Inverse scattering for N-body time-decaying harmonic oscillators

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Abstract

In the previous study [26], the author proved the uniqueness of short-range potential functions using the Enss-Weder time-dependent method [16] for a two-body quantum system described by time-decaying harmonic oscillators. In this study, we extend the result of [26] to the N-body case. We use the approaches developed in [16, 49, 53] to prove that the high-velocity limit of the scattering operator uniquely determines all the pairwise interaction potentials among the N particles, focusing respectively on each fixed pair of particles.

Keywords: N-body scattering theory, wave operator, scattering operator

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1 Introduction

We consider a quantum system governed by the N-particle harmonic oscillators and interactional potential functions described by the Hamiltonian

$$\tilde{H}(t) = -\sum_{j=1}^{N} \Delta_{r_j}/(2m_j) + k(t)\sum_{j=1}^{N} m_j r_j^2/2 + \sum_{1\leqslant j<k\leqslant N} V_{jk}(r_k - r_j) \tag{1.1}$$

acting on $L^2(\mathbb{R}^{dN})$ with $N \geqslant 2$ and $d \geqslant 2$, where $r_j \in \mathbb{R}^d$ is the position of the jth particle, $m_j > 0$ is the mass, Δ_{r_j} is the Laplacian for r_j, and V_{jk} is the interaction between jth and kth particles. The time-dependent coefficient $k(t)$ is defined as

$$k(t) = \begin{cases} \omega^2 & \text{if} \quad |t| < T, \\ \sigma/t^2 & \text{if} \quad |t| \geqslant T \end{cases} \tag{1.2}$$

for $0 < \sigma < 1/4$, $\omega > 0$ and $T > 0$.

We now introduce the Jacobi coordinates in which the center of mass of all N particles is located at the origin (see [31, Chapter 3] or [45, XI.5]). Let $\mathscr{X}$ be the subspace of $\mathbb{R}^{dN}$ described by

$$\mathscr{X} = \left\{ (r_1, \ldots, r_N) \in \mathbb{R}^{dN} \;\middle|\; \sum_{j=1}^{N} m_j r_j = 0 \right\} \simeq \mathbb{R}^{d(N-1)} \tag{1.3}$$

and $\mathscr{X}_{\mathrm{cm}}$ be the orthogonal space for $\mathscr{X}$ associated with the scalar product

$$(r, \tilde{r})_{\mathbb{R}^{dN}} = \sum_{j=1}^{N} m_j r_j \cdot \tilde{r}_j \tag{1.4}$$

on $\mathbb{R}^{dN}$ for $r = (r_1, \ldots, r_N)$ and $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_N) \in \mathbb{R}^{dN}$. When the position of the center of mass is removed, the unperturbed Hamiltonian of (1.1) is given by

$$H_0(t) = -\sum_{j=1}^{N} \Delta_{r_j}/(2m_j) + \Delta_{\mathrm{cm}}/(2M) + k(t)\sum_{j=1}^{N} m_j r_j^2/2 - k(t) M r_{\mathrm{cm}}^2/2 \tag{1.5}$$

acting on $L^2(\mathscr{X})$, where $M = \sum_{j=1}^{n} m_j$ is the total mass, $r_{\mathrm{cm}} = \sum_{j=1}^{N} m_j r_j / M \in \mathscr{X}_{\mathrm{cm}}$ is the position of the center of mass, and Δ_{cm} is the Laplace-Beltrami operator on $\mathscr{X}_{\mathrm{cm}}$. This expression (1.5) can be simplified by rewriting it in Jacobi coordinates as

$$y_j = r_{j+1} - \sum_{k=1}^{j} m_k r_k \Big/ \sum_{k=1}^{j} m_k \in \mathscr{X} \tag{1.6}$$

for $1 \leqslant j \leqslant N-1$. The reduced mass μ_j is defined as

$$1/\mu_j = 1\Big/\sum_{k=1}^{j} m_k + 1/m_{j+1}. \tag{1.7}$$

Under these notations, we have

$$\sum_{j=1}^{N} m_j r_j^2 = M r_{\mathrm{cm}}^2 + \sum_{j=1}^{N-1} \mu_j y_j^2, \tag{1.8}$$

$$\sum_{j=1}^{N} \Delta_{r_j}/m_j = \Delta_{\mathrm{cm}}/M + \sum_{j=1}^{N-1} \Delta_{y_j}/\mu_j \tag{1.9}$$

where Δ_{y_j} is the Laplace-Beltrami operator on $\mathscr{X}$. Therefore, in these coordinates, (1.5) can be written as the unperturbed Hamiltonian

$$H_0(t) = -\sum_{j=1}^{N-1} \Delta_{x_j}/2 + k(t)\sum_{j=1}^{N-1} x_j^2/2 \tag{1.10}$$

acting on $L^2(\mathscr{X})$ with $x_j = \sqrt{\mu_j} y_j$ for simplicity.

We now give the assumptions for each potential function V_{jk} in (1.1) as a perturbation of $H_0(t)$. Let the condition for λ be

$$0 < \lambda = (1 - \sqrt{1 - 4\sigma})/2 < 1/2. \tag{1.11}$$

Assumption 1.1. *Each function $V_{jk} : \mathbb{R}^d \to \mathbb{R}$ for $1 \leqslant j < k \leqslant N$ is a multiplication operator and decomposed into two parts:*

$$V_{jk} = V_{jk}^{\mathrm{bdd}} + V_{jk}^{\mathrm{sing}}. \tag{1.12}$$

The bounded part $V_{jk}^{\mathrm{bdd}} \in L^\infty(\mathbb{R}^d)$ satisfies

$$|V_{jk}^{\mathrm{bdd}}(x)| \lesssim \langle x\rangle^{-\rho} \tag{1.13}$$

with $\rho > 1/(1-\lambda)$, where $\langle\cdot\rangle = \sqrt{1+|\cdot|}$ and $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leqslant CB$. The singular part $V_{jk}^{\mathrm{sing}} \in L^q(\mathbb{R}^d)$ is compactly supported on $\mathscr{X}$, where the Lebesgue exponent q satisfies

$$\infty > q \begin{cases} = 2 & \text{if} \quad n \leqslant 3, \\ > n/2 & \text{if} \quad n \geqslant 4. \end{cases} \tag{1.14}$$

It is well-known that each V_{jk}^{sing} is $-\Delta$-bounded infinitesimally. The interacting Hamiltonian is

$$H(t) = H_0(t) + \sum_{1\leqslant j<k\leqslant N} V_{jk}(r_k - r_j) \tag{1.15}$$

acting on $L^2(\mathscr{X})$. Here we note $r_k - r_j \in \mathscr{X}$ for any $1 \leqslant j < k \leqslant N$. Under these definitions, the perturbed full Hamiltonian is decomposed into

$$\tilde{H}(t) = H(t) \otimes 1 + 1 \otimes (-\Delta_{\mathrm{cm}}/(2M) - k(t)Mx_{\mathrm{cm}}^2/2) \tag{1.16}$$

on $L^2(\mathscr{X}) \otimes L^2(\mathscr{X}_{\mathrm{cm}})$, where 1 denotes the identities on $L^2(\mathscr{X})$ and $L^2(\mathscr{X}_{\mathrm{cm}})$. Therefore, we will concentrate on studying the quantum system governed by $H(t)$. Because two parameter unitary propagators generated by $H_0(t)$ and $H(t)$ exist uniquely by the result of [54, Theorem 6 and Remark (a)], we denote these propagators as $U_0(t,s)$ and $U(t,s)$. In N-body quantum scattering theory, we have to consider the cluster decomposition $a = \{C_1, \ldots, C_m\}$ with $m \geqslant 2$, where $C_j \subset \{1, \ldots, N\}$ for $1 \leqslant j \leqslant m$, which satisfies $\cup_{j=1}^m C_j = \{1, \ldots, N\}$ and $C_j \cap C_k = \emptyset$ if $j \neq k$. Let $\mathscr{X}^a$ be the subspace of $\mathscr{X}$ given by

$$\mathscr{X}^a = \left\{ (r_1, \ldots, r_N) \in \mathscr{X} \;\middle|\; \sum_{j\in C} m_j r_j = 0, \ C \in a \right\} \tag{1.17}$$

and $\mathscr{X}_a$ be its orthogonal space. The cluster Hamiltonian is defined as

$$H_a(t) = -\sum_{j=1}^{N-1} \Delta_{x_j}/2 + k(t)\sum_{j=1}^{N-1} x_j^2/2 + \sum_{\{j,k\}\subset a} V_{jk}(r_k - r_j) \tag{1.18}$$

acting on $L^2(\mathscr{X})$. In the subspace $\mathscr{X}_a$, V_{jk} does not vanish for $\{j,k\} \subset a$. Therefore, it is natural that the cluster interaction

$$\sum_{\{j,k\} \not\subset a} V_{jk}(r_k - r_j) \tag{1.19}$$

be regarded as the perturbation and that the cluster Hamiltonian $H_a(t)$ of (1.18) be considered the unperturbed system for any cluster decomposition a. However, to discuss inverse scattering, it suffices to consider the N-cluster case where $a = \{\{1\}, \ldots, \{N\}\}$ and the wave operators defined as

$$W^{\pm} = \operatorname*{s-lim}_{t \to \infty} U(t,0)^* U_0(t,0). \tag{1.20}$$

We give the proof of existence of (1.20) in Section 2. Then the scattering operator is defined as

$$S(V) = (W^+)^* W^- \tag{1.21}$$

for $V = \sum_{1 \leqslant j < k \leqslant N} V_{jk}(r_k - r_j)$.

Theorem 1.2. *Let $V_1 = \sum_{1 \leqslant j < k \leqslant N} V_{jk,1}$ and $V_2 = \sum_{1 \leqslant j < k \leqslant N} V_{jk,2}$ satisfy Assumption 1.1. If $S(V_1) = S(V_2)$, then $V_{jk,1} = V_{jk,2}$ for $1 \leqslant j < k \leqslant N$.*

Many researchers have studied scattering theory under the N-body quantum systems, in particular, the the existence and asymptotic completeness of the wave operators, has been studied by a lot of researchers. Regarding the standard N-body Schrödinger operators, see [5, 13, 14, 18, 46, 47, 48]; the detailed history and proofs are also provided in [15, 31]. For N-body Schrödinger operators in the external electric fields, see [1, 3, 8, 9, 20]. For the external magnetic fields, see [2, 17]. However, there are no results regarding scattering theory for N-body time-decaying harmonic oscillators.

The Enss-Weder time-dependent method was invented in [16] for the standard Schödinger operators. Since then, many researchers have applied this method to other quantum models. The Stark effect and time-dependent electric fields were studied by [4, 6, 7, 10, 11, 22, 25, 42, 44, 49]. The repulsive Hamiltonians were studied by [21, 24, 44]. The fractional Laplacian and Dirac equations were studied by [23, 32]. The non-linear Schrödinger equations were studied by [50]. The Hartree-Fock equations were studied by [51, 52]. Recently, the author investigated the two-body time-decaying harmonic oscillator and repulsive Hamiltonian in [26, 27]. N-body inverse scattering was investigated by [16] for the standard Schrödinger case, and by [53, 49] for the Stark effect. Our proof also follows the strategies of these previous studies. That is, by an appropriate choice of Jacobi coordinates, the problem is reduced to the two-body case [26]. It should be emphasized that the pioneering works of [16, 53] already treated both the two-body and N-body case in a unified manner. The arguments of [16, 53, 49] show that the reduction to two-body channels is a fundamental principle.

Regarding scattering theory for the two-body time-decaying harmonic oscillator, [28] proved the existence of the wave operators under the assumption 1.1 only for the bounded part V^{bdd} and non-existence for $\rho \leqslant 1/(1-\lambda)$, and clarified that the condition $\rho > 1/(1-\lambda)$ is short-range. We can intuitively understand the threshold $1(1-\lambda)$ from the classical motion of the particle $x(t) = c_1 t^{1-\lambda} + c_2 t^{\lambda}$ for $t \geqslant T$, which satisfies the Newton equation $(\mathrm{d}^2/\mathrm{d}t^2)x(t) = -k(t)x(t)$ because λ is one of the roots of the quadratic equation $\lambda^2 - \lambda + \sigma = 0$. The critical case $\sigma = 1/4$ was studied by [29]. For the inverse square potential, [30] proved the asymptotic completeness of the wave operators. Moreover, [30] proved the Strichartz estimates and applied them to the initial-value problems for the non-linear Schrödinger equations. The non-linear Schrödinger equations and Strichartz estimates with time-decaying harmonic oscillators have also been studied by [33, 34, 35, 36, 37, 38, 39, 40].

Throughout this paper, we identify $L^2(\mathscr{X})$ with $L^2(\mathbb{R}^{d(N-1)})$ and use the following notations. $\|\cdot\|$ denotes the L^2-norm or operator norm on $L^2(\mathbb{R}^{d(N-1)})$, and $(\cdot,\cdot)$ denotes the scalar product of $L^2(\mathbb{R}^{d(N-1)})$. $F(\cdots)$ is the characteristic function of the set $\{\cdots\}$.

2 Existence of Wave Operators

In this section, we prove the existence of the N-cluster wave operators (1.20). To do that, we reduce the problem to a much simpler form of strong limits. We define $D = (D_1, \ldots, D_{N-1})$ for $D_j = -\mathrm{i}\nabla_{x_j}$ and $x = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{d(N-1)}$. Then the free Hamiltonian is written as

$$H_0(t) = D^2/2 + k(t)x^2/2 = \sum_{j=1}^{N-1} D_j^2/2 + k(t)\sum_{j=1}^{N-1} x_j^2/2. \tag{2.1}$$

The propagators $U_0(t,s)$ and $U(t,s)$ have unitary factorizations for $t,s \geqslant T$ or $t,s \leqslant -T$ that were proved by [28, Proposition 1] in the two-body case. We define

$$U_{0,\lambda}(t) = e^{\mathrm{i}\lambda x^2/(2t)} e^{-\mathrm{i}\lambda \log t A} e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))} \tag{2.2}$$

if $t \geqslant T$ and

$$U_{0,\lambda}(t) = e^{\mathrm{i}\lambda x^2/(2t)} e^{-\mathrm{i}\lambda \log(-t) A} e^{\mathrm{i}(-t)^{1-2\lambda}D^2/(2(1-2\lambda))} \tag{2.3}$$

if $t \leqslant -T$, where $A = (D\cdot x + x\cdot D)/2$. Then the factorizations

$$U_0(t,s) = U_{0,\lambda}(t)U_{0,\lambda}(s)^* \tag{2.4}$$

and

$$U(t,s) = e^{\mathrm{i}\lambda x^2/(2t)} e^{-\mathrm{i}\lambda \log|t| A} U_\lambda(t,s) e^{\mathrm{i}\lambda \log|s| A} e^{-\mathrm{i}\lambda x^2/(2s)} \tag{2.5}$$

hold for $t, s \geqslant T$ or $t, s \leqslant -T$, where $U_\lambda(t,s)$ is the propagator generated by

$$D^2/(2|t|^{2\lambda}) + \sum_{1\leqslant j<k\leqslant N} V_{jk}(|t|^\lambda (r_k - r_j)). \tag{2.6}$$

Regarding the proofs of (2.4) and (2.5), it suffices to prove the following Proposition 2.1 because $e^{\mathrm{i}\lambda\log|t|A} D_j e^{-\mathrm{i}\lambda \log|t|A} = D_j/|t|^\lambda$ for $1 \leqslant j \leqslant N$ clearly holds, and the rest of the proofs are demonstrated in [28, Proposition 1].

Proposition 2.1.

$$e^{\mathrm{i}\lambda\log|t|A}(r_k - r_j)e^{-\mathrm{i}\lambda\log|t|A} = |t|^\lambda (r_k - r_j) \tag{2.7}$$

holds for any $1 \leqslant j < k \leqslant N$.

Proof of Proposition 2.1. For $1 \leqslant j < k \leqslant N$ fixed, we construct the Jacobi coordinates from $y_1 = r_k - r_j$ and define $A_1 = (x_1 \cdot D_1 + D_1 \cdot x_1)/2$ for $x_1 = \sqrt{\mu_1} y_1$. We have

$$e^{\mathrm{i}\lambda\log|t|A} y_1 e^{-\mathrm{i}\lambda\log|t|A} = e^{\mathrm{i}\lambda\log|t|A_1} x_1 e^{-\mathrm{i}\lambda\log|t|A_1}/\sqrt{\mu_1}. \tag{2.8}$$

As in the proof of [28, Proposition 1], we have (2.7). □

To prove the existence of the wave operators (1.20), it suffices to prove the existence of the strong limits

$$\operatorname*{s-lim}_{t\to\pm\infty} U_\lambda(t,T)^* e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))} \tag{2.9}$$

by virtue of the factorizations (2.4) and (2.5), and the chain rules of the propagators. The following propagation estimate and the existence of the wave operators for the two-body time-decaying harmonic oscillator with only the bounded part V^{bdd} only were proved in [28, Proposition 1 and Theorem 1] by establishing the correspondence with (2.9) for the two-body case. Our proof is one of the extensions to the N-body case with the singular part V_{jk}^{sing}.

Here we introduce a conical region in which all relative momenta are uniformly bounded away from zero. We denote a $d(N-1)$-dimensional open ball with radius $r > 0$ centered at $\xi \in \mathbb{R}^{d(N-1)}$ as $B_{d(N-1)}(\xi, r)$ and its spherical surface as $\mathbb{S}^{d(N-1)-1}$. For any $1 \leqslant j < k \leqslant N$, there exists a full-rank matrix $L_{jk} : \mathbb{R}^{d(N-1)} \to \mathbb{R}^d$ such that any conjugate momentum $\xi_{jk} \in \mathbb{R}^d$ associated with the relative coordinate $r_k - r_j$ is given by $\xi_{jk} = L_{jk}\xi$ for some $\xi \in \mathbb{R}^{d(N-1)}$. We note that L_{jk} is surjective and that $\dim\ker L_{jk} = d(N-2)$. Let $\omega_0 \in \mathbb{S}^{d(N-1)-1} \setminus (\cup_{\leqslant j<k\leqslant N} \ker L_{jk})$ be fixed. Then $|L_{jk}\omega_0| > 0$ holds. In particular, there exist $\epsilon_0 > 0$ and $\delta > 0$ such that $|L_{jk}\omega| \geqslant \epsilon_0$ for any $\omega \in \mathbb{S}^{d(N-1)-1} \cap B_{d(N-1)}(\omega_0, \delta)$. We then define a conical region

$$\mathscr{C}_{\omega_0,\delta} = \{\xi \in \mathbb{R}^{d(N-1)} \setminus \{0\} \bigm| \xi/|\xi| \in \mathbb{S}^{d(N-1)-1} \cap B_{d(N-1)}(\omega_0, \delta)\}. \tag{2.10}$$

By this definition, if $\xi \in \mathscr{C}_{\omega_0,\delta}$, then $|\xi_{jk}| = |L_{jk}\xi| \geqslant \epsilon_0|\xi|$ holds for any $1 \leqslant j < k \leqslant N$. In the following Proposition 2.2 and proof of Proposition 2.3, we use $\omega_0 \in \mathbb{S}^{d(N-1)-1} \setminus (\cup_{\leqslant j<k\leqslant N} \ker L_{jk})$, $\epsilon_0 > 0$, and $\delta > 0$ chosen above.

Proposition 2.2. *Let $\phi \in \mathscr{S}(\mathbb{R}^{d(N-1)})$ be such that $\mathscr{F}_{d(N-1)}\phi \in C_0^\infty(\mathbb{R}^{d(N-1)})$ with $\operatorname{supp} \mathscr{F}_{d(N-1)}\phi \subset \{\xi \in \mathscr{C}_{\omega_0,\delta} \bigm| |\xi| \geqslant \epsilon\}$ for $0 < \epsilon \leqslant \epsilon_0$, where $\mathscr{F}_{d(N-1)}$ denotes the Fourier transform on $L^2(\mathbb{R}^{d(N-1)})$. Then*

$$\|F(\sqrt{\mu_{jk}}|r_k - r_j| \leqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda))e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))}\phi\| \lesssim_\nu t^{-\nu(1-2\lambda)}\|\langle r_k - r_j\rangle^\nu \phi\| \tag{2.11}$$

holds for any $1 \leqslant j < k \leqslant N$, where $\mu_{jk} = m_j m_k/(m_j + m_k)$, $\nu \in \mathbb{N}$, and $t \geqslant T$, and $\lesssim_\nu$ means that the constant depends on ν.

Proof of Proposition 2.2. We have

$$\begin{aligned} &F(\sqrt{\mu_{jk}}|r_k - r_j| \leqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda))e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))}\phi \\ &= e^{-\mathrm{i}t^{1-2\lambda}\sum_{j=2}^{N-1} D_j^2/(2(1-2\lambda))}F(|x_1| \leqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda))e^{-\mathrm{i}t^{1-2\lambda}D_1^2/(2(1-2\lambda))}\phi, \end{aligned} \tag{2.12}$$

where the Jacobi coordinates are reconstructed from $y_1 = r_k - r_j$ for $(j,k) \neq (1,2)$. Because $|\xi_1| = |L_{jlk}\xi| \geqslant \epsilon_0|\xi| \geqslant \epsilon^2$ holds on $\operatorname{supp} \mathscr{F}_{d(N-1)}\phi$, it suffices to prove that

$$\|F(|x_1| \leqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda))e^{-\mathrm{i}t^{1-2\lambda}D_1^2/(2(1-2\lambda))}\psi\|_{L^2(\mathbb{R}^d)} \lesssim_\nu t^{-\nu(1-2\lambda)}\|\langle x_1\rangle^\nu \psi\|_{L^2(\mathbb{R}^d)} \tag{2.13}$$

for $\mathscr{F}_d\psi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \mathscr{F}_d\psi \subset \{\xi \in \mathbb{R}^d \bigm| |\xi| \geqslant \epsilon^2\}$ where $\mathscr{F}_d$ denotes the Fourier transform on $L^2(\mathbb{R}^d)$. We write

$$\begin{aligned} &F(|x_1| \leqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda))e^{-\mathrm{i}t^{1-2\lambda}D_1^2/(2(1-2\lambda))}\psi \\ &= \int_{\mathbb{R}^d} e^{\mathrm{i}(x_1\cdot\xi - t^{1-2\lambda}|\xi|^2/(2(1-2\lambda)))}F(|x_1| \leqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda))\mathscr{F}_d\psi(\xi)\mathrm{d}\xi/(2\pi)^{d/2}. \end{aligned} \tag{2.14}$$

When $|x_1| \leqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda)$, we have

$$|x_1 - t^{1-2\lambda}\xi/(1-2\lambda)| \geqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda) \tag{2.15}$$

on $\operatorname{supp} \mathscr{F}_d\psi$. By the relation

$$\begin{aligned} &e^{\mathrm{i}(x_1\cdot\xi - t^{1-2\lambda}\xi^2/(2(1-2\lambda)))} \\ &= (x_1 - t^{1-2\lambda}\xi/(1-2\lambda))\cdot(-\mathrm{i}\nabla_\xi)e^{\mathrm{i}(x_1\cdot\xi - t^{1-2\lambda}\xi^2/(2(1-2\lambda)))}/|x_1 - t^{1-2\lambda}\xi/(1-2\lambda)|^2 \end{aligned} \tag{2.16}$$

and integrating by parts, we have (2.13). $\square$

Proposition 2.3. *The wave operators* (1.20) *exist.*

Proof of Proposition 2.3. We prove the existence of (2.9) for $t \to \infty$. The derivative at t of $U_\lambda(t,T)^* e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))}$ is

$$\begin{aligned}&(\mathrm{d}/\mathrm{d}t)U_\lambda(t,T)^* e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))}\\ &= \mathrm{i}\sum_{1\leqslant j<k\leqslant N} U_\lambda(t,T)^* V_{jk}(t^\lambda(r_k-r_j))e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))}.\end{aligned} \tag{2.17}$$

By Assumption 1.1,

$$\|V_{jk}^{\mathrm{bdd}}(t^\lambda(r_k-r_j))F(\sqrt{\mu_{jk}}|r_k-r_j| > \epsilon^2 t^{1-2\lambda}/(1-2\lambda))\| \lesssim t^{-\rho(1-\lambda)} \tag{2.18}$$

holds for any $0<\epsilon\leqslant\epsilon_0$. Whereas we have

$$V_{jk}^{\mathrm{sing}}(t^\lambda(r_k-r_j))F(\sqrt{\mu_{jk}}|r_k-r_j| > \epsilon^2 t^{1-2\lambda}/(1-2\lambda))\langle D/t^\lambda\rangle^{-2} = 0 \tag{2.19}$$

for $t \gg 1$ because V_{jk}^{sing} is compactly supported. By Proposition 2.2, we clearly have

$$\begin{aligned}&\|V_{jk}^{\mathrm{bdd}}(t^\lambda(r_k-r_j))F(\sqrt{\mu_{jk}}|r_k-r_j| \leqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda))e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))}\phi\|\\ &\qquad\lesssim t^{-\nu(1-2\lambda)}\|\langle x\rangle^\nu\phi\|\end{aligned} \tag{2.20}$$

because V_{jk}^{bdd} is bounded. Let $\chi \in C^\infty(\mathbb{R}^d)$ such that

$$\chi(x) = \begin{cases} 1 & \text{if} \quad |x|\geqslant 1\\ 0 & \text{if} \quad |x|\leqslant 1/2.\end{cases} \tag{2.21}$$

Reconstructing the Jacobi coordinates for $(j,k)\neq(1,2)$, we have

$$\begin{aligned}&\|V_{jk}^{\mathrm{sing}}(t^\lambda(r_k-r_j))F(\sqrt{\mu_{jk}}|r_k-r_j| \leqslant \epsilon^2 t^{1-2\lambda}/(1-2\lambda))e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))}\phi\|\\ &\leqslant \|V_{jk}^{\mathrm{sing}}(t^\lambda x_1/\sqrt{\mu_{jk}})\chi((1-2\lambda)x_1/(\epsilon^2 t^{1-2\lambda}))e^{-\mathrm{i}t^{1-2\lambda}D_1^2/(2(1-2\lambda))}\phi\|.\end{aligned} \tag{2.22}$$

Noting that $\|V_{jk}^{\mathrm{sing}}(t^\lambda x_1/\sqrt{\mu_{jk}})\langle\sqrt{\mu_{jk}}D_1/t^\lambda\rangle^{-2}\| = \|V_{jk}^{\mathrm{sing}}(x_1)\langle D_1\rangle^{-2}\|$ and that

$$\begin{aligned}\mathrm{i}[D_1^2,\chi((1-2\lambda)x_1/(\epsilon^2t^{1-2\lambda}))] &= ((1-2\lambda)/\epsilon^2 t^{1-2\lambda})(\nabla\chi)((1-2\lambda)x_1/(\epsilon^2t^{1-2\lambda}))\cdot D_1\\ &\quad -\mathrm{i}((1-2\lambda)/\epsilon^2t^{1-2\lambda})^2(\Delta\chi)((1-2\lambda)x_1/(\epsilon^2t^{1-2\lambda})),\end{aligned} \tag{2.23}$$

we have

$$\begin{aligned}&\|\langle\sqrt{\mu_{jk}}D_1/t^\lambda\rangle^2\chi((1-2\lambda)x_1/(\epsilon^2t^{1-2\lambda}))e^{-\mathrm{i}t^{1-2\lambda}D_1^2/(2(1-2\lambda))}\phi\|\\ &\lesssim \|\chi((1-2\lambda)x_1/(\epsilon^2t^{1-2\lambda}))e^{-\mathrm{i}t^{1-2\lambda}D_1^2/(2(1-2\lambda))}\phi\|\\ &+t^{-1}\|(\nabla\chi)((1-2\lambda)x_1/(\epsilon^2t^{1-2\lambda}))\cdot e^{-\mathrm{i}t^{1-2\lambda}D_1^2/(2(1-2\lambda))}D_1\phi\| + t^{-2+2\lambda}\|\phi\|.\end{aligned} \tag{2.24}$$

We can use Proposition 2.2 again for the first and second terms on the right side of (2.3). By (2.18), (2.19), (2.20), and (2.24), we therefore have

$$\|(\mathrm{d}/\mathrm{d}t)U_\lambda(t,T)^* e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))}\phi\| \lesssim t^{-\rho(1-\lambda)} + t^{-\nu(1-2\lambda)} + t^{-2+2\lambda} \tag{2.25}$$

for $t \gg 1$. We note that $\rho(1-\lambda) > 1$, $2-2\lambda > 1$, and $\nu(1-2\lambda) > 1$ for $\nu \gg 1$. By the Cook-Kuroda method ([45, Theorem XI.4]), this completes the proof. □

3 Inverse Scattering

To prove Theorem 1.2, it is convenient to choose different Jacobi coordinates from those of (1.6). According to [16, 49, 53], we focus on the pair of particles 1 and 2 and prove that $V_{12,1} = V_{12,2}$ because the proofs for the other pairs for j and k can be demonstrated in the same analogy. Let $y_{12} = r_2 - r_1$ be the relative position between r_1 and r_2, and

$$y_j = r_j - (m_1 r_1 + m_2 r_2)/(m_1+m_2) \tag{3.1}$$

for $3 \leqslant j \leqslant N$ be the relative position between r_j and the center of mass for r_1 and r_2. Their conjugate momentums are given by

$$-\mathrm{i}\nabla_{y_{12}} = \mu_{12}(-\mathrm{i}\nabla_{r_2}/m_2 - (-\mathrm{i}\nabla_{r_1})/m_1) \tag{3.2}$$

and

$$-\mathrm{i}\nabla_{y_j} = \mu_j(-\mathrm{i}\nabla_{r_j}/m_j - (-\mathrm{i}\nabla_{r_1} - \mathrm{i}\nabla_{r_2})/(m_1+m_2)) \tag{3.3}$$

for $3 \leqslant j \leqslant N$ where

$$\mu_{12} = m_1 m_2/(m_1+m_2) \tag{3.4}$$

and

$$\mu_j = m_j(m_1+m_2)/(m_j+m_1+m_2) \tag{3.5}$$

for $3 \leqslant j \leqslant N$ are the reduced masses respectively. Because of the relations

$$\sum_{j=1}^N m_j r_j^2 = M r_{\mathrm{cm}}^2 + \mu_{12} y_{12}^2 + \sum_{j=3}^N \mu_j y_j^2, \tag{3.6}$$

$$\sum_{j=1}^N \Delta_{r_j}/m_j = \Delta_{\mathrm{cm}}/M + \Delta_{y_{12}}/\mu_{12} + \sum_{j=3}^N \Delta_{y_j}/\mu_j, \tag{3.7}$$

we rewrite the free Hamiltonian $H_0(t)$ in (2.1) using $D = (D_{12}, D_3, \ldots, D_N)$ and $x = (x_{12}, x_3 \ldots, x_N)$ with $x_{12} = \sqrt{\mu_{12}}y_{12}$ and $x_j = \sqrt{\mu_j}y_j$ for $3 \leqslant j \leqslant N$. The relative momentum D_{jk} is defined such that $D_{1k} = D_{2k} = D_k - D_{12}$ for $3 \leqslant k \leqslant N$

and $D_{jk} = D_k - D_j$ for $3 \leqslant j < k \leqslant N$. These coordinates are different from those in [16, 49, 53] because of the scalings of x_{12} and x_j for $3 \leqslant j \leqslant N$. By virtue of these scalings, we omit the effects of reduced masses in our notations. Even in these coordinates, the wave operators (1.20) exist by the same proofs in Section 2. In particular, the strong limits

$$W_\lambda^\pm = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_{0,\lambda}(t) \tag{3.8}$$

also exist, and we define

$$S_\lambda(V) = (W_\lambda^+)^* W_\lambda^-. \tag{3.9}$$

Because $W^\pm = W_\lambda^\pm U_{0,\lambda}(s_\pm)^* U_0(s_\pm, 0)$ holds for $s_+ \geqslant T$ and $s_- \leqslant -T$, we have

$$S(V) = U_0(s_+,0)^* U_{0,\lambda}(s_+) S_\lambda(V) U_{0,\lambda}(s_-)^* U_0(s_-,0). \tag{3.10}$$

This implies that $S(V_1) = S(V_2)$ is equivalent to $S_\lambda(V_1) = S_\lambda(V_2)$.

$H_0(t) \equiv H_0 = D^2/2 + \omega^2 x^2/2$ is the time-independent harmonic oscillator for $|t| < T$, and its time-evolution $e^{-\mathrm{i}tH_0}$ is governed by the Mehler formula in [12, Section 2.2] and [41, Theorem 5.29], which is

$$e^{-\mathrm{i}tH_0} = \mathscr{M}_{d(N-1)}(\tan\omega t/\omega)\mathscr{D}_{d(N-1)}(\sin\omega t/\omega)\mathscr{F}_{d(N-1)}\mathscr{M}(\tan\omega t/\omega) \tag{3.11}$$

for $\omega t \notin \pi\mathbb{Z}$, where $\mathscr{M}_{d(N-1)}$ and $\mathscr{D}_{d(N-1)}$ are multiplication and dilation given by

$$\mathscr{M}_{d(N-1)}(t)\phi(x) = e^{\mathrm{i}x^2/(2t)}\phi(x), \tag{3.12}$$

$$\mathscr{D}_{d(N-1)}(t)\phi(x) = (\mathrm{i}t)^{-d(N-1)/2}\phi(x/t). \tag{3.13}$$

Moreover,

$$e^{-\mathrm{i}tH_0} = \mathrm{i}^{d(N-1)/2}\mathscr{M}_{d(N-1)}(-\cot\omega t/\omega)\mathscr{D}_{d(N-1)}(\cos\omega t)e^{-\mathrm{i}\tan\omega t D^2/(2\omega)} \tag{3.14}$$

for $\omega t \notin (\pi/2)\mathbb{Z}$.

For $v \in \mathbb{R}^d$, the normalization is $\hat{v} = v/|v|$. Let $\hat{v}, e_3, \ldots, e_N \in \mathbb{R}^d$ be unit vectors pointing in mutually different directions. We define $v_{jk} = v_k - v_j$ for $1 \leqslant j < k \leqslant N$ where $v_1 = -v/2$, $v_2 = v/2$, and $v_j = |v|^2 e_j$ for $3 \leqslant j \leqslant N$. Let $\Phi_0 \in \mathscr{S}(\mathbb{R}^{d(N-1)})$ be $\Phi_0 = \phi_1 \otimes \phi_2$ such that $\mathscr{F}_d\phi_1 \in C_0^\infty(\mathbb{R}^d)$ and $\mathscr{F}_{d(N-1)}\phi_2 \in C_0^\infty(\mathbb{R}^{d(N-2)})$ with $\|\phi_2\|_{L^2(\mathbb{R}^{d(N-2)})} = 1$. In addition, we assume that $\operatorname{supp}\mathscr{F}_d\phi_1 \subset \{\xi_{12} \in \mathbb{R}^d \bigm| |\xi_{12}| < \eta_{12}\}$ for some $\eta_{12} > 0$ and $\operatorname{supp}\mathscr{F}_{d(N-1)}\phi_2 \subset \{(\xi_3, \ldots, \xi_N) \in \mathbb{R}^{d(N-2)} \bigm| |\xi_3| < 1, \ldots, |\xi_N| < 1\}$. We define $\Phi_v = \mathscr{T}_v\Phi_0$ for

$$\mathscr{T}_v = e^{\mathrm{i}v\cdot x_{12}} \prod_{j=3}^N e^{\mathrm{i}v_j\cdot x_j}. \tag{3.15}$$

Then there exists $f_{jk} \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f_{jk} \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant \eta_{jk}\}$ where $\eta_{1k} = \eta_{2k} = 1 + \eta_{12}$ for $3 \leqslant k \leqslant N$ and $\eta_{jk} = 2$ for $3 \leqslant j < k \leqslant N$ such that

$$\Phi_v = \mathscr{T}_v f_{jk}(D_{jk})\Phi_0 = f_{jk}(D_{jk} - v_{jk})\Phi_v. \tag{3.16}$$

To apply the Enss-Weder time-dependent method [16], the following reconstruction formula is essential. By virtue of this formula, the uniqueness of the potential functions follows from the injective property of the Radon transform.

Theorem 3.1. *Let Φ_v be defined as above and Ψ_v have the same properties for $\Psi_0 = \psi_1 \otimes \phi_2$. Then*

$$\lim_{|v|\to\infty} |v|(\mathrm{i}(S_\lambda(V) - 1)\Phi_v, \Psi_v) = \int_{-\infty}^{\infty} (V_{12}(r_2 - r_1 + \hat{v}t)\phi_1, \psi_1)_{L^2(\mathbb{R}^d)} \mathrm{d}t \tag{3.17}$$

holds for $V = \sum_{1\leqslant j<k\leqslant k} V_{jk}$, which satisfies Assumption 1.1.

It seems difficult to study the time evolution of $U_0(t, 0)$ directly for all $t \in \mathbb{R}$. Instead, we analyze $e^{-\mathrm{i}tH_0}$ for $|t| < T$ and $e^{\mp \mathrm{i}|t|^{1-2\lambda}D^2/(2(1-2\lambda))}$ for $|t| \geqslant T$ according to [26]. We define $U_{0,\lambda}(t) = e^{-\mathrm{i}tH_0}$ for $|t| < T$. In the proofs below, we can assume that

$$\pi/(2\omega) \leqslant T < \pi/\omega \tag{3.18}$$

without loss of generality, as in [26].

Lemma 3.2. *Let Φ_v be as in Theorem 3.1. Then*

$$\int_{-\infty}^{\infty} \|V_{jk}^{\mathrm{bdd}}(r_k - r_j)U_{0,\lambda}(t)\Phi_v\| \mathrm{d}t = O(|v_{jk}|^{-1}) \tag{3.19}$$

holds as $|v| \to \infty$ for any $1 \leqslant j < k \leqslant N$.

Proof of Lemma 3.2. We divide the integral such that

$$\int_{-\infty}^{\infty} = \int_{|t|<T} + \int_{|t|\geqslant T} \tag{3.20}$$

and first consider the integral on $|t| < T$. By (3.14) and (3.16), we have

$$\begin{aligned}\|V_{jk}^{\mathrm{bdd}}(r_k - r_j)e^{-\mathrm{i}tH_0}\Phi_v\| &= \|V_{jk}^{\mathrm{bdd}}(\cos\omega t(r_k - r_j))e^{-\mathrm{i}\tan\omega t D^2/(2\omega)} f_{jk}(D_{jk} - v_{jk})\Phi_v\| \\ &= \|V_{jk}^{\mathrm{bdd}}(\cos\omega t x_{12}/\sqrt{\mu_{jk}})e^{-\mathrm{i}\tan\omega t D_{12}^2/(2\omega)} f_{jk}(D_{12} - v_{jk})\Phi_v\| \\ &= \|V_{jk}^{\mathrm{bdd}}((\cos\omega t x_{12} + \sin\omega t v_{jk}/\omega)/\sqrt{\mu_{jk}})e^{-\mathrm{i}\tan\omega t D_{12}^2/(2\omega)} f_{jk}(D_{12})e^{-\mathrm{i}v_{jk}\cdot x_{12}}\Phi_v\|,\end{aligned} \tag{3.21}$$

where we have reconstructed the Jacobi coordinates from $y_{12} = r_k - r_j$ for $(j,k) \neq (1,2)$ and used the relation

$$e^{-\mathrm{i}v_{jk}\cdot x_{12}} e^{-\mathrm{i}\tan\omega t D_{12}^2/(2\omega)} e^{\mathrm{i}v_{jk}\cdot x_{12}} = e^{-\mathrm{i}\tan\omega t|v_{jk}|^2/(2\omega)} e^{-\mathrm{i}\tan\omega t D_{12}\cdot v_{jk}/\omega} e^{-\mathrm{i}\tan\omega t D_{12}^2/(2\omega)}. \tag{3.22}$$

From [26, (39), (40), and (43) in the proof of Lemma 2.3], we have

$$\int_{|t|<T} \|V_{jk}^{\mathrm{bdd}}(r_k - r_j) e^{-\mathrm{i}tH_0}\Phi_v\| \mathrm{d}t = O(|v_{jk}|^{-1}). \tag{3.23}$$

We next consider the integral on $|t| \geqslant T$. In particular, we consider $t \geqslant T$. By (2.2) and (3.16), we have

$$\begin{aligned}
\|V_{jk}^{\mathrm{bdd}}(r_k - r_j)U_{0,\lambda}(t)\Phi_v\| &= \|V_{jk}^{\mathrm{bdd}}(t^\lambda(r_k - r_j)) e^{-\mathrm{i}t^{1-2\lambda}D^2/(2(1-2\lambda))} f_{jk}(D_{jk} - v_{jk})\Phi_v\| \\
&= \|V_{jk}^{\mathrm{bdd}}(t^\lambda x_{12}/\sqrt{\mu_{jk}}) e^{-\mathrm{i}t^{1-2\lambda}D_{12}^2/(2(1-2\lambda))} f_{jk}(D_{12} - v_{jk})\Phi_v\| \\
= \|V_{jk}^{\mathrm{bdd}}((t^\lambda x_{12} + t^{1-2\lambda}v_{jk}/(1-2\lambda))/\sqrt{\mu_{jk}}) & e^{-\mathrm{i}t^{1-2\lambda}D_{12}^2/(2(1-2\lambda))} f_{jk}(D_{12}) e^{-\mathrm{i}v_{jk}\cdot x_{12}}\Phi_v\|,
\end{aligned} \tag{3.24}$$

where we have reconstructed the Jacobi coordinates from $y_{12} = r_k - r_j$ for $(j,k) \neq (1,2)$ and used the relation

$$\begin{aligned}
& e^{-\mathrm{i}v_{jk}\cdot x_{12}} e^{-\mathrm{i}t^{1-2\lambda}D_{12}^2/(2(1-2\lambda))} e^{\mathrm{i}v_{jk}\cdot x_{12}} \\
&= e^{-\mathrm{i}t^{1-2\lambda}|v_{jk}|^2/(2(1-2\lambda))} e^{-\mathrm{i}t^{1-2\lambda}D_{12}\cdot v_{jk}/(1-2\lambda)} e^{-\mathrm{i}t^{1-2\lambda}D_{12}^2/(2(1-2\lambda))}.
\end{aligned} \tag{3.25}$$

By [26, (47) and (51) in the proof of Lemma 2.3], we have

$$\int_{|t|\geqslant T} \|V_{jk}^{\mathrm{bdd}}(r_k - r_j)U_{0,\lambda}(t)\Phi_v\| \mathrm{d}t = O(|v_{jk}|^{-1}). \tag{3.26}$$

(3.23) and (3.26) complete the proof. □

Lemma 3.3. *Let Φ_v be as in Theorem 3.1. Then*

$$\int_{-\infty}^{\infty} \|V_{jk}^{\mathrm{sing}}(r_k - r_j)U_{0,\lambda}(t)\Phi_v\| \mathrm{d}t = O(|v_{jk}|^{-1}) \tag{3.27}$$

holds as $|v| \to \infty$ for any $1 \leqslant j < k \leqslant N$.

Proof of Lemma 3.3. We divide the integral such that

$$\int_{-\infty}^{\infty} = \int_{|t|\leqslant\pi/(4\omega)} + \int_{\pi/(4\omega)<|t|<T} + \int_{|t|\geqslant T} \tag{3.28}$$

and first consider the integral on $|t| \leqslant \pi/(4\omega)$. As we similarly had for (3.21), we have

$$\|V_{jk}^{\mathrm{sing}}(r_k - r_j)e^{-\mathrm{i}tH_0}\Phi_v\| = \|V_{jk}^{\mathrm{sing}}((\cos\omega t x_{12} + \sin\omega t v_{jk}/\omega)/\sqrt{\mu_{jk}})\langle D_{12}/\cos\omega t\rangle^{-2} \times e^{-\mathrm{i}\tan\omega t D_{12}^2/(2\omega)} f_{jk}(D_{12})\langle D_{12}/\cos\omega t\rangle^2 e^{-\mathrm{i}v_{jk}\cdot x_{12}}\Phi_v\|. \quad (3.29)$$

We take $g_{jk} \in C_0^\infty(\mathbb{R}^d)$ such that $f_{jk} = f_{jk}g_{fk}$. Noting that

$$\|\langle x_{12}\rangle^2\langle D_{12}/\cos\omega t\rangle^2 g_{jk}(D_{12})e^{-\mathrm{i}v_{jk}\cdot x_{12}}\Phi_v\| \lesssim \|\langle D_{12}\rangle^2 g_{jk}(D_{12})\|\|\langle x_{12}\rangle^2\Phi_0\| + \|[x_{12}^2, \langle D_{12}\rangle^2 g_{jk}(D_{12})]e^{-\mathrm{i}v_{jk}\cdot x_{12}}\Phi_v\| \lesssim 1, \quad (3.30)$$

we have

$$\int_{|t|\leqslant\pi/(4\omega)} \|V_{jk}^{\mathrm{sing}}(r_k - r_j)e^{-\mathrm{i}tH_0}\Phi_v\| = O(|v_{jk}|^{-1}) \quad (3.31)$$

by [26, (74) and (79) in the proof of Lemma 3.2]. We next consider the integral on $\pi/(4\omega) < |t| < T$. We define the d-dimensional harmonic oscillator

$$H_{0,12} = D_{12}^2/2 + w^2 x_{12}^2/2 \quad (3.32)$$

and note that the Mehler formula

$$e^{-\mathrm{i}H_{0,12}} = \mathrm{i}^{d/2}\mathscr{M}_d(-\cot\omega t/\omega)\mathscr{D}_d(\cos\omega t)e^{-\mathrm{i}\tan\omega t D_{12}^2/(2\omega)} \quad (3.33)$$

also holds. As we similarly had for (3.21), we have

$$\begin{aligned} &\|V_{jk}^{\mathrm{sing}}(r_k - r_j)e^{-\mathrm{i}tH_0}\Phi_v\| \\ &= \|V_{jk}^{\mathrm{sing}}((\cos\omega t x_{12} + \sin\omega t v_{jk}/\omega)/\sqrt{\mu_{jk}})e^{-\mathrm{i}\tan\omega t D_{12}^2/(2\omega)}e^{-\mathrm{i}v_{jk}\cdot x_{12}}\Phi_v\| \\ &= \|V^{\mathrm{sing}}((x_{12} + \sin\omega t v_{jk}/\omega)/\sqrt{\mu_{jk}})e^{-\mathrm{i}H_{0,12}}e^{-\mathrm{i}v_{jk}\cdot x_{12}}\Phi_v\| \end{aligned} \quad (3.34)$$

using (3.33). We therefore have

$$\int_{\pi/(4\omega)<|t|<T} \|V_{jk}^{\mathrm{sing}}(r_k - r_j)e^{-\mathrm{i}tH_0}\Phi_v\| = O(|v_{jk}|^{-1}) \quad (3.35)$$

by [26, (88) and (93) in the proof of Lemma 3.2]. Finally, we consider the integral on $|t| \geqslant T$, in particular $t \geqslant T$. As we similarly had for (3.24), we have

$$\|V_{jk}^{\mathrm{sing}}(r_k - r_j)U_{0,\lambda}(t)\Phi_v\| = \|V_{jk}^{\mathrm{sing}}((t^\lambda x_{12} + t^{1-2\lambda}v_{jk}/(1-2\lambda))/\sqrt{\mu_{jk}})\langle D_{12}/t^\lambda\rangle^{-2} \times e^{-\mathrm{i}t^{1-2\lambda}D_{12}^2/(2(1-2\lambda))} f_{jk}(D_{12})\langle D_{12}/t^\lambda\rangle^2 e^{-\mathrm{i}v_{jk}\cdot x_{12}}\Phi_v\|. \quad (3.36)$$

As in (3.30), we have

$$\|\langle x_{12}\rangle^2\langle D_{12}/t^\lambda\rangle^2 g_{jk}(D_{12})e^{-\mathrm{i}v_{jk}\cdot x_{12}}\Phi_v\| \lesssim 1 \quad (3.37)$$

for $g_{jk} \in C_0^\infty(\mathbb{R}^d)$ such that $f_{jk} = f_{jk} g_{jk}$. We inductively have

$$\|\langle x_{12}\rangle^2 \langle D_{12}/t^\lambda\rangle^N g_{jk}(D_{12}) e^{-\mathrm{i}v_{jk}\cdot x_{12}} \Phi_v\| \lesssim 1 \tag{3.38}$$

for any $N \in \mathbb{N}$. Therefore, by [26, (97) and (99) in the proof of Lemma 3.2], we have

$$\int_{|t|\geqslant T} \|V_{jk}^{\mathrm{sing}}(r_k - r_j) U_{0,\lambda}(t)\Phi_v\| = O(|v_{jk}|^{-1}). \tag{3.39}$$

(3.31), (3.35), and (3.39) imply that Lemma 3.3 holds. □

By virtue of Lemmas 3.2 and 3.3, we immediately have the following Lemma as in [16, Corollary 2.3] (see also [4, 6, 7, 10, 11, 21, 22, 23, 26, 27, 42, 43, 44, 49, 53]). We therefore omit its proof.

Lemma 3.4. *Let Φ_v be as in Theorem 3.1. Then*

$$\sup_{t\in\mathbb{R}} \|(U(t,0)W_\lambda^- - U_{0,\lambda}(t))\Phi_v\| = O(|v|^{-1}) \tag{3.40}$$

holds as $|v| \to \infty$.

Proof of Theorem 3.1. We have

$$\begin{aligned} \mathrm{i}(S_\lambda(V) - 1) &= \mathrm{i}(W_\lambda^+ - W_\lambda^-)^* W_\lambda^- = \mathrm{i}\int_{-\infty}^{\infty} ((\mathrm{d}/\mathrm{d}t)U(t,0)^* U_{0,\lambda}(t))^* W_\lambda^- \mathrm{d}t \\ &= \sum_{1\leqslant j<k\leqslant N} \int_{-\infty}^{\infty} U_{0,\lambda}(t)^* V_{jk}(r_k - r_j) U(t,0) W_\lambda^- \mathrm{d}t \end{aligned} \tag{3.41}$$

and

$$|v|((\mathrm{i}S(V) - 1)\Phi_v, \Psi_v) = |v| \int_{-\infty}^{\infty} (V_{12}(r_2 - r_1) U_{0,\lambda}(t)\Phi_v, U_{0,\lambda}(t)\Psi_v)\mathrm{d}t + R(v), \tag{3.42}$$

where

$$\begin{aligned} R(v) &= |v| \sum_{(j,k)\neq(1,2)} \int_{-\infty}^{\infty} (V_{jk}(r_k - r_j) U_{0,\lambda}(t)\Phi_v, U_{0,\lambda}(t)\Psi_v)\mathrm{d}t \\ &+ |v| \sum_{1\leqslant j<k\leqslant N} \int_{-\infty}^{\infty} ((U(t,0)W_\lambda^- - U_{0,\lambda}(t))\Phi_v, V_{jk}(r_k - r_j) U_{0,\lambda}(t))\mathrm{d}t. \end{aligned} \tag{3.43}$$

Noting that

$$\int_{-\infty}^{\infty} \|V_{jk}(r_k - r_j) U_{0,\lambda}(t)\Phi_v\|\mathrm{d}t = O(|v|^{-2}) \tag{3.44}$$

holds as $|v| \to \infty$ for $(j,k) \neq (1,2)$, we have $R(v) = O(|v|^{-1})$ by Lemmas 3.2, 3.3, and 3.4. We now prove

$$|v| \int_{-\infty}^{\infty} (V_{12}(r_2 - r_1) U_{0,\lambda}(t)\Phi_v, U_{0,\lambda}(t)\Psi_v) \mathrm{d}t \to \int_{-\infty}^{\infty} (V_{12}(r_2 - r_1 + \hat{v}t)\phi_1, \psi_1)_{L^2(\mathbb{R}^d)} \mathrm{d}t \tag{3.45}$$

as $|v| \to \infty$. We first focus on the term V_{12}^{bdd}. We divide the integral such that

$$\int_{-\infty}^{\infty} = \int_{|t|<\pi/(2\omega)} + \int_{\pi/(2\omega)\leqslant|t|<T} + \int_{|t|\geqslant T} \tag{3.46}$$

and consider the integrals on $|t| < \pi/(2\omega)$ and $\pi/(2\omega) \leqslant |t| < T$. Recalling that ϕ_2 is normalized, we have

$$\begin{aligned}
&(V_{12}^{\mathrm{bdd}}(r_2 - r_1) e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v) \\
&= (V_{12}^{\mathrm{bdd}}(\cos\omega t x_{12}/\sqrt{\mu_{12}}) e^{-\mathrm{i}\tan\omega t D_{12}^2/(2\omega)}\Phi_v, e^{-\mathrm{i}\tan\omega t D_{12}^2/(2\omega)}\Psi_v) \\
&= (V_{12}^{\mathrm{bdd}}(x_{12}/\sqrt{\mu_{12}}) e^{-\mathrm{i}H_{0,12}} e^{\mathrm{i}v\cdot x_{12}}\phi_1, e^{-\mathrm{i}H_{0,12}} e^{\mathrm{i}v\cdot x_{12}}\psi_1)_{L^2(\mathbb{R}^d)}
\end{aligned} \tag{3.47}$$

for $|t| < T$ by (3.14) and (3.33). Therefore

$$\begin{aligned}
&|v| \int_{|t|<\pi/(2\omega)} (V_{12}^{\mathrm{bdd}}(r_2 - r_1) e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v) \mathrm{d}t \\
&\to \int_{-\infty}^{\infty} (V_{12}^{\mathrm{bdd}}(x_{12}/\sqrt{\mu_{12}} + \hat{v}t)\phi_1, \psi_1)_{L^2(\mathbb{R}^d)} \mathrm{d}t
\end{aligned} \tag{3.48}$$

and

$$|v| \int_{\pi/(2\omega)\leqslant|t|<T} (V_{12}^{\mathrm{bdd}}(r_2 - r_1) e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v) \mathrm{d}t \to 0 \tag{3.49}$$

hold as $|v| \to \infty$ by [26, (64) and (65) in the proof of Theorem 2.1]. Let $U_{0,12}(t,s)$ be the propagator for the d-dimensional time-dependent harmonic oscillator

$$H_{0,12}(t) = D_{12}^2/2 + k(t)x_{12}^2/2 \tag{3.50}$$

and define

$$U_{0,12,\lambda}(t) = e^{\mathrm{i}\lambda x_{12}^2/(2t)} e^{-\mathrm{i}\lambda \log t A_{12}} e^{-\mathrm{i}t^{1-2\lambda} D_{12}^2/(2(1-2\lambda))} \tag{3.51}$$

if $t \geqslant T$ and

$$U_{0,12,\lambda}(t) = e^{\mathrm{i}\lambda x_{12}^2/(2t)} e^{-\mathrm{i}\lambda \log(-t) A_{12}} e^{\mathrm{i}(-t)^{1-2\lambda} D_{12}^2/(2(1-2\lambda))} \tag{3.52}$$

if $t \leqslant -T$, where $A_{12} = (D_{12}\cdot x_{12} + x_{12}\cdot D_{12})/2$. By (2.2), (2.3), (3.51), and (3.52), we have

$$\begin{aligned}
&(V_{12}^{\mathrm{bdd}}(r_2 - r_1) U_{0,\lambda}(t)\Phi_v, U_{0,\lambda}(t)\Psi_v) \\
&= (V_{12}^{\mathrm{bdd}}(t^{\lambda} x_{12}/\sqrt{\mu_{12}}) e^{-\mathrm{i}t^{1-2\lambda} D_{12}^2/(2(1-2\lambda))}\Phi_v, e^{-\mathrm{i}t^{1-2\lambda} D_{12}^2/(2(1-2\lambda))}\Psi_v) \\
&= (V_{12}^{\mathrm{bdd}}(x_{12}/\sqrt{\mu_{12}}) U_{0,12,\lambda}(t) e^{\mathrm{i}v\cdot x_{12}}\phi_1, U_{0,12,\lambda}(t) e^{\mathrm{i}v\cdot x_{12}}\psi_1)_{L^2(\mathbb{R}^d)}
\end{aligned} \tag{3.53}$$

for $|t| \geqslant T$. By [26, (66) in the proof of Theorem 2.1],

$$|v| \int_{|t|\geqslant T} (V_{12}^{\mathrm{bdd}}(r_2 - r_1)U_{0,\lambda}(t)\Phi_v, U_{0,\lambda}(t)\Psi_v)\mathrm{d}t \to 0 \tag{3.54}$$

holds as $|v| \to \infty$. We next focus on the term V_{12}^{sing}. We divide the integral such that

$$\int_{-\infty}^{\infty} = \int_{|t|<\pi/(4\omega)} + \int_{\pi/(4\omega)\leqslant|t|<T} + \int_{|t|\geqslant T} \tag{3.55}$$

and consider the integrals on $|t| < \pi/(4\omega)$ and $\pi/(4\omega) \leqslant |t| < T$. Because (3.47) also holds with V_{12}^{bdd} replaced by V_{12}^{sing}, we have

$$\begin{aligned} |v| \int_{|t|<\pi/(4\omega)} &(V_{12}^{\mathrm{sing}}(r_2 - r_1)e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v)\mathrm{d}t \\ &\to \int_{-\infty}^{\infty} (V_{12}^{\mathrm{sing}}(x_{12}/\sqrt{\mu_{12}} + \hat{v}t)\phi_1, \psi_1)_{L^2(\mathbb{R}^d)}\mathrm{d}t \end{aligned} \tag{3.56}$$

and

$$|v| \int_{\pi/(4\omega)\leqslant|t|<T} (V_{12}^{\mathrm{sing}}(r_2 - r_1)e^{-\mathrm{i}tH_0}\Phi_v, e^{-\mathrm{i}tH_0}\Psi_v)\mathrm{d}t \to 0 \tag{3.57}$$

as $|v| \to \infty$ by [26, (110) and (111) in the proof of Theorem 3.1]. Finally, we consider the integral on $|t| \geqslant T$. Because (3.53) also holds with V_{12}^{bdd} replaced by V_{12}^{sing}, we have

$$|v| \int_{|t|\geqslant T} (V_{12}^{\mathrm{bdd}}(r_2 - r_1)U_{0,\lambda}(t)\Phi_v, U_{0,\lambda}(t)\Psi_v)\mathrm{d}t \to 0 \tag{3.58}$$

as $|v| \to \infty$ by [26, (112) in the proof of Theorem 3.1]. Combining (3.48), (3.49), (3.54), (3.56), (3.57), and (3.58), we have (3.45). □

Proof of Theorem 1.2. We assume that $S_\lambda(V_1) = S_\lambda(V_2)$ because (3.10) holds. By the same computation as that of [26, (77) in the proof of Lemma 3.2], we have the condition commonly referred to as the Enss condition

$$\int_0^{\infty} \|V_{12}(r_2 - r_1)\langle D_{12}\rangle^{-2}F(|r_2 - r_1| \geqslant R)\|_{\mathscr{B}(L^2(\mathbb{R}^d))}\mathrm{d}R < \infty. \tag{3.59}$$

By virtue of Theorem 3.1 and the Plancherel formula for the Radon transform ([19, Theorem 2.17 in Chapter 1]), we have $V_{12,1} = V_{12,2}$ in the same way as for the proof of [16, Theorem 1.1]. □

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