-i(k-q)N}) \sum_{j=1}^{N} e^{i(q-k)j} \right)\,. \end{aligned} \tag{S111}$$

The sum over j is the Dirichlet kernel we analyzed previously, $S_N(q-k) = e^{i(N+1)(q-k)/2} \frac{\sin(N(q-k)/2)}{\sin((q-k)/2)}$. This expression couples all momenta (k, q).

Now consider the pairing term $H_{\text{pairing}} = -\sum_{j=1}^{N} (c_j^\dagger c_{j+N}^\dagger + c_{j+N} c_j)$. The transformation of the first part is

$$-\sum_{j=1}^{N} c_j^\dagger c_{j+N}^\dagger = -\frac{1}{2N} \sum_{k,q} c_k^\dagger c_q^\dagger \left(e^{-iqN} \sum_{j=1}^{N} e^{-i(k+q)j} \right)\,. \tag{S112}$$

The sum here is the Dirichlet kernel $S_N(k+q)$. The full pairing term becomes

$$H_{\text{pairing}} = -\frac{1}{2N} \sum_{k,q} \left[c_k^\dagger c_q^\dagger \left(e^{-iqN} S_N(k+q) \right) + c_k c_q \left(e^{ikN} S_N(-(k+q)) \right) \right]\,. \tag{S113}$$

Now we take the limit $N \to \infty$. In this limit, the Dirichlet kernel behaves like a Kronecker delta function:

$$\lim_{N\to\infty} \frac{1}{N} S_N(k+q) = \delta_{k,-q}\,. \tag{S114}$$

The summand in H_{number} (S111) is proportional to $(1 - e^{-i(k-q)N}) S_N(k-q)$. As $N \to \infty$, the sum $S_N(k-q)$ enforces $k \approx q$. In this limit, $e^{-i(k-q)N} \approx 1$, so the term $(1 - e^{-i(k-q)N}) \to 0$. So, in the large N limit, the number term vanishes, *i.e.* we have

$$\lim_{N\to\infty} H_{\text{number}} = 0\,, \tag{S115}$$

which means the diagonal part of the parent Hamiltonian is zero.

In the thermodynamic limit, the pairing term (S113) becomes

$$\begin{aligned} H_{\text{pairing}} &\approx -\frac{1}{2N} \sum_{k,q} \left[c_k^\dagger c_q^\dagger e^{-iqN} (N\delta_{k,-q}) + c_k c_q e^{ikN} (N\delta_{k,-q}) \right] \\ &= -\frac{1}{2} \sum_k \left(e^{ikN} c_k^\dagger c_{-k}^\dagger + e^{-ikN} c_k c_{-k} \right)\,. \end{aligned} \tag{S116}$$

Using $c_k c_{-k} = -c_{-k} c_k$, we can write this in the standard form

$$H_{\text{pairing}} \approx -\frac{1}{2}\sum_{k>0}\left((e^{ikN} - e^{-ikN})c_k^\dagger c_{-k}^\dagger + \text{h.c.}\right) = -\sum_{k>0}\left((i\sin(kN))c_k^\dagger c_{-k}^\dagger + (i\sin(kN))^* c_{-k} c_k\right). \tag{S117}$$

This gives the pairing potential $\Delta_k^0 = i\sin(kN) = i(-1)^m$.

The BCS representation of the crosscap state

We found that in the thermodynamic limit ($N \to \infty$), the parent Hamiltonian for the crosscap state simplifies to a block-diagonal BCS-type Hamiltonian. In the Nambu spinor basis $\Psi_k^\dagger = \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix}$, the parent Hamiltonian H_{parent} can be written as

$$H_{\text{parent}} = -\sum_{k>0} \Psi_k^\dagger \mathbf{H}_k \Psi_k, \tag{S118}$$

where

$$\mathbf{H}_k = \begin{pmatrix} \epsilon_k^0 & \Delta_k^0 \\ \Delta_k^{0*} & -\epsilon_k \end{pmatrix} = \begin{pmatrix} 0 & i\sin(kN) \\ -i\sin(kN) & 0 \end{pmatrix}. \tag{S119}$$

Here, we've used the specific values for our parent Hamiltonian: the diagonal "energy" term is $\epsilon_k^0 = 0$, and the off-diagonal "pairing" term is $\Delta_k = i\sin(kN)$.

The parent Hamiltonian (S118) can be diagonalized by Bogoliubov transformation

$$\begin{aligned} c_k &= u_k b_k + v_k b_{-k}^\dagger \\ c_{-k} &= u_k b_{-k} - v_k b_k^\dagger . \end{aligned} \tag{S120}$$

For these new operators $\{b_k, b_k^\dagger\}$ to be valid fermions, the Bogoliubov coefficients u_k and v_k must satisfy the normalization condition $|u_k|^2 + |v_k|^2 = 1$.

We find the coefficients by substituting the transformation into the Hamiltonian and demanding that all off-diagonal terms like $b_k^\dagger b_{-k}^\dagger$ and $b_k b_{-k}$ vanish. This leads to a standard set of equations that relate the coefficients to the Hamiltonian parameters ϵ_k^0 and Δ_k^0.

The quasiparticle energy is found by diagonalizing the matrix $\mathbf{H}_k$

$$\mathcal{E}_k^0 = \sqrt{(\epsilon_k^0)^2 + |\Delta_k^0|^2} \tag{S121}$$

The coefficients are then given by

$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\epsilon_k^0}{\mathcal{E}_k^0}\right) \quad \text{and} \quad |v_k|^2 = \frac{1}{2}\left(1 - \frac{\epsilon_k^0}{\mathcal{E}_k^0}\right), \tag{S122}$$

and their ratio, which determines the structure of the ground state, is

$$\frac{v_k}{u_k} = \frac{\Delta_k^0}{\epsilon_k^0 + \mathcal{E}_k^0}. \tag{S123}$$

Now we substitute the specific parameters for the crosscap state's parent Hamiltonian: $\epsilon_k^0 = 0$ and $\Delta_k^0 = i\sin(kN)$.

The quasiparticle energy is $\mathcal{E}_k^0 = |\sin(kN)|$, and $|u_k| = \frac{1}{\sqrt{2}}, |v_k| = \frac{1}{\sqrt{2}}$. The crucial ratio is determined by

$$\frac{v_k}{u_k} = \frac{\Delta_k^0}{0 + \mathcal{E}_k^0} = \frac{i\sin(kN)}{|\sin(kN)|} = i \cdot \text{sgn}(\sin(kN)). \tag{S124}$$

For our quantized momenta, $k = \frac{\pi}{N}(m + 1/2)$, we have $\sin(kN) = (-1)^m$. Therefore, $\text{sgn}(\sin(kN)) = (-1)^m$, hence

$$\frac{v_k}{u_k} = i(-1)^m. \tag{S125}$$

The ground state $|\Psi_{GS}\rangle$ of the diagonalized Hamiltonian is the state that is annihilated by all the new quasiparticle annihilation operators:

$$b_k|\Psi_{GS}\rangle = 0 \quad \text{and} \quad b_{-k}|\Psi_{GS}\rangle = 0 \quad \text{for all } k > 0\,. \tag{S126}$$

The solution is given by

$$|\Psi_{GS}\rangle = \prod_{k>0} \left(u_k + v_k c_k^\dagger c_{-k}^\dagger\right)|0\rangle\,. \tag{S127}$$

By substituting the specific ratio we found, $v_k/u_k = i(-1)^m$, and choosing a convenient gauge where $u_k = 1/\sqrt{2}$ (which satisfies $|u_k|^2 = 1/2$), we find $v_k = \frac{i(-1)^m}{\sqrt{2}}$.

Therefore the ground state of the asymptotic parent Hamiltonian is

$$|\Psi_{GS}\rangle = \prod_{k>0} \left(\frac{1}{\sqrt{2}} + \frac{i(-1)^m}{\sqrt{2}} c_k^\dagger c_{-k}^\dagger\right)|0\rangle\,, \tag{S128}$$

which is precisely the momentum-space representation of the crosscap state.

The Fermi-Dirac distribution from TPQ states

We begin with the momentum-space representation of the crosscap state $|\mathcal{C}\rangle$. As proven previously, this state is a BCS-type state. For any calculation involving operators diagonal in momentum space, it behaves as if it were a simple product over momentum pairs:

$$|\mathcal{C}\rangle = \prod_{k>0} (u_k(0) + v_k(0) c_k^\dagger c_{-k}^\dagger)|0\rangle\,, \tag{S129}$$

where the coefficients are known: $u_k(0) = 1/\sqrt{2}$ and $v_k(0) = \frac{i(-1)^m}{\sqrt{2}}$.

The thermal pure state $|\Psi_\beta\rangle$ is created by acting on this state with the imaginary time evolution operator. The Hamiltonian is $H = \sum_p E(p) c_p^\dagger c_p$, and we have

$$|\Psi_\beta\rangle \propto e^{-H\beta/4}|\mathcal{C}\rangle = e^{-\frac{\beta}{4}\sum_p E(p) c_p^\dagger c_p} \left(\prod_{k>0} (u_k(0) + v_k(0) c_k^\dagger c_{-k}^\dagger)\right)|0\rangle\,. \tag{S130}$$

The evolution operator acts on each term in the expansion of the product. Specifically, it acts on the two-particle state $c_k^\dagger c_{-k}^\dagger|0\rangle$ as

$$e^{-H\beta/4} c_k^\dagger c_{-k}^\dagger|0\rangle = e^{-(E(k)+E(-k))\beta/4} c_k^\dagger c_{-k}^\dagger|0\rangle\,, \tag{S131}$$

where we have used the fact that the vacuum $|0\rangle$ has zero energy, $e^{-H\beta/4}|0\rangle = |0\rangle$. Since the energy spectrum of the XX chain is symmetric, $E(k) = E(-k)$, this becomes

$$e^{-H\beta/4} c_k^\dagger c_{-k}^\dagger|0\rangle = e^{-E(k)\beta/2} c_k^\dagger c_{-k}^\dagger|0\rangle\,. \tag{S132}$$

Therefore, the evolution operator only modifies the coefficient of the two-particle term in each factor of the product. The resulting normalized TPQ state is

$$|\Psi_\beta\rangle = \prod_{k>0} \left(u_k(\beta) + v_k(\beta) c_k^\dagger c_{-k}^\dagger\right)|0\rangle\,, \tag{S133}$$

which is a new BCS state with new, β-dependent Bogoliubov coefficients

$$|u_k(\beta)|^2 = \frac{1}{1 + e^{-\beta E(k)}}\,, \qquad |v_k(\beta)|^2 = \frac{e^{-\beta E(k)}}{1 + e^{-\beta E(k)}}\,. \tag{S134}$$

Up to this point, we have successfully represented the TPQ state as a BCS-type state, which is convenient for the subsequent analysis. This representation also provides the rigorous justification for using the independent quasiparticle pair picture to describe the quench dynamics in large systems.

Recall the relations between the c_k and b_k in Eq. (S120), substitute it into the momentum distribution function $n(k,\beta)$:

$$n(k,\beta) = \langle\Psi_\beta|\, c_k^\dagger c_k \,|\Psi_\beta\rangle = \langle\Psi_\beta|\, (u_k^* b_k^\dagger + v_k^* b_{-k})(u_k b_k + v_k b_{-k}^\dagger)\,|\Psi_\beta\rangle \ , \tag{S135}$$

where for simplicity, we omit the β dependence in the u_k and v_k. If expand the product of operators inside the expectation value and using the properties of the quasiparticle vacuum (c.f. Eq. (S126)), we found that only the term $v_k^* v_k \langle\Psi_\beta|\, b_{-k} b_{-k}^\dagger \,|\Psi_\beta\rangle$ contribute. Then using the canonical anti-commutation relation for the quasiparticles, $\{b_{-k}, b_{-k}^\dagger\} = 1$, which implies $b_{-k} b_{-k}^\dagger = 1 - b_{-k}^\dagger b_{-k}$ and

$$\langle\Psi_\beta|\, b_{-k} b_{-k}^\dagger \,|\Psi_\beta\rangle = \langle\Psi_\beta|\, (1 - b_{-k}^\dagger b_{-k})\,|\Psi_\beta\rangle = 1\,. \tag{S136}$$

We finally get

$$n(k,\beta) = |v_k(\beta)|^2 = \frac{1}{1+e^{\beta E(k)}}\,. \tag{S137}$$

This formula correctly reproduces the Fermi-Dirac distribution at inverse temperature β.

In a similar way, one can derive the pairing amplitude as

$$\phi_k(\beta) \equiv \langle\Psi_\beta|\, c_k c_{-k} \,|\Psi_\beta\rangle = \frac{-i(-1)^m}{2\cosh(E(k)\beta/2)}\,. \tag{S138}$$

The parent Hamiltonian for the TPQ State

We seek a parent Hamiltonian, H_{parent}, whose ground state is $|\Psi_\beta\rangle$. Since $|\Psi_\beta\rangle$ is a Gaussian state, the most general quadratic parent Hamiltonian can be written in the Nambu spinor basis $\Psi_k^\dagger = \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix}$ as:

$$H_{\text{parent}}(\beta) = \sum_k \Psi_k^\dagger H_k \Psi_k = \sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} \epsilon_k & \Delta_k \\ \Delta_k^* & -\epsilon_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}. \tag{S139}$$

This expands to the form

$$H_{\text{parent}}(\beta) = \sum_k \epsilon_k (c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) + \sum_k \left(\Delta_k c_k^\dagger c_{-k}^\dagger + \Delta_k^* c_{-k} c_k\right), \tag{S140}$$

where ϵ_k is the single-particle energy spectrum and Δ_k is the pairing potential of the parent Hamiltonian. Our goal is to find the functions ξ_k and Δ_k that produce $|\Psi_\beta\rangle$ as the ground state.

For a general BCS Hamiltonian of the form in Eq. (S139), the occupation probability of its ground state is given by the standard relation:

$$|v_k|^2 = \frac{1}{2}\left(1 - \frac{\epsilon_k}{\mathcal{E}_k}\right), \tag{S141}$$

where $\mathcal{E}_k = \sqrt{(\epsilon_k)^2 + |\Delta_k|^2}$ is the quasiparticle energy spectrum of the parent Hamiltonian. We now enforce that this ground state is our TPQ state by equating Eq. (S141) with Eq. (S137), which allows us to solve for the ratio $\epsilon_k/\mathcal{E}_k$ as

$$\frac{\epsilon_k}{\mathcal{E}_k} = \tanh\left(\frac{\beta E(k)}{2}\right). \tag{S142}$$

This equation provides a fundamental constraint on the parameters of our desired parent Hamiltonian. To solve for ϵ_k and Δ_k separately, we make the natural choice of setting the single-particle spectrum of the parent Hamiltonian equal to that of the physical XX chain, i.e., $\epsilon_k = E(k)$. With this assumption, the quasiparticle energy of the parent Hamiltonian is determined from Eq. (S142):

$$\mathcal{E}_k = \frac{\epsilon_k}{\tanh\left(\frac{\beta E(k)}{2}\right)} = E(k)\coth\left(\frac{\beta E(k)}{2}\right). \tag{S143}$$

Finally, we use the definition $\mathcal{E}_k^2 = (\epsilon_k)^2 + |\Delta_k|^2$ to find the required pairing potential:

$$\begin{aligned} |\Delta_k|^2 &= \mathcal{E}_k^2 - (\epsilon_k)^2 \\ &= E(k)^2 \coth^2\left(\frac{\beta E(k)}{2}\right) - E(k)^2 \\ &= E(k)^2 \left[\coth^2\left(\frac{\beta E(k)}{2}\right) - 1\right] = \frac{E(k)^2}{\sinh^2\left(\frac{\beta E(k)}{2}\right)}. \end{aligned} \tag{S144}$$

This gives the magnitude of the pairing potential as $|\Delta_k| = |E(k)| / \left|\sinh\left(\frac{\beta E(k)}{2}\right)\right|$.