STRONG HUB COVER PEBBLING NUMBER

RUNZE WANG

Abstract. In a graph G, we define a set of vertices to be a *strong hub set* if for any two vertices in G, we can find a path between them whose internal vertices are all in this set. We define the *strong hub cover pebbling number* of G, denoted by $h_s^*(G)$, to be the smallest t such that for any initial configuration with t pebbles on G, we can make some pebbling moves (a pebbling move consists of removing two pebbles from a vertex v and adding one pebble to another vertex adjacent to v) so that there is a strong hub set with every vertex in it having a pebble. We determine the strong hub cover pebbling numbers of paths, stars, and books.

1. Introduction

In this paper, we only study finite simple connected graphs.

The concept of graph pebbling was first proposed by Lagarias and Saks for giving an alternative proof of a theorem in number theory [6]. Then, in 1989, Chung formally introduced this concept and defined the *pebbling number* of a graph in [3]. In a graph $G = (V, E)$, we distribute t pebbles to the vertices. For two vertices u and v adjacent to each other, assuming that there are at least two pebbles on u, we can make a *pebbling move* by removing two pebbles from u and adding one pebble to v. The *pebbling number* of G, denoted by $\pi(G)$, is the smallest t such that for any initial configuration with t pebbles on G and any chosen *root vertex* $v \in V$, we can make some pebbling moves so that there is a pebble on v. Graph pebbling has been extensively studied since it was introduced. For some recent developments, see [1, 2, 5, 10].

Date: October 8, 2025.

2020 *Mathematics Subject Classification.* 05C57.

Key words and phrases. Pebbling number; Hub set; Path; Star; Book.

There are also some variants of graph pebbling. Isaak and Prudente defined a two-player pebbling game in [8]. (Also, see [9, 16].) Crull et al. introduced in [4] the cover pebbling number of G, which is the smallest number t such that for any initial configuration with t pebbles on G, we can make some pebbling moves so that every vertex in G has a pebble on it. In [7], Hurlbert and Munyan determined the cover pebbling numbers of the hypercubes. Lourdusamy and Tharani introduced the covering cover pebbling number in [14]. This concept has been studied on different graphs in [12, 13, 15].

Recently, Lourdusamy et al. introduced the *hub cover pubbling number* in [11]. The concept of hub sets was introduced by Walsh in [17]. In graph G, a nonempty vertex set $U \subseteq V$ is called a *hub set* if for any two vertices v_1, $v_2 \notin U$, we can find a path between v_1 and v_2 whose internal vertices are all in U. (If v_1 and v_2 are adjacent, then there is a path between them which does not have any internal vertices, but the empty set is also a subset of U, so the condition also holds for v_1 and v_2.) A hub set gives a model, where we convert some buildings to rapid-transit stations (choose some vertices to be in the hub set), so that people can walk from a building to a station, take the rapid-transit system to another station, and walk to the destination building. The *hub cover pebbling number* of G, denoted by $h^*(G)$, is the smallest number t such that for any initial configuration with t pebbles on G, we can make some pebbling moves so that there is a hub set U with every vertex in U having a pebble. In [11], Lourdusamy et al. determined the hub cover pebbling numbers on some wheel-related graphs.

In the hub set model, one disadvantage is that it is only guaranteed that people can efficiently transit between two buildings (two vertices not in the hub set), but not two stations (two vertices in the hub set) or one station and one building (one vertex in the hub set and one vertex not in the hub set). To solve this problem, we define a nonempty vertex set $U \subseteq V$ to be a *strong hub set* if for any two vertices in the graph, we can find a path between them whose internal vertices are all in U. This concept gives us a model where people can conveniently transit between any two places.

Similarly, we define the *strong hub cover pebbling number* of G, denoted by $h_s^*(G)$. Note that, a strong hub set is always a hub set, so we have $h_s^*(G) \geq h^*(G)$ for any graph G. However, a hub set is not always a strong hub set. For example, as shown in Figure 1, in P_4, the yellow vertex set $\{v_1,\ v_4\}$ is a hub set but not a strong hub set; in C_6, the yellow vertex set $\{v_2,\ v_4,\ v_6\}$ is a hub set but not a strong hub set.

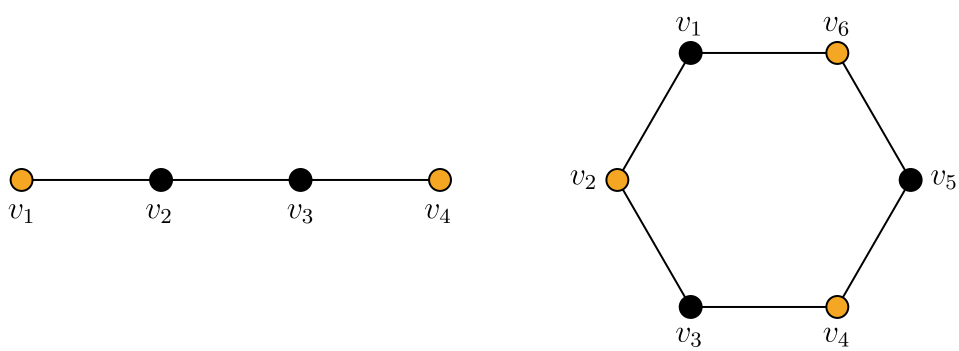

FIGURE 1. Two examples of hub sets which are not strong hub sets.

Also, we would like to mention that, a strong hub set is always a dominating set, but a dominating set is not always a strong hub set.

In this short paper, we determine the strong hub cover pebbling numbers of paths, stars, and books. Also, we suggest an open problem about cycles.

2. Paths

We denote the path on n vertices by P_n and denote the n vertices from one end to the other by $v_1,\ v_2,\ \ldots\ v_n$. Trivially, we have $h_s^*(P_1) = h_s^*(P_2) = 0$.

Theorem 2.1. *For any $n \geq 3$, we have*

$$h_s^*(P_n) = 2^{n-1} - 1.$$

We observe that for $n \geq 3$, a vertex set in P_n is a strong hub set if and only if it contains the $n-2$ vertices in the middle, so we have to let each of $v_2,\ v_3,\ \dots\ v_{n-1}$ have a pebble after some pebbling moves. If we only have $2^{n-1}-2$ pebbles, then we can put $2^{n-1}-3$ of them on v_1 and put one of them on v_n, and it is easy to see that we cannot get a pebble on v_{n-1} after any pebbling moves. So $h_s^*(P_n) \geq 2^{n-1}-1$.

To show that $2^{n-1}-1$ pebbles are enough, we will prove a slightly stronger result by induction.

Proposition 2.2. *If we have $2^{n-1}-1$ pebbles, then after some pebbling moves, we can get (at least) one pebble on each of $v_2,\ v_3,\ \dots,\ v_{n-1}$, and also get (at least) one pebble on v_1 or v_n.*

Proof. For P_3, assume that we have three pebbles. If both v_1 and v_2 have pebble(s) or both v_2 and v_3 have pebble(s), then the proof is complete. So, by symmetry, we need to handle the following three cases.

- v_1 has two pebbles and v_3 has one pebble.
- v_1 has three pebbles.
- v_2 has three pebbles.

In the first case, we can make a pebbling move by removing two pebbles from v_1 and adding a pebble to v_2. In the second case, we can also remove two pebbles from v_1 and add a pebble to v_2. In the third case, we can remove two pebbles from v_2 and add a pebble to v_1. So the base case $n=3$ is proved.

Assume that the conclusion holds true for $n=k-1$. Assume $n=k$ and we have $2^{k-1}-1$ pebbles. By the pigeonhole principle, we know that there are at least 2^{k-2} pebbles on $\{v_1,\ v_2,\ \dots,\ v_{k-1}\}$ or $\{v_2,\ v_3,\ \dots,\ v_k\}$. By symmetry, we assume that there are at least 2^{k-2} pebbles on $\{v_1,\ v_2,\ \dots,\ v_{k-1}\}$. By the induction hypothesis, we can use at most

$2^{k-2}-1$ pebbles to make some pebbling moves to get a pebble on each of $v_2,\ v_3,\ ...,\ v_{k-2}$ and also get a pebble on v_1 or v_{k-1}.

We have used at most $2^{k-2}-1$ pebbles, so there are at least 2^{k-2} pebbles not used yet. We have two cases.

Case 1. There is a pebble on v_1. In this case, we need to get a pebble on v_{k-1}.

Subcase 1.1. There are at least two pebbles on v_k. In this subcase, we just need to remove two pebbles from v_k and add a pebble to v_{k-1}.

Subcase 1.2. There is exactly one pebble on v_k. In this subcase, we still have at least $2^{k-2}-1$ pebbles not used yet, and all of them are on $\{v_1,\ v_2,\ ...,\ v_{k-1}\}$. We also take an extra pebble from v_1, so there are 2^{k-2} pebbles to use. We have the following lemma (see [6]) about the pebbling number of a path.

Lemma 2.3. $\pi(P_m) = 2^{m-1}$.

Now, $v_1,\ v_2,\ ...,\ v_{k-1}$ form a P_{k-1}, so we can use the 2^{k-2} pebbles to make some pebbling moves and get a pebble on v_{k-1}. Note that, in order to have 2^{k-2} pebbles to use, we took a pebble from v_1, so if v_1 only had one pebble on it before these pebbling moves, then after the pebbling moves, it is possible that there is no pebble on v_1. But in this subcase, we have assumed that there is a pebble on v_k. So now we have a pebble on each of $v_2,\ v_3,\ ...,\ v_k$, as desired.

Subcase 1.3. There are no pebbles on v_k. In this subcase, we have at least 2^{k-2} unused pebbles, all of which are on $\{v_1,\ v_2,\ ...,\ v_{k-1}\}$. So by Lemma 2.3, we can make some pebbling moves to get a pebble on v_{k-1}.

Case 2. There is no pebble on v_1, but there is a pebble on v_{k-1}. In this case, we need to get a pebble on v_1 or v_k. If there is already a pebble on v_k, then the proof is complete. If v_k does not have a pebble, then there are at least 2^{k-2} unused pebbles, all of which are on $\{v_1,\ v_2,\ \ldots,\ v_{k-1}\}$. By Lemma 2.3, we can make some pebbling moves to get a pebble on v_1. $\square$

3. Stars and books

For $n \geq 2$, the *star* on $n+1$ vertices, denoted by S_n, is constructed by connecting n independent vertices to a universal vertex. This universal vertex will be called the *center* of the star. It is easy to see that a vertex set in a star is a strong hub set if and only if it contains the center.

In S_n, if we only have n pebbles, then we can put one pebble on each vertex which is not the center. We cannot get a pebble on the center by making pebbling moves, so $h_s^*(S_n) \geq n+1$. If we have at least $n+1$ pebbles and none of them is on the center, then by the pigeonhole principle, there must be a vertex with two pebbles on it. We can remove these two pebbles from this vertex and place one pebble on the center. So $n+1$ pebbles are enough.

Proposition 3.1. *For any $n \geq 2$, we have*

$$h_s^*(S_n) = n+1.$$

Then, for $n \geq 2$, the *book* on $2n+2$ vertices, denoted by B_n, is defined by $S_n \square P_2$, the Cartesian product of the star on $n+1$ vertices and the path on two vertices.

Theorem 3.2. *For any $n \geq 2$, we have*

$$h_s^*(B_n) = 2n+3.$$

Proof. We construct B_n in the following way.

- Draw two copies of S_n: One has center a and n vertices $u_1,\ u_2,\ \ldots,\ u_n$ connected to a; the other has center b and n vertices $v_1,\ v_2,\ \ldots,\ v_n$ connected to b.

- Connect a with b and connect u_i with v_i for any $1 \le i \le n$.

An example of B_6 is shown in Figure 2.

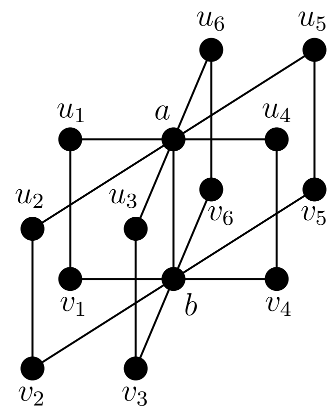

FIGURE 2. An example of B_6.

First we prove that a vertex set $S \subseteq V(B_n)$ is a strong hub set if and only if it contains one of the following three sets, which are shown by the yellow vertices in Figure 3.

- $\{a,\ b\}$;
- $\{a,\ u_1,\ u_2,\ ...,\ u_n\}$;
- $\{b,\ v_1,\ v_2,\ ...,\ v_n\}$.

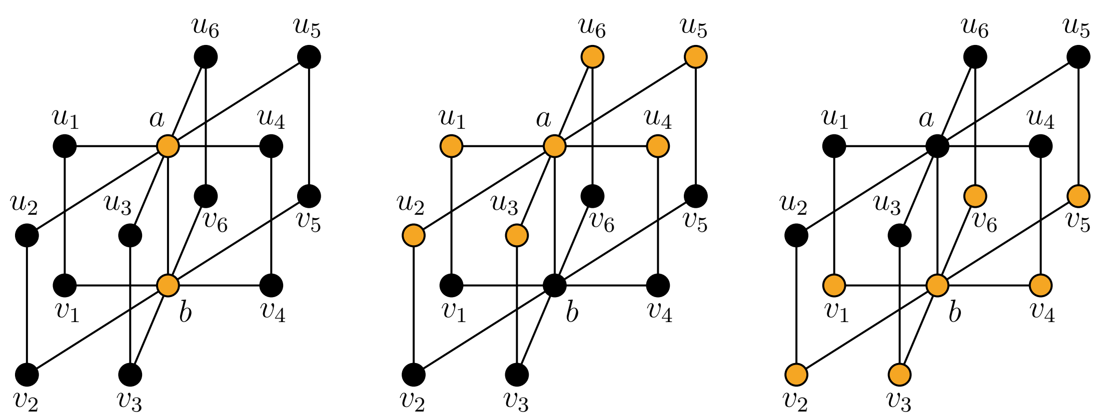

FIGURE 3. A strong hub set must contain one of the yellow sets.

For the sufficiency, it is easy to see that if S contains one of $\{a,\ b\}$, $\{a,\ u_1,\ u_2,\ ...,\ u_n\}$, and $\{b,\ v_1,\ v_2,\ ...,\ v_n\}$, then S is a strong hub set. For the necessity, assume that S does not contain any one of the three vertex set. Then, one of the following three statements must hold true.

- $S \cap \{a,\ b\} = \emptyset$.
- $a \in S$, $b \notin S$, and $u_i \notin S$ for some $1 \leq i \leq n$.
- $b \in S$, $a \notin S$, and $v_j \notin S$ for some $1 \leq j \leq n$.

If $S \cap \{a,\ b\} = \emptyset$, then for u_1 and v_2, we cannot find a path between them whose internal vertices are all in S. If $a \in S$, $b \notin S$, and $u_i \notin S$ for some $1 \leq i \leq n$, then for a and v_i, we cannot find a path between them whose internal vertices are all in S. If $b \in S$, $a \notin S$, and $v_j \notin S$ for some $1 \leq j \leq n$, then for b and u_j, we cannot find a path between them whose internal vertices are all in S. Thus, by definition, S is not a strong hub set, and the necessity is also proved.

Henceforth, if after some pebbling moves, there is at least one pebble on each vertex in $\{a,\ b\}$, we will say that "**condition 1** is fulfilled"; if after some pebbling moves, there is at least one pebble on each vertex in $\{a,\ u_1,\ u_2,\ ...,\ u_n\}$, we will say that "**condition 2** is fulfilled"; and if after some pebbling moves, there is at least one pebble on each vertex in $\{b,\ v_1,\ v_2,\ ...,\ v_n\}$, we will say that "**condition 3** is fulfilled".

Now, if we only have $2n+2$ pebbles, then we can put five pebbles on u_1, put one pebble on each of $u_2,\ u_3,\ ...,\ u_{n-1}$, and put one pebble on each of $v_1,\ v_2,\ ...,\ v_{n-1}$. It is easy to check that we cannot make pebbling moves so that one of the three conditions is fulfilled. So $h_s^*(B_n) \geq 2n+3$.

Then we show that $2n+3$ pebbles are enough. By symmetry, we need to handle three cases.

Case 1. There are at least $n+1$ pebbles on $\{a,\ u_1,\ u_2,\ ...,\ u_n\}$, and at least $n+1$ pebbles on $\{b,\ v_1,\ v_2,\ ...,\ v_n\}$.

In this case, by Proposition 3.1, we know that we can make pebbling moves to get a pebble on a and a pebble on b, so condition 1 is fulfilled.

Case 2. There are exactly $n+3$ pebbles on $\{a,\ u_1,\ u_2,\ ...,\ u_n\}$, and exactly n pebbles on $\{b,\ v_1,\ v_2,\ ...,\ v_n\}$. Depending on the pebble distribution in $\{b,\ v_1,\ v_2,\ ...,\ v_n\}$, we have three subcases.

Subcase 2.1. There is a pebble on b.

In this subcase, if there is also a pebble on a, then condition 1 is fulfilled; otherwise, there must be a u_i with at least two pebbles on it, so we can remove two pebbles from u_i and add a pebble to a, and then condition 1 is fulfilled.

Subcase 2.2. There are no pebbles on b, but there is a v_i with at least two pebbles on it.

In this subcase, we just need to remove two pebbles from v_i and add a pebble to b, and then it becomes Subcase 2.1.

Subcase 2.2. There are no pebbles on b, and there is exactly one pebble on each v_i.

If there are at least two pebbles on a, then we can remove two pebbles from a and add a pebble to b, and condition 3 is fulfilled.

If there is exactly one pebble on a, then there is a u_i with at least two pebbles on it, so we can remove two pebbles from u_i and add a pebble to a. Now there are two pebbles on a, so we can remove these two pebbles from a and add a pebble to b, and condition 3 is fulfilled.

If there are no pebbles on a, then either there is a u_i with at least four pebbles on it, or there are u_j and u_k with at least two pebbles on each of them. In either case we can remove four pebbles (either from u_i, or from u_j and u_k) and add two pebbles to a. Then we can remove these two pebbles from a and add a pebble to b, and condition 3 is fulfilled.

Case 3. There are at least $n+4$ pebbles on $\{a,\ u_1,\ u_2,\ ...,\ u_n\}$. Depending on how many pebbles there are on a, we have four subcases.

Subcase 3.1. There are at least three pebbles on a.

In this subcase, we can remove two pebbles from a and add a pebble to b, and then condition 1 is fulfilled.

Subcase 3.2. There are exactly two pebbles on a.

In this subcase, there must be a u_i with at least two pebbles on it, so we can remove two pebbles from u_i and add a pebble to a. Now there are three pebbles on a, and it becomes Subcase 3.1.

Subcase 3.3. There is exactly one pebble on a.

In this subcase, there are at least $n+3$ pebbles on $\{u_1,\ u_2,\ ...,\ u_n\}$, so either there is a u_i with at least four pebbles on it, or there are u_j and u_k with at least two pebbles on each of them. In either case we can remove four pebbles (either from u_i, or from u_j and u_k) and add two pebbles to a. Now there are three pebbles on a, and it becomes Subcase 3.1.

Subcase 3.4. There are no pebbles on a.

If there is at least one pebble on each vertex in $\{u_1,\ u_2,\ ...,\ u_n\}$, then we have two possible cases.

- There is a u_i with at least three pebbles on it. We can remove two pebbles from u_i and add one pebble to a. Now condition 2 is fulfilled.
- There are $u_i,\ u_j,\ u_k,\ u_\ell$ with exactly two pebbles on each of them. We can remove these eight pebbles and add four pebbles to a. Then we remove two pebbles from a and add one pebble to b. Now condition 1 is fulfilled.

If there is a vertex in $\{u_1,\ u_2,\ ...,\ u_n\}$, say u_1, with no pebbles on it, then there are $n+4$ pebbles on $\{u_2,\ u_3,\ ...,\ u_n\}$, and we have four possible cases.

- There is a $u_i \in \{u_2,\ u_3,\ ...,\ u_n\}$ with at least six pebbles on it. We can remove these six pebbles and add three pebbles to a. Then we remove two pebbles from a and add a pebble to b. Now condition 1 is fulfilled.

- There are $u_i,\ u_j \in \{u_2,\ u_3,\ ...,\ u_n\}$ with exactly two pebbles on u_i and exactly five pebbles on u_j. We remove two pebbles from u_i and add one pebble to a. Then we remove four pebbles from u_j and add two pebbles to a. Now there are three pebbles on a, and we can remove two pebbles from a and add a pebble to b to fulfill condition 1.
- There are $u_i,\ u_j \in \{u_2,\ u_3,\ ...,\ u_n\}$ with exactly three pebbles on u_i and exactly four pebbles on u_j. This is the same as the previous case.
- There are $u_i,\ u_j,\ u_k \in \{u_2,\ u_3,\ ...,\ u_n\}$ with at least two pebbles on each of them. We can remove these six pebbles and add three pebbles to a. Then we can remove two pebbles from a and add a pebble to b to fulfill condition 1.

□

4. An open problem

We consider the strong hub cover pebbling numbers of cycles. In a cycle C_n, we can pick a vertex, label it v_1, and counterclockwise label the other $n-1$ vertices $v_2,\ v_3,\ ...,\ v_n$. We observe that, up to isomorphism, a vertex set in C_n is a strong hub set if and only if it contains $v_1,\ v_2,\ ...,\ v_{n-2}$. Here we make a conjecture.

Conjecture 4.1.

$$h_s^*(C_n) = \begin{cases} 2^k + 2^{k-1} - 3, & if\ n = 2k\ is\ even, \\ 2^{k+1} - 3, & if\ n = 2k+1\ is\ odd. \end{cases}$$

To see the lower bound: If $n = 2k$ is even, and we only have $2^k + 2^{k-1} - 4$ pebbles, then we can put all of them on v_1. Similarly, if $n = 2k+1$ is odd, and we only have $2^{k+1} - 4$ pebbles, then we can also put all of them on v_1. It is easy to verify that in these two initial configurations, we cannot choose a strong hub set $S \subseteq V(C_n)$ and place one pebble on each vertex in S by making pebbling moves.

References

[1] M. Adauto, C. de Figueiredo, G. Hurlbert, and D. Sasaki. On the pebbling numbers of Flower, Blanuša and Watkins snarks. *Discrete Appl. Math.*, 361:336–346, 2025.

[2] L. Alcón and G. Hurlbert. Pebbling in powers of paths. *Discrete Math.*, 346(5):Paper No. 113315, 20, 2023.

[3] F. R. K. Chung. Pebbling in hypercubes. *SIAM J. Discrete Math.*, 2(4):467–472, 1989.

[4] B. Crull, T. Cundiff, P. Feltman, G. H. Hurlbert, L. Pudwell, Z. Szaniszlo, and Z. Tuza. The cover pebbling number of graphs. *Discrete Math.*, 296(1):15–23, 2005.

[5] S. Fischer. On the pebbling number of $C_m \times C_n$ in case of even parity. *Discrete Math.*, 342(2):292–313, 2019.

[6] G. H. Hurlbert. A survey of graph pebbling. In *Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999)*, volume 139, pages 41–64, 1999.

[7] G. H. Hurlbert and B. Munyan. Cover pebbling hypercubes. *Bull. Inst. Combin. Appl.*, 47:71–76, 2006.

[8] G. Isaak and M. Prudente. Two-player pebbling on diameter 2 graphs. *Internat. J. Game Theory*, 50(2):581–596, 2021.

[9] G. Isaak, M. Prudente, A. Potylycki, W. Fagley, and J. Marcinik. On two-player pebbling. *Communications on Number Theory and Combinatorial Theory*, 3:Article 3, 12pp, 2022.

[10] F. Kenter, D. Skipper, and D. Wilson. Computing bounds on product graph pebbling numbers. *Theoret. Comput. Sci.*, 803:160–177, 2020.

[11] A. Lourdusamy, F. J. Beaula, and F. Patrick. Hub cover pebbling number. *Ars Combin.*, 160:31–35, 2024.

[12] A. Lourdusamy and T. Mathivanan. Covering cover pebbling number for square of a cycle. *J. Prime Res. Math.*, 8:102–105, 2012.

[13] A. Lourdusamy and A. Punitha Tharani. Covering cover pebbling number of a hypercube & diameter d graphs. *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.*, 15(2):121–134, 2008.

[14] A. Lourdusamy and A. P. Tharani. Covering cover pebbling number. *Util. Math.*, 78:41–54, 2009.

[15] I. McGahan. Covering cover pebbling number of products of paths. *University of Minnesota M.S. thesis*, 2015.

[16] M. J. Prudente. *Two-Player Graph Pebbling*. ProQuest LLC, Ann Arbor, MI, 2015. Thesis (Ph.D.)–Lehigh University.

[17] M. Walsh. The hub number of a graph. *Int. J. Math. Comput. Sci.*, 1(1):117–124, 2006.

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA

Email address: rwang6@memphis.edu; runze.w@hotmail.com