Redicoloring some classes of circulant torunaments

Narda Cordero-Michel*, Mika Olsen

Departamento de Matemáticas Aplicadas y Sistemas,
Universidad Autónoma Metropolitana - Cuajimalpa, México

Abstract

Given a digraph D with no loops, the *dicoloring graph* of D, denoted by $\mathcal{D}_k(D)$, is the graph whose vertices are the acyclic k-colorings of D and two colorings are adjacent in $\mathcal{D}_k(D)$ if they differ in color on exactly one vertex. In this paper, we prove that there is no expression $\phi(\vec{\chi})$ in terms of the dichromatic number $\vec{\chi}$, such that the graph $\mathcal{D}_k(D)$ is connected for all graphs D and integers $k \geq \phi(\vec{\chi})$. We give conditions for the dicoloring graph of two infinite families of circulant tournaments to be connected and we provide upper bounds for its diameter. In particular, for the Payley tournament $\vec{C}_7(1,2,4)$, also known as ST_7, we prove that $\mathcal{D}_k(\vec{C}_7(1,2,4))$ is connected and has diameter 8, for each $k \geq 3$.

Keywords: Dicoloring graph, acyclic k-coloring, k-mixing, circulant tournament

1. Introduction

Cereceda ([2], 2007) focused his doctoral research on the subject of graph recolouring of a k-colorable graph G, which is a reconfiguration problem were it is analyzed if a k-coloring of G might be transformed into another k-coloring of G by recoloring one vertex at a time in such a way that a proper k-coloring is maintained at any step. This problem is represented by the k-color graph (or k-recoloring graph) $\mathcal{C}_k(G)$, it is the graph that has the set of proper k-colorings of G as its vertex set and two k-colorings are adjacent whenever they differ in color on precisely one vertex of G. He obtained significant results for this problem when $k = 3$, he determined the complexity of deciding if the coloring graph is connected for $k \geq 3$ and, when $\mathcal{C}_k(G)$ is connected, he investigated the length of a shortest path in $\mathcal{C}_k(G)$ between two colorings. Ever since, many authors have studied graph recolorings, for a survey on this matter we refer the reader to [4]. Bousquet and his collaborators [1] studied digraph redicolorings as a generalization of the graph recoloring problem. They introduced the concept of the k-dicoloring graph $\mathcal{D}_k(D)$ of a digraph D, it is the graph whose vertex set is the set of acyclic k-colorings of D, with two k-colorings being adjacent whenever

*Corresponding author.

they differ on the color of exactly one vertex. Bousquet *et al.* proved that it is PSPACE-complete to decide whether two given k-colorings of a digraph D belong to the same connected component of $\mathcal{D}_k(D)$ even when $k = 2$, when D has maximum degree equal to 5, or when D is an oriented planar graph with maximum degree equal to 6. Moreover, they also proved sufficient conditions for $\mathcal{D}_k(D)$ being connected in terms of some parameters of the maximum degree and minimum degree in D, they provided upper bounds for the diameter of $\mathcal{D}_k(D)$, as well as a lower bound for the number of edges in an oriented graph whose dicoloring graph has isolated vertices, and they established some conjectures. In [8], Picasarri-Arrieta proved sufficient conditions for $\mathcal{D}_k(D)$ being connected in terms of the maximum semi-degree in D and he gave upper bounds for the diameter of the dicoloring graph. In [7], Nisse *et. al.* obtained bounds for the diameter of $\mathcal{D}_k(D)$ in terms of the cycle degeneracy of D, the digrundy number of D, and the $\mathcal{D}$-width of D. Furthermore, they showed a connection between the dicoloring graph of a digraph D and the color graph of its underlying graph $UG(D)$.

In [3], Cereceda proved that there is no expression $\phi(\chi)$ in terms of the chromatic number χ, such that the graph $\mathcal{C}_k(G)$ is connected for all graphs G and integers $k \geq \phi(\chi)$. In this direction, we will see that there is no expression $\phi(\vec{\chi})$ in terms of the dichromatic number $\vec{\chi}$, such that the graph $\mathcal{D}_k(D)$ is connected for all digraphs D and integers $k \geq \phi(\vec{\chi})$. He also proved that if $k = \chi(G) \leq 3$, then $\mathcal{C}_k(G)$ is disconnected and if $k = \chi(G) \geq 4$, then $\mathcal{C}_k(G)$ may be connected or disconnected. In this direction, we will see that there is an infinite family of 2-mixing 2-dichromatic tournaments, an infinite family of 3-mixing 3-dichromatic tournaments and a 3-dichromatic tournament which is not 3-mixing.

In [8], Picasarri-Arrieta proved for every digraph D with $\Delta \geq 3$, where $\Delta = \max\{\max(d^+(v), d^-(v)) \colon v \in V(D)\}$, and every $k \geq \Delta + 1$, that $\mathcal{D}_k(D)$ consists of isolated vertices and at most one further component that has diameter at most $c_\Delta n^2$, where $c_\Delta = O(\Delta^2)$ is a constant depending only on Δ. In this paper, we improve this result for two families of circulant tournaments $\vec{C}_{2n+1}\langle\emptyset\rangle$ and $\vec{C}_{2n+1}\langle n\rangle$ defined in the following section. We show for $n \geq 2$ and $k \geq 3$ that $\mathcal{D}_k(\vec{C}_{2n+1}\langle\emptyset\rangle)$ is connected and has diameter at most $4n+1+\lfloor\frac{n+1}{2}\rfloor$; we also prove for $n \geq 4$ and $k \geq 3$ that $\mathcal{D}_k(\vec{C}_{2n+1}\langle n\rangle)$ is connected and has diameter at most $4n+2+\lfloor\frac{n}{2}\rfloor$, whenever $k \neq \frac{2n+1}{3}$; and $\mathcal{D}_k(\vec{C}_{2n+1}\langle n\rangle)$ consists of isolated vertices and one further component that has diameter at most $2n+1+\lfloor\frac{2n+1}{4}\rfloor$ whenever $k = \frac{2n+1}{3}$.

Along this paper we will work with families of digraphs without digons, also called *oriented graphs*, whose dichromatic numbers are known. We omit "whithout digons" from the hypotheses and we use that any pair of vertices induces an acyclic set without mention it.

2. Definitions

An *acyclic k-coloring* of a digraph D is a function $\alpha\colon V(D) \to \{1, 2, \ldots, k\}$ such that each *color class*, not necessarily nonempty, $C_i^\alpha = \{x \in V(D)\colon \alpha(x) = i\}$ with $i \in \{1, 2, \ldots, k\}$, induces an acyclic subdigraph of D. The *dichromatic number* of a digraph D, denoted by $\vec{\chi}(D)$, is the minimum number k for which D admits an acyclic k-coloring of its vertex set, and we say that D is $\vec{\chi}(D)$-*dichromatic*. In this paper, every coloring is acyclic, so we will omit the word "acyclic".

For any $k \geq \vec{\chi}(D)$, the *k-dicoloring graph of* D, denoted by $\mathcal{D}_k(D)$, is defined as the graph whose vertices are the k-colorings of D, and two k-colorings of D are *adjacent* whenever they differ on precisely one vertex. Two colorings of D, α and β, are at *distance* d, denoted by $d(\alpha, \beta)$, if there exists a sequence of colorings of D $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_d = \beta$ such that α_i and α_{i+1} are adjacent for each $i \in \{0, 1, \ldots, d-1\}$ and any other sequence with this property has at least the same number of elements. Notice that, if there exists a k-coloring of D, then it has at least $\frac{k!}{(k-p)!}$ different k-colorings, where p is the number of nonempty color classes. The digraph D is *k-mixing* if $\mathcal{D}_k(D)$ is connected and *k-freezable* if $\mathcal{D}_k(D)$ contains an isolated vertex.

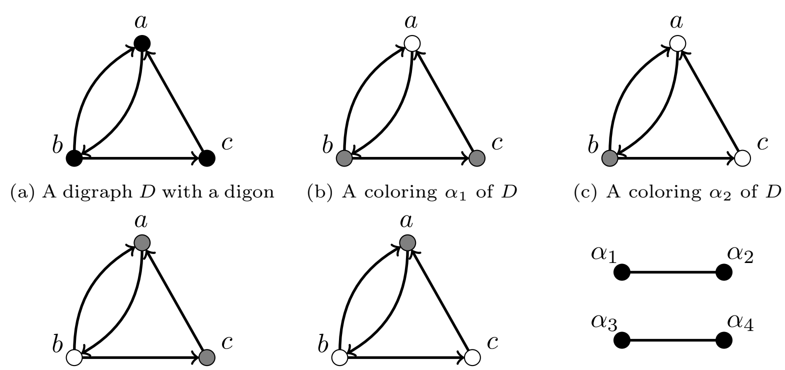
A digraph D is *uniquely k-colorable* for some k, $k \geq 2$, whenever every k-coloring of D induces the same partition of $V(D)$, this is when any two k-colorings of D differ only in a permutation of the colors used for the same set of color classes.

A digraph D on n vertices is always $(n+1)$-mixing as: 1) any k-coloring α, with $k \leq n+1$, can be converted into a $(n+1)$-coloring β simply by changing the colors of vertices, one by one, in such a way that each color class in β is a singleton; 2) if β and γ are two $(n+1)$-colorings of D such that each of their color classes is a singleton, then β can be converted into γ in the following way, for each vertex $x \in V(D)$, if x is not colored with $\gamma(x)$ and $\gamma(x)$ is not currently assigned to any vertex, then recolor x with $\gamma(x)$, and if $\gamma(x)$ is currently assigned to y, then first recolor y with the color that is not currently assigned to any vertex, and then recolor x with $\gamma(x)$. Hence, any two k-colorings, with $k \leq n+1$, can be converted into each other by recoloring one vertex at a time.

It is immediate from the definition of a dicoloring graph that if D is a digraph with digons, then its 2-dicoloring graph $\mathcal{D}_2(D)$ is disconnected (see example in Figure 1).

For $n \geq 2$, the *cyclic circulant tournament*, denoted by $\vec{C}_{2n+1}(1, 2, \ldots, n)$, is the digraph whose vertex set is $V(\vec{C}_{2n+1}(1, 2, \ldots, n)) = \mathbb{Z}_{2n+1}$ and its arc set consists of all the arcs $(a, a+j)$ for every jump $j \in \{1, 2, \ldots, n\}$ and every $a \in \mathbb{Z}_{2n+1}$.

The circulant tournament obtained from $\vec{C}_{2n+1}(1, 2, \ldots, n)$ by reversing one of its jumps, say j, is denoted by $\vec{C}_{2n+1}\langle j \rangle$, this is, for a fixed j the set of arcs $\{(a, a+j)\colon a \in \mathbb{Z}_{2n+1}\}$ of $\vec{C}_{2n+1}(1, 2, \ldots, n)$ is replaced by the set of arcs $\{(a, a-j)\colon a \in \mathbb{Z}_{2n+1}\}$ in $\vec{C}_{2n+1}\langle j \rangle$. In this way, we can denote the cyclic circulant tournament $\vec{C}_{2n+1}(1, 2, \ldots, n)$ by $\vec{C}_{2n+1}\langle \emptyset \rangle$.



(a) A digraph D with a digon (b) A coloring α_1 of D (c) A coloring α_2 of D

(d) A coloring α_3 of D (e) A coloring α_4 of D (f) $\mathcal{D}_2(D)$

Figure 1: A 2-dichromatic digraph D, with a disconnected 2-dicoloring graph.

3. Preliminaries

First of all we establish an obvious proposition that has consequences for a family of digraphs studied by Neumann-Lara and his collaborators in [5].

Proposition 1. *If a digraph D is uniquely k-colorable, then D is k-freezable and $\mathcal{D}_k(D)$ is the graph with $k!$ isolated vertices.*

In [5], Neumann-Lara *et al.* proved that there exists an infinite family of uniquely colorable k-dichromatic tournaments, for all $k \geq 2$. They also proved that the tournament $\vec{C}_{2n+1}\langle n\rangle - \{0\}$ is uniquely 2-colorable, for $n \geq 4$, with chromatic classes $\{1, 2, \ldots, n\}$ and $\{n+1, n+2, \ldots, 2n\}$.

Corollary 2. *For each $k \geq 2$ there exists a tournament T such that T is k-freezable and it's dicoloring graph has $k!$ isolated vertices. In particular, for all $n \geq 4$, $\vec{C}_{2n+1}\langle n\rangle - \{0\}$ is 2-freezable and $\mathcal{D}_2(\vec{C}_{2n+1}\langle n\rangle - \{0\})$ consists of two isolated vertices.*

In the following two lemmas, we analyze some particular cases where at least one of two colorings has a lot of singular color classes, in these cases it is easier to determine the distance between the colorings.

Lemma 3. *Let D be a digraph of order n and k and l integers with $2 \leq k \leq \min\{l, n\}$. And let α and β be two l-colorings of D, where β has k singular color classes, C_1^β, C_2^β, $\ldots$, C_k^β, and $\alpha(u) = \beta(u)$ for each $u \in V(D) \setminus \left(\bigcup_{i=1}^k C_i^\beta\right)$. Then $d(\alpha, \beta) \leq k$ in $\mathcal{D}_n(D)$.*

Proof. Proceeding by induction on k.

Whenever $k = 2$, we have that $C_1^\alpha \cup C_2^\alpha (\subseteq C_1^\beta \cup C_2^\beta$. Start at α and recolor the vertices in $C_1^\beta \cup C_2^\beta$ with the color assigned to them by β, this takes at most two steps since D has no digon and any pair of vertices induces an acyclic set.

Assume that the result is valid for $k \geq 2$. And suppose that β has $k+1$ singular color classes C_1^β, C_2^β, $\dots$, C_{k+1}^β, and $\alpha(u) = \beta(u)$ for each $u \in V(D) \setminus \left(\bigcup_{i=1}^{k+1} C_i^\beta\right)$. Hence, $\bigcup_{i=1}^{k+1} C_i^\alpha \subseteq \bigcup_{i=1}^{k+1} C_i^\beta$.

Start at α. If there is an index $j \in \{1, 2, \dots, k+1\}$ such that $C_j^\alpha = \emptyset$, then recolor the unique vertex v in C_j^β with color $\beta(v)$. Call the new coloring α' and let $J = \{1, 2, \dots, k+1\} \setminus \{j\}$. The coloring α' satisfies that $d(\alpha, \alpha') = 1$, $\alpha'(u) = \beta(u)$ for each vertex in $V(D) \setminus \left(\bigcup_{i \in J} C_i^\beta\right)$ and $|C_i^\beta| = 1$ for each $i \in J$. So, by the inductive hypothesis, $d(\alpha', \beta) \leq k$. And thus, $d(\alpha, \beta) \leq k+1$.

If $C_i^\alpha \neq \emptyset$ for each $i \in \{1, 2, \dots, k+1\}$, then C_i^α is a singleton for each $i \in \{1, 2, \dots, k+1\}$. Assume that $C_i^\alpha \neq C_i^\beta$ for each $i \in \{1, 2, \dots, k+1\}$. Call u_i the vertex in C_i^α. Recolor vertex u_1 with color $\beta(u_1)$, then recolor vertex $u_{\beta(u_1)}$ with color $\beta(u_{\beta(u_1)})$, continue in this way until each vertex has the color that β assigns to it, this takes $k+1$ steps, and each intermediary step is a coloring since D has no digon. Then, $d(\alpha, \beta) \leq k+1$. $\square$

Lemma 4. *Let D be a digraph of order n , and k and l integers with $2 \leq k \leq \min\{l, n-1\}$. And let α and β be two l-colorings of D, where β has a color class with two vertices, C_1^β, $k-1$ singular color classes, C_2^β, C_3^β, $\dots$, C_k^β, and $\alpha(u) = \beta(u)$ for each $u \in V(D) \setminus \left(\bigcup_{i=1}^{k} C_i^\beta\right)$. Then $d(\alpha, \beta) \leq k+1$ in $\mathcal{D}_n(D)$.*

Proof. Proceeding by induction on k.

Whenever $k = 2$, $C_1^\beta = \{u, v\}$ and $C_2^\beta = \{w\}$ and $C_1^\alpha \cup C_2^\alpha \subseteq \{u, v, w\}$.

Start at α. If $C_i^\alpha = \emptyset$, then $C_{3-i}^\alpha \subseteq \{u, v, w\}$ and we can recolor the vertices in $\{u, v, w\}$ to get β in at most 3 steps.

If $|C_i^\alpha| = 1$, then $|C_{3-i}^\alpha| = 2$. Assuming $\alpha \neq \beta$, we can recolor one vertex in C_{3-i}^α with the color assigned to it by β, namely i. Then, if needed, recolor the vertex in C_i^α with color $3-i$. And finally, recolor the last vertex, if needed, with color i. This takes at most 3 steps.

Assume that the result is valid for $k \geq 2$. And suppose that β has a color class with two vertices, namely C_1^β, k singular color classes, namely C_2^β, C_3^β, $\dots$, C_{k+1}^β, and $\alpha(u) = \beta(u)$ for each $u \in V(D) \setminus \left(\bigcup_{i=1}^{k+1} C_i^\beta\right)$. Hence, $\bigcup_{i=1}^{k+1} C_i^\alpha \subseteq \bigcup_{i=1}^{k+1} C_i^\beta$.

Start at α. If $C_1^\alpha = \emptyset$, then recolor the two vertices in C_1^β with color 1. The new coloring α' satisfies $d(\alpha, \alpha') = 2$ and, by Lemma 3, $d(\alpha', \beta) \leq k$. Hence, $d(\alpha, \beta) \leq k+2$. Assume that $C_1^\alpha \neq \emptyset$

If there is an index $j \in \{2, 3, \dots, k+1\}$ such that $C_j^\alpha = \emptyset$, then recolor the unique vertex v in C_j^β with color $\beta(v)$. Call the new coloring α' and let $J = \{2, 3, \dots, k+1\} \setminus \{j\}$. The coloring α' satisfies that $d(\alpha, \alpha') = 1$, $\alpha'(u) = \beta(u)$ for each vertex in $V(D) \setminus \left(\bigcup_{i \in \{1\} \cup J} C_i^\beta\right)$ and $|C_i^\beta| = 1$ for each $i \in J$. So, by the inductive hypothesis, $d(\alpha', \beta) \leq k+1$. And thus, $d(\alpha, \beta) \leq k+2$.

If $C_i^\alpha \neq \emptyset$ for each $i \in \{1, 2, \dots, k+1\}$, then C_j^α has precisely two elements for some $j \in \{1, 2, \dots, k+1\}$, and C_i^α is a singleton for each $i \in J$, where $J = \{1, 2, \dots, k+1\} \setminus \{j\}$. If $j = 1$ and $C_1^\alpha = C_1^\beta$, then $d(\alpha, \beta) \leq k$, by Lemma

3. So, we assume this is not the case. Necessarily, one of the vertices in C_j^α, say v, can be recolored with the color assigned to it by β, so we recolor v with $\beta(v)$. The new coloring α' satisfies that $C_{\beta(v)}^{\alpha'}$ has precisely two elements. If the other vertex in $C_{\beta(v)}^{\alpha'}$, say w, has $\alpha'(w) = \beta(w)$, then $d(\alpha', \beta) \leq k$, by Lemma 3. If $\alpha'(w) \neq \beta(w)$, then we recolor vertex w with $\beta(w)$. Proceeding this way, eventually, we get β by recoloring each vertex in $\bigcup_{i=1}^{k+1} C_i^\beta$ at most once. Hence, $d(\alpha, \beta) \leq k+2$. This concludes the proof. □

Corollary 5. *Any digraph D of order n is k-mixing for every $k \geq n-1$ and $\mathcal{D}_k(D)$ has diameter at most $2n$.*

4. The cyclic circulant tournament $\vec{C}_{2n+1}\langle\emptyset\rangle$

The following result is an extended version of a Theorem 1 in [6] that will help us to prove the result announced at the introduction.

Theorem 6 (Neumann-Lara and Urrutia [6])**.** *Let n be a positive integer. Then $\vec{\chi}(\vec{C}_{2n+1}\langle\emptyset\rangle) = 2$ and every 2-coloring of $\vec{C}_{2n+1}\langle\emptyset\rangle$ induces a partition of $V(\vec{C}_{2n+1}\langle\emptyset\rangle)$ into two sets of the form $\{a, a+1, a+2, \ldots, a+n\}$ and $\{a+n+1, a+n+2, \ldots, a-1\}$, with $a \in \mathbb{Z}_{2n+1}$. Moreover, if w, x and y are three different vertices in $\mathbb{Z}_{2n+1}$ satisfying that $\{w, x, y\} \nsubseteq \{a, a+1, a+2, \ldots, a+n\}$, for each $a \in \mathbb{Z}_{2n+1}$, then $\{w, x, y\}$ induces a triangle in $\vec{C}_{2n+1}\langle\emptyset\rangle$.*

Proposition 7. *The 2-dicoloring graph of the cyclic circulant tournament $\vec{C}_{2n+1}\langle\emptyset\rangle$ is the cycle C_{4n+2}. Moreover,* $\operatorname{diam}(\mathcal{D}_2(\vec{C}_{2n+1}\langle\emptyset\rangle)) = 2n+1$.

Proof. Let $T = \vec{C}_{2n+1}\langle\emptyset\rangle$. For each $a \in \mathbb{Z}_{2n+1}$, let α_a and β_a be the 2-colorings of T given by

$$\alpha_a(i) = \begin{cases} 1 & \text{if } i \in \{a, a+1, a+2, \ldots, a+n\}; \\ 2 & \text{if } i \in \{a+n+1, a+n+2, \ldots, a-1\}. \end{cases}$$

and

$$\beta_a(i) = \begin{cases} 1 & \text{if } i \in \{a, a+1, a+2, \ldots, a+n-1\}; \\ 2 & \text{if } i \in \{a+n, a+n+1, \ldots, a-1\}. \end{cases}$$

Since every 2-coloring of T induces a partition of $V(T)$ into two sets of the form $\{a, a+1, a+2, \ldots, a+n\}$ and $\{a+n+1, a+n+2, \ldots, a-1\}$, any 2-coloring of T corresponds to α_a or β_a for some $a \in \mathbb{Z}_{2n+1}$. Hence, $V(\mathcal{D}_2(T)) = \{\alpha_a : a \in \mathbb{Z}_{2n+1}\} \cup \{\beta_a : a \in \mathbb{Z}_{2n+1}\}$. In the coloring $\alpha_a(i)$ the only vertices whose color can be changed are the vertices a and $a+n$, thus the vertex $\alpha_a(i)$ has degree 2 in the graph $\mathcal{D}_2(T)$. Analogously, $\beta_a(i)$ has degree 2 in the graph $\mathcal{D}_2(T)$.

Claim 1. For each $a \in \mathbb{Z}_{2n+1}$, $N_{\mathcal{D}_2(T)}(\alpha_a) = \{\beta_a, \beta_{a+1}\}$ and $N_{\mathcal{D}_2(T)}(\beta_a) = \{\alpha_{a-1}, \alpha_a\}$.

We have that $\alpha_a(a+n) = 1 \neq 2 = \beta_a(a+n)$ and $\alpha_a(i) = \beta_a(i)$ whenever $i \neq a+n$, thus, α_a and β_a are adjacent in $\mathcal{D}_2(T)$; $\alpha_a(a) = 1 \neq 2 = \beta_{a+1}(a)$

and $\alpha_a(i) = \beta_{a+1}(i)$ whenever $i \neq a$, thus, α_a and β_{a+1} are adjacent in $\mathcal{D}_2(T)$. Hence, $\{\beta_a, \beta_{a+1}\} \subseteq N_{\mathcal{D}_2(T)}(\alpha_a)$ and analogously, $\{\alpha_{a-1}, \alpha_a\} \subseteq N_{\mathcal{D}_2(T)}(\beta_a)$. Since each vertex in $\mathcal{D}_2(T)$ has degree equal to 2, it follows that $N_{\mathcal{D}_2(T)}(\alpha_a) = \{\beta_a, \beta_{a+1}\}$ and $N_{\mathcal{D}_2(T)}(\beta_a) = \{\alpha_{a-1}, \alpha_a\}$.

From Claim 1, $W = \beta_0\alpha_0\beta_1\alpha_1 \cdots \beta_{2n}\alpha_{2n}\beta_0$ is a Hamiltonian cycle in $\mathcal{D}_2(T)$ and each vertex in $\mathcal{D}_2(T)$ has degree 2. Therefore, $\mathcal{D}_2(T)$ is a cycle of length $4n + 2$. $\square$

Theorem 8. *Let $k \geq 3$. Then $\vec{C}_{2n+1}\langle\emptyset\rangle$ is k-mixing and $\mathcal{D}_k(\vec{C}_{2n+1}\langle\emptyset\rangle)$ has diameter at most $4n + 1 + \lfloor \frac{n+1}{2} \rfloor$.*

Proof. Let $T = \vec{C}_{2n+1}\langle\emptyset\rangle$. Since T is 2-dichromatic, $\mathcal{D}_k(T)$ is defined for every $k \geq 2$. Let $k \geq 3$. By Theorem 6, for any k-coloring α of T, and any color class C_r^α with $r \in \{1, 2, \ldots, k\}$, there exists an $a \in \mathbb{Z}_{2n+1}$, such that $C_r^\alpha \subseteq \{a, a+1, \ldots, a+n\}$. Let $\alpha, \beta \colon \mathbb{Z}_{2n+1} \to \{1, 2, \ldots, k\}$ be two k-colorings of T. And let C_1^α, C_2^α, $\ldots$, C_k^α and C_1^β, C_2^β, $\ldots$, C_k^β be the color classes of α and β, respectively.

Case 1. *$C_i^\alpha = \emptyset$ for some $i \in \{1, 2, \ldots, k\}$.*

Suppose w.l.o.g. that $C_k^\alpha = \emptyset$. Let $b \in \mathbb{Z}_{2n+1}$, such that $C_k^\beta \subseteq \{b, b+1, \ldots, b+n\}$. Assume w.l.o.g. that $b = 0$, then $C_k^\beta \subseteq \{0, 1, 2, \ldots, n\}$. Start at α and recolor each vertex in $\{0, 1, 2, \ldots, n\}$ with color k, this takes $n + 1$ steps. Let α' be the current coloring. Recolor, when needed, each vertex u in $\{n + 1, n+2, \ldots, 2n\}$ with $\beta(u)$. Finally, recolor each vertex u in $\{0, 1, 2, \ldots, n\} \setminus C_k^\beta$ with $\beta(u)$. The total process takes at most $n + 1 + n + n - |C_k^\beta| \leq 3n + 1$.

Case 2. *There are two color classes C_i^α and C_j^α, with $\{i, j\} \subset \{1, 2, \ldots, k\}$ and $i \neq j$, such that $C_i^\alpha \cup C_j^\alpha \subseteq \{a, a+1, \ldots, a+n\}$ for some $a \in \mathbb{Z}_{2n+1}$.*

Suppose w.l.o.g. that $|C_i^\alpha| \geq |C_j^\alpha|$. Start at α, then recolor all vertices in C_j^α with color i. This process takes at most $\lfloor \frac{n+1}{2} \rfloor$ steps, and the new coloring α' satisfies that $C_j^{\alpha'} = \emptyset$, and thus $d(\alpha', \beta) \leq 3n + 1$, by Case 1. Hence, $d(\alpha, \beta) \leq d(\alpha, \alpha') + d(\alpha', \beta) \leq 3n + 1 + \lfloor \frac{n+1}{2} \rfloor$.

Case 3. *For each pair of color classes C_i^α and C_j^α, with $\{i, j\} \subset \{1, 2, \ldots, k\}$ and $i \neq j$, there is no $a \in \mathbb{Z}_{2n+1}$ such that $C_i^\alpha \cup C_j^\alpha \subseteq \{a, a+1, \ldots, a+n\}$.*

Consider a color class C_i^α, with $i \in \{1, 2, \ldots, k\}$, such that $C_i^\alpha \neq \emptyset$. Let $a \in \mathbb{Z}_{2n+1}$ such that $C_i^\alpha \subseteq \{a, a+1, \ldots, a+n\}$. Assume w.l.o.g. that $a = 0$. So, if we start at α we can recolor every vertex in $\{0, 1, 2 \ldots, n\} \setminus C_i^\alpha$ with color i, this process takes at most n steps. Let α' be the current coloring. We have that α' satisfies the hypothesis of Case 2, so $d(\alpha', \beta) \leq 3n+1+\lfloor \frac{n+1}{2} \rfloor$, and thus $d(\alpha, \beta) \leq d(\alpha, \alpha') + d(\alpha', \beta) \leq 4n + 1 + \lfloor \frac{n+1}{2} \rfloor$.

$\square$

5. The circulant tournament $\vec{C}_{2n+1}\langle n\rangle$

In what follows we will show a series of results in order to prove our main result on $\vec{C}_{2n+1}\langle n\rangle$ tournaments:

Theorem 9. *Let n and k be two integers, $n \geq 3$ and $k \geq 3$, and consider the cyclic tournament $\vec{C}_{2n+1}\langle n\rangle$. Then either:*

(i) *$\vec{C}_7\langle 3\rangle$ is k-mixing and $\mathcal{D}_k(\vec{C}_7\langle 3\rangle)$ has diameter at most 8,*

(ii) *for $n \geq 5$, $\vec{C}_{2n+1}\langle n\rangle$ is 3-mixing and $\mathcal{D}_3(\vec{C}_{2n+1}\langle n\rangle)$ has diameter at most $3n+4+\lfloor\frac{2n+1}{3}\rfloor$,*

(iii) *for $n \geq 4$, $k \geq 4$ and $k \neq \frac{2n+1}{3}$, $\vec{C}_{2n+1}\langle n\rangle$ is k-mixing and $\mathcal{D}_k(\vec{C}_{2n+1}\langle n\rangle)$ has diameter at most $4n+2+\lfloor\frac{n}{2}\rfloor$, or*

(iv) *for $n \geq 4$ and $k = \frac{2n+1}{3}$, $\vec{C}_{2n+1}\langle n\rangle$ is not k-mixing, $\mathcal{D}_k(\vec{C}_{2n+1}\langle n\rangle)$ has $3k!$ isolated vertices and the distance between two non-frozen colorings α and β is at most $2n+1+\lfloor\frac{2n+1}{4}\rfloor$.*

Theorem 9 follows from Theorems 14, 15, 16 and 17. In order to prove those theorems, we need the following four lemmas:

Lemma 10. *Let $\vec{C}_{2n+1}\langle n\rangle$ and $i \in \mathbb{Z}_{2n+1}$, then the subset $\{i, i+n, i+n+1\}$ of $\mathbb{Z}_{2n+1}$ induces a maximal acyclic subdigraph.*

Proof. Let $T = \vec{C}_{2n+1}\langle n\rangle$. By the definition of T, we have that $(i+n, i)$, $(i+n, i+n+1)$, and $(i, i+n+1)$ are arcs in T. Hence, the subdigraph induced by $\{i, i+n, i+n+1\}$ is acyclic.

Now take j in $\mathbb{Z}_{2n+1} \setminus \{i, i+n, i+n+1\}$ and let D be the subdigraph of T induced by $\{j, i, i+n, i+n+1\}$: if $j \in \{i+1, 1+2, \ldots, i+n-1\}$, then (i, j) and $(j, i+n)$ are arcs in D, and thus $(i, j, i+n, i)$ is a cycle in D; and if $j \in \{i+n+2, i+n+3, \ldots, i-1\}$, then $(i+n+1, j)$ and (j, i) are arcs in D, and thus $(i+n+1, j, i, i+n+1)$ is a cycle in D. $\square$

Lemma 11. *If $V \subseteq \mathbb{Z}_{2n+1}$ induces an acyclic tournament in $\vec{C}_{2n+1}\langle n\rangle$, then $|V| \leq n$. Moreover, if $|V| = n$, then, for some $a \in \mathbb{Z}_{2n+1}$, $V = \{a, a+1, \ldots, a+n-1\}$ or $V = \{a, a+1, \ldots, a+n+1\} \setminus \{a+1, a+n\}$.*

Proof. Let $T = \vec{C}_{2n+1}\langle n\rangle$. To prove the first assertion, consider a set $V \subseteq \mathbb{Z}_{2n+1}$ which induces an acyclic tournament D in T. Then V contains a source a, and thus $|V| \leq |\{a\} \cup N^+(a)| = n+1$. Since $(a+1, a+n-1, a+n+1, a+1)$ is a directed triangle in D, it follows that $|V| \leq n$.

Now, assume $|V| = n$. If $a+n+1 \notin V$, then $V = \{a, a+1, \ldots, a+n-1\}$. If $a+n+1 \in V$, then $a+1 \notin V$ or $(a+1, a+j, a+n+1, a+1)$ would be a directed triangle in D for $1 < j < n$. In this way, $V = \{a, a+1, \ldots, a+n+1\} \setminus \{a+1, a+n\}$. $\square$

We call a subset $\{i, i+n, i+n+1\}$ of $\mathbb{Z}_{2n+1}$ in $\vec{C}_{2n+1}\langle n\rangle$ a *forbidden triangle*.

Lemma 12. *Let n and k be two integers, with $n \geq 4$ and $k \geq 3$. Let $\alpha\colon \mathbb{Z}_{2n+1} \to \{1, 2, \ldots, k\}$ be a coloring of $\vec{C}_{2n+1}\langle n \rangle$. If there is a color class C_j^{α} of α which is not a forbidden triangle, then α can be transformed into a coloring $\beta\colon \mathbb{Z}_{2n+1} \to \{1, 2, \ldots, k\}$ such that its jth color class, C_j^{β}, is $C_j^{\beta} = \{a, a+1, a+2, \ldots, a+n-1\}$ for some $a \in \mathbb{Z}_{2n+1}$. Moreover, in $\mathcal{D}_k(\vec{C}_{2n+1}\langle n \rangle)$, the colorings α and β are at distance at most $n - |C_j^{\alpha}|$ whenever $|C_j^{\alpha}| \in \{0, 1\}$, and at most $n + 2 - |C_j^{\alpha}|$ otherwise.*

Proof. Let $T = \vec{C}_{2n+1}\langle n \rangle$. Since T is 3-dichromatic, $\mathcal{D}_k(T)$ is defined for every $k \geq 3$. Let C_1^{α}, C_2^{α}, $\ldots$, C_k^{α} be the color classes of α. And, suppose w.l.o.g. that $|C_1^{\alpha}|$ is not a forbidden triangle.

Case 1. $|C_1^{\alpha}| \geq 3$.

Then the subdigraph of T induced by C_1^{α}, say H, is a transitive tournament, and thus it has a source. Let a be the source of H, then $V(H) \subseteq \{a\} \cup N^+(a) = \{a, a+1, a+2, \ldots, a+n+1\} \setminus \{a+n\}$. Notice that C_1^{α} cannot contain a forbidden triangle. Hence, $\{a+n+1, a+2n+1, a+2n+2\} = \{a+n+1, a, a+1\} \not\subseteq V(H)$ and thus $|\{a+n+1, a+1\} \cap V(H)| \leq 1$.

If $a+n+1 \notin V(H)$, then recolor one by one all vertices in $\{a, a+1, a+2, \ldots, a+n-1\} \setminus V(H)$ with color 1. The current coloring β satisfies $d(\alpha, \beta) \leq n - |C_1^{\alpha}|$.

So, we assume that $a+n+1 \in V(H)$. Recolor one by one all vertices in $\{a+2, a+3, \ldots, a+n-1\} \setminus V(H)$ with color 1, this process takes at most $n - 2 - (|C_1^{\alpha}| - 2) = n - |C_1^{\alpha}|$ vertex recolorings.

If $k \geq 4$, then let j be a color in $\{1, 2, \ldots, k\} \setminus \{1, \alpha(a+1), \alpha(a+n)\}$, then $C_j^{\alpha} \subseteq \{a+n+2, a+n+3, \ldots, a+2n\}$. By Lemma 11, the subdigraph of T induced by $\{a+n+1, a+n+2, a+n+3, \ldots, a+2n\}$ is a transitive tournament. Hence, we recolor vertex $a+n+1$ with color j. Next, recolor vertex $a+1$ with color 1. The current coloring β satisfies $d(\alpha, \beta) \leq n - |C_1^{\alpha}| + 2 = n + 2 - |C_1^{\alpha}|$ from α.

If $k = 3$.

Whenever $\alpha(a+1) = \alpha(a+n) = l$, let h be the color in $\{1, 2, 3\} \setminus \{1, l\}$. Then $C_h^{\alpha} \subseteq \{a+n+2, a+n+3, \ldots, a+2n\}$. So, we recolor vertex $a+n+1$ with color h and then recolor vertex $a+1$ with color 1. The current coloring β satisfies $d(\alpha, \beta) \leq n - |C_1^{\alpha}| + 2 = n + 2 - |C_1^{\alpha}|$ (see Figure 2).

Whenever $l = \alpha(a+1) \neq \alpha(a+n) = h$, then $\alpha(a+2n) = l$. Otherwise, whenever $\alpha(a+2n) = h$ and if, for some $i \in \{a+n+2, a+n+3, \ldots, a+2n-1\}$, $\alpha(i) = h$, then $(a+n, i, a+2n, a+n)$ is a monochromatic cycle and if, for each $i \in \{a+n+2, a+n+3, \ldots, a+2n-1\}$, $\alpha(i) = l$, then $(a+n+2, a+2n-1, a+1, a+n+2)$ is a monochromatic cycle. In any case, we have a contradiction. Hence, $\alpha(a+2n) = l$. The above implies that $C_h^{\alpha} \subseteq \{a+n, a+n+1, \ldots, a+2n-1\}$. By Lemma 11, the subdigraph of T induced by $\{a+n, a+n+1, a+n+2, \ldots, a+2n-1\}$ is a transitive tournament. So, we recolor vertex $a+n+1$ with color h. Next, recolor vertex $a+1$ with color 1. The current coloring β satisfies $d(\alpha, \beta) \leq n - |C_1^{\alpha}| + 2 = n + 2 - |C_1^{\alpha}|$ (see Figure 3).

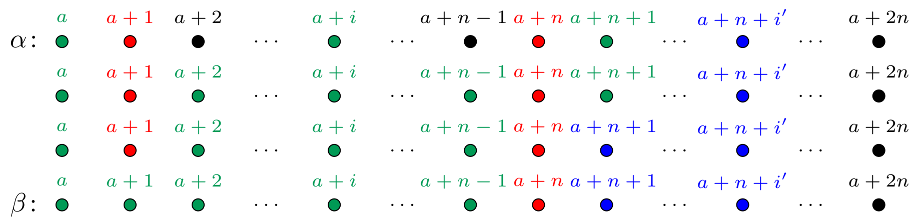


Figure 2: Case 1. $|C_1^\alpha| \geq 3$ whenever $k = 3$ and $\alpha(a+1) = \alpha(a+n)$. Color 1 is green, color l is red and color h is blue. The vertices for which we do not know the color assigned to them are represented in black.

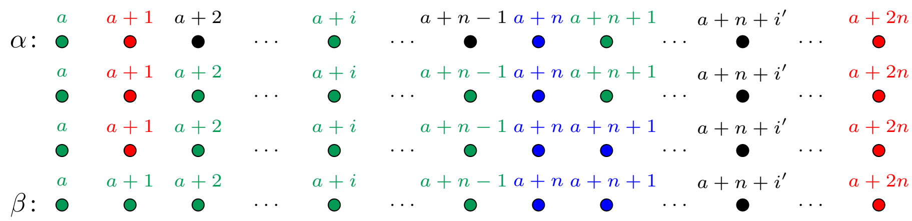


Figure 3: Case 1. $|C_1^\alpha| \geq 3$ whenever $k = 3$ and $\alpha(a+1) \neq \alpha(a+n)$. Color 1 is green, color l is red and color h is blue. The vertices for which we do not know the color assigned to them are represented in black.

Any coloring β of Case 1 is at distance at most $n + 2 - |C_1^\alpha|$.

Case 2. $|C_1^\alpha| \in \{0, 1\}$.

If $|C_1^\alpha| = 1$, let a be the only vertex in C_1^α. Otherwise, let a be any vertex in T. Recolor one by one all vertices in $\{a, a+1, a+2, \ldots, a+n-1\} \setminus C_1^\alpha$ with color 1. Thus, we obtain a coloring β with the color class$C_1^\beta = \{a, a+1, a+2, \ldots, a+n-1\}$ in n steps, and $d(\alpha, \beta) \leq n - |C_1^\alpha|$.

Case 3. $|C_1^\alpha| = 2$.

Whenever $C_1^\alpha \subseteq \{a, a+1, a+2, \ldots, a+n-1\}$ for some $a \in \mathbb{Z}_{2n+1}$. Then we recolor the vertices in $\{a, a+1, a+2, \ldots, a+n-1\} \setminus C_1^\alpha$ with color 1. In this way, we obtain a coloring β with the color class $C_1^\beta = \{a, a+1, a+2, \ldots, a+n-1\}$ in $n-2 = n-|C_1^\alpha|$ steps, and $d(\alpha, \beta) \leq n - |C_1^\alpha|$. Whenever $C_1^\alpha = \{a, a+n\}$ for some $a \in \mathbb{Z}_{2n+1}$, we have that $(a+n, a) \in A(T)$. Since $n \geq 4$, we may recolor vertex $a+n+2$ with color 1. The new coloring α' is acyclic as the subdigraph of T induced by $\{a, a+n, a+n+2, a+2n-1\}$ is a transitive tournament. Moreover, $d(\alpha', \alpha) = 1$, and its color class $C_1^{\alpha'}$ satisfies $|C_1^{\alpha'}| = 3$, so we can proceed as in Case 1 to obtain a coloring β such that $d(\alpha', \beta) \leq n + 2 - |C_1^{\alpha'}| = n + 2 - 3 = n+1-2 = n+1-|C_1^\alpha|$ and, for some $a \in \mathbb{Z}_{2n+1}$, $C_1^\beta = \{a, a+1, \ldots, a+n-1\}$. So, $d(\alpha, \beta) \leq n + 1 - |C_1^\alpha| + 1 = n + 2 - |C_1^\alpha|$. $\square$

Let n be an integer, $n \geq 3$, and let $\mathscr{C}$ be the set of colorings $\alpha\colon \mathbb{Z}_{2n+1} \to \{1,2,3\}$ of $\vec{C}_{2n+1}\langle n\rangle$ such that the color classes of α are $\{a\}$, $\{a+1, a+2, \ldots, a+n\}$ and $\{a+n+1, a+n+2, \ldots, a+2n\}$ for some $a \in \mathbb{Z}_{2n+1}$.

Lemma 13. *Let n be an integer with $n \geq 3$. If α and β are two 3-colorings in $\mathscr{C}$, then α and β are at distance at most $3n-1$ in $\mathcal{D}_3(\vec{C}_{2n+1}\langle n\rangle)$. Moreover, if the singular color classes of α and β have different colors, then α and β are at distance at most $2n+1$.*

Proof. Let α and β be two colorings in $\mathscr{C}$. Suppose w.l.o.g. that the color classes of α are $C_1^\alpha = \{0\}$, $C_2^\alpha = \{1,2,\ldots,n\}$ and $C_3^\alpha = \{n+1, n+2, \ldots, 2n\}$ and the color classes of β are $C_h^\beta = \{a\}$, $C_i^\beta = \{a+1, a+2, \ldots, a+n\}$ and $C_j^\beta = \{a+n+1, a+n+2, \ldots, a+2n\}$, where $\{h,i,j\} = \{1,2,3\}$.

Case 1. $|C_1^\beta| = 1$ *and* $\beta(0) = 1$.

The above implies that $C_1^\beta = \{0\}$. If $i = 2$, then $j = 3$, and thus α and β are the same coloring. So, we assume that $i = 3$. In this way, the color classes of β are $C_1^\beta = \{0\}$, $C_3^\beta = \{1,2,\ldots,n\}$ and $C_2^\beta = \{n+1, n+2, \ldots, 2n\}$. Start with coloring α. Recolor, one by one, the vertices in $\{1,2,\ldots,n-1\}$ with color 1. The current coloring is acyclic by Lemma 11. Then recolor, one by one, the vertices in $\{n+1, n+2, \ldots, 2n-1\}$ with color 2. After, recolor vertex n with color 3. Next, recolor vertex $2n$ with color 2. Finally, recolor, one by one, all vertices in $\{1,2,\ldots,n-1\}$ with color 3. The current coloring β satisfies $d(\alpha,\beta) \leq n-1+n-1+1+1+n-1 = 3n-1$.

Case 2. $|C_1^\beta| = 1$ *and* $\beta(0) \neq 1$.

The above implies that $C_1^\beta = \{x\}$ for some $x \in \{1,2,\ldots,2n\}$. By symmetry, we may assume that $x \in \{1,2,\ldots,n\}$.

Case 2.1. $x = n$.

Start with coloring α. If $\beta(1) = 2 = \alpha(n)$, then recolor vertex n with color 1 and, after, recolor vertex 0 with color 2. The current coloring β satisfies $d(\alpha,\beta) = 2$. If $\beta(1) = 3 \neq 2 = \alpha(n)$, then recolor all vertices in $\{n, n+1, \ldots, 2n-1\} \setminus \{n+1\}$ with color 1, this takes $n-1$ vertex recolorings. After, recolor vertex $n-1$ with color 3 and, then, recolor vertex $2n$ with color 2. Next, recolor all vertices in $\{0,1,\ldots,n-2\} \setminus \{1\}$ with color 3, this takes $n-2$ steps. Now, recolor vertex $n+1$ with color 2 and, then, recolor vertex 1 with color 3. Finally, recolor all vertices in $\{n+2, n+3, \ldots, 2n-1\}$ with color 2, this takes $n-2$ vertex recolorings. The current coloring β satisfies $d(\alpha,\beta) \leq n-1+2+n-2+2+n-2 = 3n-1$.

Case 2.2. $1 \leq x \leq n-1$.

Start with coloring α. If $\beta(1) = 3 \neq 2 = \alpha(x)$, then recolor all vertices in $\{1,2,\ldots,x\}$ with color 1, this takes x vertex recolorings. After, recolor all vertices in $\{x+1, x+2, \ldots, x+n\} \setminus C_2^\alpha$ with color 2, this takes $n-(n-x) = x$

steps. Finally, recolor all vertices in $\{x+n+1, x+n+2, \ldots, x+2n\} \setminus C_3^\alpha$ with color 3, this also takes x vertex recolorings. The current coloring β satisfies $d(\alpha,\beta) \leq 3x \leq 3(n-1) = 3n-3$. If $\beta(1) = 2 = \alpha(x)$, then recolor all vertices in $\{x+n+1, x+n+2, \ldots, 2n\}$ with color 1, this takes $n-x$ vertex recolorings. After, recolor all vertices in $\{x+1, \ldots, n\}$ with color 3, this takes $n-x$ steps. Then, recolor all vertices in $\{x+n+2, x+n+3, \ldots, 2n, 0\}$ with color 2, this also takes $n-x$ vertex recolorings. Then recolor vertex x with color 1 and, finally, recolor vertex $x+n+1$ with color 2. The current coloring β satisfies $d(\alpha,\beta) \leq 3(n-x)+2 \leq 3(n-1)+2 = 3n-1$.

Case 3. $|C_1^\beta| = n$ *and* $\beta(0) = 1$.

Let $x = \min\{b \in \{1, 2, \ldots, n\} \colon \beta(b) \neq 1\}$. Then $C_2^\beta \cup C_3^\beta = \{x, x+1, \ldots, x+n\}$ and $C_1^\beta = \{x+n+1, x+n+2, \ldots, x+2n\}$. Start with coloring α. Let α' be the coloring with color classes $C_1^{\alpha'} = \{x+n+1, x+n+2, \ldots, x+2n\}$, $C_2^{\alpha'} = \{x, \ldots, n\}$ and $C_3^{\alpha'} = \{n+1, \ldots, x+n\}$ obtained from α by recoloring all vertices in $\{x+n+1, x+n+2, \ldots, x+2n\} \setminus \{0\}$ with color 1, then $d(\alpha, \alpha') = n-1$.

Case 3.1. $\beta(x) = 2$.

Here $\alpha'(x) = \alpha(x) = 2 = \beta(x)$. Now, we transform α' into β. If $C_2^\beta = \{x\}$, then recolor all vertices in $\{x+1, x+2, \ldots, x+2n\} \setminus C_3^\beta$ with color 3. This process takes at most $n-1$ vertex recolorings, and the current coloring is β. Hence, $d(\alpha,\beta) \leq n-1+n-1 = 2n-2$. If $C_2^\beta = \{x, x+1, \ldots, x+n-1\}$, then recolor all vertices in $\{n+1, \ldots, x+n-1\}$ with color 2. This process takes at most $n-1$ vertex recolorings, and the current coloring is β. Hence, $d(\alpha,\beta) \leq n-1+n-1 = 2n-2$.

Case 3.2. $\beta(x) = 3$.

Here $\alpha'(x) = \alpha(x) = 2 \neq 3 = \beta(x)$. Now, we transform α' into β. If $C_3^\beta = \{x\}$, then recolor each vertex in $\{n+1, \ldots, x+n-1\}$ with color 2, this process takes at most $n-1$ steps. After, recolor vertex x with color 3. And, finally, recolor vertex $x+n$ with color 2. The current coloring is β and satisfies $d(\alpha,\beta) \leq n-1+n-1+1+1 = 2n$. If $C_3^\beta = \{x, x+1, \ldots, x+n-1\}$, then recolor all vertices in $\{x+1, \ldots, x+n\} \setminus C_3^{\alpha'}$ with color 3, this process takes at most $n-1$ steps. After, recolor vertex $x+n$ with color 2. And, finally, recolor vertex x with color 3. The current coloring is β and satisfies $d(\alpha,\beta) \leq n-1+n-1+1+1 = 2n$.

Case 4. $|C_1^\beta| = n$ *and* $\beta(0) \neq 1$.

The above implies that $C_1^\beta = \{x+1, x+2, \ldots, x+n\}$ for some $x \in \{0, 1, \ldots, n\}$.

Case 4.1. $C_2^\beta = \{x\}$ *or* $C_3^\beta = \{x+n+1\}$.

Whenever $C_2^\beta = \{x\}$, we have that $C_3^\beta = \{x+n+1, x+n+2, \ldots, x+2n\}$. Start with coloring α. If $x = 0$, then recolor vertex n with color 1. After, recolor vertex 0 with color 2. Next, recolor all vertices in $\{1, 2, \ldots, n-1\}$ with color 1.

The current coloring is β and satisfies $d(\alpha, \beta) \leq 1+1+n-1 = n+1$. If $x \neq 0$, we recolor vertex $n+1$ with color 1. After, we recolor vertex 0 with color 3. Next, we recolor all vertices in $\{x+1, x+2, \ldots, x+n\} \setminus \{n+1\}$ with color 1. And, finally, we recolor all vertices in $\{1, \ldots, x-1\}$ with color 3. The current coloring is β and satisfies $d(\alpha, \beta) \leq 1+1+n-1+x-1 \leq 2n$.

Analogously, $d(\alpha, \beta) \leq 2n$ whenever $C_3^{\beta} = \{x+n+1\}$.

Case 4.2. *$C_2^{\beta} = \{x+n+1\}$ or $C_3^{\beta} = \{x\}$.*

Whenever $C_3^{\beta} = \{x\}$, we have that $C_2^{\beta} = \{x+n+1, x+n+2, \ldots, x+2n\}$. Start with coloring α. If $x = 0$, then recolor all vertices in $\{1, 2, \ldots, n-1\}$ with color 1. Then, recolor all vertices in $\{n+1, n+2, \ldots, 2n-1\}$ with color 2. Next, recolor vertex 0 with color 3. After, recolor vertex n with color 1. Finally, recolor vertex $2n$ with color 2. The current coloring is β and satisfies $d(\alpha, \beta) \leq n-1+n-1+1+1+1 = 2n+1$. If $x \neq 0$, then recolor vertex n with color 1 and, after, recolor vertex 0 with color 2. Then, recolor all vertices in $\{x+1, x+2, \ldots, x+n\} \setminus \{n\}$ with color 1. Next, recolor all vertices in $\{x+n+2, x+n+3, \ldots, 2n\}$ with color 2. After, recolor vertex x with color 3. Finally, recolor vertex $x+n+1$ with color 2. The current coloring is β and satisfies $d(\alpha, \beta) \leq 1+1+n-1+n-x-1+1+1 \leq 2n+1$.

Analogously, $d(\alpha, \beta) \leq 2n+1$ whenever $C_2^{\beta} = \{x+n+1\}$. $\square$

Theorem 14. *Let n be an integer, $n \geq 5$. Then $\mathcal{D}_3(\vec{C}_{2n+1}\langle n \rangle)$ is connected and has diameter at most $3n+4+\lfloor \frac{2n+1}{3} \rfloor$.*

Proof. Let $T = \vec{C}_{2n+1}\langle n \rangle$. Let $\rho, \sigma \colon \mathbb{Z}_{2n+1} \to \{1,2,3\}$ be two 3-colorings of T. Let C_1^{ρ}, C_2^{ρ}, C_3^{ρ} and C_1^{σ}, C_2^{σ}, C_3^{σ} be the color classes of ρ and σ, respectively. Since $2n+1 \geq 11$, it follows that $|C_i^{\rho}| \geq \lceil \frac{2n+1}{3} \rceil \geq 4$ and $|C_j^{\sigma}| \geq \lceil \frac{2n+1}{3} \rceil \geq 4$, for some $i \in \{1,2,3\}$ and $j \in \{1,2,3\}$.

By Lemma 12, there are two 3-colorings ρ' and σ' such that its ith and jth color classes are $C_i^{\rho'} = \{a+1, a+2, \ldots, a+n\}$ and $C_j^{\sigma'} = \{b+1, b+2, \ldots, b+n\}$, respectively, for some $a \in \mathbb{Z}_{2n+1}$ and $b \in \mathbb{Z}_{2n+1}$. Moreover, $d(\rho, \rho') \leq n+2-|C_i^{\rho}| \leq n+2-\lceil \frac{2n+1}{3} \rceil$ and $d(\sigma', \sigma) \leq n+2-|C_j^{\sigma}| \leq n+2-\lceil \frac{2n+1}{3} \rceil$.

Case 1. $i = j$.

Let g and h be the two colors in $\{1,2,3\} \setminus \{i\}$.

Case 1.1. $\{a+n+1, a+2n+1\} \subseteq C_g^{\rho'}$.

Hence, $\{a+n+2, a+n+3, \ldots, a+2n\} = C_h^{\rho'}$ and $\{a+n+1, a+2n+1\} = C_g^{\rho'}$, otherwise T contains a monochromatic directed triangle. Then, from ρ', we obtain a coloring $\alpha \in \mathscr{C}$ by recoloring vertex $a+n+1$ with color h. Where α is such that $d(\rho, \alpha) \leq n+2-\lceil \frac{2n+1}{3} \rceil + 1$ and whose singleton color class is colored with color g.

Now, let us consider coloring σ', we have that the other color classes of σ', $C_g^{\sigma'}$ and $C_h^{\sigma'}$, satisfy that $C_g^{\sigma'} \cup C_h^{\sigma'} = \{b+n+1, b+n+2, \ldots, b+2n+1\}$ and both are nonempty.

Whenever $\{b+n+1, b+2n+1\} \subseteq C_h^{\sigma'}$, it follows that $C_g^{\sigma'} = \{b+n+2, b+n+3, \ldots, b+2n\}$. So, we recolor vertex $b+2n+1$ with color g, and thus the current coloring β is in $\mathscr{C}$ and $d(\beta,\sigma) \leq n+2-\lceil\frac{2n+1}{3}\rceil+1$. Moreover, by Lemma 13, $d(\alpha,\beta) \leq 2n+1$, and thus $d(\rho,\sigma) \leq d(\rho,\alpha)+d(\alpha,\beta)+d(\beta,\sigma) \leq n+3-\lceil\frac{2n+1}{3}\rceil+2n+1+n+3-\lceil\frac{2n+1}{3}\rceil = 4n+7-2\lceil\frac{2n+1}{3}\rceil$.

Whenever $\{b+n+1, b+2n+1\} \subseteq C_g^{\sigma'}$, it follows that $C_h^{\sigma'} = \{b+n+2, b+n+3, \ldots, b+2n\}$. So, we recolor vertex $b+2n+1$ with color h, and thus the current coloring β is in $\mathscr{C}$ and $d(\beta,\sigma) \leq n+2-\lceil\frac{2n+1}{3}\rceil+1$. Moreover, by Lemma 13, $d(\alpha,\beta) \leq 3n-1$, and thus $d(\rho,\sigma) \leq d(\rho,\alpha)+d(\alpha,\beta)+d(\beta,\sigma) \leq n+3-\lceil\frac{2n+1}{3}\rceil+3n-1+n+3-\lceil\frac{2n+1}{3}\rceil = 5n+5-2\lceil\frac{2n+1}{3}\rceil$.

Whenever $\sigma'(b+n+1) \neq \sigma'(b+2n+1)$, we may suppose w.l.o.g. that $\sigma'(b+n+1) = g$ and $\sigma'(b+2n+1) = h$. So, we recolor all vertices in $\{b+n+2, b+n+3, \ldots, b+2n\} \setminus C_g^{\sigma'}$ with color g, this takes at most $n-1$ vertex recolorings and the current coloring β is in $\mathscr{C}$ and $d(\beta,\sigma) \leq n+2-\lceil\frac{2n+1}{3}\rceil+n-1$. Moreover, by Lemma 13, $d(\alpha,\beta) \leq 2n+1$, and thus $d(\rho,\sigma) \leq d(\rho,\alpha)+d(\alpha,\beta)+d(\beta,\sigma) \leq n+3-\lceil\frac{2n+1}{3}\rceil+2n+1+2n+1-\lceil\frac{2n+1}{3}\rceil = 5n+5-2\lceil\frac{2n+1}{3}\rceil$.

In a similar way it can be proved that if $\{a+n+1, a+2n+1\} \subseteq C_h^{\rho'}$, $\{b+n+1, b+2n+1\} \subseteq C_g^{\sigma'}$, or $\{b+n+1, b+2n+1\} \subseteq C_h^{\sigma'}$, then $d(\rho,\sigma) \leq 5n+5-2\lceil\frac{2n+1}{3}\rceil$.

Case 1.2. $\rho'(a+n+1) \neq \rho'(a+2n+1)$ *and* $\sigma'(b+n+1) \neq \sigma'(b+2n+1)$.

We assume w.l.o.g. that $\rho'(a+n+1) = \sigma'(b+n+1) = g$ and $\rho'(a+2n+1) = \sigma'(b+2n+1) = h$.

Since there are other $n-1$ vertices, namely $a+n+2, a+n+3, \ldots, a+2n$, colored with colors g and h in ρ', one of the two color classes has at least $\lceil\frac{n-1}{2}\rceil+1$ vertices. The same can be said about the color classes of colors g and h in σ'. Let $M = \max\{|C_g^{\rho'}|, |C_h^{\rho'}|, |C_g^{\sigma'}|, |C_h^{\sigma'}|\}$. Suppose w.l.o.g. that $M = |C_g^{\rho'}|$. Then $M = |C_g^{\rho'}| \geq \lceil\frac{n-1}{2}\rceil+1$, and $|C_h^{\rho'}| \leq n+1-M$. Start with ρ' and recolor all vertices in $\{a+n+2, a+n+3, \ldots, a+2n\} \setminus C_g^{\rho'}$ with color g, this takes at most $n-M$ steps, and the current coloring α is in $\mathscr{C}$ and $d(\rho,\alpha) \leq n+2-\lceil\frac{2n+1}{3}\rceil+n-M$.

If $|C_h^{\sigma'}| \geq \lceil\frac{n-1}{2}\rceil+1$, then $M \geq |C_h^{\sigma'}|$ and $|C_g^{\sigma'}| \leq n+1-M$. Start with coloring σ' and recolor all vertices in $\{b+n+2, b+n+3, \ldots, b+2n\} \setminus C_h^{\sigma'}$ with color h, this takes at most $n-M$ steps, and the current coloring β is in $\mathscr{C}$ and $d(\beta,\sigma) \leq n+2-\lceil\frac{2n+1}{3}\rceil+n-M$. Moreover, by Lemma 13, $d(\alpha,\beta) \leq 2n+1$, and thus $d(\rho,\sigma) \leq d(\rho,\alpha)+d(\alpha,\beta)+d(\beta,\sigma) \leq 2n+2-\lceil\frac{2n+1}{3}\rceil-M+2n+1+2n+2-\lceil\frac{2n+1}{3}\rceil-M = 6n+5-2\lceil\frac{2n+1}{3}\rceil-2M$. Since $M \geq \lceil\frac{n-1}{2}\rceil+1$, it follows that $n-M \leq n-\lceil\frac{n-1}{2}\rceil-1 = \lfloor\frac{n-1}{2}\rfloor$. Hence, $d(\rho,\sigma) \leq 6n+5-2\lceil\frac{2n+1}{3}\rceil-2M \leq 4n+5-2\lceil\frac{2n+1}{3}\rceil+2\lfloor\frac{n-1}{2}\rfloor \leq 4n+5-2\lceil\frac{2n+1}{3}\rceil+n-1 = 5n+4-2\lceil\frac{2n+1}{3}\rceil$.

If $|C_g^{\sigma'}| \geq \lceil\frac{n-1}{2}\rceil+1$, then $M \geq |C_g^{\sigma'}|$. Now, start with coloring σ' and recolor all vertices in $\{b+n+2, b+n+3, \ldots, b+2n\} \setminus C_h^{\sigma'}$ with color h, the current coloring β is in $\mathscr{C}$ and $d(\beta,\sigma) \leq n+2-\lceil\frac{2n+1}{3}\rceil+M-1$. Moreover, by

Lemma 13, $d(\alpha,\beta) \le 2n+1$, and thus $d(\rho,\sigma) \le d(\rho,\alpha) + d(\alpha,\beta) + d(\beta,\sigma) \le 2n+2-\lceil\frac{2n+1}{3}\rceil - M + 2n+1+n+2-\lceil\frac{2n+1}{3}\rceil + M - 1 = 5n+4-2\lceil\frac{2n+1}{3}\rceil$.

Case 2. $i \ne j$.

Let h be the color in $\{1,2,3\} \setminus \{i,j\}$.

Case 2.1. $\{a+n+1, a+2n+1\} \subseteq C_j^{\rho'}$.

Hence, $\{a+n+2, a+n+3, \ldots, a+2n\} = C_h^{\rho'}$ and $\{a+n+1, a+2n+1\} = C_j^{\rho'}$. Then, from ρ', we obtain a coloring $\alpha \in \mathscr{C}$ by recoloring vertex $a+n+1$ with color h. Where α is such that $d(\rho,\alpha) \le n+2-\lceil\frac{2n+1}{3}\rceil + 1$ and whose singleton color class is colored with color j.

Now, let us consider coloring σ', we have that $C_i^{\sigma'}$ and $C_h^{\sigma'}$ satisfy that $C_i^{\sigma'} \cup C_h^{\sigma'} = \{b+n+1, b+n+2, \ldots, b+2n+1\}$ and both are nonempty.

Whenever $\{b+n+1, b+2n+1\} \subseteq C_h^{\sigma'}$, it follows that $C_i^{\sigma'} = \{b+n+2, b+n+3, \ldots, b+2n\}$. So, we recolor vertex $b+2n+1$ with color i, and thus the current coloring β is in $\mathscr{C}$, its singleton color class is colored with color h and $d(\beta,\sigma) \le n+2-\lceil\frac{2n+1}{3}\rceil+1$. Moreover, by Lemma 13, $d(\alpha,\beta) \le 2n+1$, and thus $d(\rho,\sigma) \le d(\rho,\alpha)+d(\alpha,\beta)+d(\beta,\sigma) \le n+3-\lceil\frac{2n+1}{3}\rceil+2n+1+n+3-\lceil\frac{2n+1}{3}\rceil = 4n+7-2\lceil\frac{2n+1}{3}\rceil$.

Whenever $\{b+n+1, b+2n+1\} \subseteq C_i^{\sigma'}$, it follows that $C_h^{\sigma'} = \{b+n+2, b+n+3, \ldots, b+2n\}$. So, we recolor vertex $b+2n+1$ with color h, and thus the current coloring β is in $\mathscr{C}$, its singleton color class is colored with color i and $d(\beta,\sigma) \le n+2-\lceil\frac{2n+1}{3}\rceil+1$. Moreover, by Lemma 13, $d(\alpha,\beta) \le 2n+1$, and thus $d(\rho,\sigma) \le d(\rho,\alpha)+d(\alpha,\beta)+d(\beta,\sigma) \le n+3-\lceil\frac{2n+1}{3}\rceil+2n+1+n+3-\lceil\frac{2n+1}{3}\rceil = 4n+7-2\lceil\frac{2n+1}{3}\rceil$.

Whenever $\sigma'(b+n+1) \ne \sigma'(b+2n+1)$, we may suppose w.l.o.g. that $\sigma'(b+n+1) = i$ and $\sigma'(b+2n+1) = h$. If $|C_i^{\sigma'}| \ge |C_h^{\sigma'}|$, then we recolor all vertices in $\{b+n+2, b+n+3, \ldots, b+2n\} \setminus C_i^{\sigma'}$ with color i, this takes at most $\lfloor\frac{n-1}{2}\rfloor$ steps and the current coloring β is in $\mathscr{C}$ and its singleton color class is colored with color h. And, if $|C_h^{\sigma'}| \ge |C_i^{\sigma'}|$, then we recolor all vertices in $\{b+n+2, b+n+3, \ldots, b+2n\} \setminus C_h^{\sigma'}$ with color h, this takes at most $\lfloor\frac{n-1}{2}\rfloor$ vertex recolorings and the current coloring β is in $\mathscr{C}$ and its singleton color class is colored with color i. Either way, $d(\beta,\sigma) \le n+2-\lceil\frac{2n+1}{3}\rceil+\lfloor\frac{n-1}{2}\rfloor$. Moreover, by Lemma 13, $d(\alpha,\beta) \le 2n+1$, and thus $d(\rho,\sigma) \le d(\rho,\alpha)+d(\alpha,\beta)+d(\beta,\sigma) \le n+3-\lceil\frac{2n+1}{3}\rceil+2n+1+n+2-\lceil\frac{2n+1}{3}\rceil+\lfloor\frac{n-1}{2}\rfloor = 4n+6-2\lceil\frac{2n+1}{3}\rceil+\lfloor\frac{n-1}{2}\rfloor$.

In a similar way it can be proved that if $\{b+n+1, b+2n+1\} \subseteq C_i^{\sigma'}$, then $d(\rho,\sigma) \le 4n+6-2\lceil\frac{2n+1}{3}\rceil+\lfloor\frac{n-1}{2}\rfloor$.

Case 2.2. $\{a+n+1, a+2n+1\} \subseteq C_h^{\rho'}$.

Hence, $\{a+n+2, a+n+3, \ldots, a+2n\} = C_j^{\rho'}$ and $\{a+n+1, a+2n+1\} = C_h^{\rho'}$. Then, from ρ', we obtain a coloring $\alpha \in \mathscr{C}$ by recoloring vertex $a+n+1$ with color j. Where α is such that $d(\rho,\alpha) \le n+2-\lceil\frac{2n+1}{3}\rceil+1$ and whose singleton color class is colored with color h.

Now, let us consider coloring σ', we have that $C_i^{\sigma'}$ and $C_h^{\sigma'}$ satisfy that $C_i^{\sigma'} \cup C_h^{\sigma'} = \{b+n+1, b+n+2, \ldots, b+2n+1\}$ and both are nonempty.

Whenever $\{b+n+1, b+2n+1\} \subseteq C_h^{\sigma'}$, it follows that $C_i^{\sigma'} = \{b+n+2, b+n+3, \ldots, b+2n\}$. So, we recolor vertex $b+2n+1$ with color i, and thus the current coloring β is in $\mathscr{C}$, its singleton color class is colored with color h and $d(\beta,\sigma) \leq n+2-\lceil \frac{2n+1}{3} \rceil + 1$. Moreover, by Lemma 13, $d(\alpha,\beta) \leq 3n-1$, and thus $d(\rho,\sigma) \leq d(\rho,\alpha) + d(\alpha,\beta) + d(\beta,\sigma) \leq n+3-\lceil \frac{2n+1}{3} \rceil + 3n-1+n+3-\lceil \frac{2n+1}{3} \rceil = 5n+5-2\lceil \frac{2n+1}{3} \rceil$.

Whenever $\{b+n+1, b+2n+1\} \subseteq C_i^{\sigma'}$, it follows that $C_h^{\sigma'} = \{b+n+2, b+n+3, \ldots, b+2n\}$. So, we recolor vertex $b+2n+1$ with color h, and thus the current coloring β is in $\mathscr{C}$, its singleton color class is colored with color i and $d(\beta,\sigma) \leq n+2-\lceil \frac{2n+1}{3} \rceil + 1$. Moreover, by Lemma 13, $d(\alpha,\beta) \leq 2n+1$, and thus $d(\rho,\sigma) \leq d(\rho,\alpha) + d(\alpha,\beta) + d(\beta,\sigma) \leq n+3-\lceil \frac{2n+1}{3} \rceil + 2n+1+n+3-\lceil \frac{2n+1}{3} \rceil = 4n+7-2\lceil \frac{2n+1}{3} \rceil$.

Whenever $\sigma'(b+n+1) \neq \sigma'(b+2n+1)$, we may suppose w.l.o.g. that $\sigma'(b+n+1) = i$ and $\sigma'(b+2n+1) = h$. Then we recolor all vertices in $\{b+n+2, b+n+3, \ldots, b+2n\} \setminus C_i^{\sigma'}$ with color i, this takes at most $n-1$ steps and the current coloring β is in $\mathscr{C}$ and its singleton color class is colored with color h. Hence, $d(\beta,\sigma) \leq n+2-\lceil \frac{2n+1}{3} \rceil + n - 1$. Moreover, by Lemma 13, $d(\alpha,\beta) \leq 2n+1$, and thus $d(\rho,\sigma) \leq d(\rho,\alpha) + d(\alpha,\beta) + d(\beta,\sigma) \leq n+3-\lceil \frac{2n+1}{3} \rceil + 2n+1+2n+1-\lceil \frac{2n+1}{3} \rceil = 5n+5-2\lceil \frac{2n+1}{3} \rceil$.

In a similar way it can be proved that if $\{b+n+1, b+2n+1\} \subseteq C_i^{\sigma'}$, then $d(\rho,\sigma) \leq 5n+5-2\lceil \frac{2n+1}{3} \rceil$.

Case 2.3. *$\rho'(a+n+1) \neq \rho'(a+2n+1)$ and $\sigma'(b+n+1) \neq \sigma'(b+2n+1)$.*

We may assume that $\rho'(a+n+1) = j$, $\rho'(a+2n+1) = h$, $\sigma'(b+n+1) = i$, and $\sigma'(b+2n+1) = h$. Suppose w.l.o.g. that $|C_h^{\rho'}| \geq |C_h^{\sigma'}|$. Then, starting from ρ', recolor all vertices in $\{a+n+2, a+n+3, \ldots, a+2n\} \setminus C_h^{\rho'}$ with color h, this takes at most $n-1-|C_h^{\rho'}|$ vertex recolorings. The current coloring α is in $\mathscr{C}$ and its singleton color class is colored with color j. Hence, $d(\rho,\alpha) \leq n+2-\lceil \frac{2n+1}{3} \rceil + n-1-|C_h^{\rho'}|$.

Now, starting from σ', recolor all vertices in $\{b+n+2, b+n+3, \ldots, b+2n\} \setminus C_i^{\rho'}$ with color i, this takes at most $|C_h^{\sigma'}| - 1 \leq |C_h^{\rho'}| - 1$ vertex recolorings. The current coloring β is in $\mathscr{C}$ and its singleton color class is colored with color h. Thus, $d(\beta,\sigma) \leq n+2-\lceil \frac{2n+1}{3} \rceil + |C_h^{\rho'}| - 1$. Moreover, by Lemma 13, $d(\alpha,\beta) \leq 2n+1$, and thus $d(\rho,\sigma) \leq d(\rho,\alpha) + d(\alpha,\beta) + d(\beta,\sigma) \leq 2n+1-\lceil \frac{2n+1}{3} \rceil - |C_h^{\rho'}| + 2n+1+n+1-\lceil \frac{2n+1}{3} \rceil + |C_h^{\rho'}| = 5n+3-2\lceil \frac{2n+1}{3} \rceil$.

From all cases, it follows that $d(\rho,\sigma) \leq 5n+5-2\lceil \frac{2n+1}{3} \rceil \leq 5n+5-2\left(\frac{2n+1}{3}\right) = \frac{11n+13}{3} = 3n+4+\frac{2n+1}{3}$. Hence, $\mathcal{D}_3(T)$ is connected and $\operatorname{diam}(\mathcal{D}_3(T)) \leq 3n+4+\lfloor \frac{2n+1}{3} \rfloor$. □

Using a Python program, we obtained information on $\mathcal{D}_k(\vec{C}_7\langle 3\rangle)$ for each $k \in \{3,4,5,6\}$, which is included in Table 1. Provided with all the acyclic subsets of $\vec{C}_7\langle 3\rangle$ in which 0 is the source and an empty set of edges E, the program performs the following steps:

1. Make a list L with every acyclic subset of $\vec{C}_7\langle 3\rangle$ by rotating each acyclic subset in which 0 is the source.
2. Make a list P with all the ordered partitions of $\{1,2,3,4,5,6,7\}$ with k parts such that each part is either an element of L or the empty set.
3. Decide if two k-partitions (k-colorings), $P[i]$ and $P[j]$ with $i < j$, are adjacent in $\mathcal{D}_k(\vec{C}_7\langle 3\rangle)$, and if they are, then include the pair (i,j) to a list of edges E.
4. Construct the graph $\mathcal{D}_k(\vec{C}_7\langle 3\rangle)$ with vertex set the first length(P) non-negative integers and E as the set of edges, and evaluate basic parameters of $\mathcal{D}_k(\vec{C}_7\langle 3\rangle)$.

	$\mathcal{D}_k(\vec{C}_7\langle 3\rangle)$							
k	order	size	connected	δ	Δ	diameter	radius	girth
3	504	1,512	✓	6	6	8	7	3
4	7,560	54,684	✓	13	15	8	7	3
5	47,880	536,760	✓	20	24	8	7	3
6	199,080	2,997,540	✓	27	33	8	7	3

Table 1: $\mathcal{D}_k(\vec{C}_7\langle 3\rangle)$ for $k \in \{3,4,5,6\}$

Theorem 15. *Let k be an integer, with $k \geq 3$. Then $\mathcal{D}_k(\vec{C}_7\langle 3\rangle)$ is connected and has diameter 8.*

Proof. Let $T = \vec{C}_7\langle 3\rangle$. We have that $\mathcal{D}_k(T)$ is connected and has diameter 8, for each $k \in \{3,4,5,6\}$, by the information in Table 1. Moreover, $\mathcal{D}_k(T)$ is connected for every $k \geq 7$, by Corollary 5.

Let α and β be two k-colorings of T for some $k \geq 7$. Let $A = \{i \in \{1,2,\ldots,k\} \colon C_i^\alpha \neq \emptyset\}$ and $B = \{i \in \{1,2,\ldots,k\} \colon C_i^\beta \neq \emptyset\}$. Since $3 \leq |A| \leq 7$ and $3 \leq |B| \leq 7$, it follows that $3 \leq |A \cup B| \leq 14$.

If $6 \leq |A| \leq 7$ or $6 \leq |B| \leq 7$, we have that one of the colorigs has seven nonempty color classes, all singletons, or it has six nonempty color classes, five singletons and one with two vertices. Thus, by Lemmas 3 and 4, $d(\alpha,\beta) \leq 7$. So, we assume $3 \leq |A| \leq 5$ and $3 \leq |B| \leq 5$.

If $|A \cup B| \leq 6$, then the subgraph of $\mathcal{D}_k(T)$ induced by the colorings that have empty color classes for each color in $\{1,2,\ldots,k\} \setminus (AUB)$ is isomorphic to $\mathcal{D}_h(T)$, where $h = |A \cup B|$. Hence, $d(\alpha,\beta) \leq 8$. So, we assume $|A \cup B| \geq 7$.

Moreover, if $|A \cap B| \leq 1$, then α and β have at most one nonempty color class of the same color, say color j. Then, starting at α we recolor each vertex $u \in \mathbb{Z}_7 \setminus C_j^\beta$ with color $\beta(u)$, and then give color j to vertices in C_j^β. In this way, $d(\alpha,\beta) \leq 7$. So, we assume $|A \cap B| \geq 2$. Hence, $4 \leq |A| \leq 5$ and $4 \leq |B| \leq 5$.

Whenever $|A \cap B| = 2$, it follows that α or β has 5 nonempty color classes. Suppose w.l.o.g. that $|B| = 5$, then $|B \setminus A| = 3$. Let j and j' be the two colors in $A \cap B$. Start at α. Recolor each vertex $u \in \mathbb{Z}_7 \setminus C_j^\beta \cup C_{j'}^\beta$ with color $\beta(u)$. We may assume there are four vertices in $C_j^\beta \cup C_{j'}^\beta$, and that one of those four vertices, say v, is colored with a color in $B \setminus A$, otherwise we recolor one vertex, say v, with a color in $B \setminus A$ such that it is assigned to exactly one vertex. Then we have three vertices that need to be colored with colors j and j', this can be achieved in at most 3 steps, by Lemma 4. After, recolor v with $\beta(v)$. In this way, $d(\alpha, \beta) \leq 8$.

Whenever $|A \cap B| = 3$, it follows that $|A| = |B| = 5$, then $|B \setminus A| = 2$. Let i and i' be the two colors in $B \setminus A$. Start at α. Recolor each vertex u in $C_i^\beta \cup C_{i'}^\beta$ with color $\beta(u)$, then there are at most five vertices that need to be recolored with the three colors in $A \cap B = \{j, j', j''\}$. Observe that, in the current coloring γ, one of the color classes C_h^γ with $h \in \{j, j', j''\}$ is empty or a singleton, suppose w.l.o.g. that $h = j$. Let us assume $C_j^\gamma \neq \emptyset$. If $C_j^\gamma = C_j^\beta$, the four remaining vertices can be recolored as above. So we assume $C_j^\gamma \neq C_j^\beta$, so we recolor some vertex in $C_j^\beta \setminus C_j^\gamma$ with color j. Now, one of the color classes of j' or j'' is empty or a singleton. We proceed as before, until we end up with α. Since at most one vertex is recolored twice, $d(\alpha, \beta) \leq 8$.

Therefore, $\operatorname{diam}(\mathcal{D}_k(T)) \leq 8$. Moreover, let $\alpha, \beta \colon \mathbb{Z}_7 \to \{1, 2, 3\}$ be the colorings given by $\alpha(0) = 1$, $\alpha(1) = 1$, $\alpha(2) = 1$ $\alpha(3) = 2$, $\alpha(4) = 2$, $\alpha(5) = 2$ and $\alpha(6) = 3$, and $\beta(0) = 2$, $\beta(1) = 2$, $\beta(2) = 2$ $\beta(3) = 3$, $\beta(4) = 1$, $\beta(5) = 1$ and $\beta(6) = 1$. It is easy to check that $d(\alpha, \beta) = 8$ for every $k \geq 3$. Hence, $\operatorname{diam}(\mathcal{D}_k(T) = 8$. □

Theorem 16. *Let n and k be two integers, $n \geq 4$ and $k \geq 4$. Then $\mathcal{D}_k(\vec{C}_{2n+1}\langle n\rangle)$ is connected whenever $k \neq \frac{2n+1}{3}$, and has diameter at most $4n + 2 + \lfloor \frac{n}{2} \rfloor$.*

Proof. Let $T = \vec{C}_{2n+1}\langle n\rangle$. Since T is 3-dichromatic, $\mathcal{D}_k(T)$ is defined for every $k \geq 4$. Let $\rho, \sigma \colon \mathbb{Z}_{2n+1} \to \{1, 2, \dots, k\}$ be two colorings of T. And let C_1^ρ, C_2^ρ, $\dots$, C_k^ρ and C_1^σ, C_2^σ, $\dots$, C_k^σ be the color classes of ρ and σ, respectively.

Case 1. *$C_i^\rho = \emptyset$ for some $i \in \{1, 2, \dots, k\}$.*

Supposse w.l.o.g. that $C_k^\rho = \emptyset$. Since C_k^σ induces an acyclic tournament, it has a source. Assume w.l.o.g. that 0 is the source of this tournament, then $C_k^\sigma \subseteq \{0, 1, 2, \dots, n+1\} \setminus \{n\}$. Start at ρ and recolor each vertex in $\{0, 1, 2, \dots, n-1\}$ with color k, this takes n steps. Call ρ' to the current coloring.

Whenever $\sigma(n+1) \neq k$. If $\rho'(n) = \rho'(2n)$, then recolor, if needed, one of this two vertices with the color assigned to it by σ. After, recolor each vertex u in $\{n+1, n+2, \dots, 2n-1\}$ (again, if needed) with color $\sigma(u)$. Then, recolor the last vertex in $\{n, 2n\}$. Finally, recolor each vertex u in $\{0, 1, 2, \dots, n-1\} \setminus C_k^\sigma$ with color $\sigma(u)$. The total process takes at most $n + n + 1 + n - |C_k^\sigma| \leq 3n + 1$. If $\rho'(n) \neq \rho'(2n)$, then recolor each vertex u in $\{n+1, n+2, \dots, 2n-1\}$, if needed, with color $\sigma(u)$. Then, recolor the vertices in $\{n, 2n\}$ with the color that σ gives them. Finally, recolor each vertex u in $\{0, 1, 2, \dots, n-1\} \setminus C_k^\sigma$ with color $\sigma(u)$. The total process takes at most $n + n + 1 + n - |C_k^\sigma| \leq 3n + 1$.

Whenever $\sigma(n+1)=k$. If $\rho'(n)=\rho'(2n)$, then recolor, if needed, one of this two vertices with the color assigned to it by σ. After, recolor each vertex u in $\{n+2,n+3,\ldots,2n-1\}$ with color $\sigma(u)$. Next, if needed, recolor the other vertex in $\{n,2n\}$. Then, recolor vertex 1 with color $\sigma(1)$. After, recolor vertex $n+1$ with color k. Finally, recolor each vertex u in $\{0,1,2,\ldots,n-1\}\setminus C_k^\sigma$ with color $\sigma(u)$. The total process takes at most $n+n+2+n-(|C_k^\sigma|-1)\le 3n+1$. If $\rho'(n)\neq\rho'(2n)$, then recolor each vertex u in $\{n+2,n+3,\ldots,2n-1\}$, if needed, with color $\sigma(u)$. Then, recolor the vertices in $\{n,2n\}$ with the color that σ gives them. After, recolor vertex 1 with color $\sigma(1)$ and, next, recolor vertex $n+1$ with color k. Finally, recolor each vertex u in $\{0,1,2,\ldots,n-1\}\setminus C_k^\sigma$ with color $\sigma(u)$. The total process takes at most $n+n+2+n-(|C_k^\sigma|-1)\le 3n+1$.

Case 2. *There are two color classes C_i^ρ and C_j^ρ, with $\{i,j\}\subseteq\{1,2,\ldots,k\}$ and $i\neq j$, such that $C_i^\rho\cup C_j^\rho\subseteq\{a+1,a+2,\ldots,a+n\}$ for some $a\in\mathbb{Z}_{2n+1}$.*

Suppose w.l.o.g. that $|C_i^\rho|\ge|C_j^\rho|$. Start at ρ, then recolor all vertices in C_j^ρ with color i. This process takes at most $\lfloor\frac{n}{2}\rfloor$ steps, and the new coloring ρ' satisfies that $C_j^{\rho'}=\emptyset$, and thus $d(\rho',\sigma)\le 3n+2$, by Case 1. Hence, $d(\rho,\sigma)\le d(\rho,\rho')+d(\rho',\sigma)\le 3n+2+\lfloor\frac{n}{2}\rfloor$.

Case 3. *For each pair of color classes C_i^ρ and C_j^ρ, with $\{i,j\}\subseteq\{1,2,\ldots,k\}$ and $i\neq j$, there is no $a\in\mathbb{Z}_{2n+1}$ such that $C_i^\rho\cup C_j^\rho\subseteq\{a+1,a+2,\ldots,a+n\}$.*

Case 3.1. *There is a color class C_i^ρ, with $i\in\{1,2,\ldots,k\}$, such that $C_i^\rho\subseteq\{a,a+1,a+2,\ldots,a+n-1\}$ and $C_i^\rho\neq\emptyset$, for some $a\in\mathbb{Z}_{2n+1}$.*

Then C_i^ρ is not a forbidden triangle. Assume w.l.o.g. that $a=0$. So, if we start at ρ we can recolor every vertex in $\{0,1,2\ldots,n-1\}\setminus C_i^\rho$ with color i, this process takes at most $n-1$ steps. Call ρ' to the current coloring. It follows that there must be two color classes of ρ' contained in $\{n,n+1,\ldots,2n-1\}$ or in $\{n+1,n+2,\ldots,2n\}$, so we can proceed as in Case 2. An thus, $d(\rho,\sigma)\le n-1+3n+2+\lfloor\frac{n}{2}\rfloor\le 4n+1+\lfloor\frac{n}{2}\rfloor$.

Case 3.2. *For each color class C_i^ρ, with $i\in\{1,2,\ldots,k\}$, there exists an $a\in\mathbb{Z}_{2n+1}$ such that $\{a,a+n+1\}\subseteq C_i^\rho$.*

Since $k\neq\frac{2n+1}{3}$, it follows that, for some $s\in\{1,2,\ldots,k\}$, $|C_s^\rho|\neq 3$, and thus ρ has a color class that is not a forbidden triangle. By Lemma 12, there is a coloring $\rho'\colon\mathbb{Z}_{2n+1}\to\{1,2,\ldots,k\}$ such that $C_s^{\rho'}=\{a,a+1,\ldots,a+n-1\}$ for some $a\in\mathbb{Z}_{2n+1}$ and $d(\rho,\rho')\le n$. As in Case 3.1, there must be two color classes of ρ' contained in $\{a+n,a+n+1,\ldots,n+2n-1\}$ or in $\{a+n+1,a+n+2,\ldots,a+2n\}$, so we can proceed as in Case 2. An thus, $d(\rho,\sigma)\le n+3n+2+\lfloor\frac{n}{2}\rfloor\le 4n+2+\lfloor\frac{n}{2}\rfloor$. $\square$

Theorem 17. *Let $n\ge 4$ such that $2n+1\equiv 0\pmod 3$. Then $\mathcal{D}_{\frac{2n+1}{3}}(\vec{C}_{2n+1}\langle n\rangle)$ consists of $3(\frac{2n+1}{3})!$ isolated vertices and one further component that has diameter at most $2n+1+\lfloor\frac{2n+1}{4}\rfloor$.*

Proof. Let $n' \geq 1$ be such that $2n+1 = 6n'+3$, then $\frac{2n+1}{3} = 2n'+1$ and $n = 3n'+1$. Let $T = \vec{C}_{6n'+3}\langle 3n'+1\rangle = \vec{C}_{2n+1}\langle n\rangle$. Since T is 3-dichromatic, $\mathcal{D}_{2n'+1}(T)$ is defined for each $n' \geq 1$. Let $\alpha\colon \mathbb{Z}_{6n'+3} \to \{1, 2, \ldots, 2n'+1\}$ be the coloring of T given by:

$$\alpha(i) = \begin{cases} 1 & \text{whenever } i \in \{0, 3n'+1, 3n'+2\}; \\ 2 & \text{whenever } i \in \{1, 2, 3n'+3\}; \\ 3 & \text{whenever } i \in \{3, 3n'+4, 3n'+5\}; \\ 4 & \text{whenever } i \in \{4, 5, 3n'+6\}; \\ \vdots & \qquad\qquad \vdots \\ 2n' & \text{whenever } i \in \{3n'-2, 3n'-1, 6n'\}; \\ 2n'+1 & \text{whenever } i \in \{3n', 6n'+1, 6n'+2\}. \end{cases}$$

By Lemma 10, each color class induces an acyclic subdigraph. Moreover, since each color class is a maximum acyclic set, if we recolor any vertex, then the new color function is not acyclic, and thus α is an isolated vertex in $\mathcal{D}_{2n'+1}(T)$.

We denote by $\alpha + \ell$ the coloring obtained from alpha by rotating its color classes by ℓ places, this is, α_ℓ be the coloring defined as $\alpha_\ell(i+\ell) = \alpha(i)$ for each $i \in \mathbb{Z}_{6n'+3}$.

The colorings α_1 and α_2 are also isolated vertices in $\mathcal{D}_{2n'+1}(T)$. Moreover, every permutation of the colors assigned to the color classes of α, α_1, or α_2, results in a new coloring which is also an isolated vertex in $\mathcal{D}_{2n'+1}(T)$. Thus, $\mathcal{D}_{2n'+1}(T)$ contains at least $3(2n'+1)!$ isolated vertices.

Now, suppose that ρ and σ are two colorings such that each of them contains a color class that is not a forbidden triangle. Proceeding as in the proof of Theorem 16, with Cases 1, 2 and 3 (excluding Case 3.2), we obtain that $d(\rho,\sigma) \leq 4(3n'+1) + 2 + \lfloor \frac{3n'+1}{2} \rfloor = 12n' + 6 + \lfloor \frac{3n'+1}{2} \rfloor \leq 2n+1+\lfloor \frac{2n+1}{4} \rfloor$. Moreover, as any two colorings, each containing a color class that is not a forbidden triangle, belong to the same connected component, it follows that $\mathcal{D}_{2n'+1}(T)$ has precisely $3(2n'+1)!$ isolated vertices. □

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