

Chroma+ model stellar surface intensities: Spherical formal solution

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ABSTRACT

We announce V. 2025-08-08 of the Chroma+ suite of stellar atmosphere and spectrum modelling codes for fast, approximate, effectively platform-independent stellar spectrum synthesis, written in a number of free well-supported programming languages. The Chroma+ suite now computes the emergent surface intensity and flux distributions and the hydrostatic pressure structure assuming a spherical atmosphere rather than local flatness by implementing the analytic formal solution of the 1D spherical radiative transfer equation of Chapman (1966) based on an integrating factor. We present our adaptation and discretization of the solution and demonstrate the resulting impact of our sphericity treatment on a number of computed observables, including exo-planet transit light-curves. All codes are available from the OpenStars [www](http://www.ap.smu.ca/OpenStars) site: www.ap.smu.ca/OpenStars.

Subject headings: Stars: atmospheres, imaging, planetary systems; planets and satellites: detection

1. Introduction

The Chroma+ stellar atmosphere and spectrum modelling suite (Chroma+, Short & Bennett (2021) and papers in that series) provides for quick self-consistent convergence of the total hydrostatic pressure structure, $P(\tau_{\text{Ros}})$, the gas $P_{\text{gas}}(\tau_{\text{Ros}})$, electron $P_{\text{e}}(\tau_{\text{Ros}})$, and radiation $P_{\text{rad}}(\tau_{\text{Ros}})$ pressure structures, the opacity structure, $\kappa_{\nu}(\tau_{\text{Ros}})$ and $\kappa_{\text{Ros}}(\tau_{\text{Ros}})$, the gas pressure equation-of-state, $P_{\text{gas}}(T_{\text{kin}}, \rho, P_{\text{e}}, \dots)$ (EOS), and the molecular and ionization equilibria for a given kinetic temperature structure, $T_{\text{kin}}(\tau_{\text{Ros}})$, and synthesizes the spectrum with the VALD (Pakhomov, Ryabchikova & Piskunov (2019)) atomic line list and select molecular bands in the just-overlapping-line approximation (JOLA, Zeidler-K.T. & Koester (1982)). The $T_{\text{kin}}(\tau_{\text{Ros}})$ structure, along with initial guesses at the $P_{\text{gas}}(\tau_{\text{Ros}})$ and $P_{\text{e}}(\tau_{\text{Ros}})$ structures, is scaled from a suitable starting model properly converged with V. 15 of the Phoenix (Allard & Hauschildt (1995)) atmospheric modeling code, thus allowing us to avoid the need to converge the line-blanketed radiative thermal equilibrium solution. The Chroma+ suite has been provided in effectively platform-independent languages including Python (ChromaStarPy DOI: zenodo.1095687), Java, and Javascript, making it suitable for quick, easily accessible numerical experiments in a pedagogical context, as well as for research contexts where parameter-perturbation and differential comparison are central. The Chroma+ suite may be compared to the Spectroscopy Made Easy (SME) package described in Valenti & Piskunov (1996) and Piskunov & Valenti (2017) under IDL. The suite is available from the OpenStars www site: www.ap.smu.ca/OpenStars.

The Chroma+ suite originally adopted the simplifying approximation of local flatness, or plane-parallel (PP) geometry, in which the atmosphere is treated as a planar slab of gas infinite in extent perpendicular to the line-of-sight, normally taken to be the z -axis of the Cartesian and spherical polar coordinate systems. The PP approximation is most realistic for stars of relatively large surface gravity, $\log g$, and becomes less realistic with

decreasing $\log g$ value. In modelling the outgoing surface specific intensity distribution, $I_{\lambda}^{+}(\tau = 0, \cos \theta)$, where θ is the angle between the I_{λ}^{+} beam and the surface normal of a PP model, the approximation is best for the pencil beam normal to the star’s surface ($\cos \theta = 1$) and worsens as beams approach grazing-incidence ($\cos \theta \gtrsim 0$). For a spherical star, beams of sufficiently small $\cos \theta$ value pass through a geometric path length, Δz , that corresponds to an optical depth interval, $\Delta \tau$, less than unity so that the star is optically thin for that beam, whereas in PP geometry all beams emerge from a semi-infinite medium. The effects of the PP approximation on various observables and on 1D vertical atmospheric structure have been well documented (Fieldus, Lester & Rogers (1990), Neilson & Lester (2013a)). One effect of more recent concern is that on the modelled light-curves of exo-planetary transits (Neilson *et al.* (2017), Neilson, Lester & Baron (2022)), particularly during ingress and egress during which $I_{\lambda}^{+}(\tau = 0, \cos \theta)$ beams of $\cos \theta \gtrsim 0$ are being occulted.

Because the Chroma+ suite does not converge the line blanketed radiative thermal equilibrium problem, we only need account for sphericity in the formal solution of the radiative transfer equation (RTE) for the $I_{\lambda}^{+}(\tau = 0, \mu_0)$ distribution for a given $T_{\text{kin}}(\tau)$ structure, where $\mu_0 \equiv \cos \theta$ for θ defined at the surface. As of Version 2025-08-08 (ISO 8601 versioning), the Chroma+ suite now evaluates the formal solution for the $I_{\lambda}^{+}(\tau = 0, \mu_0)$ distribution adopting spherical atmospheric geometry by discretizing the analytic formal solution of Chapman (1966) based on an integrating factor. We note that our $T_{\text{kin}}(\tau)$ structures are re-scaled from Phoenix V. 15 Allard & Hauschildt (1995) models that were calculated with spherical geometry, so the adoption of sphericity in our formal solution makes our overall procedure more self-consistent. Sphericity also affects the formal solution of the hydrostatic equilibrium equation (HSE) for the total pressure structure, $P(\tau)$ (Fieldus, Lester & Rogers (1990)), albeit in a way that is relatively straightforward to accommodate. For consistency, we also now allow for sphericity in our HSE solution.

There are other codes, written in FORTRAN, that solve the spherical radiative transfer problem, such as S3R2T (V15), the radiative transfer module of Version 15 of the PHOENIX stellar atmosphere and spectrum modelling code (Allard & Hauschildt (1995)) and SATLAS (Lester & Neilson (2008)), the spherical version of the ATLAS9 and ATLAS12 atmospheric modelling codes (Kurucz (2014), Castelli & Kurucz (2006)). However, the Chroma+ suite has the advantage of being very fast because of its approximate approach for economizing the procedure, and is suitable for more interactive environments such as the Python integrated development environment (IDE), which allow a user to more quickly extract approximate results from fitting observed spectra, and the Javascript/HTML environment which allows for web demonstrations. By contrast, the spectrum synthesis procedure in the Java version of the Chroma+ suite has been parallelized in the wavelength domain and allows for relatively fast spectrum synthesis of large spectral regions blanketed by thousands of lines. All versions have been updated with the spherical solution.

In Section 2, we review the radiative transfer formal solution of Chapman (1966) and our adaptation and discretization of the solution. In Section 3 we present the comparison of the emergent observables computed with spherical and PP geometry.

2. The spherical formal solution

2.1. Radiative transfer equation (RTE)

The monochromatic RTE for the monochromatic specific intensity distribution, $I_\lambda(r, \mu)$, in 1D spherical geometry must be formulated with absolute radial height, r , as the independent position coordinate and is

$$\frac{\mu}{\kappa_\lambda(r)\rho(r)} \frac{dI_\lambda(r, \mu)}{dr} + \frac{(1 - \mu^2)}{\kappa_\lambda(r)\rho(r)r} \frac{dI_\lambda(r, \mu)}{d\mu} = S_\lambda(r) - I_\lambda(r, \mu) \quad (1)$$

where μ is now a variable that depends on r for a given I_λ beam and is the cosine of the angle between the beam and the normal to the tangent to the local concentric spherical shell being intersected by the beam, and S_λ is the radiative source function and is assumed to be isotropic ($S_\lambda(\mu) = S_\lambda$).

Chapman (1966) (C66 henceforth) presents an analytic expression for the formal solution for an atmosphere of infinite extent that was derived using an integrating factor, $\Phi(r, \mu)$. Below we present our adaptation of the solution for a discretized atmospheric model of finite extent for outgoing surface intensity only, $I_\lambda^+(\tau = 0, \mu_0)$.

In what follows, indices i and j refer to absolute radial height, r (ξ in the notation of C66), N is the total number of discrete height points and is currently set equal to 64, and index k refers to the $\mu_{0,k} \equiv \cos \theta_k$ value between the direction of the current pencil beam (*ie.* the direction of integrating) and the radial direction at the *surface* of the model (index $i = N$). The set $\{\mu_{0,k}\}$ is the 32 positive abscissae values of a 64-point Gauss-Legendre quadrature on the domain $[0, 1]$.

We define the core to be a sphere of radius equal to the nominal stellar radius corresponding to the input $\log g$ and M parameters, so that the core (or "inner") radius, R_i in the notation of C66, is $(GM/g)^{1/2}$. Our independent depth variable is the radial Rosseland optical depth scale, τ_{Ros} , equally spaced in $\log \tau_{\text{Ros}}$ in the range $[-6, 2]$. Our established procedure computes the Rosseland mass extinction coefficient, $\kappa_{\text{Ros}}(\tau_{\text{Ros}})$, from the total continuous extinction and then computes the resulting geometric depth scale, $z(\tau_{\text{Ros}})$. To avoid unphysical values arising from the *ad hoc* upper boundary condition, we

take the atmospheric radial extent, Δz , to be $(z_0 - z_{N-2})$. We then approximate the radius of the outermost layer of the atmosphere, R_0 in the notation of C66, as $R_i + \Delta z$. We then generate a grid of absolute discrete radial heights in our atmospheric model, $\{r_i\}$, from our $\{z_i\}$ values in the i range $[2, N]$ by setting r_0 equal to R_i .

In what follows, r_b is the impact parameter of a parallel pencil beam travelling toward the observer and b is the height index of the shell grazed by a beam, k . We evaluated r_b as

$$r_b = r_N(1 - \mu_{0,k}^2)^{1/2} \quad (2)$$

Following C66, we must distinguish between core-intersecting beams ($r_b < r_0$) and non-core-intersecting beams ($r_b \geq r_0$), and μ_c is the critical value of $\mu_{0,k}$ distinguishing the two regimes in the notation of C66. For the special case of surface intensity ($r_i = r_N$), the critical value is given by

$$\mu_c = |[1 - (\frac{r_0}{r_N})^2]^{1/2}| \quad (3)$$

We replace the combination $\kappa_\lambda(\xi)\rho(\xi)d\xi$ in the integrating factor and the formal solution of C66 with our corresponding monochromatic $\Delta\tau_{\lambda,i}$ interval because our τ_λ scales are physically consistent with the model structure and our computed $\kappa_\lambda(r)$ distribution. Then, the integrating factor for height i and any local direction cosine, μ , comparable to Eq. 25 of C66, is evaluated as

$$\Phi_\lambda(r_i, \mu) \approx \exp\left\{-\sum_{j=i}^N \frac{\Delta\tau_{\lambda,j}}{[1 - (\frac{r_i}{r_j})^2(1 - \mu^2)]^{1/2}}\right\} \quad (4)$$

For the special case of the surface, $\Phi_\lambda(r_N, \mu) = \exp(-0) = 1$, simplifying the solution below.

We assume local thermodynamic equilibrium (LTE) so that the monochromatic source function, $S_\lambda(r_i)$, is given by the Planck function at the local kinetic temperature, $B_\lambda(T_{\text{kin}}(r_i))$. The surface intensity for out-going beams that are core-intersecting ($\mu_c \leq \mu_{0,k} \leq 1$), comparable to Eq. 28 of C66, is evaluated as

$$I_\lambda^+(r_N, \mu_{0,k}) \approx B_\lambda(r_0) \Phi_\lambda \left\{ r_0, \left[1 - \left(\frac{r_N}{r_0} \right)^2 (1 - \mu_{0,k}^2) \right]^{1/2} \right\} \\ + \sum_{i=1}^N \Phi_\lambda \left\{ r_i, \left[1 - \left(\frac{r_N}{r_i} \right)^2 (1 - \mu_{0,k}^2) \right]^{1/2} \right\} B_\lambda(r_i) \frac{\Delta\tau_{\lambda,i}}{\left[1 - \left(\frac{r_N}{r_i} \right)^2 (1 - \mu_{0,k}^2) \right]^{1/2}} \quad (5)$$

The surface intensity for out-going beams that are non-core-intersecting ($0 \leq \mu_{0,k} < \mu_c$), comparable to Eq. 29 of C66, is evaluated as

$$I_\lambda^+(r_N, \mu_{0,k}) \approx I_\lambda^-(r_b, 0) \Phi_\lambda \{r_b, 0\} \\ + \sum_{i=b}^N \Phi_\lambda \left\{ r_i, \left[1 - \left(\frac{r_N}{r_i} \right)^2 (1 - \mu_{0,k}^2) \right]^{1/2} \right\} B_\lambda(r_i) \frac{\Delta\tau_{\lambda,i}}{\left[1 - \left(\frac{r_N}{r_i} \right)^2 (1 - \mu_{0,k}^2) \right]^{1/2}} \quad (6)$$

where $I_\lambda^-(r_b, 0)$ on the RHS of Eq. 6, comparable to Eq. 27 of C66, is evaluated as

$$I_\lambda^-(r_b, 0) \approx \sum_{i=b}^N \left(\frac{\Phi_\lambda \{r_b, 0\}}{\Phi_\lambda \left\{ r_i, \left[1 - \left(\frac{r_b}{r_i} \right)^2 \right]^{1/2} \right\}} \right) B_\lambda(r_i) \frac{\Delta\tau_{\lambda,i}}{\left[1 - \left(\frac{r_b}{r_i} \right)^2 (1 - \mu_{0,k}^2) \right]^{1/2}} \quad (7)$$

2.2. Hydrostatic equilibrium (HSE)

We also account for the effect of sphericity on the hydrostatic pressure structure, $P(r)$. Following Lester & Neilson (2008) and Fieldus, Lester & Rogers (1990) the HSE is now

$$\frac{dP}{d\tau_{\text{Ros}}} = \frac{g(\tau_{\text{Ros}})}{\kappa(\tau_{\text{Ros}})} \quad (8)$$

where $g(r)$ is evaluated as GM/r^2 with the stellar mass, M , held constant at its total value and the $\{r_i\}$ grid is computed as described above.

We adapt the formal solution of Gray (2005) that we numerically integrate inward so that it now accounts in the integrand for the dependence of the gravitational acceleration, g , on the radial height, r ,

$$P(\tau_{\text{Ros},i}) = \left\{ \frac{3}{2} \sum_{j=N}^i \frac{\tau_{\text{Ros},j} g(\tau_{\text{Ros},j}) P_j^{1/2}}{\kappa_{\text{Ros},j}} \Delta \log \tau_{\text{Ros},j} \right\}^{2/3} \quad (9)$$

where the $\{P_i\}$ values on the RHS are the current guess of the total pressure structure.

2.3. Modeling parameters

We now need an additional independent input parameter in our modelling procedure to evaluate r_0 , and we have chosen stellar mass, M . The r_0 value is then obtained from the input value of $\log g$ as $\log r_0 = 0.5(\log G + \log M - \log g)$. We have provided a new Boolean input parameter, `ifSphere`, that allows the user to select between the spherical and PP formal solutions of the RTE and the HSE.

3. Results

Fieldus, Lester & Rogers (1990) implemented a spherical formal solution for the HSE, RTE and the radiative thermal equilibrium solution, in the ATLAS code (Kurucz (2014)) and studied the effect on the modeled structure and emergent spectral energy distribution

(SED) of a set of models of an early-type star of T_{eff} value equal to 10 000 K with varying $\log g$ and M values. In Fig. 1 we show our computed SEDs ($\log F_{\nu}$ vs $\log \nu$) for models of T_{eff} value equal to 10 000 K, M equal to $2.0 M_{\text{Sun}}$ and $\log g$ values of 4.0 and 1.5, computed in PP and spherical geometry for comparison to Fig. 2 of Fieldus, Lester & Rogers (1990). The SEDs for the models of $\log g$ value of 4.0 are indistinguishable in this plot, which is to be expected at relatively high $\log g$ values. For the models of $\log g$ value of 1.5, the SEDs are only barely distinguishable in this plot, with the most visible difference being that the PP SED is slightly greater than the spherical SED in the broad region where F_{ν} peaks. We emphasize that our models all have the same $T_{\text{kin}}(\tau)$ structure and only differ in the solution to the formal solutions of the RTE and HSE, whereas Fieldus, Lester & Rogers (1990) are comparing SEDs for atmospheric models in which the radiative equilibrium $T_{\text{kin}}(\tau)$ structure is also converged consistently with the geometry.

Neilson & Lester (2013a) compared the centre-to-limb intensity variation (CLIV) and the fitted parameters of a number of limb darkening laws for spherical and PP models based on the ATLAS code Kurucz (2014) in a number of photometric bandpasses, including that of the *Kepler* space telescope, and their investigation includes a model of T_{eff} equal to 5000 K, $\log g$ equal to 2.0, and M equal to $5.0 M_{\text{Sun}}$. In Fig. 2 we show our CLIV computed with the Chroma+ suite for a model of these parameters with the spherical and PP formal solutions at a wavelength from our background continuum λ -grid of 653.1 nm, close to the centre of the *Kepler* bandpass. This Figure may be compared with Fig. 1 of Neilson & Lester (2013a). Our results are at least qualitatively similar to theirs in that we find the greatest difference between the two geometries is near the limb where μ_0 is less than ~ 0.1 , with the spherical CLIV dropping precipitously below that from the PP model. This is to be expected - at low μ_0 values near the limb, the $I_{\lambda}^{+}(\tau = 0, \mu_0)$ beams are emerging from an optically thick path-length in a semi-infinite planar atmosphere, whereas for the spherical model they are emerging from a finite path-length through a spherical shell that does not

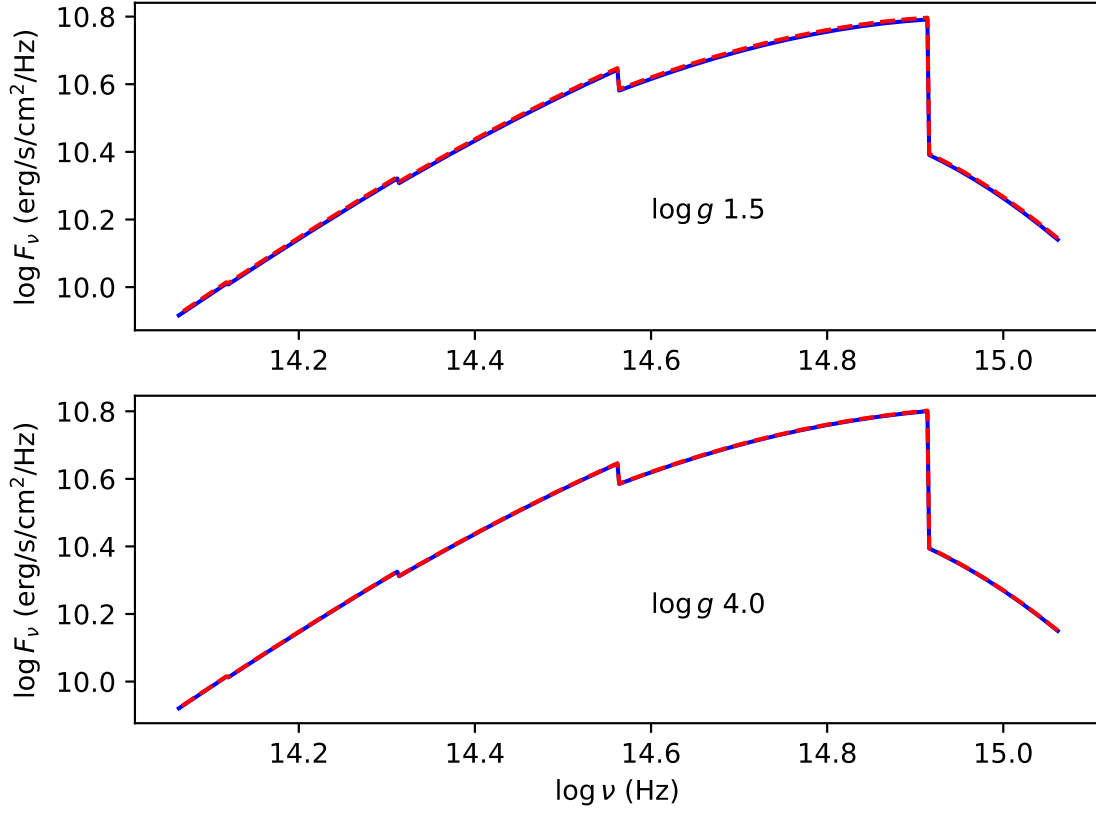


Fig. 1.— Spectral energy distribution (SED) for the test models of Fieldus, Lester & Rogers (1990) of T_{eff} of 10 000 K, M of $2.0 M_{\text{Sun}}$ and $\log g$ 4.0 (upper panel) and 1.5 (lower panel) computed with the spherical solution (blue solid line) and the PP solution (red dashed line), comparable to Fig. 2 of Fieldus, Lester & Rogers (1990).

intersect the opaque inner core.

To draw out the implications for predicted exo-planetary transit light-curves, in Fig. 3 we show the predicted transit light-curves for a host star of the same parameters and a planet of radius $1 R_{\text{Jup}}$, a circular orbit of radius 1 AU, and an orbital inclination, i , of 90° . For the spherical geometry, the apparent onset of ingress is delayed, and the light variation at ingress is more rapid, than that for PP geometry, and this is consistent with the more rapid decline in $I_\lambda^+(\tau = 0, \mu_0)$ with decreasing μ_0 at small μ_0 values near the limb in the spherical model as compared to the PP model.

4. Discussion

The Chroma+ suite may now be used for projects in which the student investigates the limb darkening of exo-planetary host stars with $I_\lambda^+(\tau = 0, \mu_0)$ distributions computed with spherical and planar geometry and the corresponding exo-planetary transit light-curves. This includes projects in which students investigate the realism of various limb-darkening laws, find best-fit limb darkening coefficients for various laws, and investigate the effect on inferred exo-planet and host star properties (Neilson & Lester (2013a), Neilson & Lester (2013b)).

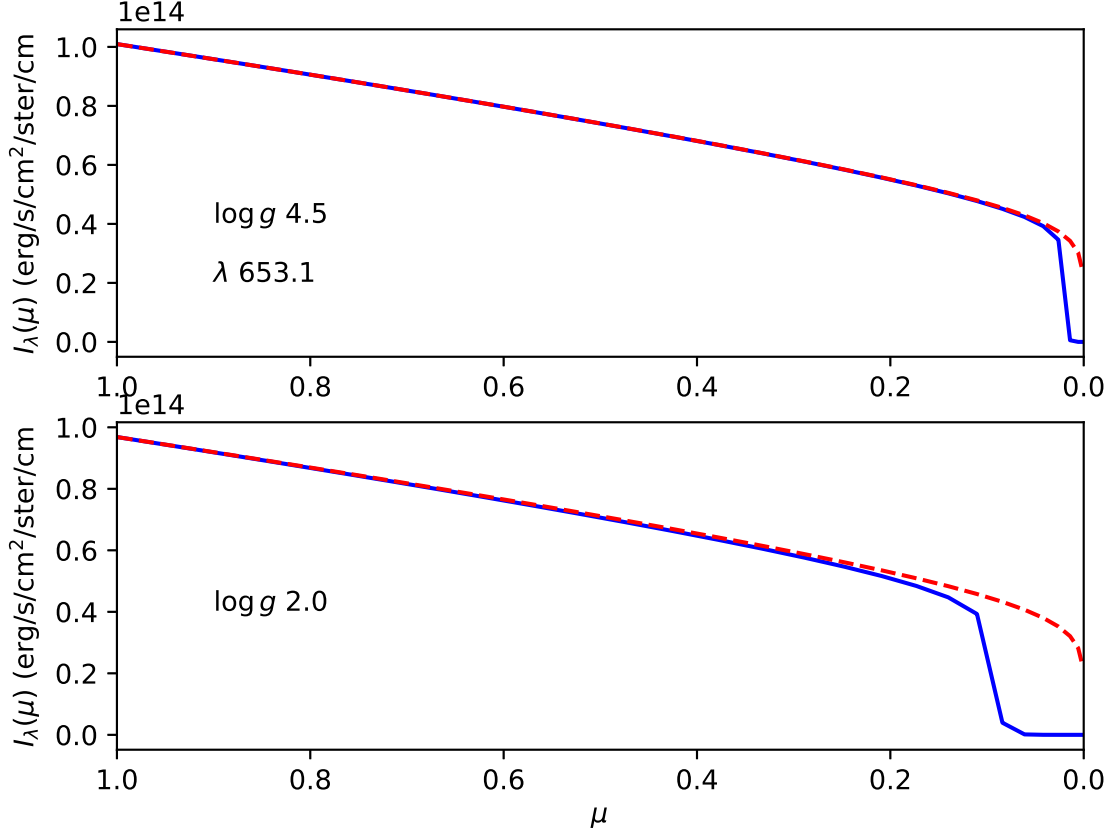


Fig. 2.— The continuum limb darkening curve for the test model of Neilson & Lester (2013a) of T_{eff} of 5000 K, $\log g$ of 2.0, and M of $5.0 M_{\text{Sun}}$ (lower panel) computed with the spherical solution (blue solid line) and the PP solution (red dashed line). For comparison we also include the limb darkening for a model of the same parameters except for a $\log g$ value of 4.5 (upper panel). The limb darkening is shown for a λ value from our background continuum λ grid of 653.1 nm, near the centre of the Kepler bandpass, and is comparable to Fig. 1 of Neilson & Lester (2013a).

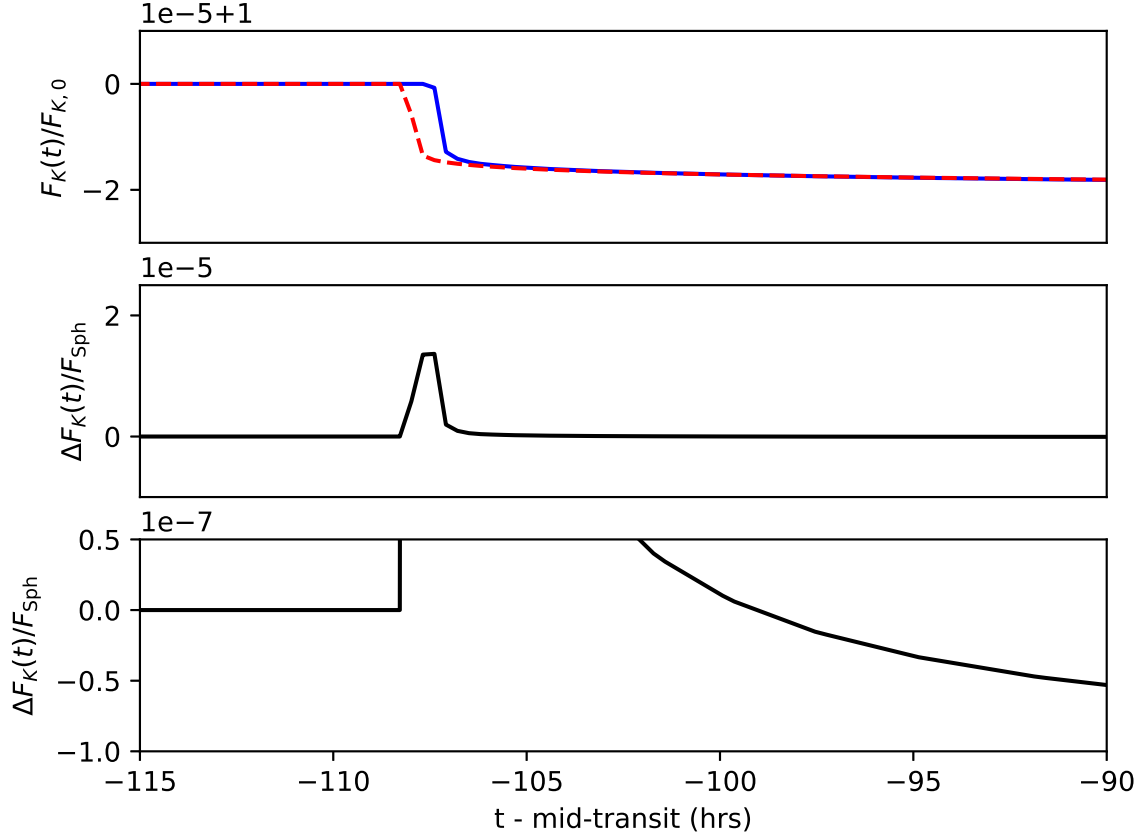


Fig. 3.— Upper panel: The transit light-curve at ingress for the model of Fig. 2 and a planet of radius equal to $1R_{\text{Jup}}$ and orbital radius equal to 1 AU computed with the spherical solution (blue solid line) and the PP solution (red dashed line). Middle and lower panels: The relative difference between the light-curve computed with the spherical and the PP model.

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