

Identification in Auctions with Truncated Transaction Prices

Tonghui Qi*

October 9, 2025

Abstract

I establish nonparametric identification results in first- and second-price auctions when transaction prices are truncated by a binding reserve price under a range of information structures. When the number of potential bidders is fixed and known across all auctions, if only the transaction price is observed, the bidders' private-value distribution is identified in second-price auctions but not in first-price auctions. Identification in first-price auctions can be achieved if either the number of active bidders or the number of auctions with no sales is observed. When the number of potential bidders varies across auctions and is unknown, the bidders' private-value distribution is identified in first-price auctions but not in second-price auctions, provided that both the transaction price and the number of active bidders are observed. I derive analogous results to auctions with entry costs, which face a similar truncation issue when data on potential bidders who do not enter are missing.

JEL CLASSIFICATION NUMBERS: C14, C57, D44

KEYWORDS: Auctions, Nonparametric identification, Truncated transaction price, Reserve price, Entry cost, Unknown number of bidders.

*Department of Economics, National University of Singapore, qi.tonghui@u.nus.edu. I am grateful to my advisor, Sorawoot Srisuma, for his guidance and detailed comments. I also thank Ming Li, Jingfeng Lu, Satoru Takahashi, and Allen Vong for their valuable suggestions. All remaining errors are my own.

1 Introduction

Identification in auctions has been extensively studied for decades, with the bidder’s value distribution being one of the most important model primitives. Within the independent private values (IPV) paradigm, the seminal papers, Guerre et al. (2000) and Athey and Haile (2002), demonstrate that the bidder’s value distribution can be nonparametrically identified from bids in first-price and second-price auctions, respectively.

Building on this foundation, subsequent research has extended the symmetric IPV framework in various directions. For instance, see Li et al. (2002) and Aradillas-López et al. (2013) for affiliated or correlated private values, Campo et al. (2003) and Somaini (2020) for asymmetric bidders, Guerre et al. (2009) for risk averse bidders, Krasnokutskaya (2011), Hu et al. (2013), and Luo and Xiao (2023) for unobserved heterogeneity, Gimenes and Guerre (2020) for interdependent values. More generally, as the studies of identification in auctions have been explored under progressively weaker assumptions, most research continue to assume bids or transaction prices are observed in *all* auctions.

However, this seemingly innocuous assumption may not always hold in practice. For instance, bids below the reserve price may simply be absent from the dataset available to the econometrician. In the extreme case where a seller sets a public reserve price, bidders with values below the reserve are unlikely to submit bids, and no bids or transaction prices will be observed in some auctions where all bidders’ values fall below the reserve. A similar issue also arises when auction participation involves a fixed cost. In such settings, only bidders with sufficiently high values will choose to enter. If all bidders’ values are low, no one participates and such auction yields no observable data.

These scenarios present two key identification challenges: (1) the observed number of bidders reflects only those who submit bids (referred to as active bidders), rather than the total number of individuals interested in the item (referred to as potential bidders); (2) the bid or transaction price is only observed conditional on the auction having at least one active bidder (referred to as valid auctions). As a result, both binding reserve prices and entry costs lead to truncation in the observed data. Ignoring this issue can result in biased and inconsistent estimation of the bidder’s value distribution.

To avoid this truncation problem, some empirical studies assume the reserve price is nonbinding, e.g., see Campo et al. (2011), Haile and Tamer (2003), and Aradillas-López et al. (2013), in the context of timber auction from U.S. Forest Service (USFS) timber auctions. However, the reserve price can indeed be binding in many other auctions, see Roberts (2013), Choi et al. (2016), and Ostrovsky and Schwarz (2023) for instance. Since traditional methods typically rely on the unconditional distribution of bids (or transaction

prices) and the number of potential bidders, they may not be applicable when a binding reserve price or entry cost is present. Despite the prevalence of this truncation phenomenon, there is very limited research on identification in auctions when observed bids or transaction prices are truncated. A notable exception is work of Shneyerov and Wong (2011), who focused on first-price auctions with binding reserve prices. A key assumption they make is that the number of active bidders is observed, while I have derived identification results across different information structures and auction formats. My work thus complements their study in filling this gap in the literature.

Throughout this paper, I assume that only transaction prices—rather than all bids—are observed, which is commonly the case in the real world, especially in descending or ascending auctions. Since descending auctions are strategically equivalent to first-price auctions, and ascending auctions are strategically equivalent to second-price auctions (Milgrom and Weber (1982)), I focus on first- and second-price auctions, but similar identification results can be directly applied to descending and ascending auctions.

To study identification in auctions with binding reserves, I first analyze the case where the number of potential bidders is fixed and known. In this setup, I show that if only the transaction price is observed, the bidder’s private value distribution is identified in second-price auctions but *not* in first-price auctions. I then demonstrate that identification in first-price auctions can be facilitated by observing either the number of invalid auctions or the number of active bidders in valid auctions. Next, I consider the case where the number of potential bidders varies across auctions. If this number is known for each auction, then the bidder’s private value distribution can be point identified from the transaction price in both first- and second-price auctions. However, if the number of potential bidders is unknown, but the transaction price and the number of active bidders are observed, then the bidder’s private value distribution is identified in first-price auctions but *not* in second-price auctions. In this case, identification in second-price auctions can be facilitated by observing the number of invalid auctions. After analyzing auctions with binding reserve prices, I extend the analysis to auctions with entry costs and show that similar—though not identical—identification results hold.

The key to identification with truncated transaction price lies in the truncation threshold. Without knowledge of this threshold, the unconditional distribution of bidders’ values cannot be recovered from the distribution of transaction prices, conditional on the auction having at least one active bidder. Therefore, identification depends on whether there is sufficient information to recover this truncation threshold. An important result used in this paper is that the seller’s optimal screening level (which is also the truncation threshold) remains invariant to the number of potential bidders. This invariance creates a link across auctions with

different numbers of potential bidders and is crucial for identification when there is a binding reserve price. In auctions with entry costs, the bidder’s bidding threshold changes with the number of potential bidders. In that case, the fixed cost is assumed to remain invariant in order to link auctions with different numbers of potential bidders. Since it is the quantile at which screening occurs—rather than the reserve price in levels—that governs identification, a quantile-based approach is adopted to simplify notation and facilitate analysis.¹

This paper contributes to the existing literature in several ways. Firstly, it contributes to the identification and estimation of auctions with a binding reserve price. Although this topic has been examined in some papers as an extension of their main results, it remains inadequately studied. This paper extends the extant results and summarizes them across different model setups. For instance, Guerre et al. (2000) show that in first-price auctions, when the number of potential bidders is fixed, the bidder’s private value distribution can be identified from the number of actual bidders in each auction and the distribution of bids conditional on exceeding the reserve price. I extend their result by showing that observing only the transaction price, rather than all bids, is sufficient for identification. Besides, Athey and Haile (2002) show that if the true number of bidders is fixed and known, then the bidder’s private value distribution conditional on being higher than the reserve price can be identified from the transaction price in second-price auctions, while I further demonstrate that the unconditional distribution of the bidder’s private value is also identified. Shneyerov and Wong (2011) prove identification in first-price auctions with a binding reserve price using the transaction price and the number of active bidders. However, they assume the reserve price is the same across auctions with different number of potential bidders, while I complement their result by showing that it is indeed optimal for the seller to do so under some regular conditions. Moreover, I also show identification results for second-price auctions and across different information structures.

Secondly, this paper contributes to studies on the identification in auctions with an unknown number of bidders. Song (2004) shows in ascending auctions with IPV bidders, observation of any two valuations of which rankings from the top is known nonparametrically identifies the bidder’s value distribution. An et al. (2010) develop a nonparametric procedure to recover the distribution of bids conditional on the unknown number of potential bidders in first-price auctions. Freyberger and Larsen (2022) consider ascending auctions with unobserved heterogeneity and an unknown number of bidders and show different identification results depending on whether the reserve price is secret or public. However, all these papers require information of at least two order statistics of bids, whereas I assume only the trans-

¹The use of a quantile-based approach for identification and estimation of auctions is not new. See Gimenes (2017) and Gimenes and Guerre (2022) for example.

action price is observed. Adams (2007) also merely uses the transaction price and shows the bidder’s value distribution can be identified if the distribution of the number of potential bidders is known, while I consider the case where this distribution is unknown.

Thirdly, this paper also sheds light on identification in auctions with entry costs. Li (2005) proposes an MSM estimator to estimate the bidder’s value distribution for first-price auctions with entry costs. However, this paper focuses on parametric estimation while I focus on nonparametric identification. Marmer et al. (2013), Gentry and Li (2014), and Chen et al. (2025) consider a two-stage auction game where each potential bidder observes a private signal of their private value and then decides whether to pay a fixed cost to enter the auction. If they enter, they learn their private values exactly and submit bids. Their identification results rely on the observation of the number of potential bidders, while I also analyze the case where the number of potential bidders is unknown. Besides, they focus on first-price auctions but I also study the identification in second-price auctions. In this paper, I simplify their selective entry model by assuming bidders know their private values exactly before making their entry decisions. Then the structure of this selective entry model is actually very similar to auctions with binding reserve prices: both lead to truncation in the observed transaction prices. By applying methods similar to those used in the binding reserve model, I show different identification results under different information structures.

The remainder of this paper is organized as follows: Section 2 provides a simple example to introduce the notation employed throughout this paper. Section 3 formally presents the model and identification results in auctions with binding reserve prices. Section 4 extends these results to auctions with entry costs. Finally, section 5 concludes.

2 Notation and Observables

In this part, I use a simple example to introduce the observables and notations used in this paper. Consider 3 auctions each with 2 bidders. Suppose their bids are as follows:

	bidder 1	bidder 2
Auction 1	3	4
Auction 2	2	3
Auction 3	1	2

Throughout this paper, I index bidders by i and auctions by l . In the absence of reserve prices, the observed bids (denoted by B_l), the numbers of observed bidders (denoted by N_l^{obs}), and the observed transaction prices (denoted by T_l^f for first-price auctions and T_l^s for second-price auctions) are as follows:

	B_l	T_l^f	T_l^s	N_l^{obs}
Auction 1	(3,4)	4	3	2
Auction 2	(2,3)	3	2	2
Auction 3	(1,2)	2	1	2

However, if the seller sets a reserve price of $R = 2.5$, then bidders with values below 2.5 will not submit bids. In this case, the econometrician observes only those auctions in which at least one bidder submits a bid. In this example, only two auctions contain actual bids: auction 1, with bids of 3 and 4, and auction 2, with a single bid of 3. Auction 3, in contrast, features no bids and is referred to as an “invalid” auction.² Depending on the way of data collection, the econometrician may observe such auctions without bids, or they may not observe them at all. Throughout this paper, I assume that only valid auctions and active bidders are observed unless stated otherwise, consistent with common empirical practice.³ For instance, in auction 2, bidder 2 is an active bidder while bidder 1 is a potential bidder who remains inactive due to the binding reserve price. Thus, the econometrician observes only one bidder—bidder 2—in this auction. Accordingly, in the presence of a binding reserve price, the bids, transaction prices, reserve prices, and numbers of bidders observed by the econometrician are as follows:⁴

	B_l	T_l^f	T_l^s	R	N_l^{obs}
Auction 1	(3,4)	4	3	2.5	2
Auction 2	(3)	3	2.5	2.5	1

In contrast, I assume bidders always know the true number of potential bidder and thus determine their bidding strategies based on the true N .

It is worth noting that, although bids and transaction prices are not observed in invalid auctions, the number of such auctions may still contain useful information. Consequently, whether or not invalid auctions are observed can have important implications for identification. In the absence of data on invalid auctions, it is important to recognize that the distributions of bids, transaction prices, and the numbers of active bidders are observed only conditional on the auction being valid—that is, conditional on there is at least one active bidder. Throughout this paper, I assume only the transaction price T_l rather than all bids B_l is observed. Let L denote the number of valid auctions (always assumed to be observed by the econometrician), L^{invalid} denote the number of invalid auctions, N_l denote the number

²In first-price auctions, bidders’ bids vary with the reserve price. This effect is disregarded here, as the example serves solely to introduce notation. It will, however, be accounted for in the formal analysis.

³See Hendricks and Porter (2007) for a discussion of the empirical relevance of this assumption.

⁴It is not important whether the reserve price R is observed, as we can always identify R by $\min\{T_l\}$.

of potential bidders in auction l , and N_l^{act} denote the number of active bidders in auction l , then we have $N_l^{\text{obs}} = N_l^{\text{act}} \mid N_l^{\text{act}} \geq 1$. When N_l varies across auctions, let $\bar{N}$ represent the maximum number of potential bidders across all auctions. In the example above, we have $L = 2$, $L^{\text{invalid}} = 1$, $N_1^{\text{obs}} = N_1^{\text{act}} = 2$, $N_2^{\text{obs}} = N_2^{\text{act}} = 1$, $N_3^{\text{act}} = 0$, and $\bar{N} = N_1 = N_2 = 2$.

3 Identification in Auctions with Binding Reserves

Throughout this paper, I consider auctions with symmetric, IPV, and risk neutral bidders. Denote the private value of bidder i by v_i , the CDF of the bidder's private value distribution by $F(\cdot)$, the PDF of the bidder's private value distribution by $f(\cdot)$, the seller's utility function by $U(\cdot)$, and the seller's outside option, i.e., the value derived if the good remains unsold, by V_0 . Assumption 1 describes the restrictions imposed on v_i , $F(\cdot)$, $U(\cdot)$ and V_0 .

Assumption 1:

- A1.1: $v_i \in [\underline{v}, \bar{v}]$ for some $0 \leq \underline{v} < \bar{v} < \infty$ for all i .
- A1.2: $F(\cdot)$ is absolutely continuous and strictly increasing.
- A1.3: $U'(\cdot) > 0$, $U''(\cdot) \leq 0$.
- A1.4: $V_0 \in [0, \bar{v})$.
- A1.5: $\psi(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing.

A1.1 means each bidder's private value is non-negative and bounded above; A1.2 guarantees $f(v)$ exists and is strictly positive in its support, which implies $\psi(v)$ in A1.5 is properly defined; A1.3 indicates the seller can be either risk neutral or risk averse; A1.4 is a necessary condition to incentivize the seller to sell the object in an auction; A1.5 is a standard regularity condition in the auction literature. $\psi(v)$ is called the virtual valuation and this assumption ensures that the second-price auction is the optimal dominant-strategy incentive-compatible (DSIC) selling mechanism. When $\psi(\cdot)$ is decreasing, the seller may want to choose the winner according to a random process.⁵ In this paper, A1.5 guarantees that the optimal reserve price exists and is unique.

As will be shown below, it is the proportion of the seller's screening level, $F(R)$, rather than the reserve price itself, R , that plays a central role in identification. Therefore, to simplify the notation, I use a quantile method in the following analysis. Denote the quantile function of bidder's value by $V(\cdot) = F^{-1}(\cdot)$ and define the bidder's type by $\alpha_i = F(v_i)$, then by construction, $\alpha_i \sim \text{Unif}[0, 1]$. Moreover, A1.2 guarantees $V(\cdot)$ is differentiable and

⁵See Myerson (1981) and McAfee and McMillan (1987).

$V'(\cdot) > 0$. Since bidders with types lower than the seller's screening level will not submit bids, the observed transaction price is truncated from below.

3.1 Bidding Strategy and Optimal Screening Level

The equilibrium bidding strategies in first- and second-price auctions with binding reserve prices are well established in the literature, see Riley and Samuelson (1981), Milgrom and Weber (1982), and Krishna (2009), etc. For completeness and to facilitate later analysis, I briefly re-derive these strategies in terms of quantiles, as this formulation will be used throughout the remainder of the paper. Suppose the seller's screening level is $\alpha_0 > 0$, then a bidder with type $\alpha_i < \alpha_0$ will not bid in both first-and second-price auctions. In first-price auctions, denote the equilibrium bidding function when the bidder's type is α by $b(\alpha; \alpha_0)$. Then a bidder's expected payoff when his true type is α but bid $b(\beta; \alpha_0)$ for some $\beta \geq \alpha_0$ is given by:

$$\pi(\beta; \alpha) = [V(\alpha) - b(\beta; \alpha_0)]\beta^{N-1}.$$

Taking derivative, we have

$$\frac{\partial \pi}{\partial \beta} = [V(\alpha) - b(\beta; \alpha_0)](N-1)\beta^{N-2} - b'(\beta; \alpha_0)\beta^{N-1}.$$

For $b(\alpha; \alpha_0)$ to be an equilibrium, it must be optimal for a bidder to bid $b(\alpha; \alpha_0)$ when his true type is $\alpha > \alpha_0$. Thus the FOC gives

$$\left. \frac{\partial \pi}{\partial \beta} \right|_{\beta=\alpha} = [V(\alpha) - b(\alpha; \alpha_0)](N-1)\alpha^{N-2} - b'(\alpha; \alpha_0)\alpha^{N-1} = 0,$$

which implies $\forall \alpha \in (\alpha_0, 1)$,

$$\frac{d}{d\alpha} [b(\alpha; \alpha_0)\alpha^{N-1}] = V(\alpha)\frac{d\alpha^{N-1}}{d\alpha}. \quad (1)$$

Therefore, by the fundamental theorem of calculus, we have

$$\begin{aligned} b(\alpha; \alpha_0)\alpha^{N-1} &= b(\alpha_0; \alpha_0)\alpha_0^{N-1} + \int_{\alpha_0}^{\alpha} V(t)dt^{N-1} \\ &= b(\alpha_0; \alpha_0)\alpha_0^{N-1} + V(t)t^{N-1}\big|_{\alpha_0}^{\alpha} - \int_{\alpha_0}^{\alpha} V'(t)t^{N-1}dt. \end{aligned}$$

Since $b(\alpha_0; \alpha_0) = V(\alpha_0)$, the equation above can be simplified to

$$b(\alpha; \alpha_0) = V(\alpha) - \frac{1}{\alpha^{N-1}} \int_{\alpha_0}^{\alpha} V'(t)t^{N-1}dt.$$

Note that

$$\frac{\partial b(\alpha; \alpha_0)}{\partial \alpha_0} = -\frac{1}{\alpha^{N-1}} [-V'(\alpha_0)\alpha_0^{N-1}] = \frac{\alpha_0^{N-1}}{\alpha^{N-1}} V'(\alpha_0) > 0,$$

which implies a bidder's equilibrium bid increases with the reserve price set by the seller.

In second-price auctions, it is always a weakly dominant strategy for a bidder with $\alpha_i > \alpha_0$ to bid his true value $V(\alpha_i)$. Throughout this paper, I assume both the bidder and the seller are rational and play the equilibrium strategy. In the next lemma, I show that under Assumption 1, the seller's optimal screening level is unique and does not change with N .

Lemma 1: Under Assumption 1, there exists a unique optimal screening level $\alpha^* \in [0, 1]$ that maximizes the seller's expected payoff in both first- and second-price auctions. Moreover, this optimal screening level does not change with N .

Proof. First consider first-price auctions. The seller's expected utility is given by:

$$\Pi_f(\alpha) = U(V_0)\alpha^N + \int_{\alpha}^1 U(b(t; \alpha))dt^N.$$

Taking derivative, we have

$$\begin{aligned} \Pi'_f(\alpha) &= U(V_0) \cdot N\alpha^{N-1} - U(b(\alpha; \alpha)) \cdot N\alpha^{N-1} + \int_{\alpha}^1 U'(b(t; \alpha)) \frac{\partial b(t; \alpha)}{\partial \alpha} dt^N \\ &= U(V_0) \cdot N\alpha^{N-1} - U(b(\alpha; \alpha)) \cdot N\alpha^{N-1} + \int_{\alpha}^1 U'(b(t; \alpha)) \frac{\alpha^{N-1}}{t^{N-1}} V'(\alpha) \cdot Nt^{N-1} dt \\ &= U(V_0) \cdot N\alpha^{N-1} - U(V(\alpha)) \cdot N\alpha^{N-1} + \int_{\alpha}^1 U'(b(t; \alpha)) V'(\alpha) \cdot N\alpha^{N-1} dt \\ &= N\alpha^{N-1} \left[U(V_0) - U(V(\alpha)) + \int_{\alpha}^1 U'(b(t; \alpha)) V'(\alpha) dt \right]. \end{aligned}$$

Let

$$g(\alpha) = U(V_0) - U(V(\alpha)) + \int_{\alpha}^1 U'(b(t; \alpha)) V'(\alpha) dt.$$

Taking derivative, we get

$$g'(\alpha) = -2U'(V(\alpha))V'(\alpha) + V'(\alpha) \int_{\alpha}^1 U''(b(t; \alpha)) \frac{\partial b(t; \alpha)}{\partial \alpha} dt + V''(\alpha) \int_{\alpha}^1 U'(b(t; \alpha)) dt.$$

Note that the quantile analog of $\psi(v)$ in A1.5 is $J(\alpha) \equiv V(\alpha) - (1 - \alpha)V'(\alpha)$, thus there must be

$$J'(\alpha) = 2V'(\alpha) - (1 - \alpha)V''(\alpha) > 0.$$

Since $U''(\cdot) \leq 0$, it can be shown that

$$\begin{aligned}
g'(\alpha) &\leq -2U'(V(\alpha))V'(\alpha) + V''(\alpha) \int_{\alpha}^1 U'(V(\alpha))dt \\
&= -2U'(V(\alpha))V'(\alpha) + V''(\alpha)U'(V(\alpha))(1 - \alpha) \\
&= -U'(V(\alpha))[2V'(\alpha) - (1 - \alpha)V''(\alpha)] \\
&= -U'(V(\alpha))J'(\alpha) \\
&< 0.
\end{aligned}$$

Since $g(1) = U(V_0) - U(V(1)) < 0$, there exists a unique optimal screening level $\alpha^* \in [0, 1)$ that maximizes $\Pi_f(\alpha)$. Moreover, if $g(0) > 0$, then $\alpha^* > 0$.

For second-price auctions. The seller's expected payoff is

$$\Pi_s(\alpha) = U(V_0)\alpha^N + U(V(\alpha))N\alpha^{N-1}(1 - \alpha) + N(N - 1) \int_{\alpha}^1 U(V(t))t^{N-2}(1 - t)dt.$$

Taking derivative, we get

$$\Pi'_s(\alpha) = N\alpha^{N-1} [U(V_0) + U'(V(\alpha))V'(\alpha)(1 - \alpha) - U(V(\alpha))].$$

Let

$$g(\alpha) = U(V_0) + U'(V(\alpha))V'(\alpha)(1 - \alpha) - U(V(\alpha)),$$

then we have

$$\begin{aligned}
g'(\alpha) &= U''(\alpha)[V'(\alpha)]^2(1 - \alpha) + U'(\alpha)V''(\alpha)(1 - \alpha) - 2U'(\alpha)V'(\alpha) \\
&\leq U'(\alpha)V''(\alpha)(1 - \alpha) - 2U'(\alpha)V'(\alpha) \\
&= U'(\alpha) [V''(\alpha)(1 - \alpha) - 2V'(\alpha)] \\
&= -U'(\alpha)J'(\alpha) \\
&< 0.
\end{aligned}$$

Since $g(1) = U(V_0) - U(V(1)) < 0$, there exists a unique optimal screening level $\alpha^* \in [0, 1)$ that maximizes $\Pi_s(\alpha)$. Moreover, if $g(0) > 0$, then $\alpha^* > 0$. To demonstrate that the optimal screening level is independent of N , simply note that N does not appear in the first-order conditions for either first- or second-price auction. $\square$

The independence of the optimal reserve price from the number of potential bidders, N , is a well-established result, originally derived by Riley and Samuelson (1981). In this paper, I extend that result to accommodate risk-averse sellers and reformulate it in terms of quantiles to facilitate the subsequent analysis. This reformulation is particularly relevant because

the seller's screening level corresponds directly to the truncation threshold in the transaction price. As shown below, the independence of this truncation threshold on N is crucial for identification when N varies across auctions, as it links auctions with different N 's together. By contrast, as demonstrated in Section 4, when truncation arises from entry costs rather than reserve prices, the truncation threshold becomes dependent on N and thus complicating identification. In that setting, it is the invariance of the fixed entry cost—rather than the screening level—that facilitates identification when N varies.

3.2 Identification when N is Fixed

I begin by considering a simplified setting in which all auctions share a common N , and this value is known to the econometrician. The next two propositions demonstrate that if only T_l is observed, then the bidder's value distribution can be identified in second-price auctions but not in first-price auctions. However, identification in first-price auctions becomes feasible if L^{invalid} is also observed.

Proposition 1: Suppose all auctions have a common and known $N \geq 2$, and the seller's screening level is strictly greater than 0. If the econometrician observes only the truncated transaction price T_l , but does not observe N_l^{obs} or L^{invalid} , then α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified in second-price auctions, but not in first-price auctions.

Proof. To begin with, I show α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ are not identified in first-price auctions. Denote the quantile function of the observed transaction price by $T(\cdot)$. Suppose $T(\cdot)$ is generated by $\alpha_1 > 0$ and $V_1(\cdot)$. Denote the bidding function for α_1 and $V_1(\cdot)$ by $b_1(\cdot)$. Recall that equation (1) gives

$$\frac{d}{d\alpha} [b_1(\alpha)\alpha^{N-1}] = V(\alpha) \cdot \frac{d}{d\alpha} (\alpha^{N-1}),$$

which is equivalent to

$$b_1(\alpha) + \frac{\alpha}{N-1} b_1'(\alpha) = V(\alpha). \quad (2)$$

Since $b_1(\alpha_1) = V(\alpha_1)$, there must be $b_1'(\alpha_1) = 0$. Also note that

$$\begin{aligned} \mathbb{P}(T_l < b_1(\alpha)) &= \mathbb{P}(b_1(\alpha_i^{N:N}) < b_1(\alpha) \mid \alpha_i^{N:N} \geq \alpha_1) \\ &= \frac{\mathbb{P}(\alpha_1 \leq \alpha_i^{N:N} < \alpha)}{\mathbb{P}(\alpha_i^{N:N} \geq \alpha_1)} \\ &= \frac{\alpha^N - \alpha_1^N}{1 - \alpha_1^N}, \end{aligned}$$

which implies

$$b_1(\alpha) = T \left(\frac{\alpha^N - \alpha_1^N}{1 - \alpha_1^N} \right).$$

Taking derivative, we have

$$b'_1(\alpha) = \frac{N\alpha^{N-1}}{1 - \alpha_1^N} T' \left(\frac{\alpha^N - \alpha_1^N}{1 - \alpha_1^N} \right),$$

Let $\alpha = \alpha_1$, we get

$$T'(0) = \frac{1 - \alpha_1^N}{N\alpha_1^{N-1}} b'_1(\alpha_1) = 0.$$

Pick $\alpha_2 \in (0, 1)$ arbitrarily and let

$$\begin{aligned} b_2(\alpha) &= T \left(\frac{\alpha^N - \alpha_2^N}{1 - \alpha_2^N} \right) \quad \forall \alpha \geq \alpha_2, \\ V_2(\alpha) &= b_2(\alpha) + \frac{\alpha}{N-1} b'_2(\alpha). \end{aligned}$$

Now I show α_2 and $V_2(\cdot)$ also generate $T(\cdot)$. To show $V_2(\cdot)$ generate $b_2(\cdot)$, note that $b'_2(\alpha_2) = T'(0) = 0$, which implies $b_2(\alpha_2) = V_2(\alpha_2)$. Therefore, $b_2(\cdot)$ satisfies both the FOC and the boundary condition, and thus is the equilibrium bidding function for $V_2(\cdot)$. To show α_2 and $b_2(\cdot)$ generate $T(\cdot)$, note that for all $\alpha \geq \alpha_2$,

$$\begin{aligned} \mathbb{P}(T_l < b_2(\alpha)) &= \mathbb{P}(b_2(\alpha_i^{N:N}) < b_2(\alpha) \mid \alpha_i^{N:N} \geq \alpha_2) \\ &= \frac{\mathbb{P}(\alpha_2 \leq \alpha_i^{N:N} < \alpha)}{\mathbb{P}(\alpha_i^{N:N} \geq \alpha_2)} \\ &= \frac{\alpha^N - \alpha_2^N}{1 - \alpha_2^N}, \end{aligned}$$

which implies

$$b_2(\alpha) = T \left(\frac{\alpha^N - \alpha_2^N}{1 - \alpha_2^N} \right).$$

Since the choice of α_2 is arbitrary, α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ are not identified in first-price auctions.

Now I show α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified in second-price auctions. Denote the number of active bidders by N^{act} , then we have $\min\{T_l\} \xrightarrow{p} R$ and

$$\begin{aligned} \frac{1}{L} \sum_{l=1}^L \mathbb{1}(T_l = R) &\xrightarrow{p} \mathbb{P}(N^{\text{obs}} = 1) \\ &= \mathbb{P}(N^{\text{act}} = 1 \mid N^{\text{act}} \geq 1) \\ &= \frac{N(1 - \alpha^*)(\alpha^*)^{N-1}}{1 - (\alpha^*)^N}. \end{aligned}$$

Let

$$f(\alpha) = \frac{(1 - \alpha)\alpha^{N-1}}{1 - \alpha^N}.$$

To show α^* is point identified, it suffices to show $f(\alpha)$ is strictly increasing in $(0, 1)$. Taking derivative, we get

$$\begin{aligned} f'(\alpha) &= \frac{1}{(1 - \alpha^N)^2} \{ [(N - 1)\alpha^{N-2} - N\alpha^{N-1}] (1 - \alpha^N) - (\alpha^{N-1} - \alpha^N)(-N\alpha^{N-1}) \} \\ &= \frac{\alpha^{N-2}}{(1 - \alpha^N)^2} [(N - 1) - N\alpha - (N - 1)\alpha^N + N\alpha^{N+1} + N\alpha^N - N\alpha^{N+1}] \\ &= \frac{\alpha^{N-2}}{(1 - \alpha^N)^2} [(N - 1) - N\alpha + \alpha^N]. \end{aligned}$$

Let

$$g(\alpha) = (N - 1) - N\alpha + \alpha^N,$$

we have

$$g'(\alpha) = -N + N\alpha^{N-1} < 0.$$

which implies $g(\alpha) > g(1) = 0$, and thus $f'(\alpha) > 0$ for $\alpha \in (0, 1)$. Now focus on auctions with $T_l > R$ and denote the quantile function of transaction price in these auctions by $T_2(\cdot)$. Then we have

$$\begin{aligned} \mathbb{P}(T_2 < V(\alpha)) &= \mathbb{P}(v^{N-1:N} < V(\alpha) \mid v^{N-1:N} > V(\alpha^*)) \\ &= \frac{\mathbb{P}(v^{N-1:N} < V(\alpha)) - \mathbb{P}(v^{N-1:N} \leq V(\alpha^*))}{1 - \mathbb{P}(v^{N-1:N} \leq V(\alpha^*))} \\ &= \frac{\phi(\alpha) - \phi(\alpha^*)}{1 - \phi(\alpha^*)}, \end{aligned}$$

where

$$\phi(\alpha) = N\alpha^{N-1} - (N - 1)\alpha^N.$$

Therefore, for $\alpha \geq \alpha^*$, $V(\alpha)$ can be identified by

$$V(\alpha) = T_2 \left(\frac{\phi(\alpha) - \phi(\alpha^*)}{1 - \phi(\alpha^*)} \right),$$

which completes the proof. $\square$

Proposition 1 establishes a disappointing yet important result: when the observed transaction prices are truncated due to binding reserves, the bidder's private value distribution in first-price auctions cannot be identified without additional information. In second-price auctions, by contrast, the proposition suggests that a two-step approach is required: one

must first recover the seller's screening level (which is also the truncation threshold) before identifying the distribution of bidders' private values. Given the prevalence of binding reserve prices in practice, this proposition has direct relevance for empirical auction studies.

The key to identification in this setting is the seller's screening level α^* . If $\alpha^* > 0$, standard methods recover only the conditional distribution of bidder values $V(\alpha \mid \alpha > \alpha^*)$ rather than the unconditional distribution $V(\alpha)$. Without knowledge of α^* , it is impossible to obtain the unconditional value distribution from the conditional distribution. Therefore, identification of the unconditional distribution of bidder's value requires external information that can pin down α^* . The identification strategy differs between auction formats. In second-price auctions, there is a mass point in the transaction price distribution at the reserve price, which provides information that can be used to recover α^* . The next proposition shows that in first-price auctions, if the number of invalid auctions are also observed, then this additional data can be used to identify α^* . As a result, $V(\alpha)$ for $\alpha \geq \alpha^*$ is also identified in first-price auctions.

Proposition 2: Suppose all auctions have a common and known $N \geq 2$. If T_l and L^{invalid} are observed, then α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified in first-price auctions.

Proof. Note that

$$\frac{L^{\text{invalid}}}{L + L^{\text{invalid}}} \xrightarrow{p} (\alpha^*)^N,$$

thus α^* is identified. Let $T(\cdot)$ denote the quantile function of T_l , then by (2), $V(\cdot)$ for all $\alpha \geq \alpha^*$ can be point identified by

$$\begin{aligned} V(\alpha) &= b(\alpha) + \frac{\alpha}{N-1} b'(\alpha) \\ &= T\left(\frac{\alpha^N - (\alpha^*)^N}{1 - (\alpha^*)^N}\right) + \frac{N}{N-1} \frac{\alpha^N}{1 - \alpha^*} T'\left(\frac{\alpha^N - (\alpha^*)^N}{1 - (\alpha^*)^N}\right), \end{aligned}$$

which completes the proof. $\square$

However, as noted earlier, invalid auctions are typically unobserved in real-world data, which limits the applicability of the identification strategy described above. Proposition 3 shows that identification of α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can still be achieved if N_l^{obs} is observed, even when N is unknown. This result holds for both first-price and second-price auctions.

Proposition 3: Suppose all auctions have a common $N \geq 2$. If the econometrician observes both T_l and N_l^{obs} , then α^* , N , and $V(\alpha)$ for $\alpha \geq \alpha^*$ are all point identified in both

first- and second-price auctions.

Proof. First note that

$$\max\{N_l^{\text{obs}}\} \xrightarrow{p} N,$$

thus N is identified. Also note that

$$\frac{1}{L} \sum_{l=1}^L \mathbb{1}(N_l^{\text{obs}} = N) \xrightarrow{p} \mathbb{P}(N^{\text{act}} = N \mid N^{\text{act}} \geq 1) = \frac{(1 - \alpha^*)^N}{1 - (\alpha^*)^N},$$

where N^{act} denotes the number of active bidders. Let

$$f(\alpha) = \frac{(1 - \alpha)^N}{1 - \alpha^N}.$$

Taking derivative, we have

$$\begin{aligned} f'(\alpha) &= \frac{-N(1 - \alpha)^{N-1}(1 - \alpha^N) + N(1 - \alpha)^N \alpha^{N-1}}{(1 - \alpha^N)^2} \\ &= \frac{N(1 - \alpha)^{N-1}}{(1 - \alpha^N)^2} [- (1 - \alpha^N) + (1 - \alpha)\alpha^{N-1}] \\ &= \frac{N(1 - \alpha)^{N-1}}{(1 - \alpha^N)^2} (\alpha^{N-1} - 1) \\ &< 0. \end{aligned}$$

Thus $f(\alpha)$ is strictly decreasing in $(0, 1)$, which implies α^* is point identified. Therefore, $V(\alpha)$ for $\alpha \geq \alpha^*$ can be point identified in both first- and second-price auctions by similar arguments as before. $\square$

One thing worth noting about Proposition 3 is, unlike Proposition 1, the mass point of the transaction price distribution at the reserve price is not required in second-price auctions. Therefore, α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified in second-price auctions even if T_l and N_l^{obs} are only observed in auctions with at least 2 active bidders.

3.3 Identification when N Varies

The previous section presented identification results under the assumption that all auctions have a common N , which may not always hold in real-world auction data. This section examines identification when N varies across auctions. Proposition 4 shows that if N varies across auctions and is known by the econometrician, then α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ are point identified in both first- and second-price auctions, even if only transaction prices strictly

above the reserve price are observed.

Proposition 4: Suppose N is known at at least two different values larger than 1. Then α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be point identified in both first- and second-price auctions if $T_l \mid T_l > R$ is observed.

Proof. Pick N_1, N_2 from the known N 's s.t. $N_1 > N_2 > 1$. Denote the corresponding quantile function of the transaction price by $T_1(\cdot)$ and $T_2(\cdot)$. From the previous part, we know that the optimal screening level doesn't change with N , thus $\alpha_1^* = \alpha_2^* = \alpha^*$. First consider first-price auctions. From equation (2), we have

$$\begin{aligned} V(\alpha) &= b_1(\alpha) + \frac{\alpha}{N_1 - 1} b_1'(\alpha) \\ &= T_1 \left(\frac{\alpha^{N_1} - (\alpha^*)^{N_1}}{1 - (\alpha^*)^{N_1}} \right) + \frac{N_1}{N_1 - 1} \frac{\alpha^{N_1}}{1 - (\alpha^*)^{N_1}} T_1' \left(\frac{\alpha^{N_1} - (\alpha^*)^{N_1}}{1 - (\alpha^*)^{N_1}} \right), \end{aligned}$$

and

$$\begin{aligned} V(\alpha) &= b_2(\alpha) + \frac{\alpha}{N_2 - 1} b_2'(\alpha) \\ &= T_2 \left(\frac{\alpha^{N_2} - (\alpha^*)^{N_2}}{1 - (\alpha^*)^{N_2}} \right) + \frac{N_2}{N_2 - 1} \frac{\alpha^{N_2}}{1 - (\alpha^*)^{N_2}} T_2' \left(\frac{\alpha^{N_2} - (\alpha^*)^{N_2}}{1 - (\alpha^*)^{N_2}} \right). \end{aligned}$$

If $T_1(\alpha^{N_1}) + \frac{N_1 \alpha^{N_1}}{N_1 - 1} T_1'(\alpha^{N_1}) = T_2(\alpha^{N_2}) + \frac{N_2 \alpha^{N_2}}{N_2 - 1} T_2'(\alpha^{N_2})$, then $\alpha^* = 0$ and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified by

$$V(\alpha) = T_1(\alpha^{N_1}) + \frac{N_1 \alpha^{N_1}}{N_1 - 1} T_1'(\alpha^{N_1}).$$

Otherwise, since $V(\alpha^*) = T_1(0) = T_2(0)$, there must be $T_1'(0) = T_2'(0) = 0$. Taking derivatives and let $\alpha = \alpha^*$, we then have

$$\begin{aligned} V'(\alpha^*) &= \frac{N_1}{N_1 - 1} \frac{(\alpha^*)^{N_1}}{1 - (\alpha^*)^{N_1}} \frac{N_1 (\alpha^*)^{N_1 - 1}}{1 - (\alpha^*)^{N_1}} T_1''(0) \\ &= \frac{N_2}{N_2 - 1} \frac{(\alpha^*)^{N_2}}{1 - (\alpha^*)^{N_2}} \frac{N_2 (\alpha^*)^{N_2 - 1}}{1 - (\alpha^*)^{N_2}} T_2''(0), \end{aligned}$$

which implies

$$\frac{N_1^2}{N_1 - 1} \left[\frac{(\alpha^*)^{N_1}}{1 - (\alpha^*)^{N_1}} \right]^2 T_1''(0) = \frac{N_2^2}{N_2 - 1} \left[\frac{(\alpha^*)^{N_2}}{1 - (\alpha^*)^{N_2}} \right]^2 T_2''(0).$$

Let

$$f(\alpha) = \alpha^{N_1 - N_2} \cdot \frac{1 - \alpha^{N_2}}{1 - \alpha^{N_1}}.$$

Since $\alpha \in (0, 1)$, $f(\alpha) > 0$, and thus

$$f(\alpha^*) = \left[\frac{N_2^2(N_1 - 1) T_2''(0)}{N_1^2(N_2 - 1) T_1''(0)} \right]^{\frac{1}{2}}.$$

Now we show f is strictly monotone on $(0, 1)$. Note that

$$f'(\alpha) = \frac{\alpha^{N_1 - N_2 - 1}}{(1 - \alpha^{N_1})^2} [N_1(1 - \alpha^{N_2}) - N_2(1 - \alpha^{N_1})],$$

thus it suffice to show $N_1(1 - \alpha^{N_2}) - N_2(1 - \alpha^{N_1})$ is strictly positive. Let

$$g(\alpha) = N_1(1 - \alpha^{N_2}) - N_2(1 - \alpha^{N_1}),$$

we have

$$g'(\alpha) = N_1 N_2 \alpha^{N_2 - 1} (\alpha^{N_1 - N_2} - 1) < 0.$$

Therefore, $g(\alpha) > g(1) = 0$ for all $\alpha \in (0, 1)$, which implies $f'(\alpha) > 0$, and thus α^* is identified. Then $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified by

$$V(\alpha) = T_1 \left(\frac{\alpha^{N_1} - (\alpha^*)^{N_1}}{1 - (\alpha^*)^{N_1}} \right) + \frac{N_1}{N_1 - 1} \frac{\alpha^{N_1}}{1 - (\alpha^*)^{N_1}} T_1' \left(\frac{\alpha^{N_1} - (\alpha^*)^{N_1}}{1 - (\alpha^*)^{N_1}} \right).$$

Now consider second-price auctions. Since we assume only auctions with transaction prices strictly higher than the reserve price are observed, we have $T_l = v_l^{N-1:N} \mid v_l^{N-1:N} > V(\alpha^*)$. Let

$$\begin{aligned} \phi_1(\alpha) &= N_1 \alpha^{N_1 - 1} - (N_1 - 1) \alpha^{N_1}, \\ \phi_2(\alpha) &= N_2 \alpha^{N_2 - 1} - (N_2 - 1) \alpha^{N_2}. \end{aligned}$$

Then for $\alpha \geq \alpha^*$, we have

$$V(\alpha) = T_1 \left(\frac{\phi_1(\alpha) - \phi_1(\alpha^*)}{1 - \phi_1(\alpha^*)} \right) = T_2 \left(\frac{\phi_2(\alpha) - \phi_2(\alpha^*)}{1 - \phi_2(\alpha^*)} \right).$$

If $T_1(\phi_1(\alpha)) = T_2(\phi_2(\alpha))$, then $\alpha^* = 0$ and $V(\alpha)$ can be identified by

$$V(\alpha) = T_1(\phi_1(\alpha)).$$

Otherwise, since $\alpha^* < 1$ (by Assumption A1.4), $\phi_1'(\alpha)$ and $\phi_2'(\alpha)$ are properly defined and strictly positive at α^* . Taking derivatives and evaluated at $\alpha = \alpha^*$, we have

$$V'(\alpha^*) = \frac{\phi_1'(\alpha^*)}{1 - \phi_1(\alpha^*)} T_1'(0) = \frac{\phi_2'(\alpha^*)}{1 - \phi_2(\alpha^*)} T_2'(0),$$

which implies

$$\frac{\phi'_1(\alpha^*)}{\phi'_2(\alpha^*)} \frac{1 - \phi_2(\alpha^*)}{1 - \phi_1(\alpha^*)} = \frac{T'_2(0)}{T'_1(0)}.$$

Note that

$$\phi'_1(\alpha) = N_1(N_1 - 1)\alpha^{N_1-2}(1 - \alpha), \quad \phi'_2(\alpha) = N_2(N_2 - 1)\alpha^{N_2-2}(1 - \alpha).$$

Then we have

$$\frac{\phi'_1(\alpha^*)}{\phi'_2(\alpha^*)} = \frac{N_1(N_1 - 1)}{N_2(N_2 - 1)}(\alpha^*)^{N_1-N_2},$$

and thus the condition above can be simplified to

$$(\alpha^*)^{N_1-N_2} \frac{1 - \phi_2(\alpha^*)}{1 - \phi_1(\alpha^*)} = \frac{N_2(N_2 - 1)}{N_1(N_1 - 1)} \frac{T'_2(0)}{T'_1(0)}.$$

Let

$$f(\alpha) = \alpha^{N_1-N_2} \frac{1 - \phi_2(\alpha)}{1 - \phi_1(\alpha)}.$$

Now we need to show $f(\alpha)$ is strictly monotone on $(0, 1)$. Note that

$$\begin{aligned} \alpha \phi'_1(\alpha) &= N_1(N_1 - 1)\alpha^{N_1-1}(1 - \alpha) = N_1 [\phi_1(\alpha) - \alpha^{N_1-1}], \\ \alpha \phi'_2(\alpha) &= N_2(N_2 - 1)\alpha^{N_2-1}(1 - \alpha) = N_2 [\phi_2(\alpha) - \alpha^{N_2-1}]. \end{aligned}$$

It can be shown

$$f'(\alpha) = \alpha^{N_1-N_2-1} \frac{1 - \phi_2(\alpha)}{1 - \phi_1(\alpha)} \left[N_1 \frac{1 - \alpha^{N_1-1}}{1 - \phi_1(\alpha)} - N_2 \frac{1 - \alpha^{N_2-1}}{1 - \phi_2(\alpha)} \right].$$

WLOG, suppose $N_1 > N_2$, then it suffices to show

$$N_1 \frac{1 - \alpha^{N_1-1}}{1 - \phi_1(\alpha)} > N_2 \frac{1 - \alpha^{N_2-1}}{1 - \phi_2(\alpha)}.$$

Let

$$g(\alpha; N) = \frac{N(1 - \alpha^{N-1})}{1 - N\alpha^{N-1} + (N-1)\alpha^N},$$

now it suffices to show $g(\alpha; N+1) > g(\alpha; N)$ for all N . Note that

$$g(\alpha; N+1) - g(\alpha; N) = \frac{1 - N^2\alpha^{N-1} + (2N^2 - 2)\alpha^N - N^2\alpha^{N+1} + \alpha^{2N}}{[1 - N\alpha^{N-1} + (N-1)\alpha^N][1 - (N+1)\alpha^N + N\alpha^{N+1}]},$$

and the numerator of $g(\alpha; N+1) - g(\alpha; N)$ can be written as

$$\begin{aligned} h(\alpha) &= (1 - \alpha^N)^2 - N^2\alpha^{N-1}(1 - \alpha)^2 \\ &= \alpha^{N-1}(1 - \alpha)^2 [(1 + \alpha + \alpha^2 + \dots + \alpha^{N-1})(1 + \alpha^{-1} + \alpha^{-2} + \dots + \alpha^{-(N-1)}) - N^2] \\ &= \alpha^{N-1}(1 - \alpha)^2 [N + (N-1)(\alpha + \alpha^{-1}) + \dots + (\alpha^{N-1} + \alpha^{-(N-1)}) - N^2] \\ &> \alpha^{N-1}(1 - \alpha)^2 [N + 2(N-1) + 2(N-2) + \dots + 2 - N^2] \\ &= 0, \end{aligned}$$

which guarantees $g(\alpha; N+1) > g(\alpha; N)$. Therefore, α^* is identified and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified by

$$V(\alpha) = T_1 \left(\frac{\phi_1(\alpha) - \phi_1(\alpha^*)}{1 - \phi_1(\alpha^*)} \right),$$

which completes the proof. $\square$

Proposition 4 shows that exogenous variation in N facilitates the identification of α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$, provided that N is known. In second price auctions, with the variation of N , the mass point at $T_l = R$ is not required. Notably, identification can be achieved with just two distinct known values of N , even if N takes on more than two values across auctions. Proposition 4 further implies if $T_l \mid T_l = R$ is also observed in second-price auctions, then the model is overidentified and thus testable.

However, in practice, the econometrician is more likely to observe the number of active bidders rather than the total number of potential bidders. When a binding reserve price is set by the seller, some bidders may be screened out, causing the number of active bidders to fall short of the true N . In such cases, using the observed number of bidders N_l^{obs} as a proxy for N_l may lead to biased estimation. The next proposition demonstrates that when only the transaction price and the number of active bidders are observed while the true N is unknown, then α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ remain point identified in first-price auctions but are not identified in second-price auctions.

Proposition 5: Assume N is random, finite, and unknown. Suppose T_l and N_l^{obs} are observed, then α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be point identified in first-price auctions, but not in second-price auctions.

Proof. First consider first-price auctions. Denote the quantile function of the transaction price conditional on $N^{\text{obs}} = k$ by $T_k(\cdot)$ and the bidder's bidding strategy when $N = k$ by $b_k(\cdot)$. Denote the maximum number of bidders by $\bar{N}$, then $\bar{N}$ can be identified by $\max\{N_l^{\text{obs}}\}$. First focus on auctions with $N_l^{\text{obs}} = \bar{N}$. Since there must be $N_l = \bar{N}$ when $N_l^{\text{obs}} = \bar{N}$, for all $\alpha > \alpha^*$, we have

$$\begin{aligned} \mathbb{P}(T_l < b_{\bar{N}}(\alpha) \mid N^{\text{obs}} = \bar{N}) &= \mathbb{P}(T_l < b_{\bar{N}}(\alpha) \mid N^{\text{obs}} = \bar{N}, N = \bar{N}) \\ &= \mathbb{P}(b_{\bar{N}}(\alpha_i^{\bar{N}:\bar{N}}) < b_{\bar{N}}(\alpha) \mid \alpha_i > \alpha^* \quad \forall i \leq \bar{N}) \\ &= [\mathbb{P}(\alpha_i < \alpha \mid \alpha_i > \alpha^*)]^{\bar{N}} \\ &= \left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}}, \end{aligned}$$

which implies

$$b_{\bar{N}}(\alpha) = T_{\bar{N}} \left(\left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}} \right).$$

Taking derivative, we get

$$b'_{\bar{N}}(\alpha) = T'_{\bar{N}} \left(\left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}} \right) \cdot \bar{N} \left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}-1} \frac{1}{1 - \alpha^*}.$$

Now focus on auctions with $N_l^{\text{obs}} = \bar{N} - 1$. Note that $b_{\bar{N}}(1) > b_{\bar{N}-1}(1)$, thus there exist $\beta \in (0, 1)$ s.t. $b_{\bar{N}}(\beta) = b_{\bar{N}-1}(1)$. Let $p = \mathbb{P}(N = \bar{N} \mid N^{\text{obs}} = \bar{N} - 1)$, then for $\alpha > \beta$, we have

$$\begin{aligned} \mathbb{P}(T < b_{\bar{N}}(\alpha) \mid N^{\text{obs}} = \bar{N} - 1) &= p \cdot \mathbb{P}(T < b_{\bar{N}}(\alpha) \mid N^{\text{obs}} = \bar{N} - 1, N = \bar{N}) + (1 - p) \\ &= p \cdot [\mathbb{P}(\alpha_i < \alpha \mid \alpha_i > \alpha^*)]^{\bar{N}-1} + (1 - p) \\ &= p \left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}-1} + (1 - p), \end{aligned}$$

which implies

$$b_{\bar{N}}(\alpha) = T_{\bar{N}-1} \left(p \left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}-1} + (1 - p) \right).$$

Taking derivative, we get

$$b'_{\bar{N}}(\alpha) = T'_{\bar{N}-1} \left(p \left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}-1} + (1 - p) \right) \cdot p(\bar{N} - 1) \left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}-2} \frac{1}{1 - \alpha^*}.$$

Let $\alpha = 1$, we then have

$$T'_{\bar{N}-1}(1) \cdot p(\bar{N} - 1) \frac{1}{1 - \alpha^*} = b'_{\bar{N}}(1) = T'_{\bar{N}}(1) \cdot \bar{N} \frac{1}{1 - \alpha^*},$$

which implies

$$p = \frac{\bar{N} T'_{\bar{N}}(1)}{(\bar{N} - 1) T'_{\bar{N}-1}(1)}.$$

Since RHS is observed, p is identified. Note that

$$p = \mathbb{P}(N = \bar{N} \mid N^{\text{act}} = \bar{N} - 1) = \frac{\mathbb{P}(N = \bar{N}, N^{\text{act}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{act}} = \bar{N} - 1)}$$

and

$$\frac{\mathbb{P}(N^{\text{obs}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{obs}} = \bar{N})} = \frac{\mathbb{P}(N^{\text{act}} = \bar{N} - 1 \mid N^{\text{act}} \geq 1)}{\mathbb{P}(N^{\text{act}} = \bar{N} \mid N^{\text{act}} \geq 1)} = \frac{\mathbb{P}(N^{\text{act}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{act}} = \bar{N})},$$

thus

$$\begin{aligned}
p \cdot \frac{\mathbb{P}(N^{\text{obs}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{obs}} = \bar{N})} &= \frac{\mathbb{P}(N = \bar{N}, N^{\text{act}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{act}} = \bar{N})} \\
&= \frac{\mathbb{P}(N = \bar{N})\mathbb{P}(N^{\text{act}} = \bar{N} - 1 \mid N = \bar{N})}{\mathbb{P}(N = \bar{N})\mathbb{P}(N^{\text{act}} = \bar{N} \mid N = \bar{N})} \\
&= \frac{\bar{N}\alpha^*(1 - \alpha^*)^{\bar{N}-1}}{(1 - \alpha^*)^{\bar{N}}} \\
&= \frac{\bar{N}\alpha^*}{1 - \alpha^*}
\end{aligned}$$

Since

$$\frac{\sum_{l=1}^L \mathbb{1}(N_l^{\text{obs}} = \bar{N} - 1)}{\sum_{l=1}^L \mathbb{1}(N_l^{\text{obs}} = \bar{N})} \xrightarrow{p} \frac{\mathbb{P}(N^{\text{obs}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{obs}} = \bar{N})}$$

and $g(\alpha) = \frac{\alpha}{1-\alpha}$ is strictly increasing, α^* is identified. Therefore, $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified by

$$\begin{aligned}
V(\alpha) &= b_{\bar{N}}(\alpha) + \frac{\alpha}{\bar{N} - 1} b'_{\bar{N}}(\alpha) \\
&= T_{\bar{N}} \left(\left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}} \right) + \frac{\bar{N}}{\bar{N} - 1} \left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}-1} \frac{\alpha}{1 - \alpha^*} T'_{\bar{N}} \left(\left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right)^{\bar{N}} \right).
\end{aligned}$$

To prove non-identification for second-price auctions, it suffices to provide a counter-example. Denote the bidder's private value by v_i and the reserve price by R . Consider the following two cases. In case 1, $N_l = 2$, $v_i \sim \text{Unif}[0, 1]$. $R = 0.5$, which implies $\alpha^* = 0.5$. Then we have

$$\begin{aligned}
\mathbb{P}(N_l^{\text{obs}} = 1) &= \frac{2(1 - \alpha^*)\alpha^*}{1 - (\alpha^*)^2} = \frac{2}{3}, \\
\mathbb{P}(N_l^{\text{obs}} = 2) &= \frac{(1 - \alpha^*)^2}{1 - (\alpha^*)^2} = \frac{1}{3}.
\end{aligned}$$

Moreover, we have

$$\mathbb{P}(T_l = 0.5 \mid N_l^{\text{obs}} = 1) = 1,$$

and

$$\mathbb{P}(T_l < t \mid N_l^{\text{obs}} = 2) = \frac{(t - 0.5)^2 + 2(t - 0.5)(1 - t)}{(1 - 0.5)^2} = -4t^2 + 8t - 3 \quad \forall t \in [0.5, 1].$$

In case 2, $N_l = 1$ with probability $p_1 = 0.4$, $N_l = 2$ with probability $p_2 = 0.6$, $v_i \sim \text{Unif}[\frac{1}{4}, 1]$. $R = 0.5$, which implies $\alpha^* = \frac{1}{3}$. Then

$$\begin{aligned}
\mathbb{P}(N_l^{\text{act}} = 0) &= p_1\alpha^* + p_2(\alpha^*)^2 = \frac{1}{5}, \\
\mathbb{P}(N_l^{\text{act}} = 1) &= p_1(1 - \alpha^*) + p_2 \cdot 2(1 - \alpha^*)\alpha^* = \frac{8}{15}, \\
\mathbb{P}(N_l^{\text{act}} = 2) &= p_2(1 - \alpha^*)^2 = \frac{4}{15},
\end{aligned}$$

which implies

$$\begin{aligned}\mathbb{P}(N_l^{\text{obs}} = 1) &= \frac{\mathbb{P}(N_l^{\text{act}} = 1)}{\mathbb{P}(N_l^{\text{act}} = 1) + \mathbb{P}(N_l^{\text{act}} = 2)} = \frac{2}{3}, \\ \mathbb{P}(N_l^{\text{obs}} = 2) &= \frac{\mathbb{P}(N_l^{\text{act}} = 2)}{\mathbb{P}(N_l^{\text{act}} = 1) + \mathbb{P}(N_l^{\text{act}} = 2)} = \frac{1}{3}.\end{aligned}$$

Besides, it can also be shown

$$\mathbb{P}(T_l = 0.5 \mid N_l^{\text{obs}} = 1) = 1,$$

and

$$\mathbb{P}(T_l < t \mid N_l^{\text{obs}} = 2) = \frac{(t - 0.5)^2 + 2(t - 0.5)(1 - t)}{(1 - 0.5)^2} = -4t^2 + 8t - 3 \quad \forall t \in [0.5, 1].$$

Therefore, case 1 and case 2 generate the same joint distribution of T_l and N_l^{obs} , which implies they are not distinguishable from the data. $\square$

In fact, $V(\alpha)$ for $\alpha \geq \alpha^*$ cannot be identified in second-price auctions even if all bids are observed. Again consider the previous example, it can be shown in both case 1 and case 2,

$$\mathbb{P}(B_{i,l} < b \mid N_l^{\text{obs}} = 1) = \mathbb{P}(B_{i,l} < b \mid N_l^{\text{obs}} = 2) = \frac{b - 0.5}{1 - 0.5} = 2b - 1 \quad \forall b \in [0.5, 1].$$

Thus in this model setup, observing all bids does not provide more information than merely observing T_l and N_l^{obs} . This non-identification result for second-price auctions is disappointing. However, the identification of α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be facilitated by observing invalid auctions, as is shown in Proposition 6.

Proposition 6: Suppose N is random, strictly positive, finite and not observed. Then α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be set-identified from T_l and N_l^{obs} in second-price auctions if L^{invalid} is also observed.

Proof. Let

$$P = \begin{pmatrix} 0 \\ \mathbb{P}(N = 1) \\ \vdots \\ \mathbb{P}(N = \bar{N}) \end{pmatrix}, \quad P^{\text{obs}} = \frac{1}{L + L^{\text{invalid}}} \begin{pmatrix} L^{\text{invalid}} \\ \sum_l \mathbb{1}(N_l^{\text{obs}} = 1) \\ \vdots \\ \sum_l \mathbb{1}(N_l^{\text{obs}} = \bar{N}) \end{pmatrix},$$

and

$$C(\alpha) = \begin{pmatrix} 1 & \alpha & \cdots & \alpha^{\bar{N}-1} & \alpha^{\bar{N}} \\ 0 & C_1^1(1-\alpha) & \cdots & C_{\bar{N}-1}^1(1-\alpha)\alpha^{\bar{N}-2} & C_{\bar{N}}^1(1-\alpha)\alpha^{\bar{N}-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & C_{\bar{N}-1}^{\bar{N}-1}(1-\alpha)^{\bar{N}-1} & C_{\bar{N}}^{\bar{N}-1}(1-\alpha)^{\bar{N}-1}\alpha \\ 0 & 0 & \cdots & 0 & C_{\bar{N}}^{\bar{N}}(1-\alpha)^{\bar{N}} \end{pmatrix}.$$

Then we have

$$P^{\text{obs}} \xrightarrow{P} C(\alpha^*)P.$$

There are $\bar{N} + 1$ equations and $\bar{N} + 1$ unknowns, thus α^* is at least set-identified and the identified set for α^* is

$$\left\{ \alpha \in [0, 1] \mid (C(\alpha)^{-1}P^{\text{obs}})_1 = 0, (C(\alpha)^{-1}P^{\text{obs}})_j \in [0, 1] \ \forall j = 2, 3, \dots, \bar{N} + 1 \right\}.$$

Let $T_{\bar{N}}(\cdot)$ denote the quantile function of the transaction price when $N^{\text{obs}} = \bar{N}$, then $V(\alpha)$ for $\alpha \geq \alpha^*$ is also at least set-identified by

$$V(\alpha) = T_{\bar{N}} \left(\phi_{\bar{N}} \left(\frac{\alpha - \alpha^*}{1 - \alpha^*} \right) \right),$$

where $\phi_{\bar{N}}(\alpha) = \bar{N}\alpha^{\bar{N}-1} - (\bar{N} - 1)\alpha^{\bar{N}}$. □

4 Identification in Auctions with Entry Costs

In this section, I study auctions with symmetric, IPV, and risk neutral bidders where bidders have to pay a fixed cost F to enter and submit bids. It is assumed that $0 < F < V(1)$, which ensures bidders with high types will choose to enter. The reserve price is assumed to be constant (if it exists) across all auctions and is normalized to 0.⁶ Similar to the binding reserve model, I refer to bidders who enter the auction and submit bids as active bidders, and I define auctions with at least one active bidder as valid auctions. Throughout this section, I assume only active bidders and the transaction prices in valid auctions can be observed, if not stated otherwise. Therefore, the structure of this entry model closely resembles that of the binding reserve price model in Section 3. This section establishes the identification results for the entry model. For consistency of notation, I also denote the observed transaction price by T_l , the number of active bidders by N_l^{act} , the observed number of bidders by N_l^{obs} ($N_l^{\text{obs}} = N_l^{\text{act}} \mid N_l^{\text{act}} \geq 1$), the number of valid auctions by L , and the number of invalid auctions by L^{invalid} .

⁶This is indeed just a normalization, as the entry decisions for bidders with private values $\{v_i\}_{i=1}^N$ in an auction with entry cost F and reserve R is equivalent to that for bidders with private values $\{v_i - R\}_{i=1}^N$ in an auction with entry cost F and reserve 0. This normalization simplifies notations in this section.

4.1 Bidding Strategy

In this part, I derive the bidder's equilibrium bidding strategy in first- and second-price auctions when such an entry cost exists. Although similar setup has been considered in Samuelson (1985), he studies a procurement model which cannot be directly applied here.

4.1.1 Bidding strategy in first-price auctions

Guess that there exists an equilibrium where bidders whose private values $\alpha_i < \alpha^*$ do not enter, and bidders whose private values $\alpha_i \geq \alpha^*$ bid $b(\alpha_i)$ which is strictly increasing in α_i with $b(\alpha^*) = 0$. Now I derive $b(\alpha)$ and show it is indeed an equilibrium. First, for bidders with $\alpha_i = \alpha^*$ to be indifferent with bidding 0 and not bidding, there must be

$$V(\alpha^*) \cdot (\alpha^*)^{N-1} = F.$$

Since LHS strictly increases with α^* and $0 < F < V(1)$, there exist a unique solution for α^* . For $b(\alpha)$ to be an equilibrium, it must be optimal for a bidder to bid $b(\alpha)$ when his true type is α . Note that by bidding $b(\beta)$ for some $\beta \geq \alpha^*$, this bidder's expected payoff is

$$\pi(\beta; \alpha) = [V(\alpha) - b(\beta)]\beta^{N-1} - F.$$

Taking derivative, we have

$$\frac{\partial \pi}{\partial \beta} = [V(\alpha) - b(\beta)](N-1)\beta^{N-2} - b'(\beta)\beta^{N-1}.$$

The FOC then gives

$$\left. \frac{\partial \pi}{\partial \beta} \right|_{\beta=\alpha} = [V(\alpha) - b(\alpha)](N-1)\alpha^{N-2} - b'(\alpha)\alpha^{N-1} = 0,$$

which implies

$$\frac{d}{d\alpha} [b(\alpha)\alpha^{N-1}] = V(\alpha) \cdot \frac{d}{d\alpha} (\alpha^{N-1}).$$

Since $b(\alpha^*) = 0$ and $V(\alpha^*)(\alpha^*)^{N-1} = F$, we have

$$\begin{aligned} b(\alpha) &= \frac{1}{\alpha^{N-1}} \int_{\alpha^*}^{\alpha} V(t) dt^{N-1} \\ &= V(\alpha) - \frac{1}{\alpha^{N-1}} \left[F + \int_{\alpha^*}^{\alpha} V'(t) t^{N-1} dt \right]. \end{aligned}$$

Now we verify this bidding strategy is indeed an equilibrium. Firstly, note that

$$\pi(\alpha; \alpha) = \int_{\alpha^*}^{\alpha} V'(t) t^{N-1} dt > 0 \quad \forall \alpha > \alpha^*.$$

Thus when $\alpha > \alpha^*$, bidding $b(\alpha)$ is better than not bidding. Secondly, since

$$\pi(\beta; \alpha) = \beta^{N-1}[V(\alpha) - V(\beta)] + \int_{\alpha^*}^{\beta} V'(t)t^{N-1}dt,$$

we have

$$\Delta(\beta; \alpha) \equiv \pi(\beta; \alpha) - \pi(\alpha; \alpha) = \beta^{N-1}[V(\alpha) - V(\beta)] + \int_{\alpha}^{\beta} V'(t)t^{N-1}dt.$$

Note that

$$\frac{\partial \Delta}{\partial \beta} = (N-1)\beta^{N-2}[V(\alpha) - V(\beta)],$$

thus

$$\begin{aligned} \frac{\partial \Delta}{\partial \beta} &> 0 \Leftrightarrow \beta < \alpha, \\ \frac{\partial \Delta}{\partial \beta} &< 0 \Leftrightarrow \beta > \alpha, \end{aligned}$$

which implies

$$\Delta(\beta; \alpha) \leq \Delta(\alpha; \alpha) = 0.$$

Therefore, when $\alpha > \alpha^*$, bidding $b(\alpha)$ is indeed optimal given that all other bidders play the same strategy. Finally, we need to show when $\alpha < \alpha^*$, it is optimal not to submit a bid. Suppose this bidder bid $b(\beta)$ for some $\beta \geq \alpha^*$, then his expected payoff

$$\pi(\beta; \alpha) < \pi(\beta; \alpha^*) = \beta^{N-1}[V(\alpha^*) - V(\beta)] \leq 0,$$

which completes the proof.

4.1.2 Bidding strategy in second-price auctions

If a bidder decides to enter a second-price auction, then it is always a weakly dominant strategy to bid his true value. Thus it suffices to determine the entry decision. Suppose only bidders with types $\alpha_i \geq \alpha^*$ enter, then the expected payoff for bidders with types $\alpha_i = \alpha^*$ equals to

$$\pi(\alpha^*) = V(\alpha^*)(\alpha^*)^{N-1} - F.$$

In equilibrium, there should be $\pi(\alpha^*) = 0$, which implies α^* is defined by

$$V(\alpha^*)(\alpha^*)^{N-1} = F.$$

By similar arguments as before, it can be checked that this is indeed an equilibrium.

4.2 Identification Results

In equilibrium, both binding reserve prices and entry costs can lead to the observed transaction prices being truncated from below. However, a key difference in the identification problem between auctions with binding reserve prices and those with entry costs is that, in the entry model, the bidder's equilibrium bidding threshold depends on the number of potential bidders N , whereas in the reserve price model, it does not. This dependence on N makes identification in this entry model significantly more challenging when N varies. In this section, I extend the identification results established for auctions with binding reserve prices to auctions with entry costs and demonstrate that analogous—though not identical—results continue to hold. Furthermore, the identification strategies also provide methods to test whether the entry cost F exists (is strictly positive).

Proposition 7: Suppose all auctions have a common and known $N \geq 2$, and a fixed entry cost F . If the econometrician only observes T_l , then α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified in second-price auctions, but not in first-price auctions. If L^{invalid} is also observed, then α^* , F and $V(\alpha)$ for $\alpha \geq \alpha^*$ can also be identified in first-price auctions.

Proof. Denote the observed quantile function of the transaction price by $T(\cdot)$. To show α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ are not identified in first-price auctions with entry costs, pick $\alpha_1 \neq \alpha_2$ and let

$$\begin{aligned} b_1(\alpha) &= T\left(\frac{\alpha^N - \alpha_1^N}{1 - \alpha_1^N}\right) \quad \forall \alpha \geq \alpha_1, \\ b_2(\alpha) &= T\left(\frac{\alpha^N - \alpha_2^N}{1 - \alpha_2^N}\right) \quad \forall \alpha \geq \alpha_2, \\ V_1(\alpha) &= b_1(\alpha) + \frac{\alpha}{N-1} b'_1(\alpha), \\ V_2(\alpha) &= b_2(\alpha) + \frac{\alpha}{N-1} b'_2(\alpha), \\ F_1 &= V_1(\alpha_1) \alpha_1^{N-1}, \\ F_2 &= V_2(\alpha_2) \alpha_2^{N-1}. \end{aligned}$$

By arguments similar to those in Proposition 1, it can be shown both $(\alpha_1, V_1(\cdot), F_1)$ and $(\alpha_2, V_2(\cdot), F_2)$ could generate $T(\cdot)$, which implies they are not distinguishable.

To show α^* and $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified in second-price auctions with entry

costs, note that

$$\begin{aligned} \frac{1}{L} \sum_{l=1}^L \mathbb{1}(T_l = 0) &\xrightarrow{p} \mathbb{P}(N^{\text{act}} = 1 \mid N^{\text{act}} \geq 1) \\ &= \frac{N(1 - \alpha^*)(\alpha^*)^{N-1}}{1 - (\alpha^*)^N}. \end{aligned}$$

By similar arguments that in Proposition 1, α^* is identified. Let $T_2(\cdot)$ denote the quantile function of $T_l \mid T_l > 0$, then $V(\alpha)$ for $\alpha \geq \alpha^*$ can be identified by

$$V(\alpha) = T_2 \left(\frac{\phi(\alpha) - \phi(\alpha^*)}{1 - \phi(\alpha^*)} \right),$$

and F can be identified by

$$F = V(\alpha^*) \cdot (\alpha^*)^{N-1},$$

where

$$\phi(\alpha) = N\alpha^{N-1} - (N-1)\alpha^N.$$

Finally, when L^{invalid} is observed in first-price auctions, α^* can be identified from

$$\frac{L^{\text{invalid}}}{L + L^{\text{invalid}}} \xrightarrow{p} (\alpha^*)^N.$$

Then $V(\cdot)$ for $\alpha \geq \alpha^*$ can be identified by

$$V(\alpha) = T \left(\frac{\alpha^N - (\alpha^*)^N}{1 - (\alpha^*)^N} \right) + \frac{N}{N-1} \frac{\alpha^N}{1 - \alpha^*} T' \left(\frac{\alpha^N - (\alpha^*)^N}{1 - (\alpha^*)^N} \right),$$

and F can be identified by

$$F = V(\alpha^*) \cdot (\alpha^*)^{N-1},$$

which completes the proof. $\square$

Proposition 8: Suppose all auctions have a common $N \geq 2$ and a fixed entry cost F . If we observe both T_l and N_l^{obs} , then N , α^* , F , and $V(\alpha)$ for $\alpha \geq \alpha^*$ are all point identified in both first- and second-price auctions.

Proof. By similar arguments as in Proposition 3, N can be identified by $\max\{N_l^{\text{obs}}\}$, and α^* can be identified from

$$\frac{1}{L} \sum_{l=1}^L \mathbb{1}(N_l^{\text{obs}} = N) \xrightarrow{p} \frac{(1 - \alpha^*)^N}{1 - (\alpha^*)^N}.$$

Thus F and $V(\alpha)$ for $\alpha \geq \alpha^*$ are identified using similar method as in Proposition 7. $\square$

Proposition 9: Suppose N is known at at least two different values larger than 1. Assume the auction has a fixed entry cost F . Then F and $V(\alpha)$ for α greater than some threshold can be at least set-identified in first- and second-price auctions if $T_l \mid T_l > 0$ is observed.

Proof. Pick two different observed numbers of bidders N_1, N_2 . Denote the bidding threshold at N_1 and N_2 by α_1 and α_2 . First consider first-price auctions. Denote the quantile function of transaction price for auctions with N_1 and N_2 observed bidders by $T_1(\alpha)$ and $T_2(\alpha)$. Denote the corresponding true bidding function by $b_1(\alpha)$ and $b_2(\alpha)$. Then we have

$$\begin{aligned} b_1(\alpha) &= T_1 \left(\frac{\alpha^{N_1} - \alpha_1^{N_1}}{1 - \alpha_1^{N_1}} \right) \quad \forall \alpha \geq \alpha_1, \\ b_2(\alpha) &= T_2 \left(\frac{\alpha^{N_2} - \alpha_2^{N_2}}{1 - \alpha_2^{N_2}} \right) \quad \forall \alpha \geq \alpha_2, \end{aligned}$$

which implies

$$\begin{aligned} V(\alpha) &= b_1(\alpha) + \frac{\alpha}{N_1 - 1} b_1'(\alpha) \\ &= T_1 \left(\frac{\alpha^{N_1} - \alpha_1^{N_1}}{1 - \alpha_1^{N_1}} \right) + \frac{\alpha}{N_1 - 1} \frac{N_1 \alpha^{N_1-1}}{1 - \alpha_1^{N_1}} T_1' \left(\frac{\alpha^{N_1} - \alpha_1^{N_1}}{1 - \alpha_1^{N_1}} \right), \end{aligned}$$

and

$$\begin{aligned} V(\alpha) &= b_2(\alpha) + \frac{\alpha}{N_2 - 1} b_2'(\alpha) \\ &= T_2 \left(\frac{\alpha^{N_2} - \alpha_2^{N_2}}{1 - \alpha_2^{N_2}} \right) + \frac{\alpha}{N_2 - 1} \frac{N_2 \alpha^{N_2-1}}{1 - \alpha_2^{N_2}} T_2' \left(\frac{\alpha^{N_2} - \alpha_2^{N_2}}{1 - \alpha_2^{N_2}} \right). \end{aligned}$$

Let $\alpha = 1$, we have

$$T_1(1) + \frac{1}{N_1 - 1} \frac{N_1}{1 - \alpha_1^{N_1}} T_1'(1) = V(1) = T_2(1) + \frac{1}{N_2 - 1} \frac{N_2}{1 - \alpha_2^{N_2}} T_2'(1).$$

Let $\alpha = \alpha_1$, we have

$$V(\alpha_1) = T_1(0) + \frac{N_1}{N_1 - 1} \frac{\alpha_1^{N_1}}{1 - \alpha_1^{N_1}} T_1'(0).$$

Let $\alpha = \alpha_2$, we have

$$V(\alpha_2) = T_2(0) + \frac{N_2}{N_2 - 1} \frac{\alpha_2^{N_2}}{1 - \alpha_2^{N_2}} T_2'(0).$$

Since

$$F = V(\alpha_1) \cdot \alpha_1^{N_1-1} = V(\alpha_2) \cdot \alpha_2^{N_2-1},$$

and $T_1(0) = T_2(0) = 0$, we have

$$\frac{N_1}{N_1 - 1} \frac{\alpha_1^{N_1}}{1 - \alpha_1^{N_1}} T_1'(0) \cdot \alpha_1^{N_1-1} = \frac{N_2}{N_2 - 1} \frac{\alpha_2^{N_2}}{1 - \alpha_2^{N_2}} T_2'(0) \cdot \alpha_2^{N_2-1}.$$

Combine them together, we have

$$\begin{cases} T_1(1) + \frac{N_1}{N_1 - 1} \frac{1}{1 - \alpha_1^{N_1}} T_1'(1) = T_2(1) + \frac{N_2}{N_2 - 1} \frac{1}{1 - \alpha_2^{N_2}} T_2'(1), \\ \frac{N_1}{N_1 - 1} \frac{\alpha_1^{2N_1-1}}{1 - \alpha_1^{N_1}} T_1'(0) = \frac{N_2}{N_2 - 1} \frac{\alpha_2^{2N_2-1}}{1 - \alpha_2^{N_2}} T_2'(0). \end{cases}$$

Since there are two equations and two unknowns, α_1 and α_2 are at least set-identified, and thus F and $V(\alpha)$ for $\alpha > \min\{\alpha_1, \alpha_2\}$ are at least set-identified. If these two equations give a unique solution, then they are point identified.

Now consider second-price auctions. Denote the quantile function of transaction price conditional on it being strictly positive for auctions with N_1 and N_2 observed bidders by $T_1(\alpha)$ and $T_2(\alpha)$. By similar arguments as above, we have

$$V(\alpha) = T_1 \left(\frac{\phi_1(\alpha) - \phi_1(\alpha_1)}{1 - \phi_1(\alpha_1)} \right) = T_2 \left(\frac{\phi_2(\alpha) - \phi_2(\alpha_2)}{1 - \phi_2(\alpha_2)} \right),$$

where

$$\begin{aligned} \phi_1(\alpha) &= N_1 \alpha^{N_1-1} - (N_1 - 1) \alpha^{N_1}, \\ \phi_2(\alpha) &= N_2 \alpha^{N_2-1} - (N_2 - 1) \alpha^{N_2}. \end{aligned}$$

WLOG, suppose $N_1 < N_2$ and thus $\alpha_1 < \alpha_2$. Let $\alpha = \alpha_2$, then

$$T_1 \left(\frac{\phi_1(\alpha_2) - \phi_1(\alpha_1)}{1 - \phi_1(\alpha_1)} \right) = T_2(0),$$

which implies

$$\frac{\phi_1(\alpha_2) - \phi_1(\alpha_1)}{1 - \phi_1(\alpha_1)} = T_1^{-1}(T_2(0)).$$

Moreover, we also have

$$T_1(0) \alpha_1^{N_1-1} = F = T_2(0) \alpha_2^{N_2-1}.$$

Thus we have two equations and two unknowns, which implies (α_1, α_2) is at least set identified, and thus F and $V(\alpha)$ for $\alpha > \alpha_2$ are also set identified. If these two equations give a unique solution, then they are point identified. $\square$

Proposition 10: Suppose N is random, finite, and not observed. Assume the auction has a fixed entry cost F . If T_l and N_l^{obs} are observed, then F and $V(\alpha)$ for α greater than

some threshold can be point identified in first-price auctions and at least set-identified in second-price auctions.

Proof. First consider first-price auctions. From the previous part, we know the equilibrium bidding strategy is

$$b_N(\alpha) = V(\alpha) - \frac{1}{\alpha^{N-1}} \left[F + \int_{\alpha_N^*}^{\alpha} V'(t) t^{N-1} dt \right],$$

thus

$$b_N(1) = V(1) - \left[F + \int_{\alpha_N^*}^1 V'(t) t^{N-1} dt \right].$$

Since α_N^* increases with N , $b_N(1)$ also increases with N . Note that $\bar{N}$ is identified by $\max\{N_l^{\text{obs}}\}$ and $b_{\bar{N}}(1)$ can be identified by $\max\{T_l \mid N^{\text{obs}} = \bar{N}\}$. Since $N^{\text{obs}} = \bar{N}$ implies $N = \bar{N}$, for $t \in [b_{\bar{N}-1}(1), b_{\bar{N}}(1)]$, we have

$$\mathbb{P}(T > t \mid N^{\text{obs}} = \bar{N}) = 1 - [\mathbb{P}(b_{\bar{N}}(\alpha_i) < t \mid \alpha_i > \alpha_{\bar{N}}^*)]^{\bar{N}}$$

and

$$\mathbb{P}(T > t \mid N^{\text{obs}} = \bar{N} - 1) = p \left\{ 1 - [\mathbb{P}(b_{\bar{N}}(\alpha_i) < t \mid \alpha_i > \alpha_{\bar{N}}^*)]^{\bar{N}-1} \right\}$$

where $p = \mathbb{P}(N = \bar{N} \mid N^{\text{obs}} = \bar{N} - 1)$. Since the LHS of these two equations are observed, p can be identified by letting $t \uparrow b_{\bar{N}}(1)$. Note that

$$p = \mathbb{P}(N = \bar{N} \mid N^{\text{act}} = \bar{N} - 1) = \frac{\mathbb{P}(N^{\text{act}} = \bar{N} - 1, N = \bar{N})}{\mathbb{P}(N^{\text{act}} = \bar{N} - 1)},$$

and

$$\frac{\mathbb{P}(N^{\text{obs}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{obs}} = \bar{N})} = \frac{\mathbb{P}(N^{\text{act}} = \bar{N} - 1 \mid N^{\text{act}} \geq 1)}{\mathbb{P}(N^{\text{act}} = \bar{N} \mid N^{\text{act}} \geq 1)} = \frac{\mathbb{P}(N^{\text{act}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{act}} = \bar{N})},$$

thus

$$\begin{aligned} p \cdot \frac{\mathbb{P}(N^{\text{obs}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{obs}} = \bar{N})} &= \frac{\mathbb{P}(N^{\text{act}} = \bar{N} - 1, N = \bar{N})}{\mathbb{P}(N^{\text{act}} = \bar{N})} \\ &= \frac{\mathbb{P}(N = \bar{N}) \mathbb{P}(N^{\text{act}} = \bar{N} - 1 \mid N = \bar{N})}{\mathbb{P}(N = \bar{N}) \mathbb{P}(N^{\text{act}} = \bar{N} \mid N = \bar{N})} \\ &= \frac{\bar{N} \alpha_{\bar{N}}^*}{1 - \alpha_{\bar{N}}^*}. \end{aligned}$$

Since LHS is observed, RHS is identified, which implies $\alpha_{\bar{N}}^*$ is identified. Therefore, F and $V(\alpha)$ for $\alpha \geq \alpha_{\bar{N}}^*$ can be identified by similar arguments in Proposition 5.

Now consider second-price auctions. Again we have α_N^* increases with N . Note that for $N \geq 2$, $V(\alpha_N^*)$ can be identified by $\min\{T_l \mid N_l^{\text{obs}} = N\}$, thus we have

$$V(\alpha_N^*) \cdot (\alpha_N^*)^{\bar{N}-1} = F = V(\alpha_{\bar{N}-1}^*) \cdot (\alpha_{\bar{N}-1}^*)^{\bar{N}-2}.$$

Again let $p = \mathbb{P}(N = \bar{N} \mid N^{\text{obs}} = \bar{N} - 1)$, then we have

$$\begin{aligned} \mathbb{P}(T_l < V(\alpha_N^*) \mid N_l^{\text{obs}} = \bar{N} - 1) &= (1 - p) \cdot \mathbb{P}(T_l < V(\alpha_N^*) \mid N_l^{\text{obs}} = \bar{N} - 1, N = \bar{N} - 1) \\ &= (1 - p) \cdot \phi_{\bar{N}-1} \left(\frac{\alpha_{\bar{N}}^* - \alpha_{\bar{N}-1}^*}{1 - \alpha_{\bar{N}-1}^*} \right) \end{aligned}$$

where

$$\phi_{\bar{N}-1}(\alpha) = (\bar{N} - 1)\alpha^{\bar{N}-2} - (\bar{N} - 2)\alpha^{\bar{N}-1}.$$

Let $Q = \frac{\mathbb{P}(N^{\text{obs}} = \bar{N} - 1)}{\mathbb{P}(N^{\text{obs}} = \bar{N})}$. By similar arguments as above, we have

$$\mathbb{P}(T_l < V(\alpha_N^*) \mid N_l^{\text{obs}} = \bar{N} - 1) \cdot Q = \left(Q - \frac{\bar{N}\alpha_{\bar{N}}^*}{1 - \alpha_{\bar{N}}^*} \right) \cdot \phi_{\bar{N}-1} \left(\frac{\alpha_{\bar{N}}^* - \alpha_{\bar{N}-1}^*}{1 - \alpha_{\bar{N}-1}^*} \right)$$

Thus we have two equations and two unknowns, which implies $(\alpha_{\bar{N}}^*, \alpha_{\bar{N}-1}^*)$ is at least set identified. By similar arguments in Proposition 6, F and $V(\alpha)$ for $\alpha > \alpha_{\bar{N}-1}^*$ are also set identified, and they will be point identified if these two equations give a unique solution. $\square$

5 Conclusion

While the presence of a binding reserve price is common in real-world auctions, research on the resulting problem of data truncation is surprisingly limited. This paper addresses this gap by establishing identification results for both first- and second-price auctions when the observed transaction price data are truncated by a binding reserve price. Moreover, it extends these results to auctions with entry costs, where only auctions with at least one entrant are observed. The identification results are summarized in the following table, where $\times$ means not identified, $\checkmark$ means point identified and set- $\checkmark$ means set-identified.

N type	Data type	Binding reserve		Entry costs	
		first-price	second-price	first-price	second-price
Fixed, known	T_l	$\times$	$\checkmark$	$\times$	$\checkmark$
Fixed, known	T_l, L^{invalid}	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Fixed, unknown	T_l, N_l^{obs}	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Varying, known	$T_l \mid T_l > R$	$\checkmark$	$\checkmark$	set- $\checkmark$	set- $\checkmark$
Varying, unknown	T_l, N_l^{obs}	$\checkmark$	$\times$	$\checkmark$	set- $\checkmark$
Varying, unknown	$T_l, N_l^{\text{obs}}, L^{\text{invalid}}$	$\checkmark$	set- $\checkmark$	$\checkmark$	set- $\checkmark$

The key to identification with truncated transaction prices hinges on knowing the truncation threshold. When the number of potential bidders N is fixed, and both N and the transaction prices are observed, the distribution of bidders' private values is identified in second-price auctions but not in first-price auctions—regardless of whether truncation arises from binding reserve prices or entry costs. The main distinction between the identification results for first- and second-price auctions lies in the behavior of the transaction price. In first-price auctions, the transaction price is continuous and provides no direct information about the bidder's truncation threshold. In contrast, in second-price auctions, the transaction price exhibit a mass point at the reserve price, which helps reveal the truncation threshold and thus facilitates identification. If invalid auctions are also observed, the distribution of private values is also identified in first-price auctions, as the fraction of invalid auctions contain information about the truncation threshold. Alternatively, if the number of active bidders is observed, then both N and the bidder's private value distribution can be identified; in this case, N can be inferred from $\max\{N_l^{\text{obs}}\}$, eliminating the need to know it ex ante.

When N varies and is known, the bidder's private value distribution can be point-identified from transaction prices in both first- and second-price auctions if truncation arises from a reserve price. However, this paper only establishes set identification for the bidder's private value distribution if truncation arises from entry costs. The difficulty in identifying the entry cost model is due to the dependence of the truncation threshold on N . When N varies and is unknown, the identification problem becomes significantly more complex. In first-price auctions, the bidder's private value distribution can be identified from T_l and N_l^{obs} regardless of whether truncation arises from binding reserve prices or entry costs. In second-price auctions, if T_l and N_l^{obs} are observed, the bidder's private value distribution is not identified if truncation is due to reserve price, but it is at least set-identified if truncation is due to entry costs. The difference in identification results for second-price auctions stems from the fact that the support of the transaction price varies with N under entry costs, but remains the same in the binding reserve case. This support variation provides information

to help identify the truncation threshold.

While this paper takes a step toward understanding identification in auctions with truncated transaction prices, there remains substantial work to be done. First, this study focuses exclusively on auctions with symmetric independent private valued (IPV) bidders. Extending the identification results to more general settings—such as asymmetric bidders, interdependent or correlated values, and auctions with unobserved heterogeneity—under truncated bids or transaction prices is left for future research. Moreover, this paper only addresses identification. The positive identification results in the paper appear constructive and should pave the way to the development of corresponding estimation methodologies.

References

- Adams, C. P. (2007). Estimating demand from ebay prices. *International Journal of Industrial Organization*, 25(6):1213–1232.
- An, Y., Hu, Y., and Shum, M. (2010). Estimating first-price auctions with an unknown number of bidders: A misclassification approach. *Journal of Econometrics*, 157(2):328–341.
- Aradillas-López, A., Gandhi, A., and Quint, D. (2013). Identification and inference in ascending auctions with correlated private values. *Econometrica*, 81(2):489–534.
- Athey, S. and Haile, P. A. (2002). Identification of standard auction models. *Econometrica*, 70(6):2107–2140.
- Campo, S., Guerre, E., Perrigne, I., and Vuong, Q. (2011). Semiparametric estimation of first-price auctions with risk-averse bidders. *The Review of Economic Studies*, 78(1):112–147.
- Campo, S., Perrigne, I., and Vuong, Q. (2003). Asymmetry in first-price auctions with affiliated private values. *Journal of Applied Econometrics*, 18(2):179–207.
- Chen, X., Gentry, M. L., Li, T., and Lu, J. (2025). Identification and inference in first-price auctions with risk averse bidders and selective entry. *The Review of Economic Studies*, 92(2).
- Choi, S., Nesheim, L., and Rasul, I. (2016). Reserve price effects in auctions: Estimates from multiple regression-discontinuity designs. *Economic Inquiry*, 54(1):294–314.
- Freyberger, J. and Larsen, B. J. (2022). Identification in ascending auctions, with an application to digital rights management. *Quantitative Economics*, 13(2):505–543.
- Gentry, M. and Li, T. (2014). Identification in auctions with selective entry. *Econometrica*, 82(1):315–344.
- Gimenes, N. (2017). Econometrics of ascending auctions by quantile regression. *Review of Economics and Statistics*, 99(5):944–953.
- Gimenes, N. and Guerre, E. (2020). Nonparametric identification of an interdependent value model with buyer covariates from first-price auction bids. *Journal of Econometrics*, 219(1):1–18.

- Gimenes, N. and Guerre, E. (2022). Quantile regression methods for first-price auctions. *Journal of Econometrics*, 226(2):224–247.
- Guerre, E., Perrigne, I., and Vuong, Q. (2000). Optimal nonparametric estimation of first-price auctions. *Econometrica*, 68(3):525–574.
- Guerre, E., Perrigne, I., and Vuong, Q. (2009). Nonparametric identification of risk aversion in first-price auctions under exclusion restrictions. *Econometrica*, 77(4):1193–1227.
- Haile, P. A. and Tamer, E. (2003). Inference with an incomplete model of english auctions. *Journal of Political Economy*, 111(1):1–51.
- Hendricks, K. and Porter, R. H. (2007). An empirical perspective on auctions. *Handbook of Industrial Organization*, 3:2073–2143.
- Hu, Y., McAdams, D., and Shum, M. (2013). Identification of first-price auctions with non-separable unobserved heterogeneity. *Journal of Econometrics*, 174(2):186–193.
- Krasnokutskaya, E. (2011). Identification and estimation of auction models with unobserved heterogeneity. *The Review of Economic Studies*, 78(1):293–327.
- Krishna, V. (2009). *Auction theory*. Academic press.
- Li, T. (2005). Econometrics of first-price auctions with entry and binding reservation prices. *Journal of Econometrics*, 126(1):173–200.
- Li, T., Perrigne, I., and Vuong, Q. (2002). Structural estimation of the affiliated private value auction model. *RAND Journal of Economics*, 33(2):171–193.
- Luo, Y. and Xiao, R. (2023). Identification of auction models using order statistics. *Journal of Econometrics*, 236(1):105457.
- Marmer, V., Shneyerov, A., and Xu, P. (2013). What model for entry in first-price auctions? A nonparametric approach. *Journal of Econometrics*, 176(1):46–58.
- McAfee, R. P. and McMillan, J. (1987). Auctions and bidding. *Journal of Economic Literature*, 25(2):699–738.
- Milgrom, P. R. and Weber, R. J. (1982). A theory of auctions and competitive bidding. *Econometrica*, 50(5):1089–1122.

- Myerson, R. B. (1981). Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73.
- Ostrovsky, M. and Schwarz, M. (2023). Reserve prices in internet advertising auctions: A field experiment. *Journal of Political Economy*, 131(12):3352–3376.
- Riley, J. G. and Samuelson, W. F. (1981). Optimal auctions. *The American Economic Review*, 71(3):381–392.
- Roberts, J. W. (2013). Unobserved heterogeneity and reserve prices in auctions. *The RAND Journal of Economics*, 44(4):712–732.
- Samuelson, W. F. (1985). Competitive bidding with entry costs. *Economics Letters*, 17(1-2):53–57.
- Shneyerov, A. and Wong, A. C. L. (2011). Identification in first-price and dutch auctions when the number of potential bidders is unobservable. *Games and Economic Behavior*, 72(2):574–582.
- Somaii, P. (2020). Identification in auction models with interdependent costs. *Journal of Political Economy*, 128(10):3820–3871.
- Song, U. (2004). *Nonparametric estimation of an eBay auction model with an unknown number of bidders*. Department of Economics, University of British Columbia.