

Multi-target quantum walk search on Johnson graph

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The discrete-time quantum walk on the Johnson graph $J(n, k)$ is a useful tool for performing target vertex searches with high success probability. This graph is defined by n distinct elements, with vertices being all the $\binom{n}{k}$ k -element subsets and two vertices are connected by an edge if they differ exactly by one element. However, most works in the literature focus solely on the search for a single target vertex on the Johnson graph. In this article, we utilize lackadaisical quantum walk—a form of discrete-time coined quantum walk with a weighted self-loop at each vertex of the graph—along with our recently proposed modified coin operator, C_g , to find multiple target vertices on the Johnson graph $J(n, k)$ for various values of k . Additionally, a comparison based on the numerical analysis of the performance of the C_g coin operator in searching for multiple target vertices on the Johnson graph, against various other frequently used coin operators by the discrete-time quantum walk search algorithms, shows that only C_g coin can search for multiple target vertices with a very high success probability in all the scenarios discussed in this article, outperforming other widely used coin operators in the literature.

Keywords: Quantum walk; Lackadaisical quantum walk; Spatial search; Johnson graph; Multi-target search

I. INTRODUCTION

Johnson graph, $J(n, k)$, [1] consists of a set of n distinct elements. The vertices of this graph are formed by taking k elements from the set, resulting in $N = \binom{n}{k}$ vertices, each having a degree of $d = k(n - k)$. Two vertices are considered nearest neighbor and connected by an edge if they differ by only one element. Several known graphs can be obtained from the Johnson graph by varying the parameter k . For example, the complete graph K_N can be obtained from the Johnson graph when $k = 1$, where it has $N = n$ vertices and each vertex has $d = n - 1$ edges. In fig. 1(left) a complete graph $K_5 = J(5, 1)$ with $n = 5$ elements $\{1, 2, 3, 4, 5\}$ is presented. In this case, each element corresponds to a vertex of the graph. Another example is the triangular graph, T_n , which can be obtained from the Johnson graph when $k = 2$. In fig. 1(right) a triangular graph $T_4 = J(4, 2)$ with $n = 4$ elements $\{1, 2, 3, 4\}$ is presented. In this case, the six vertices $\{12, 13, 14, 23, 24, 34\}$ are the 2-element subsets of the elements $\{1, 2, 3, 4\}$.

Johnson graph $J(n, k)$ is an interesting graph structure for the quantum computing community, particularly for the study of quantum walk [2]. It is the only graph that prevents the quasi-polynomial algorithm [3] for graph isomorphism from being polynomial. The Johnson graph has been used to study the element distinctness algorithm in both discrete-time [4] and continuous-time [5] quantum walk. Several studies on single target searches on Johnson graphs, $J(n, k)$, using both continuous-time [6] and discrete-time [7–10] quantum walks, have demonstrated that it is possible to conduct searches in optimal time with very high success probability.

In this paper, we study multi-target spatial search on

the Johnson graph using the discrete-time version of the quantum walk (QW) with our recently proposed coin C_g . We compare the results with the performance of other frequently used coins in the literature. It is noteworthy that the multi-target spatial search, which we study in this article, has applications in image processing [11] and the Johnson graph plays a crucial role in generating secure hash functions [12].

Quantum walk—a quantum counterpart of classical random walk—is a universal tool for quantum computation [13]. It has been used in several quantum algorithms, such as spatial search [2], element distinctness [4, 5], solving boolean formulas [14] and also in several applications, such as quantum hash function [15], and quantum edge detection [11]. Both continuous- [16] and discrete-time [17] versions of the quantum walk can be used to perform spatial search on a variety of different graphs. However, this generalization of the celebrated Grover search [18] to the spatial search on graphs in the form of a quantum walk was not that straightforward initially. This is because, in spatial search on graphs we are only allowed to shift from a vertex to the next nearest neighbor vertices at a time, where a vertex may not be a nearest neighbor to all the other vertices on the graph. It was argued by Paul Benioff [19] that the quantum search on a graph with N vertices will lose the quantum speedup, because, both the iterations and reflection need $\mathcal{O}(\sqrt{N})$ time each, making the total time complexity no better than the time for a classical exhaustive search on an unsorted database. However, the claim of Paul Benioff was later refuted in ref. [20] by showing that it is possible to do spatial search on a graph faster than a classical exhaustive search.

Note that, a classical computer takes $\mathcal{O}(N)$ time to search for a single target from an unsorted database of size N . Grover's original search and its generalizations [21] are quadratically faster [22] than the classical exhaustive search. The same quadratic speedup can also be achieved for spatial search by a quantum walk on sev-

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eral graphs [23–25].

For the quantum computer to be efficient, it is very important to have a high enough output probability of the desired result as well as fast computation time in order to accurately measure it in less time compared to its classical counterparts. Quantum walk search on graphs sometimes suffers on both counts, specifically, while searching for multiple target vertices on graphs. In general, quantum walk search on one- and two-dimensional square lattices cannot achieve the optimal speed of $\mathcal{O}(\sqrt{N})$ for quantum search. Although, time complexity of $\mathcal{O}(N)$ [26] for a one-dimensional periodic lattice is still quadratically faster than its corresponding time complexity for a classical random walk. Usually, in a two-dimensional periodic lattice, time complexity is $\sqrt{\log N}$ times greater than the optimal time for quantum search. However, in a two dimensional lattice, optimal speed can be achieved in continuous- [27] and discrete-time [28] quantum walk using extra long-range edges, such as the Hanoi network [29] of degree four, HN4.

Searching for multiple targets on graphs using a quantum walk is usually challenging in terms of the success probability of the target states and execution time. For example, a pair of adjacent target vertices [30–32] on a two-dimensional periodic lattice cannot be found by the quantum walk with the Grover coin, C_{grov} due to the existence of stationary states [33]. More generally, exceptional configurations of target vertices on a two-dimensional grid, which are of the form $2k \times m$ or $k \times 2m$, [34] for any positive k, m also cannot be found. Even a lackadaisical quantum walk [35, 36] with the Grover coin, C_l , [37] cannot find these configurations. Although, target vertices of the form $k \times m$, for both k, m being odd, can be found by a quantum walk search with the Grover coin. Similarly, the SKW coin, C_{skw} , cannot find target vertices arranged along the diagonal [37, 38] of a two-dimensional square lattice. Also configurations obtained by shifting and/or rotating the diagonal configurations by $\pi/2$ [34] cannot be found by the SKW coin. Sometimes, though the success probability is high, the running time increases when searching for multiple targets [39].

The problem related to searching for multiple targets in reasonably fast time including searching for the exceptional configurations can be solved by choosing a different coin operator, C_g [40], which searches for the self-loops of the target vertices. This operator works with only the discrete-time lackadaisical quantum walk [41]. It has been observed that C_g performs better compared to the other known coin operators previously studied in the literature. It can search for a single target as well as multiple target vertices with any configurations with high success probability.

This paper is arranged in the following fashion: A discussion on the quantum walk search on the Johnson graph is presented in Section II. Multi-target search on a complete graph is studied in Section III, on a triangular graph in Section IV and on $J(n, k \geq 3)$ graph in Section V. Finally, we conclude in Section VI with a discussion.

II. QW SEARCH ON JOHNSON GRAPH

In this section we study multi-target spatial search on a Johnson graph, $J(n, k)$ using discrete-time coined quantum walk. Because $J(n, k)$ is isomorphic to $J(n, n - k)$, we restrict ourself to $n \geq 2k$ case only. We represent N vertices and d edges of the Johnson graph as the basis states of the Hilbert space of vertices $\mathcal{H}_{vtx}$ and the space of edges $\mathcal{H}_{edg}$ respectively. The initial state of the vertex space is the uniform superposition of all the N basis states $|m\rangle$ of the Hilbert space $\mathcal{H}_{vtx}$:

$$|\psi_{vtx}\rangle = \frac{1}{\sqrt{N}} \sum_m |m\rangle. \quad (1)$$

Similarly the initial state for the coin space is the uniform superposition of all the edges associated with the vertex m

$$|\psi_{edg}\rangle = \frac{1}{\sqrt{d}} \sum_{m_n} |m_n\rangle, \quad (2)$$

where the summation is over all the d basis states $|m_n\rangle$ of the Hilbert space $\mathcal{H}_{edg}$. Note that m in $|m_n\rangle$ refers to the vertex label to which the edge basis state belongs and the suffix n refers to the nearest neighbor vertex label the edge basis state points to. Since discrete-time quantum walk evolves in the tensor product space $\mathcal{H} = \mathcal{H}_{vtx} \otimes \mathcal{H}_{edg}$, the initial state for the quantum walk process is given by

$$|\psi_{in}\rangle = |\psi_{vtx}\rangle \otimes |\psi_{edg}\rangle = \frac{1}{\sqrt{Nd}} \sum_m \sum_{m_n} |m\rangle \otimes |m_n\rangle. \quad (3)$$

The basis state $|m\rangle \otimes |m_n\rangle$, belonging to the tensor product space $\mathcal{H}$, represents state associated with the vertex m with the edge pointing towards the vertex n .

For the lackadaisical quantum walk, as mentioned before, we need to add a self-loop of weight l at each vertex of the graph, which creates an additional edge. This is the quantum analog of the lazy random walk in the classical regime. The self-loop allows the corresponding probability amplitude to stay put [35] at the same vertex, which helps to accumulate probability amplitude at the target vertices in quantum walk search. It has been observed that lackadaisical quantum walk can search for target vertices on a two-dimensional periodic lattice including other graphs with very high success probability without any additional amplitude amplification technique, which is usually required in case of standard quantum walk search (without self-loop) by Grover or SKW coins. The Hilbert space of the edge, $\mathcal{H}_{edg}$, becomes $d+1$ -dimensional, because of one additional edge of the self-loop. Initial state for the coin space at a vertex m is then given by

$$|\psi_{edg}\rangle = \frac{1}{\sqrt{d+l}} \left[\sum_{m_n \neq m_m} |m_n\rangle + \sqrt{l} |m_m\rangle \right], \quad (4)$$

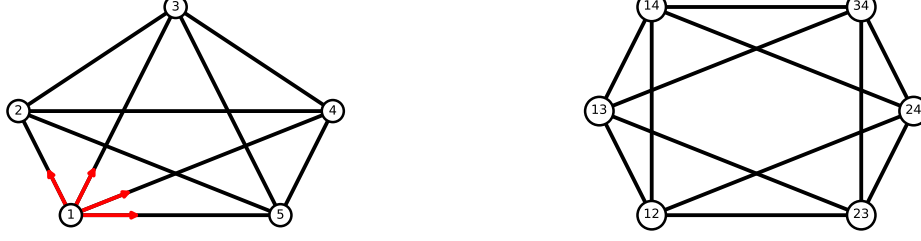


FIG. 1: Johnson graph $J(5,1)$ (left) and $J(4,2)$ (right).

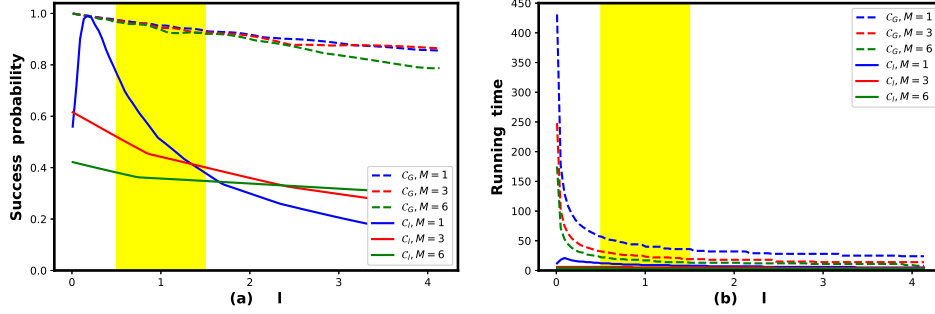


FIG. 2: (a) Variation of success probability and (b) running time as a function of the self-loop weight l for a Johnson graph $J(10,3)$

and the initial state for the quantum walk process is given by

$$|\psi_{in}\rangle = \frac{1}{\sqrt{N(d+l)}} \times \sum_m \left[\sum_{m_n \neq m_m} |m\rangle \otimes |m_n\rangle + \sqrt{l} |m\rangle \otimes |m_m\rangle \right]. \quad (5)$$

Depending on whether standard or lackadaisical quantum walk is involved in the search algorithm, we need to accordingly select the initial state between eq. (3) and eq. (5) respectively. Then the evolution operation $U = SC$, composed of modified coin operator C followed by the flip-flop shift operator S , is applied to the initial state repeatedly until the target states are obtained with high success probability.

Different types of coin operators exist, which are crucial for the quantum walk search. Let us assume that M target vertices collectively form a set $\mathcal{T}_M$, which we have to find out by quantum walk search. For simplicity of numerical evaluations, we chose first M vertices from the list of N vertices obtained from sorted tuples $combinations(n,p)$ for a Johnson graph $J(n,k)$. An example of the sorted order of vertices of the graph $J(4,2)$ in fig. 1(right) is $\{12, 13, 14, 23, 24, 34\}$.

Then the the modified coin operator C_g of ref. [40] acts

as

$$C_g |m\rangle \otimes |m_n\rangle = \begin{cases} C|m\rangle \otimes |m_n\rangle & \text{if } m \notin \mathcal{T}_M \\ C|m\rangle \otimes |m_n\rangle & \text{if } m \in \mathcal{T}_M \text{ and } m_n \neq m_m \\ -C|m\rangle \otimes |m_n\rangle & \text{if } m \in \mathcal{T}_M \text{ and } m_n = m_m \end{cases} \quad (6)$$

where $C = 2|\psi_{edg}\rangle\langle\psi_{edg}| - \mathbb{I}_{d+1 \times d+1}$ is the Grover diffusion operator. Note the difference of the above coin with the coin operator C_l of the lackadaisical quantum walk search

$$C_l |m\rangle \otimes |m_n\rangle = \begin{cases} C|m\rangle \otimes |m_n\rangle & \text{if } m \notin \mathcal{T}_M \\ -C|m\rangle \otimes |m_n\rangle & \text{if } m \in \mathcal{T}_M \end{cases} \quad (7)$$

The above two coin operators are used in lackadaisical quantum walk with a self-loop. The coin operator C_g does Grover search on the edge space of the target vertices with only the self-loop as the target, contrary to other known coin operators, which basically search for the target vertices, i.e., search for all the edge basis states of the target vertices. In effect, C_g mostly allows inward flow of the probability amplitude, with very small amount of probability amplitude going out.

There are other two well known coin operators C_{grov} and C_{skw} for the quantum walk search, which work on a graph with no self-loops. Grover coin C_{grov} acts as eq. (7) with the exception that now $C = 2|\psi_{edg}\rangle\langle\psi_{edg}| - \mathbb{I}_{d \times d}$

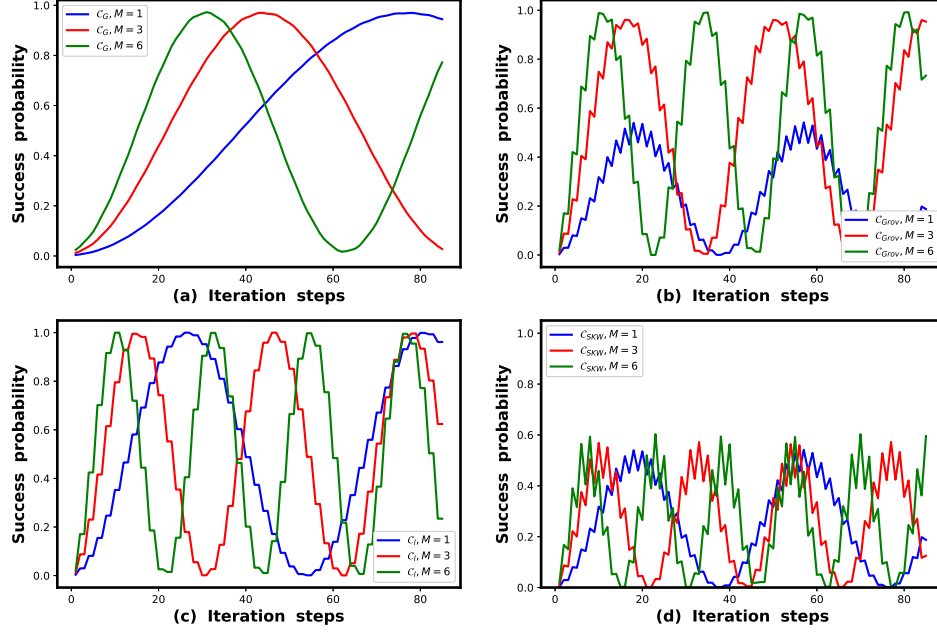


FIG. 3: Multi-target quantum walk search on a complete graph with $N = 300$ vertices and with (a) C_g , $l = 10$ (b) C_{grov} , (c) C_l , $l = 1$ and (d) C_{skw} coins for $M = 1$ (blue), 3(red), and 6(green) targets.

is obtained from eq. (2). So, C_l is the generalization of C_{grov} to lackadaisical quantum walk. Finally C_{skw} coin is given by

$$C_{skw}|m\rangle \otimes |m_n\rangle = \begin{cases} C|m\rangle \otimes |m_n\rangle & \text{if } m \notin \mathcal{T}_M \\ -I|m\rangle \otimes |m_n\rangle & \text{if } m \in \mathcal{T}_M \end{cases} \quad (8)$$

Note that, since only identity operator I acts on the target vertex state C_{skw} cannot have nontrivial generalization to lackadaisical quantum walk. The shift operator acts on the Nd basis states of the combined Hilbert space as

$$S|m\rangle \otimes |m_n\rangle = |n\rangle \otimes |m_n\rangle, \quad m \neq n. \quad (9)$$

In case of C_g and C_l , since there are self-loops in addition to the regular edges, the N basis states $|m\rangle \otimes |m_m\rangle$ with attached self-loop acted by the shift operator, remain same as

$$S|m\rangle \otimes |m_m\rangle = |m\rangle \otimes |m_m\rangle. \quad (10)$$

The action of the shift operator in eq. (9) can be better understood from its application on the complete graph $J(5,1)$ in fig. 1(left). The vertices $1, \dots, 5$ are represented by the basis states $|1\rangle \dots, |5\rangle$ of the vertex space. Each vertex has four associated edge basis states. For example, the vertex state $|1\rangle$ has four associated edge basis states $|1_2\rangle, |1_3\rangle, |1_4\rangle$ and $|1_5\rangle$, which are shown as the four red arrows from left to right respectively. Note that the suffixes 2, $\dots$, 5 of the four edge basis states represent the vertices they point to. The action of the shift operator S on the tensor product state $|1\rangle \otimes |1_2\rangle$

is as follows: $S|1\rangle \otimes |1_2\rangle = |2\rangle \otimes |2_1\rangle$. Similarly, S acts on all of the twenty basis states of the tensor product space $\mathcal{H}$. In lackadaisical quantum walk, additionally S acts on the vertex 1 with the self-loop in fig. 1(left) as $S|1\rangle \otimes |1_1\rangle = |1\rangle \otimes |1_1\rangle$. Note that in fig. 1 only graph without any self-loop has been presented. But when considering lackadaisical quantum walk, we have to keep in mind that there is a self-loop at each vertex of the graph.

The total success probability after t time steps to find one of the M targets, belonging to the set $\mathcal{T}_M$, is given by

$$p_s = \sum_{m \in \mathcal{T}_M} |\langle m | \mathcal{U}^t | \psi_{in} \rangle|^2. \quad (11)$$

In case of search with the lackadaisical quantum walk, we need to find an optimal value of the self-loop parameter l , so that the success probability is maximized. Two of the coin operators, C_g and C_l , discussed in this article are based on the lackadaisical quantum walk. In order to understand the role of the self-loop on quantum walk search on the Johnson graph $J(10,3)$, in figs. 2(a)-(b), we plot success probability and running time as a function of the self-loop weight l . We can see around the yellow strip region the success probability is very high and at the same time the running time saturates. So, we can fix a value for the self-loop in the yellow strip region and run the evolution operator to obtain the final success probability. Observe from fig. 2(a) that the success probability achieved by C_g coin is much higher than C_l . From the next section onward we will see that the success probability obtained by C_g is always higher than that obtained by C_l and others coins studied in this

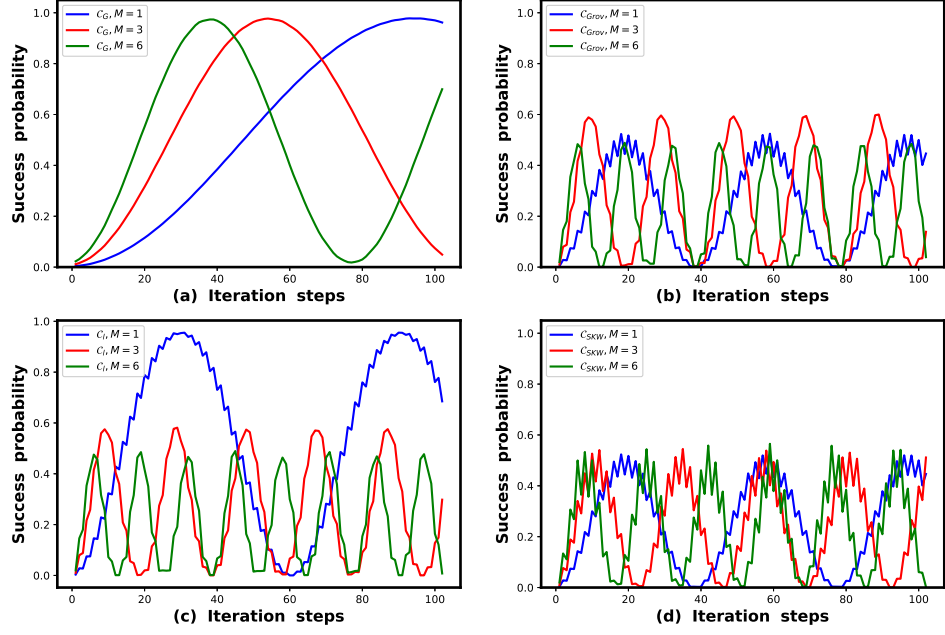


FIG. 4: Multi-target quantum walk search on a 25-triangular graph $J(25, 2)$ with $N = 300$ vertices and with (a) C_g , $l = 1$ (b) C_{grov} , (c) C_l , $l = 0.1$ and (d) C_{skw} coins for $M = 1$ (blue), 3(red), and 6(green) targets.

article.

Usually l depends on the graph, degree of the graph, number of targets M etc. Therefore, we need to obtain an optimum value of the self-loop by fixing the parameters of the Johnson graph $J(n, k)$ and the number of target states M and then running the exercise of fig. 2. However, to save time, we choose a fixed value of the self-loop parameter, that provides a reasonably high success probability, to study our multi-target quantum search.

From the next section onward we study multi-target search of the Johnson graph $J(n, k)$ for different k . Since, $k = 1$ and 2 correspond to the special graphs, known as the complete and triangular graph respectively, we devote the next two sections to study these two graphs first. Then we discuss $J(n, k \geq 3)$.

III. COMPLETE GRAPH

A complete graph K_n can be obtained by setting $k = 1$ in the Johnson graph $J(n, k)$. It has $N = n$ vertices and each vertex has $d = N - 1$ edges. It is one of the earliest graph on which quantum walk has been used to search for a target vertex. It can be shown that the quantum walk search with Grover coin on the complete graph with a self-loop at each vertices is equivalent to the Grover search on both the vertex space and the coin space [2]. Single target search on complete graph with lackadaisical quantum walk with equal wight self-loop at each vertex has been studied in ref. [42], which shows that for $l = 1$ success probability is $\mathcal{O}(1)$. In ref. [7] symmetry of the

graph is broken by using different self-loop weight in each vertex, which shows that only the wight of the self-loop at the marked vertex matters.

In this section, we consider multi-vertex search on this graph by lackadaisical quantum walk with C_g coin and compare the result with other coins. The initial state of the N -dimensional vertex space $\mathcal{H}_V = C^N$ is given by the uniform superposition of the basis states

$$|\psi_{vtx}\rangle = \frac{1}{\sqrt{N}} \sum_{m=1}^N |m\rangle. \quad (12)$$

Although, complete graph has $d = N - 1$ degree, the coin space becomes N -dimensional after we add one self-loop at each vertex of the graph. Then the initial state for the coin space $\mathcal{H}_C = C^N$ at a vertex m is given by

$$|\psi_{edg}\rangle = \frac{1}{\sqrt{N-1+l}} \left[\sum_{m_n \neq m_m} |m_n\rangle + \sqrt{l} |m_m\rangle \right]. \quad (13)$$

We have studied quantum search to find $M = 1, 3$ and 6 target vertices using four different coins and result of success probabilities to find these target vertices are plotted in fig. 3.

C_g coin: Success probabilities for $M = 1, 3$ and 6, represented by blue, red and green curves respectively, are plotted in fig. 3(a). We have fixed the self-loop weight at $l = 10$, which provides very high success probabilities in all the cases.

C_{grov} coin: As expected, running time to search a single target, $M = 1$ and success probability as represented

by the blue curve in fig. 3(b) agree with the analytical value $\pi\sqrt{N}/(2\sqrt{2})$ and 0.5 [8] respectively. Although success probabilities for $M = 3$ and 6, represented by red and green curves respectively, are very high.

C_l coin: In lackadaisical quantum walk with associated Grover coin we fix the self-loop weight at $l = 1$. Success probability for $M = 1, 3$ and 6, represented by blue, red and green curves in figs. 3(c) respectively, are all very high.

C_{skw} coin: With this coin success probabilities for $M = 1, 3$ and 6 are ~ 0.5 , represented by blue, red and green curves in figs. 3(d) respectively. Of course, we can increase the success probability further by using amplitude amplification technique [43].

IV. TRIANGULAR GRAPH

Triangular graph T_n can be obtained by setting $k = 2$ in the Johnson graph $J(n, k)$. It has $N = n(n-1)/2$ vertices and each vertex has $d = 2(n-2)$ edges. It is an example of a strongly regular graph $G(N, d, \lambda, \mu)$ [7] with the values of the four parameters being $N = n(n-1)/2$, $d = 2(n-2)$, $\lambda = n-2$, and $\mu = 4$. This system has been studied in ref. [7] to search for a single target vertex. It shows that for a specific value of the self-loop parameter the success probability of the lackadaisical quantum walk search can achieve $\mathcal{O}(1)$.

Bellow we report our result to search multiple targets on a 25-triangular/ $J(25, 2)$ graph using four coin operators. $J(25, 2)$ has $N = 300$ vertices and $d = 46$ nearest neighbor vertices at each vertex.

C_g coin: Success probabilities for $M = 1, 3$ and 6 target vertices, represented by blue, red and green curves respectively, are plotted in fig. 4(a). We have fixed the self-loop weight at $l = 1$, which provides very high success probabilities in all the three cases.

C_{grov} coin: As expected, running time to search a single target, $M = 1$ and success probability as represented by the blue curve in fig. 4(b) agree with the analytical value $\pi\sqrt{N}/(2\sqrt{2})$ and 0.5 [8] respectively. However, contrary to the complete graph case, now success probabilities for $M = 3$ and 6 target vertices, represented by red and green curves respectively, are ~ 0.5 .

C_l coin: In lackadaisical quantum walk with associated Grover coin we fix the self-loop weight at $l = 0.1$. Success probability for $M = 1$ target vertex, represented by blue curve in figs. 4(c), is very high. However, success probabilities for $M = 3$ and 6 target vertices are ~ 0.5 .

C_{skw} coin: With this coin, success probabilities for $M = 1, 3$ and 6 target vertices are ~ 0.5 , represented by blue, red and green curves in figs. 4(d) respectively. Of course, we can increase the success probability further by using amplitude amplification technique.

V. JOHNSON GRAPH $J(n, k \geq 3)$

In this section, we consider multi-target search on Johnson graphs for $k \geq 3$. Specifically, for $k = 3$, it is a tetrahedral graph with $N = n(n-1)(n-2)/6$ vertices and each vertex having $d = 3(n-3)$ nearest-neighbor vertices. It has been shown in ref. [6] that, like complete and triangular graph, this graph also allows fast quantum walk search for a single target with high success probability.

Bellow we report our result to search multiple targets on $J(13, 3)$, $J(13, 4)$, $J(13, 5)$, and $J(13, 6)$ Johnson graphs using four coin operators. From top to bottom, first row corresponds to $J(13, 3)$ with $N = 286$ vertices and $d = 30$ degree, second row corresponds to $J(13, 4)$ with $N = 715$ vertices and $d = 36$ degree, third row corresponds to $J(13, 5)$ with $N = 1287$ vertices and $d = 40$ degree, and fourth row corresponds to $J(13, 6)$ with $N = 1716$ vertices and $d = 42$ degree respectively. Left to right columns correspond to C_g , C_{grov} , C_l , and C_{skw} coins respectively.

C_g coin: Success probabilities for $M = 1, 3$ and 6, represented by blue, red and green curves respectively, are plotted in first column of fig. 5. We have fixed the self-loop weight at $l = 1$, which provides very high success probabilities in all the cases for all the four Johnson graphs.

C_{grov} coin: As expected, running time to search a single target, $M = 1$ and success probability as represented by the blue curve in second column of fig. 5 agree with the analytical value $\pi\sqrt{N}/(2\sqrt{2})$ and 0.5 [8] respectively. However, contrary to the complete graph case, now the success probabilities for $M = 3$ and 6 target vertices, represented by red and green curves respectively, gradually decrease bellow ~ 0.5 as $k \geq 3$ increases.

C_l coin: In lackadaisical quantum walk with associated Grover coin we fix the self-loop weight at $l = 1$ for all the four Johnson graphs. Success probabilities for $M = 1$, represented by blue curves in third column of figs. 5 are high. Note that the gradual decrease of success probability can be improved by choosing optimum self-loop weight for that particular Johnson graph. However, it is observed that the success probabilities for $M = 3$ and 6 target vertices, represented by red and green curves respectively, gradually decrease bellow ~ 0.5 as $k \geq 3$ increases—similar to the case observed by C_{grov} coin.

C_{skw} coin: Using this coin success probabilities for $M = 1, 3$ and 6 target vertices are ~ 0.5 , represented by blue, red and green curves in fourth column of fig. 5 respectively. Of course, we can increase the success probability further by using amplitude amplification technique.

VI. CONCLUSIONS

Discrete-time quantum walk is a widely used tool to perform spatial search for target vertices on several graphs. Although single target search has been successfully implemented on various graphs, searching for mul-

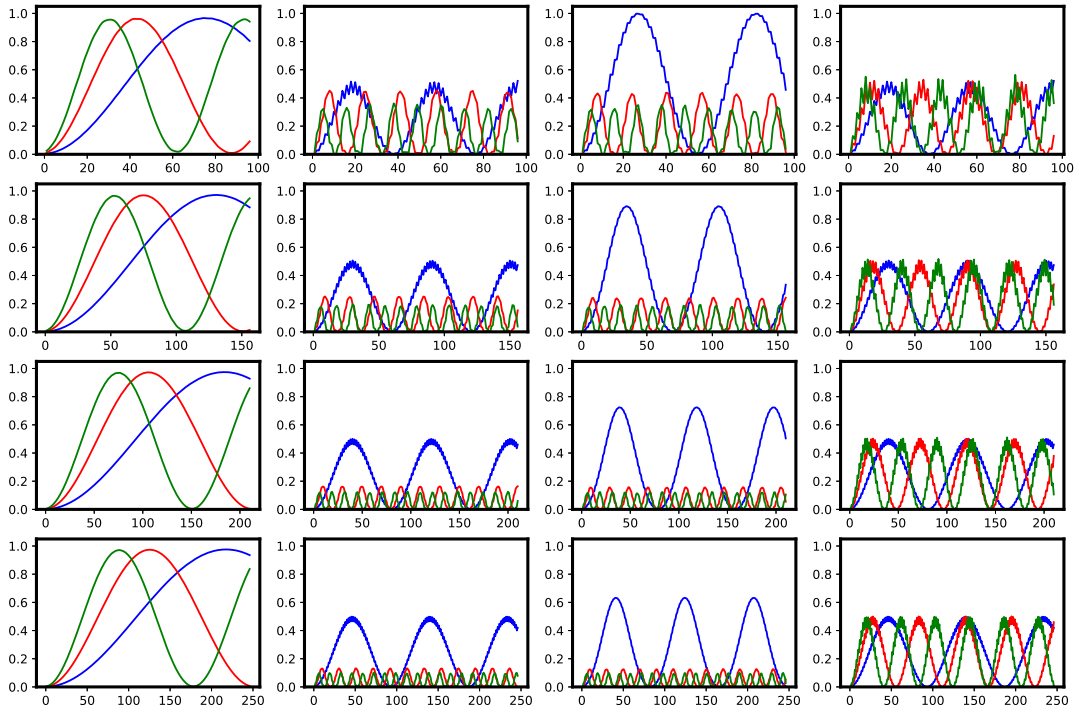


FIG. 5: Multi-target quantum walk search on Johnson graph for $M = 1$ (blue), 3(red), and 6(green) targets. Top to bottom rows correspond to the Johnson graphs $J(13,3)$, $J(13,4)$, $J(13,5)$ and $J(13,6)$ respectively. Left to right columns correspond to C_g , $l = 1.0$, C_{grov} , C_l , $l = 0.1$ and C_{skw} coins respectively.

multiple targets on graphs comes with challenges. For example, certain types of configurations of target vertices, known as exceptional configurations, are hard to search in both standard and lackadaisical quantum walk. Also, different coin operators, such as C_g , C_{grov} , C_l , and C_{skw} coins, behave differently while searching of target vertices on a graph.

In this article we explore multi-target spatial search on Johnson graphs $J(n, k)$ for different values of the parameter k . We numerically analyze and compare the performances of four different coin operators to search multiple target vertices on Johnson graphs. We observe that the C_g coin can search multiple targets on any Johnson graphs $J(n, k)$ discussed in this article with very high success probability. In the case of the C_{grov} coin, running time to search $M = 1$ target and its corresponding success probability agree with the analytical value $\pi\sqrt{N}/(2\sqrt{2})$ and 0.5 respectively for the the Johnson graphs discussed in this article. Although success probabilities for $M = 3$ and 6, represented by red and green curves respectively, become very high for $k = 1$ (complete graph), it gradually decrease below ~ 0.5 as k increases.

For $M = 3$ and 6 targets, a similar behavior is observed for the C_l coin as well. However, an $M = 1$ target can be searched by the C_l coin with high success probability for all the Johnson graphs. In the case of the C_{skw} coin, success probability to search $M = 1, 3$, and 6 is 0.5 for all the Johnson graphs. Observations from our numerical analysis suggest that among the four coins discussed in this article the C_g coin can search multiple targets with very high success probability on all the Johnson graphs.

As a future work, it would be interesting, though challenging, to analytically calculate the time complexity and success probability for multi-target search by quantum walk using the coin operators discussed in this article.

Data availability Statement: The data generated during and/or analyzed during the current study is included in the article.

Conflict of interest: The authors have no competing interests to declare that are relevant to the content of this article.

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