A NEW BOOLEAN MATRIX REPRESENTATION FOR CATALAN SEMIRINGS

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ABSTRACT. We construct a faithful representation of the semiring of all order-preserving decreasing transformations of a chain with $n+1$ elements by Boolean upper triangular $n\times n$-matrices.

Catalan monoids. Let $[n]$ stand for the set $\{1,2,\dots,n\}$ of the first n positive integers. We consider the set $[n]$ with its usual order: $1<2<\dots<n$. A transformation $\alpha\colon [n]\to[n]$ is *order-preserving* if $i\le j$ implies $i\alpha\le j\alpha$ for all $i,j\in[n]$ and *extensive* if $i\le i\alpha$ for every $i\in[n]$. Clearly, if two transformations have either of the properties of being order-preserving or extensive, then so does their product, and the identity transformation has both properties. Hence, the set C_n of all extensive order-preserving transformations of $[n]$ forms a submonoid in the monoid of all transformations of $[n]$. This submonoid often appears in the literature as the *Catalan* monoid. The name is justified by the cardinality of C_n being the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$; see [2, Theorem 14.2.8(i)].

A transformation $\alpha\colon [n]\to[n]$ is called *decreasing* (or *parking*) if $i\alpha\le i$ for all $i\in[n]$. The set C_n^- of all decreasing order-preserving transformations of $[n]$ also forms a submonoid in the monoid of all transformations of $[n]$. The submonoid C_n^- differs from the submonoid C_n if $n>1$, but the two monoids are isomorphic because C_n^- is nothing more than the monoid of all extensive order-preserving transformations of the set $\{1,2,\dots,n\}$ equipped with the 'opposite' order $1>2>\dots>n$. Slightly abusing terminology, we collectively refer to monoids in both series $\{C_n\}_{n=1,2,\dots}$ and $\{C_n^-\}_{n=1,2,\dots}$ as Catalan monoids.

Catalan monoids have been intensively studied from various viewpoint (and under various names); we mention [1, 3–5, 8–10, 12, 13] as samples of such studies.

Catalan semirings. An *additively idempotent semiring* (ai-semiring, for short) is an algebra $(A,+,\cdot)$ with binary addition $+$ and multiplication $\cdot$ such that the additive reduct $(A,+)$ is a commutative and idempotent semigroup, the multiplicative reduct $(A,\cdot)$ is a semigroup, and multiplication distributes over addition on the left and right.

The set O_n of all order-preserving transformations of $[n]$ is a submonoid in the monoid of all transformations of $[n]$; clearly, both C_n and C_n^- are submonoids of the monoid O_n. We define, for all $\alpha,\beta\in O_n$ and $i\in[n]$,

$$i(\alpha+\beta):=\max\{i\alpha,i\beta\}.$$

(Here and below the sign := stands for equality by definition; thus, $A:=B$ means that A is defined as B.) It is well known (and easy to verify) that $\alpha+\beta\in O_n$. Equipped with this addition, O_n becomes an ai-semiring in which both C_n and C_n^- form subsemirings. Hence,

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2020 *Mathematics Subject Classification.* 16Y60, 15B34, 06F05, 20M20.
Key words and phrases. Catalan monoid, Catalan semiring, Boolean matrix representation.

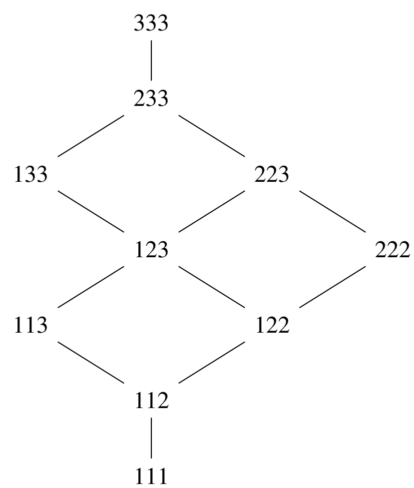
Catalan monoids admit a natural ai-semiring structure. We denote the ai-semirings defined on C_n and C_n^- by $\mathscr{C}_n$ and $\mathscr{C}_n^-$, respectively, and refer to them as *Catalan* semirings. Notice that while for each n, the monoids C_n and C_n^- are isomorphic, the semirings $\mathscr{C}_n$ and $\mathscr{C}_n^-$ are not isomorphic if $n > 1$, and moreover, their additive reducts are not isomorphic if $n > 2$.

For $\alpha, \beta \in O_n$, let

$$\alpha \leqslant \beta \iff \alpha + \beta = \beta \iff i\alpha \leq i\beta \text{ for all } i \in [n]. \tag{1}$$

The relation $\leqslant$ is a partial order on the set O_n, under which O_n becomes a distributive lattice. The addition on O_n can be recovered from the order (1)—for all $\alpha, \beta \in O_n$, their sum $\alpha + \beta$ coincides with the least upper bound of α and β with respect to $\leqslant$.

For illustration, the next picture shows the Hasse diagram of the lattice $(O_3, \leqslant)$. Each transformation $\alpha \in O_3$ is encoded by the triple $1\alpha 2\alpha 3\alpha$.



In particular, the identity transformation ε is encoded by the triple 123. The five transformations α with $\alpha \geqslant \varepsilon$ form the semiring $\mathscr{C}_3$, while the five transformations β with $\beta \leqslant \varepsilon$ constitute the semiring $\mathscr{C}_3^-$. The general picture looks the same: the transformations $\alpha \in O_n$ with $\alpha \geqslant \varepsilon$ form the semiring $\mathscr{C}_n$, while the transformations $\beta \in O_n$ with $\beta \leqslant \varepsilon$ constitute the semiring $\mathscr{C}_n^-$.

Semirings of Boolean matrices. A *Boolean* matrix is a matrix with entries 0 and 1 only. The addition and multiplication of such matrices are as usual, except that addition and multiplication of the entries are defined as $x + y := \max\{x, y\}$ and $x \cdot y := \min\{x, y\}$. For each n, the set of all Boolean $n \times n$-matrices forms an ai-semiring under matrix addition and multiplication.

A Boolean matrix $\left(a_{ij}\right)_{n\times n}$ is called *upper triangular* if $a_{ij} = 0$ for all $1 \leq j < i \leq n$. The set of all upper triangular Boolean $n \times n$-matrices is closed under matrix addition and multiplication so it forms a subsemiring of the ai-semiring of all Boolean $n \times n$-matrices. We denote by T_n and $\mathscr{T}_n$ the monoid and, respectively, the ai-semiring of all upper triangular Boolean $n \times n$-matrices.

By a *faithful representation* of a semigroup $(A, \cdot)$, respectively, ai-semiring $(A, +, \cdot)$ in the monoid T_n, respectively, in the ai-semiring $\mathscr{T}_n$, we mean an injective map $A \to T_n$ which is a semigroup, respectively, semiring homomorphism.

A faithful representation $\mathscr{C}_n \to \mathscr{T}_n$. Any transformation $\alpha\colon [n] \to [n]$ can be represented as the Boolean $n \times n$-matrix $B(\alpha)$ whose entry in position (i,j) is 1 if $i\alpha = j$ and 0 otherwise. The map $\alpha \mapsto B(\alpha)$ is an injective homomorphism from the monoid of all transformations of $[n]$ in the monoid of all Boolean $n\times n$-matrices. If α is extensive, then $i \le i\alpha$ whence the matrix $B(\alpha)$ is upper triangular. Therefore, the restriction of $\alpha \mapsto B(\alpha)$ to the monoid C_n is faithful representation of C_n in the monoid T_n of all upper triangular Boolean $n\times n$-matrices. However, the map $\alpha \mapsto B(\alpha)$ fails to respect addition so that it is not a representation of the semiring $\mathscr{C}_n$ in the semiring $\mathscr{T}_n$.

Ondřej Klíma and Libor Polák (see [7, Section 5]) found another faithful representation of the monoid C_n in the monoid T_n. (The same representation was rediscovered by Marianne Johnson and Peter Fenner [6, Lemma 5.1] and again by Yan Feng Luo, Zhen Feng Jin, and Wen Ting Zhang [8, Theorem 5.4].) Namely, let S_n stand for the set of all *stair triangular* matrices, that is, matrices $\left(a_{ij}\right)_{n\times n}$ satisfying: $a_{ii} = 1$ for all $i = 1,\dots,n$, and if $a_{ij} = 1$ for some $1 \le i < j \le n$, then

$$a_{ii+1} = \cdots = a_{ij} = a_{i+1j} = \cdots = a_{j-1j} = 1.$$

The set S_n is closed under the usual multiplication of Boolean matrices and contains the identity matrix. Thus, S_n forms a monoid and the map $\alpha \mapsto S(\alpha)$, where the (i,j)-th entry of the matrix $S(\alpha)$ equals 1 if and only if $i \le j \le i\alpha$, is an isomorphism between the monoids C_n and S_n.

It is easy to see that the set S_n is also closed under the entry-wise addition of Boolean matrices, so it also forms a subsemiring $\mathscr{S}_n$ in the semiring $\mathscr{T}_n$. Although this was not explicitly mentioned in [7], the above isomorphism $\alpha \mapsto S(\alpha)$ between the monoids C_n and S_n is, in fact, an isomorphism between the ai-semirings $\mathscr{C}_n$ and $\mathscr{S}_n$, hence, a faithful representation of the semiring $\mathscr{C}_n$ in the semiring $\mathscr{T}_n$.

A faithful representation $\mathscr{C}^-_{n+1} \to \mathscr{T}_n$. A Boolean matrix $\left(a_{ij}\right)_{n\times n}$ is called *lower triangular* if $a_{ij} = 0$ for all $1 \le i < j \le n$. The set of all lower triangular Boolean $n\times n$-matrices also forms a subsemiring of the ai-semiring of all Boolean $n\times n$-matrices; we denote this subsemiring by $\mathscr{T}^-_n$. If P is the $n\times n$-matrix with 1s on the antidiagonal and 0s elsewhere,

$$P := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & 1 & 0\\ \vdots & \vdots & .\cdot\dot{} & \vdots & \vdots\\ 0 & 1 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

then it is known (and easy to verify) that the map $M \mapsto PMP$ is an isomorphism between the ai-semirings $\mathscr{T}^-_n$ and $\mathscr{T}_n$. Therefore, to obtain a faithful representation of the semiring $\mathscr{C}^-_{n+1}$ in $\mathscr{T}_n$, it suffices to give a faithful representation in $\mathscr{T}^-_n$, which is notationally easier.

To each transformation $\alpha \in \mathscr{C}^-_{n+1}$, we assign a Boolean $n\times n$-matrix $M(\alpha)$ whose i-th row consists of $(i+1)\alpha - 1$ left-justified 1s; in other words, the i-th row is the unary representation of the number $(i+1)\alpha - 1$, left-justified. The rule can also be expressed by saying that the (i,j)-th entry of the matrix $M(\alpha)$ equals 1 if and only if $(i+1)\alpha - 1 \ge j$.

For illustration, the next table shows the map $\alpha \mapsto M(\alpha)$ for transformations in $\mathscr{C}^-_3$.

Transformation $\alpha \in \mathscr{C}^-_3$ (encoded as $1\alpha 2\alpha 3\alpha$)	111 $\downarrow$	112 $\downarrow$	113 $\downarrow$	122 $\downarrow$	123 $\downarrow$
Boolean matrix $M(\alpha)$	$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$

Theorem 1. *The map $\alpha \mapsto M(\alpha)$ is a faithful representation of the semiring $\mathscr{C}_{n+1}^{-}$ in the semiring $\mathscr{T}_n^{-}$.*

Proof. As $\alpha \in \mathscr{C}_{n+1}^{-}$ is a decreasing transformation, $(i+1)\alpha \le i+1$ for each $i = 1,\ldots,n$. Hence, $(i+1)\alpha - 1 \le i$ and one must have $j \le i$ for the condition $(i+1)\alpha - 1 \ge j$ to hold. Thus, only the i first entries in the i-th row of the matrix $M(\alpha)$ may be non-zero, and therefore, $M(\alpha)$ is a lower triangular matrix.

It is clear that the map $\alpha \mapsto M(\alpha)$ is one-to-one and preserves addition. To verify that $M(\alpha\beta) = M(\alpha)M(\beta)$ for all transformations $\alpha,\beta \in \mathscr{C}_{n+1}^{-}$, let $M(\alpha) := \big(a_{ij}\big)_{n\times n}$, $M(\beta) := \big(b_{ij}\big)_{n\times n}$, $M(\alpha\beta) := \big(c_{ij}\big)_{n\times n}$, and $M(\alpha)M(\beta) := \big(d_{ij}\big)_{n\times n}$. For all $i, j \in \{1,\ldots,n\}$,

$$
\begin{aligned}
d_{ij} = \sum_{k}^{n} a_{ik}b_{kj} = 1 &\iff \exists k\ (a_{ik} = 1\ \&\ b_{kj} = 1)\\
&\iff \exists k\ \big((i+1)\alpha - 1 \ge k\ \&\ (k+1)\beta - 1 \ge j\big)\\
&\iff \exists k\ \big((i+1)\alpha \ge k+1\ \&\ (k+1)\beta \ge j+1\big). \qquad (2)
\end{aligned}
$$

The statement (2) is equivalent to the inequality

$$(i+1)\alpha\beta \ge j+1. \qquad (3)$$

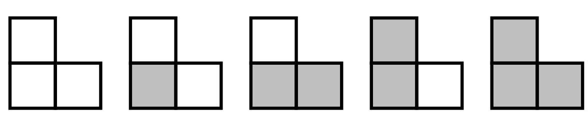
Indeed, if $(i+1)\alpha \ge k+1$ for some k, then $(i+1)\alpha\beta \ge (k+1)\beta$ since β is order-preserving. Combining the latter inequality with $(k+1)\beta \ge j+1$ yields (3). Conversely, (3) implies that (2) holds with $k := (i+1)\alpha - 1$.

The inequality (3) is equivalent to $(i+1)\alpha\beta - 1 \ge j$, and the latter inequality is equivalent to $c_{ij} = 1$ by the definition of the map $\alpha \mapsto M(\alpha)$. Hence $d_{ij} = 1 \iff c_{ij} = 1$, and the matrices $M(\alpha)M(\beta)$ and $M(\alpha\beta)$ coincide. □

Remark. The lower triangular Boolean matrices in the image of bijection $\alpha \mapsto M(\alpha)$ are in a natural one-to-one correspondence with the Young diagrams contained in the staircase shape $(n, n-1, \ldots, 1)$. The next illustration shows these matrices together with the corresponding Young diagrams (drawn in the French convention—rows are left-justified and counted from bottom to top) for $n = 2$.

$$
\begin{pmatrix} 0 & 0\\ 0 & 0 \end{pmatrix} \quad
\begin{pmatrix} 0 & 0\\ 1 & 0 \end{pmatrix} \quad
\begin{pmatrix} 0 & 0\\ 1 & 1 \end{pmatrix} \quad
\begin{pmatrix} 1 & 0\\ 1 & 0 \end{pmatrix} \quad
\begin{pmatrix} 1 & 0\\ 1 & 1 \end{pmatrix}
$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$



The established bijection with $\mathscr{C}_{n+1}^{-}$ shows that the number of Young diagrams contained in the staircase shape $(n, n-1, \ldots, 1)$ equals the $(n+1)$-st Catalan number. This fact is known; see [11, Exercise 167].

Corollary 2. *The map $\alpha \mapsto PM(\alpha)P$ is a faithful representation of the semiring $\mathscr{C}_{n+1}^{-}$ in the semiring $\mathscr{T}_n$.*

Optimality. The faithful representation of Corollary 2 is somewhat unexpected because of its ‘squeezing’ feature: an object of a larger dimension embeds into one of smaller dimension. This raises the question: Can Catalan monoids or semirings be faithfully represented by Boolean triangular matrices of even smaller size? Our next results show that both Klíma–Polák's embedding $\mathscr{C}_n \hookrightarrow \mathscr{T}_n$ and our embedding $\mathscr{C}^-_{n+1} \hookrightarrow \mathscr{T}_n$ already use triangular matrices of the minimum possible size.

Theorem 3. (i) *The Catalan monoid C_{n+2} does not embed into the monoid T_n.*
(ii) *Neither of the Catalan semirings $\mathscr{C}_{n+1}$ and $\mathscr{C}^-_{n+2}$ embeds into the semiring $\mathscr{T}_n$.*

Proof. Our argument uses the fact that the identities

$$x^n = x^{n+1}, \tag{4}$$

$$x^{n-1}y^{n-1} = x^n y^{n-1} + x^{n-1}y^n \tag{5}$$

hold in the semiring $\mathscr{T}_n$ [14, Example 2.4].

Let $\alpha \in C_{n+2}$ be defined by

$$i\alpha := \begin{cases} i+1 & \text{if } i \le n+1, \\ n+2 & \text{if } i = n+2. \end{cases}$$

Then $1\alpha^n = n+1$ and $1\alpha^{n+1} = n+2$ whence $\alpha^n \ne \alpha^{n+1}$. Therefore, the identity (4) fails in the monoid C_{n+2}. As (4) holds in the monoid T_n and identities are inherited by submonoids, C_{n+2} does not embed in T_n and the ai-semiring $\mathscr{C}^-_{n+2}$ whose multiplicative monoid is isomorphic to C_{n+2} does not embed in $\mathscr{T}_n$.

Let $\beta, \gamma \in C_{n+1}$ be defined by

$$i\beta := \begin{cases} i+1 & \text{if } i \le n, \\ n+1 & \text{if } i = n+1; \end{cases} \qquad i\gamma := \begin{cases} n & \text{if } i \le n, \\ n+1 & \text{if } i = n+1. \end{cases}$$

Then $1\beta^{n-1}\gamma^{n-1} = n\gamma^{n-1} = n$, whereas

$$1\left(\beta^n\gamma^{n-1} + \beta^{n-1}\gamma^n\right) = \max\{1\beta^n\gamma^{n-1}, 1\beta^{n-1}\gamma^n\} = \max\{n+1, n\} = n+1.$$

Therefore, the identity (5) fails in the ai-semiring $\mathscr{C}_{n+1}$. As (5) holds in $\mathscr{T}_n$ and identities are inherited by subsemirings, $\mathscr{C}_{n+1}$ does not embed in $\mathscr{T}_n$. □

Remark. A *divisor* of a monoid [semiring] is defined as a homomorphic image of one of its submonoids [subsemirings]. Since identities are inherited by divisors, the above proof of Theorem 3 shows that the results (i) and (ii) remain valid in the stronger form with embedding replaced by division.

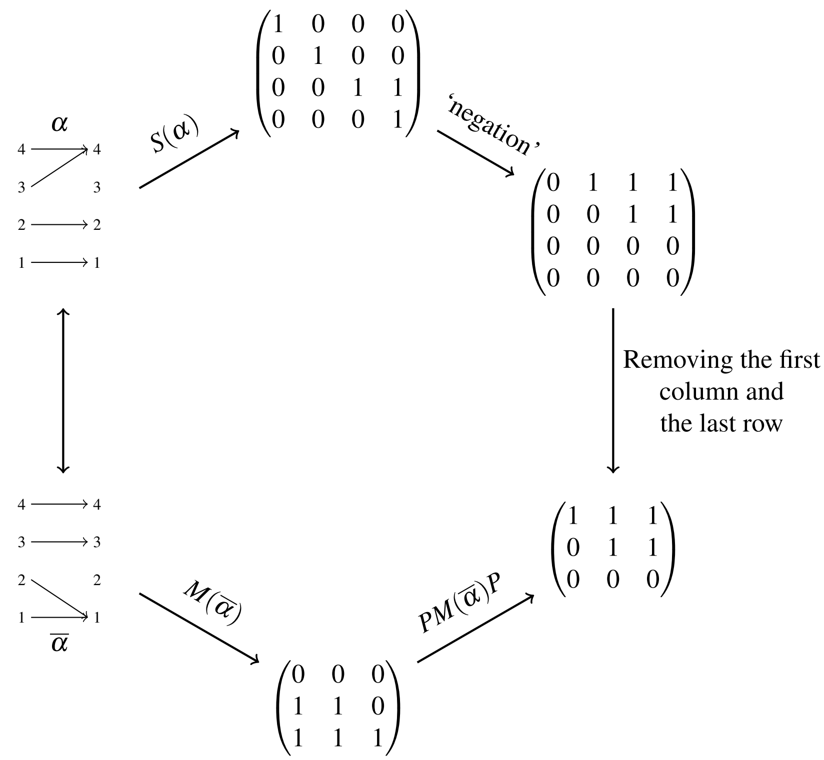
Complementarity. While the embedding $C^-_{n+1} \hookrightarrow T_n$ defined by $\alpha \mapsto PM(\alpha)P$ appears to be new, it is closely related to Klíma–Polák's embedding and may be viewed, in a sense, as the complement of the embedding $C_{n+1} \hookrightarrow T_{n+1}$ defined by $\alpha \mapsto S(\alpha)$. Let us explain how this complementarity works.

The monoid O_{n+1} of all order-preserving transformations of the chain $[n+1]$ admits the following automorphism $\alpha \mapsto \overline{\alpha}$ induced by reversing the order on $[n+1]$: for $\alpha \in O_{n+1}$, the transformation $\overline{\alpha}$ is defined by

$$i\overline{\alpha} := n+2-(n+2-i)\alpha \ \text{ for each } \ i = 1, 2, \dots, n+1.$$

The restriction of $\alpha \mapsto \overline{\alpha}$ to each of the Catalan monoids C_{n+1} and C^-_{n+1} is an isomorphism onto the other monoid. For $\alpha \in C_{n+1}$, 'negate' the upper triangle of the $(n+1)\times(n+1)$-matrix $S(\alpha)$ by replacing all 1s in this triangle by 0s and all 0s with 1; the 0s below the main diagonal remain unchanged. Removing the first column and the last row from the resulting matrix yields an $n \times n$-matrix, which coincides with matrix $PM(\overline{\alpha})P$.

The following illustration shows all steps of the process for $\alpha = 1244 \in C_4$. We restrict ourselves to this 'proof by example' since a formal verification would merely require unfolding the definitions of all the maps involved.



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