

DIVISIBILITY AMONG POWER GCD AND POWER LCM MATRICES ON CERTAIN GCD-CLOSED SETS

JIXIANG WAN AND GUANGYAN ZHU*

ABSTRACT. Let (x, y) and $[x, y]$ denote the greatest common divisor and the least common multiple of the integers x and y respectively. We denote by $|T|$ the number of elements of a finite set T . Let a, b and n be positive integers and let $S = \{x_1, \dots, x_n\}$ be a set of n distinct positive integers. We denote by (S^a) (resp. $[S^a]$) the $n \times n$ matrix whose (i, j) -entry is the a th power of (x_i, x_j) (resp. $[x_i, x_j]$). For any $x \in S$, define $G_S(x) := \{d \in S : d < x, d|x \text{ and } (d|y|x, y \in S) \Rightarrow y \in \{d, x\}\}$. In this paper, we show that if $a|b$ and S is gcd closed (namely, $(x_i, x_j) \in S$ for all integers i and j with $1 \leq i, j \leq n$) and $\max_{x \in S} \{|G_S(x)|\} = 2$ and the condition $\mathcal{G}$ being satisfied (i.e., any element $x \in S$ satisfies that either $|G_S(x)| \leq 1$, or $G_S(x) = \{y_1, y_2\}$ satisfying that $[y_1, y_2] = x$ and $(y_1, y_2) \in G_S(y_1) \cap G_S(y_2)$), then $(S^a)|(S^b)$, $(S^a)[[S^b]]$ and $[S^a][[S^b]]$ hold in the ring $M_n(\mathbf{Z})$. Furthermore, we show the existences of gcd-closed sets S such that S does not satisfy the condition $\mathcal{G}$ and such factorizations are true. Our result extends the Feng-Hong-Zhao theorem gotten in 2009. This also partially confirms a conjecture raised by Hong in [S.F. Hong, Divisibility among power GCD matrices and power LCM matrices, *Bull. Aust. Math. Soc.*, doi:10.1017/S0004972725100361].

1. INTRODUCTION

Let $\mathbf{Z}$ and $\mathbf{Z}^+$ denote the ring of integers and the set of positive integers. Let $|T|$ stand for the cardinality of a finite set T of integers. For any $x, y \in \mathbf{Z}^+$, let (x, y) and $[x, y]$ denote their greatest common divisor and least common multiple, respectively. Let f be an arithmetic function and let $S = \{x_1, \dots, x_n\}$ be a set of $n \in \mathbf{Z}^+$ distinct positive integers. Let $(f(S))$ and $(f[S])$ denote the $n \times n$ matrices having $f((x_i, x_j))$ and $(f[x_i, x_j])$ as its (i, j) -entry, respectively. We say that S is *factor closed* (FC) if $[x \in S, d|x, d > 0] \Rightarrow [d \in S]$, and that S is *gcd closed* if $(x_i, x_j) \in S \forall 1 \leq i, j \leq n$. Then any FC set is gcd closed but not conversely. Smith [34] proved that $\det(f(x_i, x_j)) = \prod_{k=1}^n (f * \mu)(x_k)$, where S is FC and $(f * \mu)(x) := \sum_{d|x} f(d)\mu(\frac{x}{d})$ for any positive integer x . Since then lots of generalizations of Smith's theorem and related results have published (see, for example, [1]- [33] and [35]- [44]). The function ξ_a is defined for any positive integer x by $\xi_a(x) := x^a$. The $n \times n$ matrix $(\xi_a(x_i, x_j))$ (abbreviated by (S^a)) and $(\xi_a[x_i, x_j])$ (abbreviated by $[S^a]$) are called *ath power GCD matrix* on S and *ath power LCM matrix* on S , respectively. In 1993, Bourque and Ligh [6] extended the Beslin-Ligh result [3] and Smith's determinant by proving that if S is gcd closed, then $\det(S^a) = \prod_{k=1}^n \alpha_{a,k}$, where

$$\alpha_{a,k} := \sum_{\substack{d|x_k \\ d \nmid x_t, x_t < x_k}} (\xi_a * \mu)(d). \quad (1.1)$$

2020 *Mathematics Subject Classification.* Primary 11C20; Secondary 11A05, 15B36.

Key words and phrases. Divisibility; power GCD matrix; power LCM matrix; gcd-closed set; greatest-type divisor.

* G.Y. Zhu is the corresponding author. The research was supported in part by the Startup Research Fund of Hubei Minzu University for Doctoral Scholars (Grant No. BS25008).

For any $x, y \in S$ with $x < y$, we say that x is a *greatest-type divisor* of y in S if $[x|y, x|d|y, d \in S] \Rightarrow [d \in \{x, y\}]$. For $x \in S$, $G_S(x)$ denotes the set of all greatest-type divisors of x in S . The concept of greatest-type divisor was introduced by Hong and played a key role in his solution [17] of the Bourque-Ligh conjecture [4]. Bourque and Ligh [7] showed that if S is FC and $a \geq 1$ is an integer, then the a th power GCD matrix (S^a) divides the a th power LCM matrix $[S^a]$ in the ring $M_n(\mathbf{Z})$ of $n \times n$ matrices over the integers. That is, $\exists A \in M_n(\mathbf{Z})$ such that $[S^a] = (S^a)A$ or $[S^a] = A(S^a)$. Hong [19] showed that such a factorization is no longer true in general if S is gcd closed and $\max_{x \in S} \{|G_S(x)|\} = 2$. These results were extended by Korkee and Haukkanen [30] and Chen et al. [8]. Hong [23] is the first one investigating the divisibility properties among power GCD matrices and among power LCM matrices. In fact, he showed that $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$ hold in $M_n(\mathbf{Z})$ if $a|b$ and S is a divisor chain (that is, $x_{\sigma(1)}|\dots|x_{\sigma(n)}$ for a permutation σ of $\{1, \dots, n\}$), and such factorizations are no longer true if $a \nmid b$ and $|S| \geq 2$. Evidently, a divisor chain is gcd closed but not conversely. In 2022 and 2023, Zhu [41] and Zhu and Li [43] confirmed three conjectures of Hong [23] by showing that $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$ hold in $M_n(\mathbf{Z})$ when $a|b$ and S is a gcd-closed set with $\max_{x \in S} \{|G_S(x)|\} = 1$. In 2022, Zhu, Li and Xu [44] showed the existences of gcd-closed sets S with $\max_{x \in S} \{|G_S(x)|\} = 2$ and infinitely many integers $b \geq 2$ such that $(S) \nmid (S^b)$ (resp. $(S) \nmid [S^b]$ and $[S] \nmid [S^b]$). As shown in [24], for any set S of positive integers and for any $x \in S$ with $|G_S(x)| \geq 2$, we say that the two distinct greatest-type divisors y_1 and y_2 of x in S *satisfy the condition $\mathcal{G}$* if $[y_1, y_2] = x$ and $(y_1, y_2) \in G_S(y_1) \cap G_S(y_2)$. We say that x *satisfies the condition $\mathcal{G}$* if any two distinct greatest-type divisors of x in S satisfy the condition $\mathcal{G}$. Moreover, we say that a set S of positive integers *satisfies the condition $\mathcal{G}$* if any element x in S satisfies that either $|G_S(x)| \leq 1$, or $|G_S(x)| \geq 2$ and x satisfies the condition $\mathcal{G}$. The following conjecture was proposed in the last section of [24].

Conjecture 1.1. [24, Conjecture 3.4] *Let a and b be positive integers with $a|b$ and let S be a gcd-closed set satisfying the condition $\mathcal{G}$. Then $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$ in the ring $M_{|S|}(\mathbf{Z})$.*

By Zhu's theorem [41] and the Zhu-Li result [43], one knows that this conjecture is true when $\max_{x \in S} \{|G_S(x)|\} = 1$. For $\max_{x \in S} \{|G_S(x)|\} = 2$, notice that the condition $\mathcal{G}$ is the condition $\mathcal{C}$ in [11]. When $a = b$ and $\max_{x \in S} \{|G_S(x)|\} = 2$, Feng, Hong and Zhao [11] verified this conjecture. For the case when $a|b$ and $a < b$, and $\max_{x \in S} \{|G_S(x)|\} \geq 2$, Conjecture 1.1 remains widely open. In this paper, our main goal is to explore Conjecture 1.1 for the case $\max_{x \in S} \{|G_S(x)|\} = 2$. The main results of this paper can be stated as follows.

Theorem 1.2. *Let a and b be positive integers with $a|b$ and let S be a gcd-closed set with $\max_{x \in S} \{|G_S(x)|\} = 2$ and the condition $\mathcal{G}$ being satisfied. Then $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$ hold in the ring $M_n(\mathbf{Z})$.*

Theorem 1.3. *Each of the following is true.*

- (i). *For any positive integer $b \neq 2$, there exist gcd-closed sets S_1 with $|S_1| = 4$ and $\max_{x \in S_1} \{|G_{S_1}(x)|\} = 2$ and the condition $\mathcal{G}$ not being satisfied such that $(S_1)|(S_1^b)$ holds in the ring $M_4(\mathbf{Z})$.*
- (ii). *For $b = 3$, or any integer $b \geq 4$ and $b \not\equiv 1, 5 \pmod{6}$, there exist gcd-closed sets S_2 with $|S_2| = 4$ and $\max_{x \in S_2} \{|G_{S_2}(x)|\} = 2$ and the condition $\mathcal{G}$ not being satisfied such that $(S_2)|[S_2^b]$ holds in the ring $M_4(\mathbf{Z})$.*

(iii). *There exist integers $b > 1$ and gcd-closed sets S_3 with $\max_{x \in S_3} \{|G_{S_3}(x)|\} = 2$, $|S_3| \in \{4, 5\}$ and the condition $\mathcal{G}$ not being satisfied such that $[S_3][[S_3^b]]$ holds in the ring $M_n(\mathbf{Z})$.*

Obviously, letting $a = b$, Theorem 1.2 reduces to the main result of Feng, Hong and Zhao [11]. The proofs of Theorems 1.2 and 1.3 use combinatorial and number-theoretic methods. We organize this paper as follows. In Section 2, we supply some preliminary lemmas that are needed in the proof of Theorem 1.2. In Section 3, we present the proofs of Theorems 1.2 and 1.3. The final section is devoted to some remarks.

As in [23], for any permutation σ on the set $\{1, \dots, n\}$, $(S^a)|(S^b)$ if and only if $(S_\sigma^a)|(S_\sigma^b)$. Likewise, $(S^a)[[S^b]]$ if and only if $(S_\sigma^a)[[S_\sigma^b]]$, and $[S^a][[S^b]]$ if and only if $[S_\sigma^a][[S_\sigma^b]]$, where $S_\sigma := \{x_{\sigma(1)}, \dots, x_{\sigma(n)}\}$. So, without loss of any generality (WLOG), throughout we always assume that $S = \{x_1, \dots, x_n\}$ satisfies $x_1 < \dots < x_n$.

2. AUXILIARY RESULTS

In this section, we supply several lemmas that will be needed in the proof of Theorem 1.2. We begin with a result due to Bourque and Ligh which gives the inverse of the power GCD matrix on a gcd-closed set.

Lemma 2.1. [6, Theorem 3] *If S is gcd closed and $(f(S))$ is nonsingular, then for any integers i and j with $1 \leq i, j \leq n$, we have*

$$((f(S))^{-1})_{ij} := \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik} c_{jk}}{\delta_k}$$

with

$$\delta_k := \sum_{\substack{d | x_k \\ d \nmid x_t, x_t < x_k}} (f * \mu)(d)$$

and

$$c_{ij} := \sum_{\substack{dx_i | x_j \\ dx_i \nmid x_t, x_t < x_j}} \mu(d). \quad (2.1)$$

Lemma 2.2. *If S is gcd closed, then the power GCD matrix (S^a) is nonsingular and for arbitrary integers i and j with $1 \leq i, j \leq n$, one has*

$$((S^a)^{-1})_{ij} := \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik} c_{jk}}{\alpha_{a,k}}$$

with c_{ij} being defined as in (2.1) and $\alpha_{a,k}$ being defined as in (1.1).

Proof. From [6, Example 1 (ii)], one knows that the power GCD matrix (S^a) is positive definite, and so is nonsingular. Then Lemma 2.1 applied to $f = \xi_a$ gives us the expected result. $\square$

We also need the following Hong's formula for the determinant of the power LCM matrix on a gcd-closed set. For any positive integer x , the function $\frac{1}{\xi_a}$ is defined by $\frac{1}{\xi_a}(x) = \frac{1}{x^a}$.

Lemma 2.3. [21, Lemma 2.1] *If S is gcd closed, then*

$$\det[S^a] = \prod_{k=1}^n x_k^{2a} \beta_{a,k}, \quad (2.2)$$

where

$$\beta_{a,k} := \sum_{\substack{d|x_k \\ d \nmid x_t, x_t < x_k}} \left(\frac{1}{\xi_a} * \mu \right)(d). \quad (2.3)$$

Lemma 2.4. *Let S be gcd closed and $\max_{x \in S} \{|G_S(x)|\} = 2$. Let $\alpha_{a,k}$ and $\beta_{a,k}$ be given as in (1.1) and (2.3), respectively. Then $\alpha_{a,1} = x_1^a$ and $\beta_{a,1} = x_1^{-a}$, and for any integer m with $2 \leq m \leq n$, we have*

$$\alpha_{a,m} = \begin{cases} x_m^a - x_{m_0}^a & \text{if } G_S(x_m) = \{x_{m_0}\}, \\ x_m^a - x_{m_1}^a - x_{m_2}^a + x_{m_3}^a & \text{if } G_S(x_m) = \{x_{m_1}, x_{m_2}\} \text{ and } x_{m_3} = (x_{m_1}, x_{m_2}) \end{cases}$$

and

$$\beta_{a,m} = \begin{cases} x_m^{-a} - x_{m_0}^{-a} & \text{if } G_S(x_m) = \{x_{m_0}\}, \\ x_m^{-a} - x_{m_1}^{-a} - x_{m_2}^{-a} + x_{m_3}^{-a} & \text{if } G_S(x_m) = \{x_{m_1}, x_{m_2}\} \text{ and } x_{m_3} = (x_{m_1}, x_{m_2}). \end{cases}$$

Proof. Employing [22, Theorem 1.2], we directly get Lemma 2.4. $\square$

In what follows, we recall several basic results on the gcd-closed sets.

Lemma 2.5. [11, Lemma 2.3] *Let S be a gcd-closed set of $n \geq 2$ distinct positive integers and let c_{ij} be defined as in (2.1). Then*

$$c_{r1} = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $2 \leq m \leq n$ and $G_S(x_m) = \{x_{m_0}\}$, then

$$c_{rm} = \begin{cases} -1 & \text{if } r = m_0, \\ 1 & \text{if } r = m, \\ 0 & \text{otherwise.} \end{cases}$$

If $3 \leq m \leq n$ and $G_S(x_m) = \{x_{m_1}, x_{m_2}\}$ and $x_{m_3} = (x_{m_1}, x_{m_2})$, then

$$c_{rm} = \begin{cases} -1 & \text{if } r = m_1 \text{ or } r = m_2, \\ 1 & \text{if } r = m \text{ or } m_3, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.6. [11] *Let S be a gcd-closed set satisfying $\max_{x \in S} \{|G_S(x)|\} = 2$ and let $x \in S$ satisfy $|G_S(x)| = 2$ and $y \in G_S(x)$. Let $z \in S$ be such that $z|x$, $z \neq x$ and $z \nmid y$. If $A := \{u \in S : z|u|x, u \neq z\}$ satisfies the condition $\mathcal{G}$, then $[y, z] = x$.*

Lemma 2.7. [25, Lemma 2.2] *Let S be gcd closed such that $\max_{x \in S} \{|G_S(x)|\} = 2$ and $|S| = n$. Let $\beta_{a,k}$ be defined as in (2.3). Then $\beta_{a,k} \neq 0$ for any integer k with $1 \leq k \leq n$.*

Lemma 2.8. *Let S be a gcd-closed set satisfying $\max_{x \in S} \{|G_S(x)|\} = 2$. Then the a th power LCM matrix $[S^a]$ is nonsingular and for all integers i and j with $1 \leq i, j \leq n$, one has*

$$([S^a]^{-1})_{ij} := \frac{1}{x_i^a x_j^a} \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik} c_{jk}}{\beta_{a,k}}$$

with c_{ij} being defined as in (2.1) and $\beta_{a,k}$ being defined as in (2.3).

Proof. Since $[x_i, x_j]^a = x_i^a x_j^a / (x_i, x_j)^a$, we have

$$[S^a] = \text{diag}(x_1^a, \dots, x_n^a) \cdot \left(\frac{1}{\xi_a}(x_i, x_j) \right) \cdot \text{diag}(x_1^a, \dots, x_n^a). \quad (2.4)$$

Hence

$$\det[S^a] = \det\left(\frac{1}{\xi_a}(x_i, x_j)\right) \cdot \prod_{k=1}^n x_k^{2a}. \quad (2.5)$$

Then from (2.2) and (2.5), we can derive that

$$\det\left(\frac{1}{\xi_a}(x_i, x_j)\right) = \prod_{k=1}^n \beta_{a,k}.$$

Lemma 2.7 tells us that $\beta_{a,k} \neq 0$ for all positive integers k ($\leq n$). So the matrix $\left(\frac{1}{\xi_a}(x_i, x_j)\right)$ is nonsingular.

Now applying Lemma 2.1 to $f = \frac{1}{\xi_a}$, one gets that

$$\left(\left(\frac{1}{\xi_a}(x_i, x_j)\right)^{-1}\right)_{ij} = \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik} c_{jk}}{\beta_{a,k}}. \quad (2.6)$$

Using (2.4) and (2.6) gives the required result. $\square$

Lemma 2.9. *Let a and b be positive integers such that $a|b$. Let S be a gcd-closed set and $x, y, z \in S$ with $G_S(x) = \{y\}$. Then each of the following is true:*

- (i). [41, Lemma 2.5] *The integer $x^a - y^a$ divides each of $(x, z)^b - (y, z)^b$ and $[x, z]^b - [y, z]^b$.*
- (ii). [43, Lemma 2.8] *If $r \in S$ and $r|x$, then $y^a[z, x]^b - x^a[z, y]^b$ is divisible by $r^a(y^a - x^a)$.*

Lemma 2.10. *Let a and b be positive integers with $a|b$. Let S be a gcd-closed set with $\max_{x \in S} \{|G_S(x)|\} = 2$ and $z \in S$. For $x \in S$ with $|G_S(x)| = 2$, let $G_S(x) = \{y_1, y_2\}$ and $y_3 := (y_1, y_2)$. Assume that the set $\{u \in S : (x, z)|u\}$ satisfies the condition $\mathcal{G}$. Then each of the following is true:*

- (i). *$x^a + y_3^a - y_1^a - y_2^a$ divides each of $(z, x)^b + (z, y_3)^b - (z, y_1)^b - (z, y_2)^b$ and $[z, x]^b + [z, y_3]^b - [z, y_1]^b - [z, y_2]^b$.*
- (ii). *For any $r \in S$ with $r|x$, $x^a[z, y_3]^b + y_3^a[z, x]^b - y_1^a[z, y_2]^b - y_2^a[z, y_1]^b$ is divisible by $r^a(x^a + y_3^a - y_1^a - y_2^a)$.*

Proof. Let $d = (x, z)$. If $x|z$, then

$$\begin{aligned} (z, x)^b + (z, y_3)^b - (z, y_1)^b - (z, y_2)^b &= x^b + y_3^b - y_1^b - y_2^b, \\ [z, x]^b + [z, y_3]^b - [z, y_1]^b - [z, y_2]^b &= z^b + z^b - z^b - z^b = 0 \end{aligned}$$

and

$$x^a[z, y_3]^b + y_3^a[z, x]^b - y_1^a[z, y_2]^b - y_2^a[z, y_1]^b = (x^a + y_3^a - y_1^a - y_2^a)z^b.$$

Since $G_S(x) = \{y_1, y_2\}$, $y_3 := (y_1, y_2)$ and x satisfies the condition $\mathcal{G}$, it follows that $xy_3 = y_1y_2$. This implies that for any positive integer l , one has

$$x^l + y_3^l - y_1^l - y_2^l = (y_2^l - y_3^l) \left(\left(\frac{x}{y_2} \right)^l - 1 \right).$$

So

$$x^b + y_3^b - y_1^b - y_2^b = (y_2^b - y_3^b) \left(\left(\frac{x}{y_2} \right)^b - 1 \right).$$

and

$$x^a + y_3^a - y_1^a - y_2^a = (y_2^a - y_3^a) \left(\left(\frac{x}{y_2} \right)^a - 1 \right). \quad (2.7)$$

Since $a|b$, we have

$$\frac{x^b + y_3^b - y_1^b - y_2^b}{x^a + y_3^a - y_1^a - y_2^a} = \frac{y_2^b - y_3^b}{y_2^a - y_3^a} \frac{\left(\frac{x}{y_2} \right)^b - 1}{\left(\frac{x}{y_2} \right)^a - 1} \in \mathbf{Z}.$$

The statements for parts (i) and (ii) are clearly true. In what follows, we let $x \nmid z$. Then $d < x$ and $d \in S$ since S is gcd closed. The conditions $d|x$ and $G_S(x) = \{y_1, y_2\}$ yield that either $d|y_1$ or $d|y_2$. One needs only to consider the following two cases.

CASE 1. $d|y_1$ and $d|y_2$. Then $d|y_3$. Since $d|z$, one has $d|(y_3, z)$. However, $y_3|y_1|x$ and $y_3|y_2|x$. One then derives that $(y_3, z)|(y_1, z)|(x, z) = d$ and $(y_3, z)|(y_2, z)|(x, z) = d$. Then $(y_3, z) = (y_1, z) = (y_2, z) = (x, z)$ which infers that $(z, x)^b + (z, y_3)^b - (z, y_1)^b - (z, y_2)^b = 0$. Hence $x^a + y_3^a - y_1^a - y_2^a$ divides $(z, x)^b + (z, y_3)^b - (z, y_1)^b - (z, y_2)^b$. So the first statement for part (i) is true in this case. Moreover, one has

$$\begin{aligned} & [z, x]^b + [z, y_3]^b - [z, y_2]^b - [z, y_1]^b \\ &= \frac{z^b x^b}{(z, x)^b} + \frac{z^b y_3^b}{(z, y_3)^b} - \frac{z^b y_2^b}{(z, y_2)^b} - \frac{z^b y_1^b}{(z, y_1)^b} \\ &= \frac{z^b}{(z, x)^b} (x^b + y_3^b - y_1^b - y_2^b) \\ &= \frac{z^b}{(z, x)^b} (y_2^b - y_3^b) \left(\left(\frac{x}{y_2} \right)^b - 1 \right) \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} & x^a [z, y_3]^b + y_3^a [z, x]^b - y_1^a [z, y_2]^b - y_2^a [z, y_1]^b \\ &= x^a \frac{z^b y_3^b}{(z, y_3)^b} + y_3^a \frac{z^b x^b}{(z, x)^b} - y_1^a \frac{z^b y_2^b}{(z, y_2)^b} - y_2^a \frac{z^b y_1^b}{(z, y_1)^b} \\ &= \frac{z^b}{(z, x)^b} (x^a y_3^b + y_3^a x^b - y_1^a y_2^b - y_2^a y_1^b) \\ &= \frac{z^b}{(z, x)^b} x^a y_3^a (x^{b-a} + y_3^{b-a} - y_1^{b-a} - y_2^{b-a}) \\ &= \frac{z^b}{(z, x)^b} x^a y_3^a (y_2^{b-a} - y_3^{b-a}) \left(\left(\frac{x}{y_2} \right)^{b-a} - 1 \right). \end{aligned} \quad (2.9)$$

It follows from (2.7) and (2.8) that

$$\frac{[z, x]^b + [z, y_3]^b - [z, y_2]^b - [z, y_1]^b}{x^a + y_3^a - y_1^a - y_2^a} = \left(\frac{z}{(z, x)} \right)^b \cdot \frac{y_2^b - y_3^b}{y_2^a - y_3^a} \cdot \frac{\left(\frac{x}{y_2} \right)^b - 1}{\left(\frac{x}{y_2} \right)^a - 1}. \quad (2.10)$$

And from (2.7) and (2.9), one derives that

$$\frac{x^a [z, y_3]^b + y_3^a [z, x]^b - y_1^a [z, y_2]^b - y_2^a [z, y_1]^b}{r^a (x^a + y_3^a - y_1^a - y_2^a)}$$

$$= \left(\frac{z}{(z, x)} \right)^b y_3^a \cdot \left(\frac{x}{r} \right)^a \frac{y_2^{b-a} - y_3^{b-a}}{y_2^a - y_3^a} \cdot \frac{\left(\frac{x}{y_2} \right)^{b-a} - 1}{\left(\frac{x}{y_2} \right)^a - 1}. \quad (2.11)$$

Since $(z, x)|z$, $r|x$, $y_2|x$ and $a|b$, one knows that all the rational numbers

$$\left(\frac{z}{(z, x)} \right)^b, \frac{y_2^b - y_3^b}{y_2^a - y_3^a}, \frac{\left(\frac{x}{y_2} \right)^b - 1}{\left(\frac{x}{y_2} \right)^a - 1}, \left(\frac{x}{r} \right)^a, \frac{y_2^{b-a} - y_3^{b-a}}{y_2^a - y_3^a} \text{ and } \frac{\left(\frac{x}{y_2} \right)^{b-a} - 1}{\left(\frac{x}{y_2} \right)^a - 1}$$

are integers. It then follows from (2.10) and (2.11) that

$$\frac{[z, x]^b + [z, y_3]^b - [z, y_2]^b - [z, y_1]^b}{x^a + y_3^a - y_1^a - y_2^a} \in \mathbf{Z},$$

and

$$\frac{x^a[z, y_3]^b + y_3^a[z, x]^b - y_1^a[z, y_2]^b - y_2^a[z, y_1]^b}{r^a(x^a + y_3^a - y_1^a - y_2^a)} \in \mathbf{Z}.$$

In other words, $x^a + y_3^a - y_1^a - y_2^a$ divides $[z, x]^b + [z, y_3]^b - [z, y_1]^b - [z, y_2]^b$, and $x^a[z, y_3]^b + y_3^a[z, x]^b - y_1^a[z, y_2]^b - y_2^a[z, y_1]^b$ is divisible by $r^a(x^a + y_3^a - y_1^a - y_2^a)$. So the second statement of part (i) and part (ii) are true in this case. Lemma 2.10 is proved in this case.

CASE 2. d divides exactly one of y_1 and y_2 . WLOG, one may let $d|y_1$ and $d \nmid y_2$. Since the set $\{u \in S : (x, z)|u|x\}$ satisfies the condition $\mathcal{G}$, $d|x$, $d \neq x$, $y_2 \in G_S(x)$ and $d \nmid y_2$, applying Lemma 2.6 gives us that $[d, y_2] = x$. Likewise, we have $[d, y_3] = y_1$. In fact, if $d = y_1$, then by $y_3|y_1$, we know that $y_3|d$ and so $[d, y_3] = d = y_1$. Now we let $d \neq y_1$. Since $d|y_1$, $y_3 \in G_S(y_1)$, and $d \nmid y_2$ implying that $d \nmid y_3$, by Lemma 2.6 we derive that $[d, y_3] = y_1$. But $d = (x, z)|z$. Then one can deduce that

$$[z, x] = [z, [d, y_2]] = [[z, d], y_2] = [z, y_2] \text{ and } [z, y_1] = [z, [d, y_3]] = [z, y_3]. \quad (2.12)$$

It readily follows from (2.12) that

$$[z, x]^b + [z, y_3]^b - [z, y_1]^b - [z, y_2]^b = 0. \quad (2.13)$$

On the one hand, since $y_1|x$, we have $(z, y_1)|(z, x)$. On the other hand, $(z, x) = d|y_1$ together with $d = (x, z)|z$ yields that $(z, x)|(z, y_1)$. Therefore

$$(z, x) = (z, y_1). \quad (2.14)$$

Since $xy_3 = y_1y_2$, by (2.12) and (2.14), we have

$$\begin{aligned} & (z, x)^b + (z, y_3)^b - (z, y_1)^b - (z, y_2)^b \\ &= (z, y_3)^b - (z, y_2)^b \\ &= \frac{z^b y_3^b}{[z, y_3]^b} - \frac{z^b y_2^b}{[z, y_2]^b} \\ &= \frac{z^b y_3^b}{[z, y_1]^b} - \frac{z^b y_2^b}{[z, x]^b} \\ &= (z, y_1)^b \frac{y_3^b}{y_1^b} - (z, x)^b \frac{y_2^b}{x^b} = 0. \end{aligned} \quad (2.15)$$

So by (2.13) and (2.15), we know that $x^a + y_3^a - y_1^a - y_2^a$ divides both of $(z, x)^b + (z, y_3)^b - (z, y_1)^b - (z, y_2)^b$ and $[z, x]^b + [z, y_3]^b - [z, y_1]^b - [z, y_2]^b$. Part (i) holds in this case.

Likewise, by (2.12) and (2.14), we can deduce that

$$\begin{aligned} & x^a[z, y_3]^b + y_3^a[z, x]^b - y_1^a[z, y_2]^b - y_2^a[z, y_1]^b \\ &= x^a[z, y_1]^b + y_3^a[z, x]^b - y_1^a[z, x]^b - y_2^a[z, y_1]^b \end{aligned}$$

$$\begin{aligned}
&= x^a \frac{z^b y_1^b}{(z, y_1)^b} + y_3^a \frac{z^b x^b}{(z, x)^b} - y_1^a \frac{z^b x^b}{(z, x)^b} - y_2^a \frac{z^b y_1^b}{(z, y_1)^b} \\
&= \frac{z^b}{(z, x)^b} (x^a y_1^b + y_3^a x^b - y_1^a x^b - y_2^a y_1^b) \\
&= \frac{z^b}{(z, x)^b} (x^a y_1^b + y_3^a x^b - y_1^a x^b - x^a y_3^a y_1^{b-a}) \\
&= \left(\frac{z}{(z, x)} \right)^b x^a (y_1^{b-a} - x^{b-a}) (y_1^a - y_3^a) \\
&= \left(\frac{z}{(z, x)} \right)^b x^a y_1^{b-a} \left(1 - \left(\frac{x}{y_1} \right)^{b-a} \right) (y_1^a - y_3^a). \tag{2.16}
\end{aligned}$$

Since $xy_3 = y_1 y_2$, we have

$$x^a + y_3^a - y_1^a - y_2^a = \left(\left(\frac{x}{y_1} \right)^a - 1 \right) (y_1^a - y_3^a). \tag{2.17}$$

So by (2.16) and (2.17), one has

$$\begin{aligned}
&\frac{x^a [z, y_3]^b + y_3^a [z, x]^b - y_1^a [z, y_2]^b - y_2^a [z, y_1]^b}{r^a (x^a + y_3^a - y_1^a - y_2^a)} \\
&= - \left(\frac{x}{r} \right)^a \left(\frac{z}{(z, x)} \right)^b y_1^{b-a} \frac{\left(\frac{x}{y_1} \right)^{b-a} - 1}{\left(\frac{x}{y_1} \right)^a - 1}.
\end{aligned}$$

But the condition $a|(b-a)$ implies that

$$\frac{\left(\frac{x}{y_1} \right)^{b-a} - 1}{\left(\frac{x}{y_1} \right)^a - 1} \in \mathbf{Z}.$$

So

$$\frac{x^a [z, y_3]^b + y_3^a [z, x]^b - y_1^a [z, y_2]^b - y_2^a [z, y_1]^b}{r^a (x^a + y_3^a - y_1^a - y_2^a)} \in \mathbf{Z}.$$

That is, $r^a (x^a + y_3^a - y_1^a - y_2^a)$ divides $x^a [z, y_3]^b + y_3^a [z, x]^b - y_1^a [z, y_2]^b - y_2^a [z, y_1]^b$ as desired. Thus part (ii) is proved in this case.

This concludes the proof of Lemma 2.10. $\square$

Lemma 2.11. *Let a and b be positive integers with $a|b$. Let S be a gcd-closed set satisfying $\max_{x \in S} \{|G_S(x)|\} = 2$. If S satisfies the condition $\mathcal{G}$, then all the elements of the n -th column and the n -th row of the matrices $(S^b)(S^a)^{-1}$, $[S^b](S^a)^{-1}$ and $[S^b][S^a]^{-1}$ are integers.*

Proof. We divide the proof into the following two cases:

CASE 1. $1 \leq i \leq n$ and $j = n$. By Lemmas 2.2 and 2.8, we have

$$\begin{aligned}
((S^b)(S^a)^{-1})_{in} &= \sum_{m=1}^n (x_i, x_m)^b \sum_{\substack{x_m | x_k \\ x_n | x_k}} \frac{c_{mk} c_{nk}}{\alpha_{a,k}} \\
&= \frac{1}{\alpha_{a,n}} \sum_{m=1}^n (x_i, x_m)^b c_{mn}, \tag{2.18}
\end{aligned}$$

$$\begin{aligned}
([S^b](S^a)^{-1})_{in} &= \sum_{m=1}^n [x_i, x_m]^b \sum_{\substack{x_m | x_k \\ x_n | x_k}} \frac{c_{mk} c_{nk}}{\alpha_{a,k}} \\
&= \frac{1}{\alpha_{a,n}} \sum_{m=1}^n [x_i, x_m]^b c_{mn}
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
([S^b][S^a]^{-1})_{in} &= \sum_{m=1}^n [x_i, x_m]^b \frac{1}{x_m^a x_n^a} \sum_{\substack{x_m | x_k \\ x_n | x_k}} \frac{c_{mk} c_{nk}}{\beta_{a,k}} \\
&= \frac{1}{x_n^a \beta_{a,n}} \sum_{m=1}^n \frac{[x_i, x_m]^b c_{mn}}{x_m^a}.
\end{aligned} \tag{2.20}$$

If $|G_S(x_n)| = 1$, we may let $G_S(x_n) = \{x_{n_1}\}$. Then by (2.18) to (2.20), Lemmas 2.4, 2.5 and 2.9, one deduces that

$$\begin{aligned}
((S^b)(S^a)^{-1})_{in} &= \frac{(x_i, x_n)^b - (x_i, x_{n_1})^b}{x_n^a - x_{n_1}^a} \in \mathbf{Z}, \\
([S^b])(S^a)^{-1})_{in} &= \frac{[x_i, x_n]^b - [x_i, x_{n_1}]^b}{x_n^a - x_{n_1}^a} \in \mathbf{Z}
\end{aligned}$$

and

$$([S^b])[S^a]^{-1})_{in} = \frac{x_{n_1}^a [x_i, x_n]^b - x_n^a [x_i, x_{n_1}]^b}{x_n^a (x_{n_1}^a - x_n^a)} \in \mathbf{Z}$$

as required.

If $|G_S(x_n)| = 2$, we may let $G_S(x_n) = \{x_{n_1}, x_{n_2}\}$ and $x_{n_3} = (x_{n_1}, x_{n_2})$. Then $x_n x_{n_3} = x_{n_1} x_{n_2}$. Since S satisfies the condition $\mathcal{G}$, by (2.18) to (2.20), Lemmas 2.4, 2.5 and 2.10, one derives that

$$\begin{aligned}
((S^b)(S^a)^{-1})_{in} &= \frac{(x_i, x_n)^b - (x_i, x_{n_1})^b - (x_i, x_{n_2})^b + (x_i, x_{n_3})^b}{x_n^a - x_{n_1}^a - x_{n_2}^a + x_{n_3}^a} \in \mathbf{Z}, \\
([S^b])(S^a)^{-1})_{in} &= \frac{[x_i, x_n]^b - [x_i, x_{n_1}]^b - [x_i, x_{n_2}]^b + [x_i, x_{n_3}]^b}{x_n^a - x_{n_1}^a - x_{n_2}^a + x_{n_3}^a} \in \mathbf{Z}
\end{aligned}$$

and

$$\begin{aligned}
([S^b])[S^a]^{-1})_{in} &= \frac{\frac{[x_n, x_i]^b}{x_n^a} - \frac{[x_{n_1}, x_i]^b}{x_{n_1}^a} - \frac{[x_{n_2}, x_i]^b}{x_{n_2}^a} + \frac{[x_{n_3}, x_i]^b}{x_{n_3}^a}}{x_n^a \left(\frac{1}{x_n^a} - \frac{1}{x_{n_1}^a} - \frac{1}{x_{n_2}^a} + \frac{1}{x_{n_3}^a} \right)} \\
&= \frac{x_n^a [x_i, x_{n_3}]^b + x_{n_3}^a [x_i, x_n]^b - x_{n_2}^a [x_i, x_{n_1}]^b - x_{n_1}^a [x_i, x_{n_2}]^b}{x_n^a (x_n^a + x_{n_3}^a - x_{n_1}^a - x_{n_2}^a)} \in \mathbf{Z}
\end{aligned}$$

as required. So Lemma 2.11 is proved in Case 1.

CASE 2. $i = n$ and $1 \leq j \leq n - 1$. By Lemmas 2.2 and 2.8, one has

$$\begin{aligned}
((S^b)(S^a)^{-1})_{nj} &= \sum_{m=1}^n (x_n, x_m)^b \sum_{\substack{x_m | x_k \\ x_j | x_k}} \frac{c_{mk} c_{jk}}{\alpha_{a,k}} \\
&= \sum_{x_j | x_k} \frac{c_{jk}}{\alpha_{a,k}} \sum_{x_m | x_k} c_{mk} (x_m, x_n)^b
\end{aligned}$$

$$\begin{aligned}
&:= \sum_{x_j | x_k} c_{jk} \omega_k, \\
([S^b](S^a)^{-1})_{nj} &= \sum_{m=1}^n [x_n, x_m]^b \sum_{\substack{x_m | x_k \\ x_j | x_k}} \frac{c_{mk} c_{jk}}{\alpha_{a,k}} \\
&= \sum_{x_j | x_k} \frac{c_{jk}}{\alpha_{a,k}} \sum_{x_m | x_k} c_{mk} [x_m, x_n]^b \\
&:= \sum_{x_j | x_k} c_{jk} \gamma_k
\end{aligned}$$

and

$$\begin{aligned}
([S^b][S^a]^{-1})_{nj} &= \sum_{m=1}^n [x_n, x_m]^b \frac{1}{x_m^a x_j^a} \sum_{\substack{x_m | x_k \\ x_j | x_k}} \frac{c_{mk} c_{jk}}{\beta_{a,k}} \\
&= \sum_{x_j | x_k} \frac{c_{jk}}{x_j^a \beta_{a,k}} \sum_{x_m | x_k} \frac{1}{x_m^a} c_{mk} [x_m, x_n]^b \\
&:= \sum_{x_j | x_k} c_{jk} \eta_k.
\end{aligned}$$

Claim that for any positive integer k with $x_j | x_k$, one has $\omega_k \in \mathbf{Z}$, $\gamma_k \in \mathbf{Z}$ and $\eta_k \in \mathbf{Z}$.

If $k = 1$, then we must have $m = j = 1$. In this case, one has

$$\begin{aligned}
\omega_1 &= \frac{1}{\alpha_{a,1}} \cdot c_{11} \cdot (x_1, x_n)^b = \frac{(x_1, x_n)^b}{x_1^a} = x_1^{b-a} \in \mathbf{Z}, \\
\gamma_1 &= \frac{1}{\alpha_{a,1}} \cdot c_{11} \cdot [x_1, x_n]^b = \frac{[x_1, x_n]^b}{x_1^a} = \frac{x_1^{b-a} x_n^b}{(x_1, x_n)^b} \in \mathbf{Z}
\end{aligned}$$

and

$$\eta_1 = \frac{1}{\beta_{a,1}} \cdot \frac{1}{x_1^{2a}} \cdot c_{11} \cdot [x_1, x_n]^b = \frac{[x_1, x_n]^b}{x_1^a} \in \mathbf{Z}$$

since $\alpha_{a,1} = x_1^a$ and $\beta_{a,1} = x_1^{-a}$. So the claim is true when $k = 1$.

Now let $k > 1$. If $|G_S(x_k)| = 1$, one can set $G_S(x_k) = \{x_{k_1}\}$ with $1 \leq k_1 < k$. By Lemmas 2.4, 2.5 and 2.9, we have

$$\begin{aligned}
\omega_k &= \frac{1}{\alpha_{a,k}} \sum_{x_m | x_k} c_{mk} (x_m, x_n)^b = \frac{(x_k, x_n)^b - (x_{k_1}, x_n)^b}{x_k^a - x_{k_1}^a} \in \mathbf{Z}, \\
\gamma_k &= \frac{1}{\alpha_{a,k}} \sum_{x_m | x_k} c_{mk} [x_m, x_n]^b = \frac{[x_k, x_n]^b - [x_{k_1}, x_n]^b}{x_k^a - x_{k_1}^a} \in \mathbf{Z}
\end{aligned}$$

and

$$\eta_k = \frac{1}{x_j^a \beta_{a,k}} \sum_{x_m | x_k} \frac{1}{x_m^a} c_{mk} [x_m, x_n]^b = \frac{x_{k_1}^a [x_k, x_n]^b - x_k^a [x_{k_1}, x_n]^b}{x_j^a (x_{k_1}^a - x_k^a)} \in \mathbf{Z}$$

as claimed. So we need only to treat the remaining case $|G_S(x_k)| = 2$. Now let $G_S(x_k) = \{x_{k_1}, x_{k_2}\}$ and $x_{k_3} := (x_{k_1}, x_{k_2})$. Then by Lemmas 2.4, 2.5 and 2.10, we have

$$\omega_k = \frac{(x_k, x_n)^b - (x_{k_1}, x_n)^b - (x_{k_2}, x_n)^b + (x_{k_3}, x_n)^b}{x_k^a - x_{k_1}^a - x_{k_2}^a + x_{k_3}^a} \in \mathbf{Z},$$

$$\gamma_k = \frac{[x_k, x_n]^b - [x_{k_1}, x_n]^b - [x_{k_2}, x_n]^b + [x_{k_3}, x_n]^b}{x_k^a - x_{k_1}^a - x_{k_2}^a + x_{k_3}^a} \in \mathbf{Z}$$

and

$$\begin{aligned} \eta_k &= \frac{\frac{[x_k, x_n]^b}{x_k^a} - \frac{[x_{k_1}, x_n]^b}{x_{k_1}^a} - \frac{[x_{k_2}, x_n]^b}{x_{k_2}^a} + \frac{[x_{k_3}, x_n]^b}{x_{k_3}^a}}{x_j^a \left(\frac{1}{x_k^a} - \frac{1}{x_{k_1}^a} - \frac{1}{x_{k_2}^a} + \frac{1}{x_{k_3}^a} \right)} \\ &= \frac{x_k^a [x_n, x_{k_3}]^b + x_{k_3}^a [x_n, x_k]^b - x_{k_2}^a [x_n, x_{k_1}]^b - x_{k_1}^a [x_n, x_{k_2}]^b}{x_j^a (x_k^a + x_{k_3}^a - x_{k_1}^a - x_{k_2}^a)} \in \mathbf{Z} \end{aligned}$$

as desired.

This completes the proof of Case 2 and that of Lemma 2.11. $\square$

Lemma 2.12. [41, Theorem 1.3] [43, Theorem 1.1] *Let a and b be positive integers with $a|b$ and let S be a gcd-closed set satisfying $\max_{x \in S} \{|G_S(x)|\} = 1$. Then in the ring $M_{|S|}(\mathbf{Z})$, we have $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$.*

Finally, we can use Lemma 2.11 to show the following main result of this section.

Lemma 2.13. *Let S be a gcd-closed set satisfying $\max_{x \in S} \{|G_S(x)|\} = 2$ and let a and b be positive integers such that $a|b$. Let $S_0 := S \setminus \{\max(S)\}$. If S satisfies the condition $\mathcal{G}$, then*

$$(S^b)(S^a)^{-1} \in M_n(\mathbf{Z}) \Leftrightarrow (S_0^b)(S_0^a)^{-1} \in M_{n-1}(\mathbf{Z}),$$

$$[S^b](S^a)^{-1} \in M_n(\mathbf{Z}) \Leftrightarrow [S_0^b](S_0^a)^{-1} \in M_{n-1}(\mathbf{Z})$$

and

$$[S^b][S^a]^{-1} \in M_n(\mathbf{Z}) \Leftrightarrow [S_0^b][S_0^a]^{-1} \in M_{n-1}(\mathbf{Z}).$$

Proof. It is clear that $S_0 := S \setminus \{x_n\} = \{x_1, \dots, x_{n-1}\}$. At first, by Lemma 2.11, one knows that all the elements of the n -th column and the n -th row of the matrices $(S^b)(S^a)^{-1}$, $[S^b](S^a)^{-1}$ and $[S^b][S^a]^{-1}$ are integers. So it is sufficient to show that $\forall i, j$ ($1 \leq i, j \leq n-1$), one has

$$\mathcal{D}_{ij} := ((S^b)(S^a)^{-1})_{ij} - ((S_0^b)(S_0^a)^{-1})_{ij} \in \mathbf{Z}, \quad (2.21)$$

$$\mathcal{E}_{ij} := ([S^b](S^a)^{-1})_{ij} - ([S_0^b](S_0^a)^{-1})_{ij} \in \mathbf{Z} \quad (2.22)$$

and

$$\mathcal{F}_{ij} := ([S^b][S^a]^{-1})_{ij} - ([S_0^b][S_0^a]^{-1})_{ij} \in \mathbf{Z}. \quad (2.23)$$

For this, we define the following function:

$$e_{uv} := \begin{cases} 1 & \text{if } x_v | x_u \\ 0 & \text{if } x_v \nmid x_u \end{cases}$$

for any positive integers u, v ($u, v \leq n$). Then for any positive integer m ($\leq n-1$), we have $e_{nm} = 1$ if $x_m | x_n$, and $e_{nm} = 0$ otherwise. We can deduce that

$$\mathcal{D}_{ij} = \sum_{m=1}^n (x_i, x_m)^b \sum_{\substack{x_m | x_k \\ x_j \nmid x_k}} \frac{c_{mk} c_{jk}}{\alpha_{a,k}} - \sum_{m=1}^{n-1} (x_i, x_m)^b \sum_{\substack{x_m | x_k \\ x_j | x_k, x_k \neq x_n}} \frac{c_{mk} c_{jk}}{\alpha_{a,k}}$$

$$\begin{aligned}
&= \frac{c_{nn}c_{jn}}{\alpha_{a,n}}(x_i, x_n)^b e_{nj} + \sum_{m=1}^{n-1} \frac{c_{mn}c_{jn}}{\alpha_{a,n}}(x_i, x_m)^b e_{nj}e_{nm} \\
&= e_{nj} \frac{c_{jn}}{\alpha_{a,n}} \left((x_i, x_n)^b + \sum_{m=1}^{n-1} (x_i, x_m)^b c_{mn}e_{nm} \right) \\
&:= e_{nj} D_{ij}.
\end{aligned} \tag{2.24}$$

Likewise, we have

$$\begin{aligned}
\varepsilon_{ij} &= \sum_{m=1}^n [x_i, x_m]^b \sum_{\substack{x_m | x_k \\ x_j | x_k}} \frac{c_{mk}c_{jk}}{\alpha_{a,k}} - \sum_{m=1}^{n-1} [x_i, x_m]^b \sum_{\substack{x_m | x_k \\ x_j | x_k, x_k \neq x_n}} \frac{c_{mk}c_{jk}}{\alpha_{a,k}} \\
&= \frac{c_{nn}c_{jn}}{\alpha_{a,n}} [x_i, x_n]^b e_{nj} + \sum_{m=1}^{n-1} \frac{c_{mn}c_{jn}}{\alpha_{a,n}} [x_i, x_m]^b e_{nj}e_{nm} \\
&= e_{nj} \frac{c_{jn}}{\alpha_{a,n}} \left([x_i, x_n]^b + \sum_{m=1}^{n-1} [x_i, x_m]^b c_{mn}e_{nm} \right) \\
&:= e_{nj} E_{ij}.
\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
\mathcal{F}_{ij} &= \sum_{m=1}^n [x_i, x_m]^b \sum_{\substack{x_m | x_k \\ x_j | x_k}} \frac{c_{mk}c_{jk}}{x_m^a x_j^a \beta_{a,k}} - \sum_{m=1}^{n-1} [x_i, x_m]^b \sum_{\substack{x_m | x_k \\ x_j | x_k, x_k \neq x_n}} \frac{c_{mk}c_{jk}}{x_m^a x_j^a \beta_{a,k}} \\
&= \frac{c_{nn}c_{jn}}{x_n^a x_j^a \beta_{a,n}} [x_i, x_n]^b e_{nj} + \sum_{m=1}^{n-1} \frac{c_{mn}c_{jn}}{x_m^a x_j^a \beta_{a,n}} [x_i, x_m]^b e_{nj}e_{nm} \\
&= e_{nj} \frac{c_{jn}}{x_j^a \beta_{a,n}} \left(\frac{[x_i, x_n]^b}{x_n^a} + \sum_{m=1}^{n-1} \frac{[x_i, x_m]^b c_{mn}e_{nm}}{x_m^a} \right) \\
&:= e_{nj} F_{ij}.
\end{aligned} \tag{2.26}$$

In what follows, we show that $D_{ij} \in \mathbf{Z}$, $E_{ij} \in \mathbf{Z}$ and $F_{ij} \in \mathbf{Z}$. Consider the following two cases:

CASE 1. $|G_S(x_n)| = 1$. One may let $G_S(x_n) = \{x_{n_0}\}$, By Lemma 2.4, one has

$$\alpha_{a,n} = x_n^a - x_{n_0}^a \text{ and } \beta_{a,n} = x_n^{-a} - x_{n_0}^{-a}.$$

However, for any positive integer m ($\leq n-1$), by Lemma 2.5, $c_{mn} = -1$ if $m = n_0$ and $c_{mn} = 0$ otherwise. So from (2.24) to (2.26) and Lemma 2.9 one can derive that

$$D_{ij} = \frac{(x_i, x_n)^b - (x_i, x_{n_0})^b}{x_n^a - x_{n_0}^a} \cdot c_{jn} \in \mathbf{Z}, \tag{2.27}$$

$$E_{ij} = \frac{[x_i, x_n]^b - [x_i, x_{n_0}]^b}{x_n^a - x_{n_0}^a} \cdot c_{jn} \in \mathbf{Z} \tag{2.28}$$

and

$$F_{ij} = \frac{x_{n_0}^a [x_i, x_n]^b - x_n^a [x_i, x_{n_0}]^b}{x_j^a (x_{n_0}^a - x_n^a)} \cdot c_{jn} \in \mathbf{Z}. \tag{2.29}$$

Since $e_{nj} \in \{0, 1\}$, (2.21) to (2.23) follow immediately from (2.24) and (2.27), (2.25) and (2.28), and (2.26) and (2.29), respectively.

CASE 2. $|G_S(x_n)| = 2$. Let $G_S(x_n) = \{x_{n_1}, x_{n_2}\}$ and $x_{n_3} = (x_{n_1}, x_{n_2})$. It then follows from (2.24) to (2.26) and Lemmas 2.4, 2.5 and 2.10 that

$$D_{ij} = \frac{(x_i, x_n)^b - (x_i, x_{n_1})^b - (x_i, x_{n_2})^b + (x_i, x_{n_3})^b}{x_n^a - x_{n_1}^a - x_{n_2}^a + x_{n_3}^a} \cdot c_{jn} \in \mathbf{Z},$$

$$E_{ij} = \frac{[x_i, x_n]^b - [x_i, x_{n_1}]^b - [x_i, x_{n_2}]^b + [x_i, x_{n_3}]^b}{x_n^a - x_{n_1}^a - x_{n_2}^a + x_{n_3}^a} \cdot c_{jn} \in \mathbf{Z}$$

and

$$\begin{aligned} F_{ij} &= \frac{c_{jn}}{x_j^a \alpha_{a,n}} \left(\frac{[x_n, x_i]^b}{x_n^a} - \frac{[x_{n_1}, x_i]^b}{x_{n_1}^a} - \frac{[x_{n_2}, x_i]^b}{x_{n_2}^a} + \frac{[x_{n_3}, x_i]^b}{x_{n_3}^a} \right) \\ &= c_{jn} \cdot \frac{x_n^a [x_i, x_{n_3}]^b + x_{n_3}^a [x_i, x_n]^b - x_{n_2}^a [x_i, x_{n_1}]^b - x_{n_1}^a [x_i, x_{n_2}]^b}{x_j^a (x_n^a + x_{n_3}^a - x_{n_1}^a - x_{n_2}^a)} \in \mathbf{Z}. \end{aligned}$$

Hence (2.21) to (2.23) hold in this case.

This finishes the proof of Lemma 2.13. $\square$

3. PROOFS OF THEOREMS 1.2 AND 1.3

In this section, we first use the lemmas presented in the previous section to show Theorem 1.2.

Proof of Theorem 1.2. We prove Theorem 1.2 by using induction on $n = |S|$.

Let $n \leq 3$. Since S is gcd closed, the set S satisfies $\max_{x \in S} \{|G_S(x)|\} = 1$. It then follows immediately from Lemma 2.12 that Theorem 1.2 (i) holds when $n \leq 3$.

Now let $n \geq 4$. Assume that the result is true for the $n - 1$ case. In what follows, we show that the result is true for the n case. Since S is a gcd-closed set with $\max_{x \in S} \{|G_S(x)|\} = 2$ and S satisfies the condition $\mathcal{G}$, it follows that $S_0 := S \setminus \{\max(S)\}$ is gcd closed and $\max_{x \in S_0} \{|G_{S_0}(x)|\} \leq 2$ and S_0 also satisfies the condition $\mathcal{G}$. One asserts that

$$(S_0^b)(S_0^a)^{-1} \in M_{n-1}(\mathbf{Z}), [S_0^b](S_0^a)^{-1} \in M_{n-1}(\mathbf{Z}) \text{ and } [S_0^b][S_0^a]^{-1} \in M_{n-1}(\mathbf{Z}). \quad (3.1)$$

We divide its proof into the following two cases.

CASE 1. $\max_{x \in S_0} \{|G_{S_0}(x)|\} = 1$. Then by Lemma 2.12, we know that (3.1) holds in this case.

CASE 2. $\max_{x \in S_0} \{|G_{S_0}(x)|\} = 2$. Then it follows from the inductive hypothesis that (3.1) is true. The assertion is proved in this case.

Now we can apply Lemma 2.13. One arrives at

$$(S^b)(S^a)^{-1} \in M_n(\mathbf{Z}), [S^b](S^a)^{-1} \in M_n(\mathbf{Z}) \text{ and } [S^b][S^a]^{-1} \in M_n(\mathbf{Z}).$$

In other words, in the ring $M_n(\mathbf{Z})$, we have $(S^a)|(S^b)$, $(S^a)[S^b]$ and $[S^a][S^b]$ as desired. Hence Theorem 1.2 is true for the n case. So Theorem 1.2 is proved. $\square$

Finally, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. (i). Let

$$S_1 := \{1, u, v, uvw\} \text{ with } (u, v) = 1 \text{ and } w > 1. \quad (3.2)$$

Evidently, $\max_{x \in S_1} \{|G_{S_1}(x)|\} = 2$ and the condition $\mathcal{G}$ is not satisfied. We can compute and get that

$$\begin{aligned}
 (S_1)^{-1}(S_1^b) &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & u & 1 & u \\ 1 & 1 & v & v \\ 1 & u & v & uvw \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & u^b & 1 & u^b \\ 1 & 1 & v^b & v^b \\ 1 & u^b & v^b & (uvw)^b \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 1 - \frac{u^b-1}{u-1} & 1 - \frac{v^b-1}{v-1} & 1 - \frac{u^b-1}{u-1} - \frac{v^b-1}{v-1} + \frac{\Delta_b}{\Delta_1} \\ 0 & \frac{u^b-1}{u-1} & 0 & \frac{u^b-1}{u-1} - \frac{\Delta_b}{\Delta_1} \\ 0 & 0 & \frac{v^b-1}{v-1} & \frac{v^b-1}{v-1} - \frac{\Delta_b}{\Delta_1} \\ 0 & 0 & 0 & \frac{\Delta_b}{\Delta_1} \end{pmatrix}, \quad (3.3)
 \end{aligned}$$

where

$$\Delta_b := (uvw)^b - u^b - v^b + 1. \quad (3.4)$$

It follows from (3.3) that $(S_1)^{-1}(S_1^b) \in M_4(\mathbf{Z})$ if and only if $\frac{\Delta_b}{\Delta_1} \in \mathbf{Z}$. Let us continue the proof of part (i) of Theorem 1.3, which is divided into four cases.

CASE 1-1. Picking $u = 2, v = 3, w = 2$, one has

$$\frac{\Delta_b}{\Delta_1} = \frac{2^b 3^b 2^b - 2^b - 3^b + 1}{2 \times 3 \times 2 - 2 - 3 + 1} = \frac{2^b(6^b - 1)}{2^3} - \frac{3^b - 1}{2^3}.$$

Since $3^2 \equiv 1 \pmod{8}$, we know that if b is even and $b \geq 4$, then $\frac{\Delta_b}{\Delta_1} \in \mathbf{Z}$. Hence $(S_1)|(S_1^b)$ in this case.

CASE 1-2. Taking $u = 3, v = 4, w = 4$, one attains that $\Delta_1 = 3 \times 4 \times 4 - 3 - 4 + 1 = 42$.

Now let $b \equiv 1 \pmod{6}$. By Fermat's little theorem, one knows that $3^6 \equiv 1 \pmod{7}$. One then derives that $3^b \equiv 3 \pmod{7}$. Evidently, we have $3^b \equiv 3 \pmod{6}$. Thus $3^b \equiv 3 \pmod{42}$. Likewise, we have $4^b \equiv 4 \pmod{42}$. Therefore, $\Delta_b = 3^b 4^b 4^b - 3^b - 4^b + 1 \equiv 3 \times 4 \times 4 - 3 - 4 + 1 \equiv 0 \pmod{42}$. Thus $\frac{\Delta_b}{\Delta_1} \in \mathbf{Z}$ implies that $(S_1)|(S_1^b)$ as desired.

CASE 1-3. Letting $u = 3, v = 4, w = 2$ gives that $\Delta_1 = 3 \times 4 \times 2 - 3 - 4 + 1 = 18$. Let $b \equiv 3 \pmod{6}$. By Euler's theorem, one has $4^6 \equiv 1 \pmod{9}$. One can deduce that $4^b \equiv 10 \pmod{9}$. Clearly, $4^b \equiv 10 \pmod{2}$. It follows that $4^b \equiv 10 \pmod{18}$. Similarly, we can get that $3^b \equiv 9 \pmod{18}$ and $2^b \equiv 8 \pmod{18}$. So $\Delta_b = 3^b 4^b 2^b - 3^b - 4^b + 1 \equiv 9 \times 10 \times 8 - 9 - 10 + 1 \equiv 0 \pmod{18}$. Thus $\frac{\Delta_b}{\Delta_1} \in \mathbf{Z}$. Hence $(S_1)|(S_1^b)$ holds in this case.

CASE 1-4. Picking $u = 2, v = 5, w = 2$, we have $\Delta_1 = 2 \times 5 \times 2 - 2 - 5 + 1 = 14$. Let $b \equiv 5 \pmod{6}$. It follows from Fermat little theorem that $2^6 \equiv 1 \pmod{7}$. Then one derives that $2^b \equiv 4 \pmod{7}$. Evidently, $2^b \equiv 4 \pmod{2}$. Therefore we obtain that $2^b \equiv 4 \pmod{14}$. By Euler's theorem, we can directly deduce that $5^6 \equiv 1 \pmod{14}$. So $5^b \equiv 3 \pmod{14}$. Thus $\Delta_b = 2^b 5^b 2^b - 2^b - 5^b + 1 \equiv 4 \times 3 \times 4 - 4 - 3 + 1 \equiv 0 \pmod{14}$. Hence $\frac{\Delta_b}{\Delta_1} \in \mathbf{Z}$ and $(S_1)|(S_1^b)$ in this case. Part (i) is proved.

(ii). Let $S_2 = S_1$ with S_1 being given as in (3.2). We can calculate and obtain that

$$\begin{aligned}
& (S_2)^{-1}[S_2^b] \\
&= \begin{pmatrix} 1 + \frac{1}{u-1} + \frac{1}{v-1} + \frac{1}{\Delta_1} & \frac{1}{1-u} - \frac{1}{\Delta_1} & \frac{1}{1-v} - \frac{1}{\Delta_1} & \frac{1}{\Delta_1} \\ \frac{1}{1-u} - \frac{1}{\Delta_1} & \frac{1}{u-1} + \frac{1}{\Delta_1} & \frac{1}{\Delta_1} & -\frac{1}{\Delta_1} \\ \frac{1}{1-v} - \frac{1}{\Delta_1} & \frac{1}{\Delta_1} & \frac{1}{v-1} + \frac{1}{\Delta_1} & -\frac{1}{\Delta_1} \\ \frac{1}{\Delta_1} & -\frac{1}{\Delta_1} & -\frac{1}{\Delta_1} & \frac{1}{\Delta_1} \end{pmatrix} \cdot \begin{pmatrix} 1 & u^b & v^b & (uvw)^b \\ u^b & u^b & (uv)^b & (uvw)^b \\ v^b & (uv)^b & v^b & (uvw)^b \\ (uvw)^b & (uvw)^b & (uvw)^b & (uvw)^b \end{pmatrix} \\
&= \begin{pmatrix} 1 + \frac{1-u^b}{u-1} + \frac{1-v^b}{v-1} + \frac{\Delta_b}{\Delta_1} & u^b + u^b \cdot \frac{1-v^b}{v-1} + \frac{\Gamma_b}{\Delta_1} & v^b + v^b \cdot \frac{1-u^b}{u-1} + \frac{\Gamma_b}{\Delta_1} & (uvw)^b \\ \frac{1-u^b}{1-u} - \frac{\Delta_b}{\Delta_1} & -\frac{\Gamma_b}{\Delta_1} & v^b \cdot \frac{1-u^b}{1-u} - \frac{\Gamma_b}{\Delta_1} & 0 \\ \frac{1-v^b}{1-v} - \frac{\Delta_b}{\Delta_1} & u^b \cdot \frac{1-v^b}{1-v} - \frac{\Gamma_b}{\Delta_1} & -\frac{\Gamma_b}{\Delta_1} & 0 \\ \frac{\Delta_b}{\Delta_1} & \frac{\Gamma_b}{\Delta_1} & \frac{\Gamma_b}{\Delta_1} & 0 \end{pmatrix}, \tag{3.5}
\end{aligned}$$

where Δ_b is given as in (3.4) and $\Gamma_b := u^b v^b (w^b - 1)$. By (3.5), we know that the necessary and sufficient conditions for $(S_2)^{-1}[S_2^b] \in M_4(\mathbf{Z})$ are $\frac{\Delta_b}{\Delta_1} \in \mathbf{Z}$ and $\frac{\Gamma_b}{\Delta_1} \in \mathbf{Z}$. We divide the following proof of part (ii) into two cases.

CASE 2-1. Picking $u = 2, v = 3, w = 2$ gives us that

$$\frac{\Delta_b}{\Delta_1} = \frac{2^b 3^b 2^b - 2^b - 3^b + 1}{2 \times 3 \times 2 - 2 - 3 + 1} = \frac{2^b(6^b - 1)}{2^3} - \frac{3^b - 1}{2^3}$$

and

$$\frac{\Gamma_b}{\Delta_1} = \frac{2^b 3^b (2^b - 1)}{2^3}.$$

Since $3^2 \equiv 1 \pmod{8}$, we know that if b is even and $b \geq 4$, then $\frac{\Delta_b}{\Delta_1} \in \mathbf{Z}$ and $\frac{\Gamma_b}{\Delta_1} \in \mathbf{Z}$. Hence $(S_2)[S_2^b]$ in this case.

CASE 2-2. Letting $u = 3, v = 4, w = 2$, we have $\Delta_1 = 3 \times 4 \times 2 - 3 - 4 + 1 = 18$. Let $b \equiv 3 \pmod{6}$. As the proof of case 2-3 of part (ii), we arrive at $3^b \equiv 9 \pmod{18}$, $4^b \equiv 10 \pmod{18}$ and $2^b \equiv 8 \pmod{18}$. So

$$\Delta_b = 3^b 4^b 2^b - 3^b - 4^b + 1 \equiv 9 \times 10 \times 8 - 9 - 10 + 1 \equiv 0 \pmod{18}$$

and

$$\Gamma_b = 3^b 4^b (2^b - 1) \equiv 9 \times 10 \times (8 - 1) \equiv 0 \pmod{18}.$$

Thus $\frac{\Delta_b}{\Delta_1} \in \mathbf{Z}$ and $\frac{\Gamma_b}{\Delta_1} \in \mathbf{Z}$. So $(S_2)[S_2^b]$ in this case. Part (ii) is proved.

(iii). Let $S_3 = \{1, 3, 5, 45\}$. Then S_3 is a gcd-closed set with $\max_{x \in S_3} \{|G_{S_3}(x)|\} = 2$ and S_3 does not satisfy the condition $\mathcal{G}$ since $G_{S_3}(45) = \{3, 5\}$ and $[3, 5] = 15 < 45$. We calculate and get that

$$[S_3^5][S_3]^{-1} = \begin{pmatrix} 1 & 3^5 & 5^5 & 45^5 \\ 3^5 & 3^5 & 15^5 & 45^5 \\ 5^5 & 15^5 & 5^5 & 45^5 \\ 45^5 & 45^5 & 45^5 & 45^5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 5 & 45 \\ 3 & 3 & 15 & 45 \\ 5 & 15 & 5 & 45 \\ 45 & 45 & 45 & 45 \end{pmatrix}^{-1}$$

$$\begin{aligned}
&= \begin{pmatrix} 1 & 243 & 3125 & 184528125 \\ 243 & 243 & 759375 & 184528125 \\ 3125 & 759375 & 3125 & 184528125 \\ 184528125 & 184528125 & 184528125 & 184528125 \end{pmatrix} \cdot \begin{pmatrix} \frac{13}{44} & -\frac{2}{11} & -\frac{7}{44} & \frac{1}{22} \\ -\frac{2}{11} & \frac{2}{33} & \frac{3}{22} & -\frac{1}{66} \\ -\frac{7}{44} & \frac{3}{22} & \frac{7}{220} & -\frac{1}{110} \\ \frac{1}{22} & -\frac{1}{66} & -\frac{1}{110} & \frac{1}{990} \end{pmatrix} \\
&= \begin{pmatrix} 8387101 & -2795440 & -1677396 & 186360 \\ 8266860 & -2692359 & -1653372 & 179496 \\ 8250000 & -2750000 & -1574375 & 175000 \\ 0 & 0 & 0 & 4100625 \end{pmatrix} \in M_4(\mathbf{Z}).
\end{aligned}$$

Hence $[S_3][S_3^5]$ holds in the ring $M_4(\mathbf{Z})$.

Let $S_3 = \{1, 2, 3, 4, 24\}$. Then S_3 is gcd closed and $\max_{x \in S_3} \{|G_{S_3}(x)|\} = 2$. Since $G_{S_3}(24) = \{3, 4\}$ and $[3, 4] = 12 < 24$, the set S_3 does not satisfy the condition $\mathcal{G}$. But

$$\begin{aligned}
&[S_3^{11}][S_3]^{-1} \\
&= \begin{pmatrix} 1 & 2^{11} & 3^{11} & 4^{11} & 24^{11} \\ 2^{11} & 2^{11} & 6^{11} & 4^{11} & 24^{11} \\ 3^{11} & 6^{11} & 3^{11} & 12^{11} & 24^{11} \\ 4^{11} & 4^{11} & 12^{11} & 4^{11} & 24^{11} \\ 24^{11} & 24^{11} & 24^{11} & 24^{11} & 24^{11} \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 24 \\ 2 & 2 & 6 & 4 & 24 \\ 3 & 6 & 3 & 12 & 24 \\ 4 & 4 & 12 & 4 & 24 \\ 24 & 24 & 24 & 24 & 24 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 2048 & 177147 & 4194304 & 1521681143169024 \\ 2048 & 2048 & 362797056 & 4194304 & 1521681143169024 \\ 177147 & 362797056 & 177147 & 743008370688 & 1521681143169024 \\ 4194304 & 4194304 & 743008370688 & 4194304 & 1521681143169024 \\ 1521681143169024 & 1521681143169024 & 1521681143169024 & 1521681143169024 & 1521681143169024 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -\frac{7}{22} & 1 & -\frac{5}{22} & -\frac{6}{11} & \frac{1}{11} \\ 1 & -\frac{3}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{5}{22} & 0 & \frac{5}{66} & \frac{2}{11} & -\frac{1}{33} \\ -\frac{6}{11} & \frac{1}{2} & \frac{2}{11} & -\frac{5}{44} & -\frac{1}{44} \\ \frac{1}{11} & 0 & -\frac{1}{33} & -\frac{1}{44} & \frac{1}{264} \end{pmatrix} \\
&= \begin{pmatrix} 138334647052987 & 2094081 & -46111549016980 & -34583662788144 & 5763943623432 \\ 138334564638720 & 2096128 & -46111521546240 & -34583596858368 & 5763932635136 \\ 137929734786330 & 370960166907 & -45976457388807 & -34667913780036 & 5747057180982 \\ 138165784412160 & 0 & -46055261470720 & -34448570580992 & 5741428604928 \\ 0 & 0 & 0 & 0 & 63403380965376 \end{pmatrix} \in M_5(\mathbf{Z}).
\end{aligned}$$

Thus $[S_3][S_3^{11}]$ holds in the ring $M_5(\mathbf{Z})$. Part (iii) is proved.

This concludes the proof of Theorem 1.3. $\square$

4. FINAL REMARKS

Let S be a gcd-closed set and let a and b be positive integers such that $a|b$. If $\max_{x \in S}\{|G_S(x)|\} = 1$, then by Zhu's theorem [41] and the Zhu-Li theorem [43], one knows that $(S^a)|(S^b)$, $(S^a)[S^b]$ and $[S^a][S^b]$ hold in the ring $M_n(\mathbf{Z})$. From Theorem 1.2 of this paper we know that such factorizations are true if $\max_{x \in S}\{|G_S(x)|\} = 2$ and the set S satisfies the condition $\mathcal{G}$. When $a = b$, for any gcd-closed sets S with $\max_{x \in S}\{|G_S(x)|\} \geq 2$, it was conjectured in [40] that such factorizations are true if and only if the set S satisfies the condition $\mathcal{G}$. By Theorem 1.3, one knows the existences of positive integers $b > 1$ and gcd-closed sets S with $\max_{x \in S}\{|G_S(x)|\} = 2$ and the condition $\mathcal{G}$ not being satisfied, such that $(S)|(S^b)$ (resp. $(S)[S^b]$ and $[S][S^b]$) holds in the ring $M_n(\mathbf{Z})$. In other words, when $a|b$ and $a < b$, the condition $\mathcal{G}$ is a sufficient and unnecessary condition for the truth of Theorem 1.2. However, it is not clear that for each integer $b > 1$, there is a gcd-closed set S with $\max_{x \in S}\{|G_S(x)|\} \geq 2$ and the condition $\mathcal{G}$ not being satisfied such that $(S)|(S^b)$ (resp. $(S)[S^b]$ and $[S][S^b]$) holds in the ring $M_n(\mathbf{Z})$. This problem is still kept open.

REFERENCES

- [1] E. Altınışık, M. Yildiz and A. Keskin, Non-divisibility of LCM matrices by GCD matrices on gcd-closed sets, *Linear Algebra Appl.* **516** (2017), 47-68.
- [2] T.M. Apostol, Arithmetical properties of generalized Ramanujan sums, *Pacific J. Math.* **41** (1972), 281-293.
- [3] S. Beslin and S. Ligh, Another generalization of Smith's determinant, *Bull. Aust. Math. Soc.* **40** (1989), 413-415.
- [4] K. Bourque and S. Ligh, On GCD and LCM matrices, *Linear Algebra Appl.* **174** (1992), 65-74.
- [5] K. Bourque and S. Ligh, Matrices associated with classes of arithmetical functions, *J. Number Theory* **45** (1993), 367-376.
- [6] K. Bourque and S. Ligh, Matrices associated with arithmetical functions, *Linear Multilinear Algebra* **34** (1993), 261-267.
- [7] K. Bourque and S. Ligh, Matrices associated with classes of multiplicative functions, *Linear Algebra Appl.* **216** (1995), 267-275.
- [8] L. Chen, Y.L. Feng, S.F. Hong and M. Qiu, On the divisibility of matrices associated with multiplicative functions, *Publ. Math. Debrecen* **100** (2022), 323-335.
- [9] L. Chen, Z.B. Lin and Q.R. Tan, Divisibility properties of power matrices associated with arithmetic functions on a divisor chain, *Algebra Colloq.* **29** (2022), 527-540.
- [10] P. Codecá and M. Nair, Calculating a determinant associated with multiplicative functions, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* **5** (2002), 545-555.
- [11] W.D. Feng, S.F. Hong and J.R. Zhao, Divisibility properties of power LCM matrices by power GCD matrices on gcd-closed sets, *Discrete Math.* **309** (2009), 2627-2639.
- [12] Y.L. Feng, M. Qiu, G.Y. Zhu and S.F. Hong, Divisibility among power matrices associated with classes of arithmetic functions, *Discrete Math.* **345** (2022), Paper No. 112993, 11 pp.
- [13] P. Haukkanen and I. Korkee, Notes on the divisibility of GCD and LCM matrices, *Int. J. Math. Math. Sci.* **28** (2005), 925-935.
- [14] S.A. Hong, S.N. Hu and Z.B. Lin, On a certain arithmetical determinant, *Acta Math. Hungar.* **150** (2016), 372-382.
- [15] S.A. Hong and Z.B. Lin, New results on the value of a certain arithmetical determinant, *Publ. Math. Debrecen* **93** (2018), 171-187.
- [16] S.A. Hong and G.Y. Zhu, Divisibility among power matrices associated with multiplicative functions, *Linear Multilinear Algebra* **72** (2024), 3152-3165.
- [17] S.F. Hong, On the Bourque-Ligh conjecture of least common multiple matrices, *J. Algebra* **218** (1999), 216-228.
- [18] S.F. Hong, Gcd-closed sets and determinants of matrices associated with arithmetical functions, *Acta Arith.* **101** (2002), 321-332.

- [19] S.F. Hong, On the factorization of LCM matrices on gcd-closed sets, *Linear Algebra Appl.* **345** (2002), 225-233.
- [20] S.F. Hong, Factorization of matrices associated with classes of arithmetical functions, *Colloq. Math.* **98** (2003), 113-123.
- [21] S.F. Hong, Notes on power LCM matrices, *Acta Arith.* **111** (2004), 165-177.
- [22] S.F. Hong, Nonsingularity of matrices associated with classes of arithmetical functions, *J. Algebra* **281** (2004), 1-14.
- [23] S.F. Hong, Divisibility properties of power GCD matrices and power LCM matrices, *Linear Algebra Appl.* **428** (2008), 1001-1008.
- [24] S.F. Hong, Divisibility among power GCD matrices and power LCM matrices, *Bull. Aust. Math. Soc.*, published online September 1, 2025, doi:10.1017/S0004972725100361.
- [25] S.F. Hong, K.P. Shum and Q. Sun, On nonsingular power LCM matrices, *Algebra Colloq.* **13** (2006), 689-704.
- [26] S.F. Hong and K.S. Enoch Lee, Asymptotic behavior of eigenvalues of reciprocal power LCM matrices, *Glasgow Math. J.* **50** (2008), 163-174.
- [27] S.F. Hong and R. Loewy, Asymptotic behavior of eigenvalues of greatest common divisor matrices, *Glasgow Math. J.* **46** (2004), 551-569.
- [28] S.F. Hong and R. Loewy, Asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions (mod r), *Int. J. Number Theory* **7** (2011), 1681-1704.
- [29] S.F. Hong, J.R. Zhao and Y.Z. Yin, Divisibility properties of Smith matrices, *Acta Arith.* **132** (2008), 161-175.
- [30] I. Korkee and P. Haukkanen, On the divisibility of meet and join matrices, *Linear Algebra Appl.* **429** (2008), 1929-1943.
- [31] Z.B. Lin and S.A. Hong, More on a certain arithmetical determinant, *Bull. Aust. Math. Soc.* **97** (2018), 15-25.
- [32] M. Mattila and P. Haukkanen, On the positive definiteness and eigenvalues of meet and join matrices, *Discrete Math.* **326** (2014), 9-19.
- [33] P.J. McCarthy, A generalization of Smith's determinant *Canad. Math. Bull.* **29** (1986), 109-113.
- [34] H.J.S. Smith, On the value of a certain arithmetical determinant, *Proc. London Math. Soc.* **7** (1875-1876), 208-212.
- [35] Q.R. Tan, Divisibility among power GCD matrices and among power LCM matrices on two coprime divisor chains, *Linear Multilinear Algebra* **58** (2010), 659-671.
- [36] Q.R. Tan, Z.B. Lin and L. Liu, Divisibility among power GCD matrices and among power LCM matrices on two coprime divisor chains II, *Linear Multilinear Algebra* **59** (2011), 969-983.
- [37] Q.R. Tan, M. Luo and Z.B. Lin, Determinants and divisibility of power GCD and power LCM matrices on finitely many coprime divisor chains, *Appl. Math. Comput.* **219** (2013), 8112-8120.
- [38] J.X. Wan, S.N. Hu and Q.R. Tan, New results on nonsingular power LCM matrices, *Electron. J. Linear Algebra* **27** (2014), 652-669.
- [39] A. Wintner, Diophantine approximations and Hilbert's space *Amer. J. Math.* **66** (1944), 564-578.
- [40] J.R. Zhao, L. Chen and S.F. Hong, Gcd-closed sets and divisibility of Smith matrices, *J. Combin. Theory, Ser. A* **188** (2022), Paper No. 105581, 23 pp.
- [41] G.Y. Zhu, On the divisibility among power GCD and power LCM matrices on gcd-closed sets, *Int. J. Number Theory* **18** (2022), 1397-1408.
- [42] G.Y. Zhu, On a certain determinant for a U.F.D., *Colloq. Math.* **171** (2023), 49-59.
- [43] G.Y. Zhu and M. Li, On the divisibility among power LCM matrices on gcd-closed sets, *Bull. Aust. Math. Soc.* **107** (2023), 31-39.
- [44] G.Y. Zhu, M. Li and X.F. Xu, New results on the divisibility of power GCD and power LCM matrices, *AIMS Math.* **7** (2022), 18239-18252.

COLLEGE OF MATHEMATICS AND PHYSICS, MIANYANG TEACHERS' COLLEGE, MIANYANG, 621000, P.R. CHINA

Email address: xiangxiangwanli@163.com; 2873555@qq.com

SCHOOL OF MATHEMATICS AND STATISTICS, HUBEI MINZU UNIVERSITY, ENSHI 445000, P.R. CHINA

Email address: 2009043@hbmzu.edu.cn