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NON-MONOTONE TRAVELING WAVES OF THE WEAK COMPETITION LOTKA-VOLTERRA SYSTEM

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ABSTRACT. We investigate traveling wave solutions in the two-species reaction-diffusion Lotka-Volterra competition system under weak competition. For the strict weak competition regime ($b < a < 1/c$, $d > 0$), we construct refined upper and lower solutions combined with the Schauder fixed point theorem to establish the existence of traveling waves for all wave speeds $s \geq s^* := \max\{2, 2\sqrt{ad}\}$, and provide verifiable sufficient conditions for the emergence of non-monotone waves. Such conditions for non-monotonic waves have not been explicitly addressed in previous studies. It is interesting to point out that our result for non-monotone waves also hold for the critical speed case $s = s^*$. In addition, in the critical weak competition case ($b < a = 1/c$, $d > 0$), we rigorously prove, for the first time, the existence of front-pulse traveling waves.

1. INTRODUCTION

In population biology, the Lotka-Volterra competition equations are widely accepted as a fundamental model for describing the interactions between competing species. By incorporating spatial diffusion into these equations, one arrives at the Lotka-Volterra competition-diffusion system, which provides a natural framework for studying the spatial propagation and coexistence of biological populations. Mathematically, such systems belong to the class of reaction-diffusion equations that admit traveling wave solutions, a central object in the study of spatial ecology and pattern formation.

For the two-species case, the system can be written in the form

$$(1.1) \quad \begin{cases} u_t = u_{xx} + u(1 - u - cv), \\ v_t = dv_{xx} + v(a - bu - v), \end{cases} \quad x \in \mathbb{R}, \quad t > 0,$$

where $u(x, t)$ and $v(x, t)$ denote the population densities of two competing species, and $a, b, c, d > 0$ are parameters reflecting the competition intensity and diffusion rate. Depending on the coefficients, the system admits several equilibria, among which the coexistence equilibrium plays a decisive role under the weak competition condition.

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We consider the traveling wave ansatz for (1.1) of the form

$$(1.2) \quad (u(x, t), v(x, t)) = (u(\xi), v(\xi)), \quad \xi = x + st,$$

where s denotes the wave speed. We write ∂_ξ as a prime. A direct calculation shows that (u, v) satisfies

$$(1.3) \quad \begin{cases} u'' - su' + u(1 - u - cv) = 0, \\ dv'' - sv' + v(a - bu - v) = 0, \end{cases} \quad \xi \in \mathbb{R}.$$

A classical problem, first posed by Tang and Fife [TF80], is to determine whether, in the strict weak competition regime

$$(1.4) \quad b < a < \frac{1}{c},$$

there exist traveling wave solutions $(u(\xi), v(\xi), s)$ connecting the extinction state $(0, 0)$ to the coexistence equilibrium

$$(u^*, v^*) = \left(\frac{1 - ac}{1 - bc}, \frac{a - b}{1 - bc} \right).$$

Equivalently, Tang and Fife investigated solutions of (1.3) subject to the boundary conditions

$$(1.5) \quad \lim_{\xi \rightarrow -\infty} (u, v)(\xi) = (0, 0), \quad \lim_{\xi \rightarrow +\infty} (u, v)(\xi) = (u^*, v^*).$$

They proved that whenever the wave speed s exceeds the critical threshold

$$s^* := \max\{2, 2\sqrt{ad}\},$$

there exists a traveling wave with strictly monotone profiles. This result provided the first rigorous demonstration that competitive interactions can generate spatial invasion dynamics governed by monotone wave fronts.

The weak competition regime has played one of the central roles in the study of traveling wave solutions for Lotka-Volterra competition-diffusion systems. Following the work of Tang and Fife, related research has made further progress. In particular, Ma [Ma01] introduced comparison principles and super-subsolution techniques that provided more flexible sufficient conditions for the existence of monotone traveling fronts under weak competition assumptions. Building on this framework, the work [SZ24] designed a boundary control scheme driven by traveling waves, explicitly exploiting the monotonicity guaranteed in the weak competition setting. More recently, Chang and Wu [CW25] extended this direction to three-species systems: by assuming weak competition between two species, they were able to preserve the monotonicity of two-species subsystems and consequently establish the existence of three-species traveling fronts. These contributions highlight that weak competition not only ensures coexistence equilibria but also provides the structural monotonicity necessary for rigorous analysis and further applications of traveling fronts.

By contrast, the study of non-monotone traveling waves remains comparatively limited. Hung [Hun12] constructed exact traveling waves under specially chosen parameters, thereby demonstrating the existence of non-monotone solutions in certain cases. Lin and Ruan [LR14] further observed the possibility of non-monotone waves and supported their existence by concrete examples and numerical simulations. Nevertheless, broad sufficient conditions for non-monotone fronts are still

lacking, and fundamental issues, such as the construction of front-pulse type non-monotone solutions or the existence of non-monotone waves at the critical wave speed $s = s^*$, remain open. We also point out related developments: in [CHY22], the authors investigated the stability of front-pulse solutions; in [Yan22], Yang constructed front-pulse solutions for the case where one species dominates the other; and in [Hun12], exact front-pulse solutions were provided. These studies suggest that while monotone waves in weak competition have been better understood, a systematic description of non-monotone traveling waves is still far from complete.

We emphasize that the present work exclusively focuses on the weak competition regime. For the strong competition case, see [KO95] and references therein, where the existence of traveling waves was established and the wave speed was shown to depend analytically on the competition coefficients. In addition, Morita and Tachibana [MT09] constructed entire solutions in the strong competition setting, providing a broader dynamical picture of invasion phenomena beyond classical fronts. Further developments on spreading speeds and traveling fronts under strong competition can be found in [GL13, CCW23, PWZ21, JW25], where qualitatively different behaviors such as bistability arise. For more details concerning the N-barrier maximum principle and its applications in reaction-diffusion systems, we refer to [CH16a, CH16b, CHL16, HLC16, CHH20, Hsi22].

1.1. Assumptions and notations. Before stating our main results, we introduce several notions. The critical wave speed is defined by

$$s^* := \max\{2, 2\sqrt{ad}\}.$$

We distinguish two parameter regimes:

- *Strict weak competition:* $b < a < \frac{1}{c}$, in which case the coexistence equilibrium (u^*, v^*) is strictly positive.
- *Critical weak competition:* $b < a = \frac{1}{c}$, where the equilibrium degenerates to $(0, v^*)$.

In the strict weak competition case, a front traveling wave refers to a solution of (1.3) satisfying (1.5). In the critical weak competition case, a front-pulse traveling wave is a solution of (1.3) satisfying the same boundary condition but with pulse-like behavior in one component.

1.2. Main results. We summarize our main theorems concerning the existence of traveling wave solutions to (1.3) under the weak competition assumption. Our results cover both the strict weak competition regime $b < a < 1/c$ and the critical weak competition regime $b < a = 1/c$. Depending on the wave speed s , we establish precise sufficient conditions for the existence of monotone and non-monotone fronts, as well as front-pulse solutions in the degenerate setting. First, we reprove the classic result of Tang and Fife via a different approach, i.e., the sub-sup solution method.

Theorem 1.1. *Given $d > 0$. Assume a, b, c satisfy (1.4). For any $s \geq s^*$, there exists a traveling wave solution $(u, v)(\xi)$ satisfying (1.3) and (1.5).*

Theorem 1.1 ensures the existence of a traveling wave solution, but does not address its monotonicity. We next turn to the refined question of when the profiles become non-monotone, and provide explicit sufficient conditions. Before that, we state a simple observation which assure the monotonicity of a wave.

Theorem 1.2. *For any $s \in \mathbb{R}$, if the wave profile (u, v) satisfies (1.3) with $0 < u(x) < u^*$ and $0 < v(x) < v^*$ for any $x \in \mathbb{R}$, then u, v are monotonic functions.*

We now turn to sufficient conditions for non-monotone solutions as follows.

Theorem 1.3. *For any $d > 0$, $b < a$, $s \geq s^*$ there exists $\delta(a, b, s) > 0$ such that under the condition $\delta(a, b, s) < c < \frac{1}{a}$, there exists a non-monotone traveling wave solution $(u, v)(\xi)$ satisfying (1.3) and (1.5).*

Theorem 1.4. *For any $d > 0$, $a < \frac{1}{c}$, $s \geq s^*$ there exists $\delta(a, c, s) > 0$ such that under the condition $\delta(a, c, s) < b < a$, there exists a non-monotone traveling wave solution $(u, v)(\xi)$ satisfying (1.3) and (1.5).*

One can consider $\delta(a, b, s)$ in Theorem 1.3 as a number close to but smaller than $1/a$ and $\delta(a, c, s)$ in Theorem 1.4 as a number close to but smaller than a . Theorems 1.3 and 1.4 establish, for the strict weak competition regime, explicit sufficient conditions for the existence of non-monotone fronts when $s \geq s^*$. A natural next step is to investigate the borderline case $a = 1/c$, namely the critical weak competition regime. In this degenerate setting, the coexistence equilibrium reduces to a semi-trivial state, and the corresponding wave dynamics give rise to a new type of solution, which we call a front-pulse. Our final result characterizes the existence of such front-pulse solutions.

Theorem 1.5. *For any $b < a$, $ac = 1$, $d > 0$. If $s \geq s^*$, there exists a non-trivial front-pulse solution of (4.1) with $u(\xi) \rightarrow 0$ as $|\xi| \rightarrow +\infty$.*

We pause to remark that if we consider the other degenerate case, $b = a < \frac{1}{c}$, we can still obtain a front-pulse solution of (4.3) with $v(\xi) \rightarrow 0$ as $|\xi| \rightarrow +\infty$. Please see Proposition 4.1. Besides classical traveling waves, we note recent progress on the critical weak competition case for the related system; see [CMX25, AX23].

We summarize our main results as follows:

- (1) For $b < a < \frac{1}{c}$, $d > 0$, we construct refined upper and lower solutions and employ the Schauder fixed point theorem to reprove the existence of traveling waves for all $s \geq s^* = \max\{2, 2\sqrt{ad}\}$ in [TF80]. Moreover, we establish verifiable sufficient conditions for non-monotone waves, that were previously missing in the literature. We also confirm the nonexistence of traveling waves for $s < s^*$.
- (2) At the threshold case $s = s^*$, we provide the first theoretical construction of non-monotone traveling waves, filling a long-standing gap where only numerical or example-based evidence had been available.
- (3) For the degenerate case $b < a = \frac{1}{c}$, $d > 0$, the classical lower-solution approach of Lin-Ruan [LR14], which relies on the Fisher-KPP asymptotics, breaks down due to the critical competition balance $a = \frac{1}{c}$. To overcome this obstacle, we introduce a refined lower solution tailored to this regime, which enables the rigorous construction of a very interesting new class of front-pulse traveling waves.

1.3. Organization. We organize the paper in the following order. In Section 2, we introduce the functional setting and establish the framework of sub- and super-solutions. Using Schauder's fixed point theorem, we prove the existence of traveling wave solutions for $s \geq s^* = \max\{2, 2\sqrt{ad}\}$ via constructing explicit sub- and super-solutions both for $s > s^*$ and the critical case $s = s^*$ and apply the shrinking-box

argument to show convergence to the coexistence equilibrium at $+\infty$. In Section 3, we develop verifiable sufficient conditions for traveling waves: we first present conditions guaranteeing monotone fronts (Theorem 1.2), and then derive explicit, checkable sufficient conditions for non-monotone waves for both cases $s \geq s^*$, thereby proving Theorems 1.3 and 1.4. We also establish a cooperative oscillation property at $+\infty$: oscillation of one component forces oscillation of the other. In Section 4, we address the critical weak competition regime by analyzing two degenerate reductions of the original system. Using the non-monotone wave constructions from Section 3, together with compactness and elliptic estimates, we prove Theorem 1.5: the existence of non-trivial front-pulse traveling waves.

2. THE EXISTENCE OF THE TRAVELING WAVE SOLUTION

2.1. Preliminaries. In this section, we give some preliminaries for our main purpose. First, we define our function space.

$$(2.1) \quad X = \{(u, v) : \mathbb{R} \rightarrow \mathbb{R}^2 \text{ is continuous function, } 0 \leq u(\xi) \leq 1, 0 \leq v(\xi) \leq a\}.$$

Also, according to the source term of (1.3), we define the functions as

$$F_1(u, v) = \beta u + u(1 - u - cv),$$

and

$$F_2(u, v) = \beta v + v(a - bu - v),$$

for some constant $\beta > 0$. Then it is easy to see that for any $(u, v) \in X$, $F(u, v)$ is uniformly Lipschitz in X . We choose $\beta > 0$ large enough so that $\frac{\partial F_1}{\partial u} \geq 0$, $\frac{\partial F_2}{\partial v} \geq 0$ in X . For this $\beta > 0$, we can rewrite equation (1.3) in

$$(2.2) \quad \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} u'' \\ v'' \end{pmatrix} - s \begin{pmatrix} u' \\ v' \end{pmatrix} - \beta \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(u, v) \\ F_2(u, v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $d_1 = 1$, and $d_2 = d$. Now we define $\lambda_{i1} < 0 < \lambda_{i2}$ to be the solution of the quadratic equation

$$d_i r^2 - sr - \beta = 0, \quad i = 1, 2.$$

For given $(u, v) \in X$, we consider the operator $P = (P_1, P_2) : X \rightarrow X$ defined as following

$$(2.3) \quad P_i(u, v)(\xi) = \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \left[\int_{-\infty}^{\xi} e^{\lambda_{i1}(\xi-s)} + \int_{\xi}^{+\infty} e^{\lambda_{i2}(\xi-s)} \right] F_i(u, v)(s) ds,$$

for $i = 1, 2$, $\xi \in \mathbb{R}$. By the variation of constant formula it is easy to see that P satisfies the equation (2.2). Next, we give the definition of super-solution and sub-solution of (1.3) as follows.

Definition 2.1. The continuous functions $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ are called a pair of super and sub solutions of (1.3) if

$$(2.4) \quad \begin{cases} \bar{u}'' - s\bar{u}' + \bar{u}(1 - \bar{u} - c\bar{v}) \leq 0, \\ \underline{u}'' - s\underline{u}' + \underline{u}(1 - \underline{u} - c\underline{v}) \geq 0, \\ d\bar{v}'' - s\bar{v}' + \bar{v}(a - b\underline{u} - \bar{v}) \leq 0, \\ d\underline{v}'' - s\underline{v}' + \underline{v}(a - b\bar{u} - \underline{v}) \geq 0, \end{cases}$$

for all $\xi \in \mathbb{R} \setminus D$ with $D = \{\xi_1, \xi_2, \dots, \xi_N\}$.

Finally, we present the most important theorem of existence of the solution in the next subsection.

2.2. The Existence Theorem. By a standard argument such as that in, e.g., [Ma01], [HL14], we have the following existence theorem for the system (1.3). The key idea is to use the super-solution and sub-solution to construct a weight subspace such that $P = (P_1, P_2)$ is a continuous, compact self-map, and apply Schauder's fixed point theory. We omit its detail here safely.

Lemma 2.1. *Let $s > 0$. Suppose (1.3) has a pair of positive super-solution and sub-solution in X satisfying*

$$(1): \bar{u}(\xi) \geq \underline{u}(\xi), \quad \bar{v}(\xi) \geq \underline{v}(\xi) \text{ for all } \xi \in \mathbb{R}.$$

(2): $\bar{u}'(\xi-) \geq \bar{u}'(\xi+), \quad \bar{v}'(\xi-) \geq \bar{v}'(\xi+), \quad \underline{u}'(\xi-) \leq \underline{u}'(\xi+), \quad \underline{v}'(\xi-) \leq \underline{v}'(\xi+),$
for all $\xi \in D$, where

$$\bar{u}'(\xi\pm) := \lim_{z \rightarrow \xi\pm} \bar{u}'(z), \quad \underline{u}'(\xi\pm) := \lim_{z \rightarrow \xi\pm} \underline{u}'(z),$$

$$\bar{v}'(\xi\pm) := \lim_{z \rightarrow \xi\pm} \bar{v}'(z), \quad \underline{v}'(\xi\pm) := \lim_{z \rightarrow \xi\pm} \underline{v}'(z).$$

Then (1.3) has a positive solution (u, v) such that $\underline{u}(\xi) \leq u(\xi) \leq \bar{u}(\xi)$, $\underline{v}(\xi) \leq v(\xi) \leq \bar{v}(\xi)$ for all $\xi \in \mathbb{R}$.

2.3. The super-solution and sub-solution of $s > s^*$. We shall derive the existence of the traveling wave solution for $s > s^* = \max\{2, 2\sqrt{ad}\}$. Given $s > s^*$, we define the following positive constants.

$$(2.5) \quad \lambda_1 = \frac{s - \sqrt{s^2 - 4}}{2}, \quad \lambda_2 = \frac{s - \sqrt{s^2 - 4ad}}{2d}, \quad \lambda_3 = \frac{s + \sqrt{s^2 - 4}}{2}, \quad \lambda_4 = \frac{s + \sqrt{s^2 - 4ad}}{2d}.$$

In fact, λ_1 and λ_3 are the positive solution of

$$x^2 - sx + 1 = 0.$$

The λ_2 and λ_4 are the positive solution of

$$dx^2 - sx + a = 0.$$

According to Lemma 2.1, we need to construct a pair of super and sub solutions of (1.3).

First, given any $\lambda > 0$, $\mu, q > 1$, it is easy to check that the function

$$(2.6) \quad f(\xi) = e^{\lambda\xi} - qe^{\mu\lambda\xi}$$

has a unique zero $\xi_0 = \frac{-\log q}{(\mu-1)\lambda} < 0$ and a unique maximum point at $\xi_M = \frac{-\log q\mu}{(\mu-1)\lambda} < \xi_0$. Moreover, we have

$$(2.7) \quad \|f\|_\infty = f(\xi_M) = (1 - \frac{1}{\mu})(q\mu)^{\frac{-1}{\mu-1}}.$$

Since f is continuous on $\mathbb{R}$ and positive on $(-\infty, \xi_0)$, for any small $\delta > 0$ there exists $\tilde{\xi} \in (\xi_M, \xi_0)$ such that $f(\tilde{\xi}) = \delta$ with $f'(\tilde{\xi}) < 0$.

Next, we sequentially select the constants $\mu_1, \mu_2, q_1, q_2, \delta_1$ and δ_2 based on the following (A1) – (A3).

(A1) Let $\mu_1 \in (1, \min\{\frac{\lambda_3}{\lambda_1}, \frac{\lambda_1 + \lambda_2}{\lambda_1}, 2\})$, $\mu_2 \in (1, \min\{\frac{\lambda_4}{\lambda_2}, \frac{\lambda_1 + \lambda_2}{\lambda_2}, 2\})$ are very close to 1.

(A2) Let $q_1 > \max\{1, \frac{1+ac}{-(\mu_1\lambda_1)^2 + s(\mu_1\lambda_1) - 1}\}$, $q_2 > \max\{1, \frac{a^2+ab}{-d(\mu_2\lambda_2)^2 + s(\mu_2\lambda_2) - a}\}$.

(A3) Pick $\delta_1 > 0$ such that $0 < \delta_1 < \min\{1 - ac, \|f_1\|_\infty\}$, where $f_1(\xi)$ is defined by (2.6) in parameters (λ_1, μ_1, q_1) . Pick $\delta_2 > 0$ such that $0 < \delta_2 < \min\{a - b, \|f_2\|_\infty\}$, where $f_2(\xi)$ is defined by (2.6) in parameters $(a, \lambda_2, \mu_2, q_2)$. Note that there exists $\xi_i \in (\xi_M^i, \xi_0^i)$ such that $f(\xi_i) = \delta_i, i = 1, 2$.

Finally, we introduce the functions $\bar{u}(\xi), \underline{u}(\xi), \bar{v}(\xi), \underline{v}(\xi)$ as follows:

$$\begin{aligned} \bar{u}(\xi) &= \begin{cases} 1 & \text{if } \xi \geq 0, \\ e^{\lambda_1 \xi} & \text{if } \xi \leq 0, \end{cases} \\ \underline{u}(\xi) &= \begin{cases} \delta_1 & \text{if } \xi \geq \xi_1, \\ e^{\lambda_1 \xi} - q_1 e^{\mu_1 \lambda_1 \xi} & \text{if } \xi \leq \xi_1, \end{cases} \\ \bar{v}(\xi) &= \begin{cases} a & \text{if } \xi \geq 0, \\ a e^{\lambda_2 \xi} & \text{if } \xi \leq 0, \end{cases} \\ \underline{v}(\xi) &= \begin{cases} \delta_2 & \text{if } \xi \geq \xi_2, \\ a e^{\lambda_2 \xi} - q_2 e^{\mu_2 \lambda_2 \xi} & \text{if } \xi \leq \xi_2, \end{cases} \end{aligned}$$

where $\xi_i < 0$ is a point such that $\underline{u}$ and $\underline{v}$ are continuous functions on $\mathbb{R}$. It is easy to see that $(\bar{u}, \underline{u}, \bar{v}, \underline{v})$ meets the assumption (1) and (2) in Lemma 2.1. In fact, we have the following theorem.

Lemma 2.2. *For each $s > s^*$, there exists a positive solution $(u, v)(\xi)$ of (1.3) with $\underline{u}(\xi) \leq u(\xi) \leq \bar{u}(\xi)$ and $\underline{v}(\xi) \leq v(\xi) \leq \bar{v}(\xi)$ for all $\xi \in \mathbb{R}$ such that*

$$\lim_{\xi \rightarrow -\infty} (u, v) = (0, 0).$$

Proof. By Lemma 2.1, it is sufficient to check that $(\bar{u}, \underline{u}, \bar{v}, \underline{v})$ satisfies the definition of sup-sub solutions. It is easy to check that $\bar{u}(\xi), \underline{u}(\xi), \bar{v}(\xi), \underline{v}(\xi)$ satisfy the conditions of Lemma 2.1. Therefore, we only need to check the differential inequalities. Without loss of generality, we check the differential inequalities of $\bar{u}(\xi)$ and $\underline{u}(\xi)$. The proofs of the other two differential inequalities are similar and are omitted for brevity. First, we claim that

$$\bar{u}''(\xi) - s\bar{u}'(\xi) + \bar{u}(\xi)(1 - \bar{u}(\xi) - c\underline{v}(\xi)) \leq 0$$

holds for $\xi \in \mathbb{R} \setminus \{0\}$. For $\xi > 0$, $\bar{u}(\xi) = 1$ and

$$\bar{u}''(\xi) - s\bar{u}'(\xi) + \bar{u}(\xi)(1 - \bar{u}(\xi) - c\underline{v}(\xi)) = -c\underline{v}(\xi) \leq 0.$$

When $\xi < 0$, $\bar{u}(\xi) = e^{\lambda_1 \xi}$ and

$$\begin{aligned} &\bar{u}''(\xi) - s\bar{u}'(\xi) + \bar{u}(\xi)(1 - \bar{u}(\xi) - c\underline{v}(\xi)) \\ &= e^{\lambda_1 \xi}[\lambda_1^2 - s\lambda_1 + 1] - \bar{u}^2(\xi) - c\bar{u}(\xi)\underline{v}(\xi) = -\bar{u}^2(\xi) - c\bar{u}(\xi)\underline{v}(\xi) \leq 0. \end{aligned}$$

Next, we claim that

$$\underline{u}''(\xi) - s\underline{u}'(\xi) + \underline{u}(\xi)(1 - \underline{u}(\xi) - c\bar{v}(\xi)) \geq 0$$

holds for $\xi \in \mathbb{R} \setminus \{\xi_1\}$. In the case $\xi > \xi_1$, we have $\underline{u}(\xi) = \delta_1$ and

$$\underline{u}''(\xi) - s\underline{u}'(\xi) + \underline{u}(\xi)(1 - \underline{u}(\xi) - c\bar{v}(\xi)) \geq \delta_1(1 - \delta_1 - ac) \geq 0.$$

For $\xi < \xi_1$, we have $\underline{u}(\xi) = e^{\lambda_1 \xi} - q_1 e^{\mu_1 \lambda_1 \xi}$ and

$$\begin{aligned} & \underline{u}''(\xi) - s\underline{u}'(\xi) + \underline{u}(\xi)(1 - \underline{u}(\xi) - c\bar{v}(\xi)) \\ &= -q_1[(\mu_1 \lambda_1)^2 - s(\mu_1 \lambda_1) + 1]e^{\mu_1 \lambda_1 \xi} - (e^{\lambda_1 \xi} - q_1 e^{\mu_1 \lambda_1 \xi})^2 - ac(e^{\lambda_1 \xi} - q_1 e^{\mu_1 \lambda_1 \xi})e^{\lambda_2 \xi} \\ &\geq -q_1[(\mu_1 \lambda_1)^2 - s(\mu_1 \lambda_1) + 1]e^{\mu_1 \lambda_1 \xi} - e^{2\lambda_1 \xi} - ace^{(\lambda_1 + \lambda_2)\xi} \\ &\geq e^{\mu_1 \lambda_1 \xi}[-q_1[(\mu_1 \lambda_1)^2 - s(\mu_1 \lambda_1) + 1] - e^{((2-\mu_1)\lambda_1)\xi} - ace^{((\lambda_1 + \lambda_2) - \mu_1 \lambda_1)\xi}] \\ &\geq e^{\mu_1 \lambda_1 \xi}[-q_1[(\mu_1 \lambda_1)^2 - s(\mu_1 \lambda_1) + 1] - 1 - ac] > 0, \end{aligned}$$

where we use the definition of λ_1, μ_1 and q_1 in (A1) – (A3).

Finally, the limit

$$\lim_{\xi \rightarrow -\infty} (u, v) = (0, 0),$$

can be proved by Squeeze Theorem. Therefore, the proof of this theorem has been complete. $\square$

2.4. The super-solution and sub-solution of $s = s^*$. We shall derive the existence of the traveling wave solution for $s = s^* = \max\{2, 2\sqrt{ad}\}$. Since the system is symmetry we only need to consider the case $ad \leq 1$. Another case, with a similar construction, will be omitted here. Consider $ad \leq 1$, let $s = s^*$, define the following positive constants

$$(2.8) \quad \hat{\lambda}_1 = \hat{\lambda}_3 = \frac{s}{2}, \quad \hat{\lambda}_2 = \frac{s - \sqrt{s^2 - 4ad}}{2d}, \quad \hat{\lambda}_4 = \frac{s + \sqrt{s^2 - 4ad}}{2d}.$$

First, given any $\lambda > 0, h > 0, q > 1$, it is easy to check that the non-negative function

$$(2.9) \quad g(\xi) = (-h\xi - q\sqrt{-\xi})e^{\lambda\xi}, \quad \xi \leq -\left(\frac{q}{h}\right)^2$$

has a unique zero $\hat{\xi}_0 = -(\frac{q}{h})^2$ and a unique maximum point at $\hat{\xi}_M < \hat{\xi}_0$. Since g is continuous and positive on $(-\infty, \hat{\xi}_0)$, for any small $\hat{\delta} > 0$ there exists $\hat{\xi} \in (\hat{\xi}_M, \hat{\xi}_0)$ such that $g(\hat{\xi}) = \hat{\delta}$ with $g'(\hat{\xi}) < 0$. Note that the unique zero $\hat{\xi}_0 \rightarrow -\infty$ when $q \rightarrow +\infty$.

We will divide the discussion into two cases. Given $s = s^*$ with $ad = 1$. We sequentially select the constants $\hat{h}_1, \hat{h}_2, \hat{q}_1, \hat{q}_2, \hat{\delta}_1$ and $\hat{\delta}_2$ based on the following (B1) – (B4).

$$(B1) \text{ Let } \hat{h}_1 = \frac{\hat{\lambda}_1}{\hat{\lambda}_1 + 1} e^{\hat{\lambda}_1 + 1}, \hat{h}_2 = \frac{a\hat{\lambda}_2}{\hat{\lambda}_2 + 1} e^{\hat{\lambda}_2 + 1}.$$

$$(B2) \text{ Let } \hat{q}_1 > \max \left\{ \sqrt{\hat{h}_1 \left(\frac{1}{\lambda_1} + 1 \right)}, 4 \left(c\hat{h}_1\hat{h}_2 \left(\frac{7}{2e\hat{\lambda}_2} \right)^{\frac{7}{2}} + \hat{h}_1^2 \left(\frac{7}{2e\hat{\lambda}_1} \right)^{\frac{7}{2}} \right) \right\}.$$

$$(B3) \text{ Let } \hat{q}_2 > \max \left\{ \sqrt{\hat{h}_2 \left(\frac{1}{\lambda_2} + 1 \right)}, \frac{4}{d} \left(b\hat{h}_1\hat{h}_2 \left(\frac{7}{2e\hat{\lambda}_1} \right)^{\frac{7}{2}} + \hat{h}_2^2 \left(\frac{7}{2e\hat{\lambda}_2} \right)^{\frac{7}{2}} \right) \right\}.$$

(B4) Pick $\hat{\delta}_1 > 0$ so small such that $0 < \hat{\delta}_1 < \min\{1 - ac, \|g_1\|_\infty\}$, where $g_1(\xi)$ is defined by (2.9). Pick $\hat{\delta}_2 > 0$ so small such that $0 < \hat{\delta}_2 < \min\{a - b, \|g_2\|_\infty\}$, where $g_2(\xi)$ is defined by (2.9). Note that there exists $\hat{\xi}_i \in (\hat{\xi}_M^i, \hat{\xi}_0^i)$ such that $g_i(\hat{\xi}_i) = \hat{\delta}_i, i = 1, 2$.

In this case, we introduce the following functions $\bar{u}(\xi), \underline{u}(\xi), \bar{v}(\xi), \underline{v}(\xi)$ as follows.

$$\begin{aligned} \bar{u}(\xi) &= \begin{cases} 1 & \text{if } \xi \geq \frac{-1}{\hat{\lambda}_1} - 1, \\ -\hat{h}_1 \xi e^{\hat{\lambda}_1 \xi} & \text{if } \xi \leq \frac{-1}{\hat{\lambda}_1} - 1, \end{cases} \\ \underline{u}(\xi) &= \begin{cases} \hat{\delta}_1 & \text{if } \xi \geq \hat{\xi}_1, \\ [-\hat{h}_1 \xi - \hat{q}_1 \sqrt{-\xi}] e^{\hat{\lambda}_1 \xi} & \text{if } \xi \leq \hat{\xi}_1, \end{cases} \\ \bar{v}(\xi) &= \begin{cases} a & \text{if } \xi \geq \frac{-1}{\hat{\lambda}_2} - 1, \\ -\hat{h}_2 \xi e^{\hat{\lambda}_2 \xi} & \text{if } \xi \leq \frac{-1}{\hat{\lambda}_2} - 1, \end{cases} \\ \underline{v}(\xi) &= \begin{cases} \hat{\delta}_2 & \text{if } \xi \geq \hat{\xi}_2, \\ [-\hat{h}_2 \xi - \hat{q}_2 \sqrt{-\xi}] e^{\hat{\lambda}_2 \xi} & \text{if } \xi \leq \hat{\xi}_2, \end{cases} \end{aligned}$$

where $\hat{\xi}_i < -(\frac{\hat{q}_i}{\hat{h}_i})^2 < \frac{-1}{\hat{\lambda}_i} - 1 < 0$ is a point such that $\underline{u}$ and $\underline{v}$ are continuous functions on $\mathbb{R}$ and $\hat{h}_1 = \frac{\lambda_1}{\lambda_1 + 1} e^{\lambda_1 + 1}$, $\hat{h}_2 = \frac{a\lambda_2}{\lambda_2 + 1} e^{\lambda_2 + 1}$. It is easy to see that $(\bar{u}, \underline{u}, \bar{v}, \underline{v})$ meets the assumption (1) and (2) in Lemma 2.1.

On the other hand, when $s = s^*$ with $ad < 1$. We select the constants $\hat{h}_1, \hat{q}_1, \hat{\delta}_1, \hat{\xi}_1$ in a similar way to the above and choose $\hat{Q}_2, \hat{\delta}_2, \hat{\mu}_2$ and $\hat{\xi}_2$ according to the following (B5) – (B7).

$$(B5) \text{ Let } \hat{\mu}_2 \in (1, \min\{\frac{\lambda_4}{\lambda_2}, 1 + \frac{\hat{\lambda}_1}{2\lambda_2}, 2\}), \text{ is very close to } 1.$$

$$(B6) \text{ Let } \hat{Q}_2 > \max\{1, \frac{a^2 + \frac{2ab\hat{h}_1 e^{-1}}{\hat{\lambda}_1}}{-d(\hat{\mu}_2\lambda_2)^2 + s(\hat{\mu}_2\lambda_2) - a}\}.$$

$$(B7) \text{ Pick } \hat{\delta}_2 > 0 \text{ such that } 0 < \hat{\delta}_2 < \min\{a - b, \|f_2\|_\infty\},$$

where λ_4, λ_2 are defined by (2.5) and $f_2(\xi)$ is defined by (2.6) in parameters $(a, \lambda_2, \hat{\mu}_2, \hat{Q}_2)$.

And choose $\hat{\xi}_2$ such that $\underline{v}(\xi)$ be $C^0(\mathbb{R})$.

In this case, we consider the functions $\bar{u}(\xi), \underline{u}(\xi), \bar{v}(\xi), \underline{v}(\xi)$ as follows.

$$\begin{aligned}
\bar{u}(\xi) &= \begin{cases} 1 & \text{if } \xi \geq \frac{-1}{\lambda_1} - 1, \\ -\hat{h}_1 \xi e^{\hat{\lambda}_1 \xi} & \text{if } \xi \leq \frac{-1}{\lambda_1} - 1, \end{cases} \\
\underline{u}(\xi) &= \begin{cases} \hat{\delta}_1 & \text{if } \xi \geq \hat{\xi}_1, \\ [-\hat{h}_1 \xi - \hat{q}_1 \sqrt{-\xi}] e^{\hat{\lambda}_1 \xi} & \text{if } \xi \leq \hat{\xi}_1, \end{cases} \\
\bar{v}(\xi) &= \begin{cases} a & \text{if } \xi \geq 0, \\ a e^{\lambda_2 \xi} & \text{if } \xi \leq 0, \end{cases} \\
\underline{v}(\xi) &= \begin{cases} \hat{\delta}_2 & \text{if } \xi \geq \hat{\xi}_2, \\ a e^{\lambda_2 \xi} - \hat{Q}_2 e^{\hat{\mu}_2 \lambda_2 \xi} & \text{if } \xi \leq \hat{\xi}_2, \end{cases}
\end{aligned}$$

where λ_2 is defined in (2.5). It is easy to see that $(\bar{u}, \underline{u}, \bar{v}, \underline{v})$ satisfying the assumption (1) and (2) in Lemma 2.1. Combining these two cases, we obtain the following theorem.

Lemma 2.3. *For $s = s^*$, there exists a positive solution $(u, v)(\xi)$ of (1.3) with $\underline{u}(\xi) \leq u(\xi) \leq \bar{u}(\xi)$ and $\underline{v}(\xi) \leq v(\xi) \leq \bar{v}(\xi)$ for all $\xi \in \mathbb{R}$ such that*

$$\lim_{\xi \rightarrow -\infty} (u, v) = (0, 0).$$

Proof. Consider the case $ad = 1$. By Lemma 2.1, it is sufficient to check that $(\bar{u}, \underline{u}, \bar{v}, \underline{v})$ satisfy the definition of sup-sub solutions. Moreover, we only need to check the differential inequalities. It is easy to see that $0 < \bar{u}(\xi) < \underline{u}(\xi) \leq 1$, $0 < \bar{v}(\xi) < \underline{v}(\xi) \leq a$ for all $\xi \in \mathbb{R}$. Without loss of generality, we check the differential inequalities of $\bar{u}(\xi)$ and $\underline{u}(\xi)$. The other two differential inequalities can be proved in a similar way. First, we claim that

$$\bar{u}''(\xi) - s\bar{u}'(\xi) + \bar{u}(\xi)(1 - \bar{u}(\xi) - c\underline{v}(\xi)) \leq 0$$

holds for $\xi \in \mathbb{R} \setminus \{\frac{-1}{\lambda_1} - 1\}$. For $\xi > \frac{-1}{\lambda_1} - 1$, $\bar{u}(\xi) = 1$ and

$$\bar{u}''(\xi) - s\bar{u}'(\xi) + \bar{u}(\xi)(1 - \bar{u}(\xi) - c\underline{v}(\xi)) = -c\underline{v}(\xi) \leq 0.$$

When $\xi < \frac{-1}{\lambda_1} - 1$, $\bar{u}(\xi) = -\hat{h}_1 \xi e^{\hat{\lambda}_1 \xi}$ and

$$\bar{u}''(\xi) - s\bar{u}'(\xi) + \bar{u}(\xi)(1 - \bar{u}(\xi) - c\underline{v}(\xi)) = -c\bar{u}(\xi)\underline{v}(\xi) \leq 0.$$

Next, we prove that

$$\underline{u}''(\xi) - s\underline{u}'(\xi) + \underline{u}(\xi)(1 - \underline{u}(\xi) - c\bar{v}(\xi)) \geq 0$$

holds for $\xi \in \mathbb{R} \setminus \{\hat{\xi}_1\}$. For $\xi > \hat{\xi}_1$, the inequality holds by a similar argument as in the case $s > s^*$. When $\xi < \hat{\xi}_1$, we have $\underline{u}(\xi) = [-\hat{h}_1\xi - \hat{q}_1\sqrt{-\xi}]e^{\hat{\lambda}_1\xi}$ and

$$\begin{aligned}
& \underline{u}''(\xi) - s\underline{u}'(\xi) + \underline{u}(\xi)(1 - \underline{u}(\xi) - c\bar{v}(\xi)) \\
&= -s\left(\frac{\hat{q}_1}{2}(-\xi)^{-\frac{1}{2}}\right)e^{\hat{\lambda}_1\xi} + \frac{\hat{q}_1}{4}(-\xi)^{-\frac{3}{2}}e^{\hat{\lambda}_1\xi} + \hat{\lambda}_1\hat{q}_1(-\xi)^{-\frac{1}{2}}e^{\hat{\lambda}_1\xi} \\
&\quad - c\bar{v}(\xi)(-\hat{h}_1\xi - \hat{q}_1(-\xi)^{\frac{1}{2}})e^{\hat{\lambda}_1\xi} - (-\hat{h}_1\xi - \hat{q}_1(-\xi)^{\frac{1}{2}})^2e^{2\hat{\lambda}_1\xi} \\
&= e^{\hat{\lambda}_1\xi} \left[-\frac{s\hat{q}_1}{2}(-\xi)^{-\frac{1}{2}} + \frac{\hat{q}_1}{4}(-\xi)^{-\frac{3}{2}} + \hat{\lambda}_1\hat{q}_1(-\xi)^{-\frac{1}{2}} - c\bar{v}(\xi)(-\hat{h}_1\xi - \hat{q}_1(-\xi)^{\frac{1}{2}}) \right] \\
&\quad - (-\hat{h}_1\xi - \hat{q}_1(-\xi)^{\frac{1}{2}})^2e^{2\hat{\lambda}_1\xi} \\
&\geq e^{\hat{\lambda}_1\xi} \left[\frac{\hat{q}_1}{4}(-\xi)^{-\frac{3}{2}} - c\hat{h}_1\hat{h}_2|\xi|^2e^{\hat{\lambda}_2\xi} - \hat{h}_1^2|\xi|^2e^{\hat{\lambda}_1\xi} \right] \\
&\geq (-\xi)^{-\frac{3}{2}}e^{\hat{\lambda}_1\xi} \left[\frac{\hat{q}_1}{4} - c\hat{h}_1\hat{h}_2\left(\frac{7}{2e\hat{\lambda}_2}\right)^{\frac{7}{2}} - \hat{h}_1^2\left(\frac{7}{2e\hat{\lambda}_1}\right)^{\frac{7}{2}} \right] \geq 0,
\end{aligned}$$

where we use the definition of $\hat{\lambda}_1$, and the fact that given $\lambda > 0$,

$$(-\xi)^{\frac{7}{2}}e^{\lambda\xi} \leq \left(\frac{7}{2e\lambda}\right)^{\frac{7}{2}} \text{ for } \xi \leq 0.$$

The proof of case $ad < 1$ is very similar to the proof of case $s > s^*$, we only check

$$\underline{v}''(\xi) - s\underline{v}'(\xi) + \underline{v}(\xi)(1 - \underline{v}(\xi) - c\bar{v}(\xi)) \geq 0$$

here. For $\xi > \hat{\xi}_2$, the inequality holds by a similar argument as in the case $s > s^*$. When $\xi < \hat{\xi}_2$, we have

$$\begin{aligned}
& d\underline{v}''(\xi) - s\underline{v}'(\xi) + \underline{v}(\xi)(a - b\bar{u}(\xi) - \underline{v}(\xi)) \\
&= -\hat{Q}_2[d(\hat{\mu}_2\lambda_2)^2 - s(\hat{\mu}_2\lambda_2) + a]e^{\hat{\mu}_2\lambda_2\xi} - b(-\hat{h}_1\xi e^{\hat{\lambda}_1\xi})(ae^{\lambda_2\xi} - \hat{Q}_2e^{\hat{\mu}_2\lambda_2\xi}) - (ae^{\lambda_2\xi} - \hat{Q}_2e^{\hat{\mu}_2\lambda_2\xi})^2 \\
&\geq -\hat{Q}_2[d(\hat{\mu}_2\lambda_2)^2 - s(\hat{\mu}_2\lambda_2) + a]e^{\hat{\mu}_2\lambda_2\xi} - b(-\hat{h}_1\xi e^{\hat{\lambda}_1\xi})(ae^{\lambda_2\xi}) - a^2e^{2\lambda_2\xi} \\
&\geq e^{\hat{\mu}_2\lambda_2\xi} \left[-\hat{Q}_2[d(\hat{\mu}_2\lambda_2)^2 - s(\hat{\mu}_2\lambda_2) + 1] + ab\hat{h}_1(\xi e^{\frac{\hat{\lambda}_1}{2}\xi})e^{(\frac{\hat{\lambda}_1}{2} + (1-\hat{\mu}_2)\lambda_2)\xi} - a^2e^{(2-\hat{\mu}_2)\lambda_2\xi} \right] \\
&\geq e^{\hat{\mu}_2\lambda_2\xi} \left[-\hat{Q}_2[d(\hat{\mu}_2\lambda_2)^2 - s(\hat{\mu}_2\lambda_2) + 1] + ab\hat{h}_1\left(\frac{-2e^{-1}}{\hat{\lambda}_1}\right) - a^2 \right] \geq 0.
\end{aligned}$$

Here we use the fact that

$$\xi e^{\frac{\alpha}{2}\xi} \geq \frac{-2e^{-1}}{\alpha} \text{ for all } \xi \leq 0,$$

where $\alpha > 0$.

Finally, the limit

$$\lim_{\xi \rightarrow -\infty} (u, v) = (0, 0),$$

can be proved by the Squeeze Theorem. Therefore, the proof of this theorem is complete. $\square$

2.5. The shrinking box argument. To analyze the asymptotic behavior of the positive solution obtained in Lemma 2.2 and Lemma 2.3 at $\xi \rightarrow +\infty$. We introduce the shrinking-box argument. This method can be refer to, for instance [LLR06], [GG22], [HL14], and [CHW25]. We define the functions $m_u(\theta)$, $m_v(\theta)$ and $M_u(\theta)$, $M_v(\theta)$ for $\theta \in [0, 1]$ as follows:

$$(2.10) \quad m_u(\theta) = \theta u^*, \quad M_u(\theta) = \theta u^* + (1 - \theta)(1 + \varepsilon),$$

$$(2.11) \quad m_v(\theta) = \theta v^*, \quad M_v(\theta) = \theta v^* + (1 - \theta)(a + \varepsilon),$$

where ε is small enough such that $0 < \varepsilon < \min\{\frac{1-ac}{c}, \frac{a-b}{b}\}$. For $0 < \theta_1 < \theta_2 < 1$, it is easy to see that

$$\begin{aligned} 0 = m_u(0) < m_u(\theta_1) < m_u(\theta_2) < m_u(1) = u^* = M_u(1) < M_u(\theta_2) < M_u(\theta_1) < M_u(0) = 1 + \varepsilon, \\ 0 = m_v(0) < m_v(\theta_1) < m_v(\theta_2) < m_v(1) = v^* = M_v(1) < M_v(\theta_2) < M_v(\theta_1) < M_v(0) = a + \varepsilon. \end{aligned}$$

We are ready to show the tail behavior of the traveling wave solution at $+\infty$ as follows.

Lemma 2.4. *Let $(u, v)(\xi)$ be a positive solution obtained in Lemma 2.2 or Lemma 2.3. Then*

$$(2.12) \quad \lim_{\xi \rightarrow +\infty} (u, v)(\xi) = (u^*, v^*).$$

Proof. By the fact that $\delta_1 \leq \underline{u}(\xi) \leq u(\xi) \leq \bar{u}(\xi) \leq 1$ and $\delta_2 \leq \underline{v}(\xi) \leq v(\xi) \leq \bar{v}(\xi) \leq a$ for all $\xi > 0$, we obtain that

$$\begin{aligned} \limsup_{\xi \rightarrow +\infty} u(\xi) &\leq 1, \quad \limsup_{\xi \rightarrow +\infty} v(\xi) \leq a \\ \liminf_{\xi \rightarrow +\infty} u(\xi) &\geq \delta_1, \quad \liminf_{\xi \rightarrow +\infty} v(\xi) \geq \delta_2. \end{aligned}$$

Now we denote

$$\begin{aligned} u^- &= \liminf_{\xi \rightarrow +\infty} u(\xi), \quad u^+ = \limsup_{\xi \rightarrow +\infty} u(\xi) \\ v^- &= \liminf_{\xi \rightarrow +\infty} v(\xi), \quad v^+ = \limsup_{\xi \rightarrow +\infty} v(\xi). \end{aligned}$$

It is easy to see that

$$\begin{aligned} m_u(0) = 0 < u^- \leq u^+ < 1 + \varepsilon = M_u(0) \\ m_v(0) = 0 < v^- \leq v^+ < a + \varepsilon = M_v(0). \end{aligned}$$

Note that (2.12) holds if we can show that

$$(2.13) \quad m_u(\theta) < u^- \leq u^+ < M_u(\theta) \quad \text{and} \quad m_v(\theta) < v^- \leq v^+ < M_v(\theta)$$

for all $\theta \in [0, 1)$. Set $\theta_0 := \sup\{\theta \in [0, 1) : (2.13) \text{ holds}\}$. Then θ_0 is well-defined and it suffices to claim that $\theta_0 = 1$. Suppose $\theta_0 < 1$. Then passing to the limit, we have

$$\begin{aligned} m_u(\theta_0) &\leq u^- \leq u^+ \leq M_u(\theta_0), \\ m_v(\theta_0) &\leq v^- \leq v^+ \leq M_v(\theta_0). \end{aligned}$$

Moreover, by the definition of θ_0 , at least one of the following conditions holds:

$$u^- = m_u(\theta_0), u^+ = M_u(\theta_0), v^- = m_v(\theta_0), v^+ = M_v(\theta_0).$$

We only prove the first case; the proofs for the others are similar. Assume $u^- = m_u(\theta_0)$. We know that $v^+ \leq M_v(\theta_0)$. If $u(\xi)$ is eventually monotone, then $u(+\infty)$ exist by $u(\xi)$ is bounded on $\mathbb{R}$. By the property of $\liminf$, we have

$$(2.14) \quad \begin{aligned} \liminf_{\xi \rightarrow +\infty} [1 - u(\xi) - cv(\xi)] &\geq [1 - \theta_0 u^* - \theta_0 cv^* - (1 - \theta_0)c(a + \varepsilon)] \\ &= (1 - \theta_0)(1 - ca - c\varepsilon) > 0. \end{aligned}$$

On the other hand, we have $\int_0^\infty u'(s)ds = u(+\infty) - u(0)$ is finite. Then we choose $\{\xi_n\}$, $\xi_n \rightarrow +\infty$ such that $\lim_{n \rightarrow +\infty} u(\xi_n) = m_u(\theta_0)$ and $\lim_{n \rightarrow +\infty} u'(\xi_n) = 0$. Integrating the first equation of the system (1.3) from 0 to ξ_n , we obtain that

$$(2.15) \quad u'(\xi_n) - u'(0) - s[u(\xi_n) - u(0)] = - \int_0^{\xi_n} u(s)[1 - u(s) - cv(s)]ds.$$

Letting $n \rightarrow +\infty$, we get a contradiction, since the left-hand side of (2.15) remains bounded and the right-hand side of (2.15) tends to $-\infty$.

If $u(\xi)$ is oscillatory at $+\infty$, let $\xi_n \rightarrow +\infty$ be all the local minimum points of $u(\xi)$. It is easy to prove that $\liminf_{\xi \rightarrow \infty} u(\xi) = \liminf_{n \rightarrow \infty} u(\xi_n) = m_u(\theta_0)$. By taking the subsequence, we can choose $\{\xi_{n_k}\}$ such that $\lim_{k \rightarrow +\infty} u(\xi_{n_k}) = m_u(\theta_0)$. Note that $u''(\xi_{n_k}) - su'(\xi_{n_k}) \geq 0$ for all k . By (2.14), we obtain that

$$\liminf_{k \rightarrow +\infty} [u''(\xi_{n_k}) - su'(\xi_{n_k}) + u(\xi_{n_k})(1 - u(\xi_{n_k}) - cv(\xi_{n_k}))] > 0,$$

a contradiction. Hence $u^- = m_u(\theta_0)$ is impossible. Using similar argument as above we can still arrive other cases are impossible. Consequently, we must have $\theta_0 = 1$ and (2.12) follows. $\square$

The proof of Theorem 1.1 is complete.

2.6. Determination of the minimum speed. In this subsection, we would like to show that there is no positive solution of (1.3) that satisfies (1.5) for $s < s^*$. We have the following theorem.

Lemma 2.5. *Under the strict or critical competition cases, for $s < s^*$ there exists no positive solution to (1.3) with the boundary condition*

$$\lim_{\xi \rightarrow -\infty} (u, v)(\xi) = (0, 0).$$

Proof. Without loss of generality, we assume $s^* = 2$. We first claim that $s \leq 0$ does not have a positive solution. Assume for some $\bar{s} \leq 0$ there exists a positive solution of (1.3) and (1.5). Then it follows from (1.5) that $\Psi(\xi) := 1 - u(\xi) - cv(\xi) \rightarrow 1$ as $\xi \rightarrow -\infty$. Hence there is a large $K > 0$ such that

$$\Psi(\xi) \geq \frac{1}{2}, \quad \text{for all } \xi \leq -K.$$

If $u(\xi)$ is not oscillate near $-\infty$, then $u'(-\infty) = 0$. An integration of the u -equation in (1.3) from $-\infty$ to $\xi \leq -K$ gives

$$0 = u'(\xi) - \bar{s}u(\xi) + \int_{-\infty}^{\xi} u(\eta)(1 - u(\eta) - cv(\eta))d\eta \geq u'(\xi) + \frac{1}{2} \int_{-\infty}^{\xi} u(\eta)d\eta.$$

This implies, by an integration from $-\infty$ to $-K$ and using $u(-\infty) = 0$, that

$$u(-K) = \int_{-\infty}^{-K} u'(\eta) d\eta \leq \frac{-1}{2} \int_{-\infty}^{-K} \int_{-\infty}^{\xi} u(\eta) d\eta d\xi < 0,$$

a contradiction to the positivity of u in $\mathbb{R}$. If $u(\xi)$ oscillate near $-\infty$, pick $\xi_n \rightarrow -\infty$ be the local minimum points of $u(\xi)$ with $u'(\xi_n) = 0$ for all n . Apply the similar argument as above, and we also have $u(-K) < 0$, a contradiction. Hence, we must have $s > 0$.

Next, we can use the standard Sturm-Liouville argument to claim that for any $s \in (0, 2)$ the equation (1.3) and (1.5) has no positive solution. We left the proof in the Appendix A. Therefore, the proof of this theorem is complete. $\square$

For the monotone solution, we quote the interesting results obtained by Tang and Fife [TF80] and Ma [Ma01]. For the non-monotone solution, we describe our construction with details.

3. THE PROFILE OF THE SOLUTION

3.1. The monotone solution. First, we give the definition of the monotone solution.

Definition 3.1. The solution pair (u, v) of (1.3) and (1.5) is called the monotone solution if both $u(\xi)$ and $v(\xi)$ are monotonic functions. Otherwise, we call (u, v) is the non-monotone solution.

Let us first prove Theorem 1.2.

Proof. For the sake of convenience, we have the following equation after the scaling $u(\xi) \rightarrow u(\frac{\xi}{s})$ and $v(\xi) \rightarrow v(\frac{\xi}{s})$:

$$\begin{cases} s^{-2}u'' - u' + u(1 - u - cv) = 0, \\ s^{-2}dv'' - v' + v(a - bu - v) = 0. \end{cases}$$

Assume, for the sake of contradiction and translation invariance, that $u'(0) = 0$. Since u satisfies

$$(3.1) \quad s^{-2}u'' - u' + u(1 - u - cv) = 0$$

we see that

$$s^{-2}(u'e^{-s^2\xi})' = -e^{-s^2\xi}u(1 - u - cv) \leq -e^{-s^2\xi}u(1 - u^* - cv^*) = 0.$$

Hence, $u'(\xi)e^{-s^2\xi}$ is non-increasing, which together with $u'(0) = 0$ implies that

$$u'(\xi) \leq 0, \quad \forall \xi \geq 0, \quad u'(\xi) \geq 0, \quad \forall \xi \leq 0.$$

We claim that $u(\xi) = u(0)$ for all $\xi \geq 0$. As u is bounded, $u(+\infty) \leq u(0)$ exists and $u'(+\infty) = 0$. If $u(+\infty) < u(0)$ which implies there exists $\xi_1 > 0$ such that

$$u''(\xi_1) = 0, \quad u'(\xi_1) < 0, \quad \text{and} \quad 0 < u(\xi_1) < u(0).$$

Choose $\xi = \xi_1$ into (3.1), we obtain the contradiction

$$0 = -u'(\xi_1) + u(\xi_1)(1 - u(\xi_1) - cv(\xi_1)) \geq -u'(\xi_1) + u(\xi_1)(1 - u^* - cv^*) = -u'(\xi_1) > 0.$$

Hence $u(+\infty) = u(0)$, that is $u(\xi) = u(0)$ for all $\xi \geq 0$. If $v(\xi)$ is a monotone function, then $u(0)$ and $v(+\infty)$ are the constant equilibrium of (1.3) which leads to a contradiction. Otherwise, we can use the same techniques to show that $v(\xi) = v(\xi_v)$

for all $\xi \geq \xi_v$ and $v'(\xi) \geq 0$ for all $\xi \leq \xi_v$. But the only positive constant equilibrium of (1.3) is (u^*, v^*) which is also lead to a contradiction. Therefore, $u(\xi)$ and $v(\xi)$ are monotone functions. $\square$

For the existence of the monotonic solution, we have the following proposition. A complete discussion of the proposition, its proof is given by Ma [Ma01, Theorem 2.1]).

Proposition 3.1. *If (1.3) has a non-constant super-solution $(\bar{u}(\xi), \bar{v}(\xi))$ and sub-solution $(\underline{u}(\xi), \underline{v}(\xi))$ satisfy*

- (1) $0 \leq \underline{u}(\xi) \leq \bar{u}(\xi) \leq u^*$ and $0 \leq \underline{v}(\xi) \leq \bar{v}(\xi) \leq v^*$ for all $\xi \in \mathbb{R}$;
- (2) $\sup_{t \leq \xi} \underline{u}(t) \leq \bar{u}(\xi)$, $\sup_{t \leq \xi} \underline{v}(t) \leq \bar{v}(\xi)$ for all $\xi \in \mathbb{R}$;
- (3) *There is no constant equilibrium in the product set*

$$[(0, \inf_{\xi \in \mathbb{R}} \bar{u}(\xi)) \cup [\sup_{\xi \in \mathbb{R}} \underline{u}(\xi), u^*)] \times [(0, \inf_{\xi \in \mathbb{R}} \bar{v}(\xi)) \cup [\sup_{\xi \in \mathbb{R}} \underline{v}(\xi), v^*)].$$

Then (1.3) and (1.5) have a monotone solution. That is, (1.3) has a traveling wavefront solution.

3.2. The non-monotone solution. We are interested to see whether there exists a non-monotonic solution. In fact, the increase or decrease of the solution is very important. Knowing whether the solution is increasing or decreasing can lead to a better understanding of certain phenomena. According to Proposition 1.2, we must find solutions whose range is not contain in $(0, u^*)$ or $(0, v^*)$. We give a special example to show that there exists a non-monotone traveling wave solution of (1.3) and satisfies (1.5). We note that the existence theorem in Theorem 1.1 has yet to express the monotonicity of the solution. Let us prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. For any $s > s^*$ be fixed, we can pick μ_1 close to 1 such that

$$q_1 = \frac{2}{-(\mu_1 \lambda_1)^2 + s(\mu_1 \lambda_1) - 1} > \max\{1, \frac{1+ac}{-(\mu_1 \lambda_1)^2 + s(\mu_1 \lambda_1) - 1}\}.$$

Then by (2.7) we obtain that the global maximum of $\underline{u}(\xi)$ is

$$\|\underline{u}\|_\infty = (1 - \frac{1}{\mu_1})(q_1 \mu_1)^{\frac{-1}{\mu_1 - 1}}$$

which is a constant depending on $s > s^*$. Then let c very close to $\frac{1}{a}$ such that

$$u^* := \frac{1-ac}{1-bc} < (1 - \frac{1}{\mu_1})(q_1 \mu_1)^{\frac{-1}{\mu_1 - 1}} = \|\underline{u}\|_\infty.$$

By the asymptotic behavior of (u, v) , Theorem 2.4, we can see that the profile of $u(\xi)$ is a non-monotone function.

If $s = s^*$, the maximum of $g(\xi)$,

$$g(\xi) = (-h\xi - q\sqrt{-\xi})e^{\lambda\xi}, \quad \xi \leq -(\frac{q}{h})^2$$

defined in (2.9), depends only on λ, h and q . Therefore, for fixed $s = s^*$, applying the same technique, we also have the same conclusion. $\square$

The proof of Theorem 1.4 follows by a similar argument.

The following is an example demonstrating a traveling wave solution that is not monotonically increasing.

Example 3.1. Set $d = 1, a = 1, c = \frac{1}{2}$, and given any $s > 2$. Then set $b_n = 1 - \frac{1}{n}$ for $n > 2(k+1)(q_2(k)^{\frac{k+1}{k}})^k - 1$ where

$$k > \max\{1, (\min\{\frac{\lambda_4}{\lambda_2}, \frac{\lambda_1 + \lambda_2}{\lambda_2}, 2\} - 1)^{-1}\}$$

and

$$q_2(k) = \frac{2}{-((1 + \frac{1}{k})\lambda_2)^2 + s((1 + \frac{1}{k})\lambda_2) - 1}.$$

If (u, v) is a solution in Lemma 2.1 with such s , then the profile of $v(\xi)$ is a non-monotone wave.

Proof. Setting $a = 1, b_n = 1 - \frac{1}{n}, c = \frac{1}{2}, d = 1$ so that the parameter satisfies the weak competition condition (1.4) for some very large $n \in \mathbb{R}^+$. The equilibrium (u^*, v^*) is equal to

$$(u^*, v^*) = (\frac{n}{n+1}, \frac{2}{n+1}).$$

For $s > s^* = 2$, the values λ_1, λ_2 are defined as

$$\lambda_1 = \lambda_2 = \frac{s - \sqrt{s^2 - 4}}{2} > 0.$$

Set $\mu_2 = 1 + \frac{1}{k} < \min\{\frac{\lambda_4}{\lambda_2}, \frac{\lambda_1 + \lambda_2}{\lambda_2}, 2\}$ for some $k > 1$ and $q_2 = q_2(k) = \frac{2}{-((1 + \frac{1}{k})\lambda_2)^2 + s((1 + \frac{1}{k})\lambda_2) - 1}$. Then we obtain that

$$\|\underline{v}\|_\infty = (\frac{1}{k+1})(q_2(k)^{\frac{k+1}{k}})^{-k}.$$

Therefore, if

$$v^* = \frac{2}{n+1} < (\frac{1}{k+1})(q_2(k)^{\frac{k+1}{k}})^{-k}$$

then the profile of $v(\xi)$ is a non-monotone wave. □

Remark 3.1. Let $s = 4.5$. Then for $\lambda_1 = \lambda_2 = \frac{s - \sqrt{s^2 - 4}}{2} \approx 0.234$, the pair of functions

$$\bar{v}(\xi) = \begin{cases} 1 & \text{if } \xi \geq 0, \\ e^{\lambda_2 \xi} & \text{if } \xi < 0, \end{cases}$$

and

$$\underline{v}(\xi) = \begin{cases} \delta_2 & \text{if } \xi \geq -5, \\ e^{\lambda_2 \xi} - q_2 e^{\mu_2 \lambda_2 \xi} & \text{if } \xi \leq -5, \end{cases}$$

are the sup-sub solutions of (2.4) where we pick $q_2 = 2.6, \mu_2 = 1 + \frac{1}{1.1} \approx 1.91$.

By direct calculation, we have $\sup_{\xi \in \mathbb{R}} \underline{v}(\xi) \approx 0.0817 > v^* = \frac{2}{n+1}$ if $n \geq 24$. That is,

for any $n \geq 24$, there exists a non-monotone wave. For Figure 1 is the profile of $\bar{v}(\xi), \underline{v}(\xi)$ and v^* .

Remark 3.2. Note that given any $\underline{s} > s^*$, for any $s \geq \underline{s}$ we can choose μ_2 very close to 1 such that (1.3) and (1.5) has a non-monotone wave whenever b is close to a . Figure 2. is the illustration.

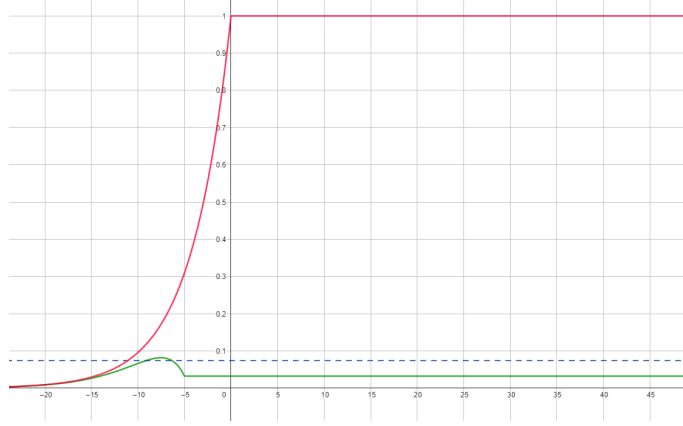


FIGURE 1. When $n = 26$, $\bar{v}(\xi)$ (red line), $v(\xi)$ (green line), $v^* = \frac{2}{27}$ (blue dash line) are all labeled on the figure. There exists $v(\xi)$ lying between the red line and green line with $\lim_{\xi \rightarrow +\infty} v(\xi) = v^*$.

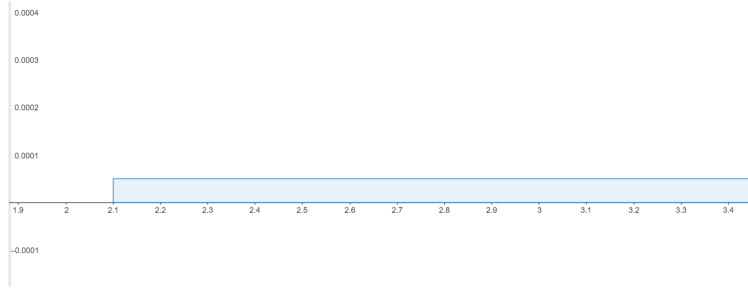


FIGURE 2. Set $d = 1, a = 1, c = \frac{1}{2}, \underline{s} = 2.1, \mu_2 = 1.001$. The horizontal axis is the wave speed s , and the vertical axis represents the difference $a - b > 0$. The blue area illustrates the region where non-monotone solutions $v(\xi)$ exist.

Moreover, on the non-monotone solution, we can prove the following basic property.

Proposition 3.2. *If $u(\xi)$ (or $v(\xi)$) is oscillating near $+\infty$, then so is $v(\xi)$ ($u(\xi)$, respectively).*

Proof. If $u(\xi)$ is oscillating near $+\infty$ and $v(\xi)$ is eventually monotone. If $v(\xi)$ is eventually monotone increasing at $+\infty$, then by assumption, there exists a sequence $I_k = [\xi_k^m, \xi_k^M]$ such that $u(\xi)$ is strictly monotonic increasing on I_k with $u(\xi_k^m)$ is a local minimum, $u(\xi_k^M)$ a local maximum and $0 < u(\xi_k^m) < u(\xi_k^M)$. By (1.3), we have

$$\begin{cases} u(\xi_k^M)(1 - u(\xi_k^M) - cv(\xi_k^M)) \geq 0, \\ u(\xi_k^m)(1 - u(\xi_k^m) - cv(\xi_k^m)) \leq 0, \end{cases}$$

which is equivalent to

$$(3.2) \quad \begin{cases} v(\xi_k^M) \leq \frac{1-u(\xi_k^M)}{c}, \\ v(\xi_k^m) \geq \frac{1-u(\xi_k^m)}{c}. \end{cases}$$

Since v is monotone increasing, (3.2) implies that $u(\xi_k^M) \leq u(\xi_k^m)$, a contradiction. Similarly, we can also show that $v(\xi)$ is not eventually monotone decreasing at $+\infty$. $\square$

At present, for non-monotone wave, we can only characterize the above property. To determine the precise increasing—decreasing behavior, more refined tools are required, which is one of our future research directions. In fact, we can use this type of non-monotone wave to construct the front-pulse solution.

4. FRONT-PULSE WAVES AT CRITICAL WEAK COMPETITION

We now consider two degenerate cases of our model under specific parameter settings.

First, when $b < a$ and $c = \frac{1}{a}$, the original system reduces to the following degenerate model:

$$(4.1) \quad \begin{cases} u'' - su' + u(1 - u - \frac{1}{a}v) = 0, & \xi \in \mathbb{R}, \\ dv'' - sv' + v(a - bu - v) = 0, \\ (u, v)(-\infty) = (0, 0), & (u, v)(+\infty) = (0, a). \end{cases}$$

It is straightforward to verify that for any $s \geq 2\sqrt{ad}$, the system (4.1) admits a trivial traveling wave solution $(u(\xi), v(\xi))$, where $u(\xi) = 0$ for all $\xi \in \mathbb{R}$ and $v(\xi)$ satisfies the classical Fisher-KPP equation:

$$(4.2) \quad \begin{cases} dv'' - sv' + v(a - v) = 0, \\ v(-\infty) = 0, & v(+\infty) = a. \end{cases}$$

On the other hand, when $b = a$ and $a < \frac{1}{c}$, the original system reduces to another degenerate model:

$$(4.3) \quad \begin{cases} u'' - su' + u(1 - u - cv) = 0, & \xi \in \mathbb{R}, \\ dv'' - sv' + v(a(1 - u) - v) = 0, \\ (u, v)(-\infty) = (0, 0), & (u, v)(+\infty) = (1, 0). \end{cases}$$

Similarly, for any $s \geq 2$, (4.3) admits a trivial traveling wave solution $(u(\xi), v(\xi))$, where $v(\xi) = 0$ for all $\xi \in \mathbb{R}$ and $u(\xi)$ satisfies the classical Fisher-KPP equation:

$$(4.4) \quad \begin{cases} u'' - su' + u(1 - u) = 0, \\ u(-\infty) = 0, & u(+\infty) = 1. \end{cases}$$

We call a non-trivial solution of (4.1) or (4.3) front-pulse solution. This type of front-pulse solution is very rare and interesting in biological models. We can obtain such type of solutions using Theorem 1.3 and Theorem 1.4. To proceed with this construction, we first introduce the following lemmas.

Lemma 4.1. *Given any compact set $\Omega \subset \mathbb{R}$. Under the same assumption of Theorem 1.3, the non-monotone solution $(u(\xi), v(\xi))$ of (1.3) and (1.5) satisfies the following interior estimate*

$$\|u\|_{C^{3,\alpha}(\Omega)}, \|v\|_{C^{3,\alpha}(\Omega)} \leq C(\Omega, s, d, a, b),$$

where the constant is independent of c .

Proof. It is easy to see that $0 < u(\xi) < 1$ and $0 < v(\xi) < a$. By the assumption, $\delta(a, b, s) < c < \frac{1}{a}$, the source term of the u equation

$$|u(\xi)(1 - u(\xi) - cv(\xi))| \leq |u(\xi)(1 - u(\xi))| + \frac{1}{a}|u(\xi)v(\xi)|$$

belongs to $L^p(\Omega)$ for any $1 < p < \infty$ and the upper bound is independent of c . Similarly, the source term in the v equation

$$v(\xi)(a - bu(\xi) - v(\xi))$$

is belong in $L^p(\Omega)$ for all $1 < p < \infty$, too. By applying elliptic estimate, we obtain that $u, v \in W^{2,p}(\Omega)$ for all $1 < p < \infty$. Then, by the Sobolev Embedding Theorem, it follows that $u, v \in C^{1,\alpha}(\Omega)$ for all $0 < \alpha < 1$. Now, since the nonlinearity involves terms like $u(1 - u - cv)$ and $v(a - bu - v)$ and we already have $u, v \in C^{1,\alpha}(\Omega)$. Therefore, we can apply the elliptic estimate once again to conclude that $u, v \in C^{3,\alpha}(\Omega)$. Consequently, there exists a constant $C = C(\Omega, s, d, a, b)$ such that

$$\|u\|_{C^{3,\alpha}(\Omega)}, \|v\|_{C^{3,\alpha}(\Omega)} \leq C(\Omega, s, d, a, b),$$

where the constant is independent of c . □

Using a similar argument, we can establish the following result.

Lemma 4.2. *Given any compact set $\Omega \subset \mathbb{R}$. Under the same assumption of Theorem 1.4, the non-monotone solution $(u(\xi), v(\xi))$ of (1.3) and (1.5) satisfies the following interior estimate*

$$\|u\|_{C^{3,\alpha}(\Omega)}, \|v\|_{C^{3,\alpha}(\Omega)} \leq C(\Omega, s, d, a, c),$$

where the constant is independent of b .

We now turn to the proof of Theorem 1.5.

Proof of Theorem 1.5. Fix any $b < a$, $d > 0$, $s \geq s^*$. By Theorem 1.3, there exist a pair of positive sup-sub solution $(\bar{u}, \underline{u}), (\bar{v}, \underline{v})$ and $\delta > 0$ such that if $\delta = \delta(a, b, s) < c < \frac{1}{a}$, then there exist a solution $(u(\xi), v(\xi))$ of (1.3) and (1.5), where $u(\xi)$ is a non-monotone function. For any $c_k \in (\delta, \frac{1}{a})$ with $c_k \rightarrow \frac{1}{a}$, there exists a non-monotone solution $(u_k(\xi), v_k(\xi))$ with

$$\|u_k\|_\infty > \|\underline{u}\|_\infty > 0$$

which implies that $u_k(\xi)$ has a uniformly positive lower bounded function, $\underline{u}(\xi)$. By the boundedness of the $C^{3,\alpha}$ norm of (u_k, v_k) , there exists a subsequence, $(u_{k_j}(\xi), v_{k_j}(\xi), c_{k_j})$ converges to $(U(\xi), V(\xi), \frac{1}{a})$ in C^2 on any compact set $[-M, M]$ as $j \rightarrow +\infty$, for some (U, V) satisfies the limiting equation

$$(4.5) \quad \begin{cases} U'' - sU' + U(1 - U - \frac{1}{a}V) = 0, & \xi \in [-M, M] \\ dV'' - sV' + V(a - bU - V) = 0, \end{cases}$$

with $\|U\|_\infty \geq \|\underline{u}\|_\infty > 0$, and $\|U\|_{C^2(\Omega)} \leq C(\Omega, s, d, a, b)$. Note that (u_{k_j}, v_{k_j}) satisfies (1.5). When $M \rightarrow +\infty$, we can see that (U, V) satisfies

$$\lim_{\xi \rightarrow -\infty} (U(\xi), V(\xi)) = (0, 0),$$

and $0 \leq U(\xi) \leq 1$, $\delta_2 \leq V(\xi) \leq a$ for ξ very large, where $U(\xi)$ is a non-trivial function. Now we claim that $\lim_{\xi \rightarrow +\infty} (U(\xi), V(\xi)) = (0, a)$.

Case 1. If $U(\xi)$ and $V(\xi)$ are eventually monotone, then assume $\lim_{\xi \rightarrow +\infty} (U(\xi), V(\xi)) = (\alpha, \beta)$. It is easy to show that the derivatives are vanish. Therefore, α, β satisfy

$$\begin{cases} \alpha(1 - \alpha - \frac{1}{a}\beta) = 0, \\ \beta(a - b\alpha - \beta) = 0, \end{cases}$$

with $0 \leq \alpha \leq 1$, and $\delta_2 \leq \beta \leq a$. This implies that $(\alpha, \beta) = (0, a)$.

Case 2. If $V(\xi)$ oscillate and $U(\xi)$ is eventually monotonic at $+\infty$, and let $\lim_{\xi \rightarrow +\infty} U(\xi) = \alpha$. By assumption, there exist a sequence $I_k = [\xi_k^m, \xi_k^M]$ such that $V(\xi)$ is monotone increasing on I_k , $V(\xi_k^m)$ is a local minimum, $V(\xi_k^M)$ is a local maximum and $V(\xi_k^m) \leq V(\xi_k^M)$. By (1.3), we have

$$\begin{cases} V(\xi_k^M)(a - bU(\xi_k^M) - V(\xi_k^M)) \geq 0, \\ V(\xi_k^m)(a - bU(\xi_k^m) - V(\xi_k^m)) \leq 0, \end{cases}$$

which is equivalent to

$$(4.6) \quad \begin{cases} U(\xi_k^M) \leq \frac{a - V(\xi_k^M)}{b}, \\ U(\xi_k^m) \geq \frac{a - V(\xi_k^m)}{b}. \end{cases}$$

Note that (4.6) implies that $U(\xi)$ is monotone decreasing. If not, we have

$$\frac{a - V(\xi_k^m)}{b} \leq U(\xi_k^m) \leq U(\xi_k^M) \leq \frac{a - V(\xi_k^M)}{b},$$

and we have $V(\xi_k^M) \leq V(\xi_k^m)$, a contradiction. Since $\lim_{\xi \rightarrow +\infty} U(\xi) = \alpha$, (4.6) shows that $\lim_{\xi \rightarrow +\infty} V(\xi) = a - b\alpha := \beta$ exist. If $\alpha > 0$, then

$$\lim_{\xi \rightarrow +\infty} U(\xi)(1 - U(\xi) - \frac{1}{a}V(\xi)) = \frac{(b - a)\alpha^2}{a} < 0.$$

On the other hand, since $U(\xi)$ is monotone decreasing, we can show that $\lim_{\xi \rightarrow +\infty} U''(\xi) = 0$ and $\lim_{\xi \rightarrow +\infty} U'(\xi) = 0$. Therefore,

$$0 = \lim_{\xi \rightarrow +\infty} (U''(\xi) - sU'(\xi) + U(\xi)(1 - U(\xi) - \frac{1}{a}V(\xi))) < 0,$$

a contradiction. Thus, $\alpha = 0$ and $\beta = a$.

Case 3. If $U(\xi)$ oscillate and $V(\xi)$ is eventually monotonic at $+\infty$, and let $\lim_{\xi \rightarrow +\infty} V(\xi) = \beta$.

Case 3.1. When $\beta = a$, then we can apply similar argument to show that $\lim_{\xi \rightarrow +\infty} U(\xi) = 0$.

Case 3.2. If $\beta < a$, then either V is monotone increasing or decreasing, we can show that $\lim_{\xi \rightarrow +\infty} V''(\xi) = \lim_{\xi \rightarrow +\infty} V'(\xi) = 0$. Thus, by (1.3), we have

$$0 = \lim_{\xi \rightarrow +\infty} V(\xi)(a - bU(\xi) - V(\xi)).$$

This implies

$$\lim_{\xi \rightarrow +\infty} U(\xi) = \lim_{\xi \rightarrow +\infty} (V(\xi)U(\xi)) \left(\frac{1}{V(\xi)} \right) = \frac{a - \beta}{b} > 0.$$

Since $\lim_{\xi \rightarrow +\infty} U(\xi)$ exist, and it deduce to Case 1.

Case 4. If $U(\xi)$ and $V(\xi)$ are oscillate at $+\infty$.

Case 4.1. If $\lim_{\xi \rightarrow +\infty} V(\xi) = a$. For any local maximum points ξ_k^M , of U . By (1.3), we have

$$U(\xi_k^M)(1 - U(\xi_k^M)) - \frac{1}{a}V(\xi_k^M) \geq 0,$$

which is equivalent to

$$0 \geq -U^2(\xi_k^M) \geq U(\xi_k^M) \left(\frac{1}{a}V(\xi_k^M) - 1 \right).$$

By the Squeeze Theorem, we have $\limsup_{\xi \rightarrow +\infty} U(\xi) = 0$. This implies that $\lim_{\xi \rightarrow +\infty} U(\xi) = 0$.

Case 4.2. If $\liminf_{\xi \rightarrow +\infty} V(\xi) = \beta < a$, and $\limsup_{\xi \rightarrow +\infty} U(\xi) = \alpha > 0$. For any small $\frac{1}{n} > 0$ there exist ξ_n which be the local minimum points of V such that $V(\xi_n) \leq \beta + \frac{1}{n}$. By (1.3), we have

$$0 \geq a - bU(\xi_n) - V(\xi_n) \geq a - bU(\xi_n) - \beta - \frac{1}{n},$$

which implies that

$$U(\xi_n) \geq \frac{a - \beta - \frac{1}{n}}{b}.$$

Therefore, $\limsup_{\xi \rightarrow +\infty} U(\xi) = \alpha \geq \frac{a - \beta}{b}$. On the other hand, up to subsequence, for any small $\frac{1}{k} > 0$, there exist the local maximum points, ξ_k , of U such that $0 < \alpha - \frac{1}{k} < U(\xi_k) < \alpha + \frac{1}{k}$. Again, by (1.3), we have

$$0 \leq 1 - U(\xi_k) - \frac{1}{a}V(\xi_k) < 1 - \alpha + \frac{1}{k} - \frac{1}{a}V(\xi_k) \leq 1 - \frac{a - \beta}{b} + \frac{1}{k} - \frac{\beta}{a}.$$

Let $k \rightarrow +\infty$, we then obtain that

$$0 \leq 1 - \frac{a}{b} + \frac{\beta}{b} - \frac{\beta}{a},$$

which is equivalent to

$$a \left(\frac{1}{b} - \frac{1}{a} \right) \leq \beta \left(\frac{1}{b} - \frac{1}{a} \right),$$

a contradiction. Hence, $\lim_{\xi \rightarrow +\infty} V(\xi) = a$ and $\lim_{\xi \rightarrow +\infty} U(\xi) = 0$.

□

Remark 4.1. As $c_{k_j} \rightarrow \frac{1}{a}$, the definition of $\underline{u}(\xi)$ change. Due to the inequality, $0 < \delta_{1_{k_j}} < u_{k_j}^*$, we in fact have $\lim_{j \rightarrow +\infty} \delta_{k_j} = 0$. However, this does not affect the maximum value $\|\underline{u}\|_\infty$.

Proposition 4.1. *For any $a < \frac{1}{c}$, $a = b$, $d > 0$. If $s \geq s^*$, there exists a non-trivial front-pulse solution of (4.3) with $v(\xi) \rightarrow 0$ as $|\xi| \rightarrow +\infty$.*

The proof is very similar to the proof of Theorem 1.5. We omit the details here.

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APPENDIX A. PROOF OF NONEXISTENCE FOR $s \in (0, s^*)$

For readers' convenience, we provide a direct proof of the nonexistence of traveling waves when $0 < s < s^*$.

Theorem A.1. *When $0 < s < s^*$, there is no positive solution (u, v) of the equation (1.3) and (1.5).*

Proof. Without loss of generality, we assume $s^* = 2$. For contradiction, if for some $0 < s < 2$ there exists a positive solution $(u, v)(\xi)$ of (1.3) and (1.5). Now, we define the positive function $w(\xi) = e^{\frac{s\xi}{2}} u(\xi)$. Then by the u -equation of (1.3), $w(\xi)$ satisfies

$$(A.1) \quad w''(\xi) + w(\xi)(1 - \frac{s^2}{4} - u(\xi) - cv(\xi)) = 0.$$

Since $0 < s < 2$ and (u, v) tends to zero as $\xi \rightarrow -\infty$, there exist small $\epsilon > 0$ and $-L < 0$ such that

$$(1 - \frac{s^2}{4} - u(\xi) - cv(\xi)) > \epsilon, \quad \text{for all } \xi < -L.$$

We define an auxiliary function $\phi(\xi) = \sin(\sqrt{\epsilon}\xi)$ which is a positive solution of the following linear boundary value problem

$$(A.2) \quad \begin{cases} \phi'' + \epsilon\phi = 0, \\ \phi(\frac{-2M\pi}{\sqrt{\epsilon}}) = \phi(\frac{-(2M-1)\pi}{\sqrt{\epsilon}}) = 0, \end{cases}$$

where $M \in \mathbb{N}$ is a large number such that $\frac{-(2M-1)\pi}{\sqrt{\epsilon}} < -L$. Set $\frac{-2M\pi}{\sqrt{\epsilon}} = \xi_1$, $\frac{-(2M-1)\pi}{\sqrt{\epsilon}} = \xi_2$. Multiply equation (A.1) by $\phi(\xi)$ and multiply equation (A.2) by $w(\xi)$, then subtract equation (A.2) from equation (A.1). We have

$$w''(\xi)\phi(\xi) - \phi''(\xi)w(\xi) + w(\xi)\phi(\xi)(1 - \frac{s^2}{4} - u(\xi) - cv(\xi) - \epsilon) = 0$$

Integrating both sides of the equation from ξ_1 to ξ_2 , we then obtain that

$$[w'(\xi)\phi(\xi) - \phi'(\xi)w(\xi)] \Big|_{\xi_1}^{\xi_2} + \int_{\xi_1}^{\xi_2} w(\xi)\phi(\xi)(1 - \frac{s^2}{4} - u(\xi) - cv(\xi) - \epsilon)d\xi = 0.$$

The first term is positive, and the second term is also positive, a contradiction. $\square$

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