

NON-ALGEBRAICITY OF NON-ABUNDANT FOLIATIONS AND ABUNDANCE FOR ADJOINT FOLIATED STRUCTURES

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ABSTRACT. Assuming the abundance conjecture in dimension d , we establish a non-algebraicity criterion of foliations: any log canonical foliation of rank $\leq d$ with $\nu \neq \kappa$ is not algebraically integrable, answering question of Ambro–Cascini–Shokurov–Spicer. Under the same hypothesis, we prove abundance for klt algebraically integrable adjoint foliated structures of dimension $\leq d$ and show the existence of good minimal models or Mori fiber spaces. In particular, when $d = 3$, all these results hold unconditionally.

Using similar arguments, we solve a problem proposed by Lu and Wu on abundance of surface adjoint foliated structures that are not necessarily algebraically integrable.

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1. INTRODUCTION

We work over the field of complex numbers $\mathbb{C}$.

This paper studies two central problems in the study of foliations on complex manifolds: the *algebraicity* of the foliation and the *birational classification* of foliations.

Algebraicity of foliations. The algebraicity problem seeks criteria for the algebraic integrability of a foliation, i.e., when the general leaves of foliations are algebraic. For foliated surfaces, this question is also known as the “Poincaré problem” and its study can be traced back to the 19th century, when Darboux ([Dar78a, Dar78b]), Poincaré ([Poi85, Poi91]), Painlevé ([Pai92]), etc. studied the orbits of vector fields on $\mathbb{P}^2$. A famous theorem of Darboux–Jouanolou–Ghys [Dar78a, Jou78, Ghy00] indicates that a surface foliation $\mathcal{F}$ is algebraically integrable as long as it admits sufficiently many invariant algebraic curves. Moreover, it is known that a surface foliation $\mathcal{F}$ with non-pseudo-effective canonical divisor $K_{\mathcal{F}}$ is uniruled, hence algebraically integrable [Miy87]. A significant advancement on the Poincaré problem was recently obtained by Lü and Tan [LT24], where new birational invariants on foliated surfaces, motivated by the formulas for the modular invariants of

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families of algebraic curves (cf. [Tan96, Tan10]), were introduced, and several criteria for the algebraicity of surface foliations were established.

However, the algebraicity problem is substantially more difficult in higher dimensions. After [Miy87], the uniruled property for foliations with non-pseudo-effective canonical divisor was generalized to higher dimensions [Bos01, BM16, CP19], and very recently to the Kähler setting [Ou25a, CP25]. There are also significant advancements on the algebraicity of foliations with numerically trivial or anti-nef canonical bundles in recent years [LPT18, Dru18, HP19, Dru21, DO22, Ou25b]. Yet, we know very little about the algebraicity of foliations with “positive” (but not trivial) canonical classes.

Birational classification of foliations. The birational classification of foliations aims to classify foliations up to birational equivalence by considering the behavior of their birational invariants. The classification is in the same spirit as the Enriques classification of algebraic surfaces (going back to the Italian school) and the minimal model program: we first run a foliated minimal model program for the foliation, obtain a minimal model (or canonical model), and classify the foliations according to the Kodaira dimension $\kappa(\mathcal{F})$ and the numerical dimension $\nu(\mathcal{F})$. For foliated surfaces, thanks to Brunella [Bru15] and McQuillan [McQ08], the minimal model program and such a classification have been completed, and the canonical models (in the sense of [McQ08, Definition III.3.1]) of all non-general type canonical foliations on surfaces are explicitly described. In more recent works, the existence of the minimal model program has been established for threefolds (cf. [CS20, CS21, SS22]), algebraically integrable foliations (cf. [ACSS21, CHLX23, LMX24a, CS25a]), and partially for rank one foliations in arbitrary dimensions (cf. [McQ24, CS25b, CS25c]). See also [CM24, Li25].

Despite the aforementioned progress, one major issue for the classification of foliations is the failure of abundance, even for surfaces. More precisely:

- (1) (cf. [McQ08, Theorem IV.2.2]) There exists a canonical foliation $\mathcal{F}$ on a smooth projective surface X such that $K_{\mathcal{F}}$ is big and nef but $K_{\mathcal{F}}$ is not semi-ample. In particular, the canonical ring

$$R(X, K_{\mathcal{F}}) := \bigoplus_{m=0}^{+\infty} H^0(X, \mathcal{O}_X(mK_{\mathcal{F}}))$$

is not finitely generated.

- (2) (cf. [McQ08, Theorem IV.5.11]) There exists a canonical foliation $\mathcal{F}$ on a smooth projective surface X such that $K_{\mathcal{F}}$ is nef but

$$1 = \nu(\mathcal{F}) \neq \kappa(\mathcal{F}) = -\infty.$$

This is very different from the case of the birational geometry of varieties, as the finite generation and base-point-freeness theorem are both known (cf. [BCHM10, Theorem 1.2], [KMM87, Theorem 3-1-1]), while the abundance conjecture (Conjecture 2.30) predicts that $\nu(X) = \kappa(X)$ and has been proved in dimension ≤ 3 (cf. [Kaw92, KMM94]).

Main Theorems. The first main theorem of this paper is the following non-algebraicity criterion for foliations that are not abundant, i.e., $\nu(\mathcal{F}) \neq \kappa(\mathcal{F})$.

Theorem 1.1. *Assume the abundance conjecture (Conjecture 2.30) in dimension d . Let $\mathcal{F}$ be a log canonical foliation of rank $\leq d$ such that $\nu(\mathcal{F}) \neq \kappa(\mathcal{F})$.*

Then $\mathcal{F}$ is not algebraically integrable.

We refer the reader to Theorem 3.7 for a more general result.

Theorem 1.1 provides a positive answer to a question of Ambro, Cascini, Shokurov, and Spicer [ACSS21, Subsection 5.4] assuming the abundance conjecture. It also provides a new (conditional) non-algebraicity criterion: Note that, by the classification of surface foliations [Bru15, McQ08], non-abundant canonical foliations form a non-empty (although possibly small) class. In particular, we can show that surface canonical foliations $\mathcal{F}$ with $\nu(\mathcal{F}) = 1$ and $\kappa(\mathcal{F}) = -\infty$ are not algebraically integrable. This is already known via the classification [McQ08, Theorem IV.5.11(3)], yet by applying Theorem 1.1 we obtain a proof without classification. Moreover, since the abundance conjecture holds in dimension ≤ 3 [KMM94, 1.1 Theorem], we immediately obtain the following:

Corollary 1.2. *Let $\mathcal{F}$ be a log canonical foliation of rank ≤ 3 such that $\nu(\mathcal{F}) \neq \kappa(\mathcal{F})$. Then $\mathcal{F}$ is not algebraically integrable.*

Our next theorem concerns the existence of good minimal models of foliations. For smooth projective varieties X , the abundance conjecture, which predicts that $\nu(X) = \kappa(X)$, implies the existence of good minimal model or Mori fiber spaces of X (cf. [GL13, Theorem 4.3], [Lai11, Theorem 4.4]). However, this implication fails for foliations, even for algebraically integrable ones (cf. [McQ08, Theorem IV.2.2], [ACSS21, Examples 5.4, 5.5]). This is a big problem as we can no longer consider the ample models foliations and hence cannot deduce a nice boundedness and moduli theory (cf. [Pas24, Example 4.1]).

One very recent approach to tackle this issue is to consider *adjoint foliated structures*, i.e., instead of considering $K_{\mathcal{F}}$, we consider the structure $(X, \mathcal{F}, t)$ associated with the canonical divisor $K_t := tK_{\mathcal{F}} + (1 - t)K_X$. A lot of works have recently been done towards the birational geometry of adjoint foliated structures [PS19, SS23, Cas+24, LW24, Cas+25a, CC25, LWX25], and now there is substantial evidence that the birational geometry of K_t (with $t < 1$) behaves much better than the birational geometry of $K_{\mathcal{F}}$ and is closer to that of K_X . The most important piece of evidence is the existence of good minimal models, and hence the finite generation of the canonical ring, for adjoint foliated structures of general type in arbitrary dimensions [Cas+25a, Theorem A].

In this paper, we show that abundance holds for algebraically integrable adjoint foliated structures and that good minimal models and Mori fiber spaces always exist, provided that the usual abundance conjecture holds:

Theorem 1.3. *Assume the abundance conjecture (Conjecture 2.30) in dimension d . Let X be a smooth projective variety of dimension $\leq d$ and let $\mathcal{F}$ be a log canonical algebraically integrable foliation on X . We let $K_t := tK_{\mathcal{F}} + (1 - t)K_X$ for any $t \in [0, 1]$. Then:*

- (1) *For any $t \in [0, 1]$, $\nu(K_t) = \kappa_t(K_t)$. Here κ_t stands for the invariant Iitaka dimension (Definition 2.28) and we have $\kappa_t(K_t) = \kappa(K_t)$ when $t \in \mathbb{Q}$.*
- (2) *For any $t \in [0, 1]$, we may run a K_t -MMP which terminates with a good minimal model or a Mori fiber space of $(X, \mathcal{F}, t)$.*

We refer the reader to Theorem 4.4 for a more general result.

It is interesting to ask whether abundance is expected to hold for adjoint foliated structures that are not necessarily algebraically integrable. Based on the proof of Theorem 1.3, adjoint foliated structures $(X, \mathcal{F}, t)$ with abundant $K_{\mathcal{F}}$ can be treated in a similar way modulo the necessary MMPs, and the main difficulty comes from the non-algebraically integrable ones. For surfaces, however, thanks to the classification of non-abundant foliations [McQ08, Theorem IV.5.11], we are able to deduce abundance in full generality:

Theorem 1.4. *Let X be a smooth projective surface, $\mathcal{F}$ a canonical foliation on X , $t \in [0, 1)$ a real number, and $K_t := tK_{\mathcal{F}} + (1 - t)K_X$. Then $\nu(K_t) = \kappa_t(K_t)$, and if K_t is pseudo-effective, then $(X, \mathcal{F}, t)$ has a good minimal model.*

We remark that [LW24, Problem 1.9] asks whether there exists a relatively minimal foliation $\mathcal{F}$ on a smooth projective surface X such that $\kappa(K_X + K_{\mathcal{F}}) \neq \nu(K_X + K_{\mathcal{F}})$. Since any relatively minimal foliation is always canonical, Theorem 1.4 provides a negative answer to [LW24, Problem 1.9] by considering the special case of $t = \frac{1}{2}$.

Sketch of the proof. We first prove Theorem 1.1. For simplicity, assume that $\mathcal{F}$ is canonical. By [ACSS21, CHLX23] we may reduce to the case where $\mathcal{F}$ is induced by an equidimensional contraction $f: X \rightarrow Z$, X is BP stable/ Z , and $K_{\mathcal{F}}$ is semi-ample/ Z and nef. After taking a finite cover, we may further reduce to the case where f is a locally stable family, hence $K_{\mathcal{F}} = K_{X/Z}$. Let $g: X \rightarrow Y$ be the ample model/ Z of $K_{\mathcal{F}}$. Then

$$K_{\mathcal{F}} \sim_{\mathbb{Q}} g^*(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y)$$

where $(Y, \mathcal{F}_Y, B_Y, \mathbf{M})$ is a generalized foliated quadruple with $K_{\mathcal{F}_Y} = K_{Y/Z}$ and $g: (Y, B_Y, \mathbf{M}) \rightarrow Z$ is a stable family of generalized pairs.

If $\mathbf{M} = \mathbf{0}$, then by [KP17, Corollary 6.20] the $\mathbb{R}$ -divisor $K_{Y/Z} + B_Y$ is crepant to $K_{T/Z'} + B_T$, where $(T, B_T) \rightarrow Z'$ is a stable family of maximal variation with projective base Z' and lc general fiber. By [PX17, Proposition 2.15] we have that $K_{T/Z'} + B_T$ is big; hence $K_{Y/Z} + B_Y$ is abundant, and so is $K_{\mathcal{F}}$. For the general case, we follow carefully [FS22, Proof of Theorem 4.6]. Thanks to the recent proof of Prokhorov–Shokurov’s $\mathbf{b}$ -semi-ampleness conjecture [BFMT25, Theorem 1.5], we can take a finite open cover $\{U_i\}$ of Z , let V_i be the inverse image of U_i in Y , and choose $L_i \in |\mathbf{M}_Y|_{\mathbb{Q}}$ so that $(Y, B_Y + L_i)|_{V_i} \rightarrow U_i$ is a stable family of lc pairs. For any prime divisor P on Y we may choose $V = V_i$ so that the generic point of P is contained in V . By [KP17, Corollary 6.18], there exists a finite cover $V' \rightarrow V$ which compactifies over $U = U_i$ to $Y' \rightarrow \bar{U}$ (after a base change), and we deduce

$$p^*(K_{Y/Z} + B_Y + D) = q^*(K_{Y'/Z} + B' + L') + E$$

for some $E \geq 0$ with p_*E supported in $Y \setminus V$, where (p, q) is a resolution of indeterminacy of $Y \dashrightarrow Y'$ up to a cover of Y , $L = L_i$, and $(Y', B' + L') \rightarrow \bar{U}$ is a stable family of pairs. Since $K_{Y'/Z} + B' + L'$ is abundant, the asymptotic vanishing order of $K_{Y/Z} + B_Y + L$ along P is 0. This implies that $K_{Y/Z} + B_Y + L \sim_{\mathbb{Q}} K_{Y/Z} + B_Y + \mathbf{M}_Y$ is abundant, and therefore $K_{\mathcal{F}}$ is abundant. See Proposition 3.6 for details.

To deduce Theorems 1.3 and 1.4, we adapt the idea of [Cas+25a, Proof of Theorem 5.2], with minor modifications. We may assume that K_t is pseudo-effective. Let $\mathfrak{A}_s := (X, \mathcal{F}, s)$ for any $s \in [0, 1]$. Consider the sets

$$\mathcal{P} := \{s \in [t, 1] \mid \mathfrak{A}_s \text{ has a } \mathbb{Q}\text{-factorial good minimal model}\}$$

and

$$\mathcal{Q} := \{s \in [t, 1] \mid \mathfrak{A}_s \text{ has a } \mathbb{Q}\text{-factorial minimal model and } K_{\mathfrak{A}_s} \text{ is abundant}\}.$$

We prove the following:

- (i) If $\mu \in \mathcal{Q}$ and $\mu > t$, then there exists $\epsilon > 0$ such that $[\mu - \epsilon, \mu) \subset \mathcal{P}$.
- (ii) If $\mu < 1$ and $\mu + \epsilon \in \mathcal{P}$ for any $0 < \epsilon \ll 1$, then $\mu \in \mathcal{P}$.

The key input for both (i) and (ii) is the following observation: if $K_{\mathfrak{A}_s}$ is nef and abundant for some s , then for any $s' < s$,

$$\frac{s}{s-s'}K_{\mathfrak{A}_{s'}} \sim_{\mathbb{R}} K_{\mathfrak{A}_0} + \frac{s'}{s-s'}K_{\mathfrak{A}_s}$$

admits a klt generalized pair structure whose nef part is nef and abundant, and hence underlies a klt pair structure (cf. Lemma 4.2). In this situation, the usual abundance conjecture implies the abundance of $K_{\mathfrak{A}_{s'}}$. See Theorem 4.3 for details.

Combining (i) and (ii) with the inclusion $\mathcal{P} \subset \mathcal{Q}$, we deduce that if $\mathcal{Q} \cap (t, 1] \neq \emptyset$, then $t \in \mathcal{P}$, and we are done by using the fact that the existence of good minimal models implies the termination of the MMP with scaling (cf. [Cas+25b, Theorem 4.5]).

Theorem 1.3 now follows immediately from Theorem 1.1, since $1 \in \mathcal{Q}$. The proof of Theorem 1.4 is similar. If $1 \in \mathcal{Q}$ then we are done. If $1 \notin \mathcal{Q}$, then by the classification [McQ08, Theorem IV.5.11] we may assume that X is of general type, so $1 - \epsilon \in \mathcal{Q}$ for any $0 < \epsilon \ll 1$ as $K_{\mathfrak{A}_{1-\epsilon}}$ is big, and we are done. Of course, several preparatory results are needed for non-algebraically integrable adjoint foliated structures on surfaces (cf. Subsection 5.1).

Structure of the paper. In Section 2 we recall preliminary material and recall useful results we need later. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.3. Log versions of Theorems 1.1 and 1.3 are also established in the corresponding sections. In Section 5, we first develop auxiliary results on surface adjoint foliated structures, and then prove Theorem 1.4.

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2. PRELIMINARIES

We will adopt the standard notation and definitions on MMP in [Sho92, KM98, BCHM10] and use them freely. For adjoint foliated structures, generalized foliated quadruples, foliated triples, foliations, and generalized pairs, we adopt the notation and definitions in [Cas+24, Cas+25a] which generally align with [LLM23, CHLX23] (for generalized foliated quadruples), [CS20, ACSS21, CS21] (for foliations and foliated triples), and [BZ16, HL23] (for generalized pairs and $\mathbf{b}$ -divisors), possibly with minor differences. We refer the reader to [Kol23] for basic notations and properties of moduli theory but we shall recall some important concepts here.

2.1. Basic notations.

Definition 2.1. Let $\Gamma \subset [0, 1]$ be a set. We define

$$\Gamma_+ := \{0\} \cup \left(\left\{ \sum_{i=1}^n \gamma_i \mid n \in \mathbb{N}^+, \gamma_1, \dots, \gamma_n \in \Gamma \right\} \cap [0, 1] \right).$$

Definition 2.2. A *contraction* is a projective morphism $f : X \rightarrow Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. A *finite cover* is a finite surjective morphism.

Notation 2.3. Let $f : X \dashrightarrow X'$ be a birational map between normal schemes. We denote by $\text{Exc}(f)$ the reduced divisor supported on the codimension one part of the exceptional locus of f .

Definition 2.4. Let m be a positive integer and $\mathbf{v} \in \mathbb{R}^m$. The *rational envelope* of $\mathbf{v}$ is the minimal rational affine subspace of $\mathbb{R}^m$ which contains $\mathbf{v}$. For example, if $m = 2$ and $\mathbf{v} = \left(\frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right)$, then the rational envelope of $\mathbf{v}$ is $(x_1 + x_2 = 1) \subset \mathbb{R}_{x_1 x_2}^2$.

Definition 2.5. Let $X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a variety. Let D be an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X and $\phi : X \dashrightarrow X'$ a birational map/ U . We say that ϕ is *D-negative* (resp. *D-trivial*) if the following conditions hold:

- (1) ϕ does not extract any divisor.
- (2) $D' := \phi_* D$ is $\mathbb{R}$ -Cartier.
- (3) There exists a resolution of indeterminacy $p : W \rightarrow X$ and $q : W \rightarrow X'$, such that

$$p^* D = q^* D' + F$$

where $F \geq 0$ and $\text{Supp } p_* F = \text{Exc}(\phi)$ (resp. $F = 0$).

2.2. Foliation and adjoint foliated structures.

Definition 2.6 (Foliations, cf. [ACSS21, CS21]). Let X be a normal variety. A *foliation* on X is a coherent subsheaf $T_{\mathcal{F}} \subset T_X$ such that

- (1) $T_{\mathcal{F}}$ is saturated in T_X , i.e. $T_X/T_{\mathcal{F}}$ is torsion free, and
- (2) $T_{\mathcal{F}}$ is closed under the Lie bracket.

The *canonical divisor* of $\mathcal{F}$ is a divisor $K_{\mathcal{F}}$ such that $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(T_{\mathcal{F}})$. If $T_{\mathcal{F}} = 0$, then we say that $\mathcal{F}$ is a *foliation by points*.

Given any dominant map $h : Y \dashrightarrow X$ and a foliation $\mathcal{F}$ on X , we denote by $h^{-1}\mathcal{F}$ the *pullback* of $\mathcal{F}$ on Y as constructed in [Dru21, 3.2]. Given any birational map $g : X \dashrightarrow X'$, we denote by $g_*\mathcal{F} := (g^{-1})^{-1}\mathcal{F}$ the *pushforward* of $\mathcal{F}$ on X' . We say that $\mathcal{F}$ is an *algebraically integrable foliation* if there exists a dominant map $f : X \dashrightarrow Z$ such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$, where $\mathcal{F}_Z$ is the foliation by points on Z , and we say that $\mathcal{F}$ is *induced by* f .

A subvariety $S \subset X$ is called *$\mathcal{F}$ -invariant* if for any open subset $U \subset X$ and any section $\partial \in H^0(U, \mathcal{F})$, we have $\partial(\mathcal{I}_{S \cap U}) \subset \mathcal{I}_{S \cap U}$, where $\mathcal{I}_{S \cap U}$ denotes the ideal sheaf of $S \cap U$ in U . For any prime divisor P on X , we define $\epsilon_{\mathcal{F}}(P) := 1$ if P is not $\mathcal{F}$ -invariant and $\epsilon_{\mathcal{F}}(P) := 0$ if P is $\mathcal{F}$ -invariant. For any prime divisor E over X , we define $\epsilon_{\mathcal{F}}(E) := \epsilon_{\mathcal{F}_Y}(E)$ where $h : Y \dashrightarrow X$ is a birational map such that E is on Y and $\mathcal{F}_Y := h^{-1}\mathcal{F}$.

Suppose that the foliation structure $\mathcal{F}$ on X is clear in the context. Then, given an $\mathbb{R}$ -divisor $D = \sum a_i D_i$ where each D_i is a prime divisor, we denote by $D^{\text{nin}} := \sum \epsilon_{\mathcal{F}}(D_i) a_i D_i$ and $D^{\text{inv}} := D - D^{\text{nin}}$.

Definition 2.7. An *adjoint foliated structure* $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ is the datum of a normal quasi-projective variety X and a projective morphism $X \rightarrow U$, a foliation $\mathcal{F}$ on X , an $\mathbb{R}$ -divisor $B \geq 0$ on X , a nef/ U $\mathbf{b}$ -divisor $\mathbf{M}$, and a real number $t \in [0, 1]$ such that $K_{\mathfrak{A}} := tK_{\mathcal{F}} + (1 - t)K_X + B + \mathbf{M}_X$ is $\mathbb{R}$ -Cartier. We may simply say that “ $\mathfrak{A}/U$ is an adjoint foliated structure” without mentioning $X, \mathcal{F}, B, \mathbf{M}, t$ at all. X, t are called the *ambient variety* and *parameter* of $\mathfrak{A}$ respectively.

We say that $\mathfrak{A}/U$ is of *general type* if $K_{\mathfrak{A}}$ is big/ U . We say that $\mathfrak{A}$ is *algebraically integrable* if $\mathcal{F}$ is algebraically integrable.

For any $\mathbb{R}$ -divisor D on X and nef/ U $\mathbf{b}$ -divisor $\mathbf{N}$ on X such that $D + \mathbf{N}_X$ is $\mathbb{R}$ -Cartier, we denote by $(\mathfrak{A}, D, \mathbf{N}) := (X, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N}, t)$. If $D = 0$ then we may drop D , and if $\mathbf{N} = \mathbf{0}$ then we may drop $\mathbf{N}$.

When $t = 0$ or $\mathcal{F} = T_X$, we call $(X, B, \mathbf{M})/U$ a *generalized pair*, and in addition, if $\mathbf{M} = \mathbf{0}$, then we call $(X, B)/U$ a pair. If $B = 0$, or if $\mathbf{M} = \mathbf{0}$, or if U is not important, then we may drop $B, \mathbf{M}, U$ respectively. If $U = \{pt\}$ then we also drop U and say that $(X, \mathcal{F}, B, \mathbf{M}, t)$ is *projective*.

For any birational map/ U $\phi : X \dashrightarrow X'$, we define $\phi_*\mathfrak{A} := (X', \phi_*\mathcal{F}, \phi_*B, \mathbf{M}, t)$ and say that $\phi_*\mathfrak{A}$ is the *image* of $\mathfrak{A}$ on X' . For any projective birational morphism $h : X' \rightarrow X$, we define

$$h^*\mathfrak{A} := (X', \mathcal{F}', B', \mathbf{M}, t)$$

where $\mathcal{F}' := h^{-1}\mathcal{F}$ and B' is the unique $\mathbb{R}$ -divisor such that $K_{h^*\mathfrak{A}} = h^*K_{\mathfrak{A}}$. For any prime divisor E on X' , we denote by

$$a(E, \mathfrak{A}) := -\text{mult}_E B'$$

the *discrepancy* of E with respect to $\mathfrak{A}$. The *total minimal log discrepancy* of $\mathfrak{A}$ is

$$\text{tml}(\mathfrak{A}) := \inf\{a(E, \mathfrak{A}) + t\epsilon_{\mathcal{F}}(E) + (1 - t) \mid E \text{ is over } X\}.$$

For any non-negative real number ϵ , we say that $\mathfrak{A}$ is ϵ -lc (resp. ϵ -klt) if $\text{tml}(\mathfrak{A}) \geq \epsilon$ (resp. $> \epsilon$). We say that $\mathfrak{A}$ is lc (resp. klt) if $\mathfrak{A}$ is 0-lc (resp. 0-klt). We say that $\mathfrak{A}$ is *canonical* if $a(E, \mathfrak{A}) \geq 0$ for any prime divisor E that is exceptional/ X .

Definition 2.8 (Potentially klt). Let X be a normal quasi-projective variety. We say that X is *potentially klt* if (X, Δ) is klt for some $\mathbb{R}$ -divisor $\Delta \geq 0$.

Definition 2.9 ([Cas+25a, Definition 3.11]). Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ be an algebraically integrable adjoint foliated structure. We say that $\mathfrak{A}$ is *foliated log smooth* if there exists a contraction $f : X \rightarrow Z$ satisfying the following.

- (1) X has at most quotient singularities.
- (2) $\mathcal{F}$ is induced by f .
- (3) (X, Σ_X) is toroidal for some reduced divisor Σ_X such that $\text{Supp } B \subset \Sigma_X$. In particular, $(X, \text{Supp } B)$ is toroidal, and X is $\mathbb{Q}$ -factorial klt.
- (4) There exists a log smooth pair (Z, Σ_Z) such that

$$f : (X, \Sigma_X) \rightarrow (Z, \Sigma_Z)$$

is an equidimensional toroidal contraction.

- (5) $\mathbf{M}$ descends to X .

We say that $f : (X, \Sigma_X) \rightarrow (Z, \Sigma_Z)$ is *associated with* $\mathfrak{A}$.

Definition 2.10 (Foliated log resolutions). Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ be an algebraically integrable adjoint foliated structure. A *foliated log resolution* of $\mathfrak{A}$ is a birational morphism $h : X' \rightarrow X$ such that $(h_*^{-1}\mathfrak{A}, \text{Exc}(h))$ is foliated log smooth. By [CHLX23, Lemma 6.2.4], a foliated log resolution for $\mathfrak{A}$ always exists.

Definition 2.11 (Property $(*)$ foliations, [ACSS21, Definition 3.8], [CHLX23, Definition 7.2.2]). Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M})/U$ be a generalized foliated quadruple, $G \geq 0$ be a reduced divisor on X , and $f : X \rightarrow Z$ a contraction. We say that $(\mathfrak{A}; G)/Z$ satisfies *Property $(*)$* if the following conditions hold.

- (1) $\mathcal{F}$ is induced by f and G is an $\mathcal{F}$ -invariant divisor.
- (2) $f(G)$ is of pure codimension 1, $(Z, f(G))$ is log smooth, and $G = f^{-1}(f(G))$.

- (3) For any closed point $z \in Z$ and any reduced divisor $\Sigma \geq f(G)$ on Z such that (Z, Σ) is log smooth near z , $(X, B + G + f^*(\Sigma - f(G)), \mathbf{M})$ is lc over a neighborhood of z .

We say that f , Z , and G are *associated* with $\mathfrak{A}$.

Definition 2.12 (ACSS, cf. [CHLX23, Definitions 5.4.2, 7.2.2, 7.2.3]). Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M})/U$ be an lc generalized foliated quadruple, $G \geq 0$ a reduced divisor on X , and $f : X \rightarrow Z$ a contraction. We say that $(\mathfrak{A}; G)/Z$ is *ACSS* if the following conditions hold:

- (1) $(\mathfrak{A}; G)/Z$ satisfies Property (*).
- (2) f is equidimensional.
- (3) There exists an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor $D \geq 0$ on X and a nef/ X $\mathbf{b}$ -divisor $\mathbf{N}$ on X , such that $\text{Supp}\{B\} \subset \text{Supp } D$, $\mathbf{N} - \alpha \mathbf{M}$ is nef/ X for some $\alpha > 1$, and for any reduced divisor $\Sigma \geq f(G)$ such that (Z, Σ) is log smooth,

$$(X, B + D + G + f^*(\Sigma - f(G)), \mathbf{N})$$

is qdlt (cf. [CHLX23, Definition 7.1.1], [dFKX17, Definition 35]).

- (4) For any lc center of $\mathfrak{A}$ with generic point η , over a neighborhood of η ,
 - (a) η is the generic point of an lc center of $(X, \mathcal{F}, \lfloor B \rfloor)$, and
 - (b) $f : (X, B + G) \rightarrow (Z, f(G))$ is a toroidal morphism.

If $(\mathfrak{A}; G)/Z$ is ACSS, then we say that $\mathfrak{A}/Z$ and $\mathfrak{A}$ are ACSS.

Definition 2.13. Let X', X, Z be normal quasi-projective varieties and $h : X' \rightarrow X$, $f : X' \rightarrow Z$ contractions. The *core model* of (h, f) associated with $(\bar{h}, \bar{f})$ is the unique normal quasi-projective variety $\bar{X}$ up to isomorphism with two contractions $\bar{h} : \bar{X} \rightarrow X$ and $\bar{f} : \bar{X} \rightarrow Z$ satisfying the following.

- (1) For any ample $\mathbb{R}$ -divisor A on X , $\bar{h}^* A$ is ample/ Z .
- (2) There exists a contraction $g : X' \rightarrow \bar{X}$ such that $\bar{h} \circ g = h$ and $\bar{f} \circ g = f$.

The variety $\bar{X}$ is called the *core model* of (h, f) associated with $(\bar{h}, \bar{f})$. Existence of core model is guaranteed by [LMX24a, Definition-Lemma 3.1].

Definition 2.14 ([LMX24a, Subsection 3.2]). Let $\mathfrak{A} := (X, \mathcal{F}, B, \mathbf{M})$ and $\mathfrak{A}' := (X', \mathcal{F}', B', \mathbf{M})$ be two algebraically integrable generalized foliated quadruples and $h : X' \rightarrow X$ a birational morphism. Let $f : X' \rightarrow Z$ be a contraction and G a reduced divisor on X' . We say that $h : (\mathfrak{A}'; G)/Z \rightarrow \mathfrak{A}$ is a *simple modification* if the following conditions hold.

- $\mathfrak{A}' = (h_*^{-1} \mathfrak{A}, \text{Exc}(h)^{\text{nin}})$ and $\mathfrak{A}'$ is lc.
- $a(E, \mathfrak{A}) \leq -\epsilon_{\mathcal{F}}(E)$ for any h -exceptional prime divisor E .
- $K_{\mathfrak{A}'} \sim_Z K_{X'} + B' + \mathbf{M}_{X'} + G$.
- $(\mathfrak{A}'; G)/Z$ satisfies Property (*).

We say that $h : (X', \mathcal{F}', B', G)/Z \rightarrow \mathfrak{A}$ is

- (1) an *ACSS modification* if it is a simple modification and $(\mathfrak{A}'; G)/Z$ is ACSS, and we say that $\mathfrak{A}'$ is an *ACSS model* of $\mathfrak{A}$.
- (2) a *core modification* if it is a simple modification and $h^* A$ is ample/ Z for any ample $\mathbb{R}$ -divisor A on X , and we say that $\mathfrak{A}'$ is a *core model* of $\mathfrak{A}$, and
- (3) $\mathbb{Q}$ -factorial if X' is $\mathbb{Q}$ -factorial.

2.3. Models in the minimal model program.

Definition 2.15 (Log birational models). Let $\mathfrak{A}/U$ be an adjoint foliated structure with ambient variety X , $\phi : X \dashrightarrow X'$ a birational map/ U , and $E := \text{Exc}(\phi^{-1})$ the reduced ϕ^{-1} -exceptional divisor. Assume that $a(D, \mathfrak{A}) \leq -t\epsilon_{\mathcal{F}}(D) - (1-t)$ for any component D of E . We let

$$\mathfrak{A}' := \left(\phi_* \mathfrak{A}, - \sum_D a(D, \mathfrak{A}) D \right)$$

where the sum runs through all irreducible components of E . If $K_{\mathfrak{A}'}$ is $\mathbb{R}$ -Cartier then we say that $\mathfrak{A}'/U$ is a *log birational model* of $\mathfrak{A}/U$.

Definition 2.16 (Minimal models). Let $\mathfrak{A}/U$ be an adjoint foliated structure with ambient variety X and $\mathfrak{A}'/U$ a log birational model of $\mathfrak{A}/U$ with ambient variety X' and associated birational map $\phi : X \dashrightarrow X'$, such that $K_{\mathfrak{A}'}$ is nef/ U .

Let t be the parameter of $\mathfrak{A}$.

- (1) We say that $\mathfrak{A}'/U$ is a *bs-weak lc model* or *weak lc model in the sense of Birkar-Shokurov* of $\mathfrak{A}/U$, if for any prime divisor D on X which is exceptional over X' ,

$$a(D, \mathfrak{A}) \leq a(D, \mathfrak{A}').$$

We also say that ϕ is a bs-weak lc model of $\mathfrak{A}/U$.

- (2) We say that $\mathfrak{A}'/U$ is a *bs-minimal model* or *minimal model in the sense of Birkar-Shokurov* of $\mathfrak{A}/U$, if for any prime divisor D on X which is exceptional over X' ,

$$a(D, \mathfrak{A}) < a(D, \mathfrak{A}').$$

We also say that ϕ is a bs-minimal model of $\mathfrak{A}/U$.

- (3) We say that $\mathfrak{A}'/U$ is a *bs-semi-ample model* or *semi-ample model in the sense of Birkar-Shokurov* of $\mathfrak{A}/U$ if it is a bs-weak lc model of $\mathfrak{A}/U$ and $K_{\mathfrak{A}'}$ is semi-ample/ U . We also say that ϕ is a bs-semi-ample model of $\mathfrak{A}/U$.

- (4) We say that $\mathfrak{A}'/U$ is a *bs-good minimal model* or *good minimal model in the sense of Birkar-Shokurov* of $\mathfrak{A}/U$ if it is a bs-minimal model of $\mathfrak{A}/U$ and $K_{\mathfrak{A}'}$ is semi-ample/ U . We also say that ϕ is a bs-good minimal model of $\mathfrak{A}/U$.

If, in addition, the induced birational map $X \dashrightarrow X'$ does not extract any divisor, then we remove the initial “bs-” or the phrase “in the sense of Birkar-Shokurov” in the previous definitions.

Definition 2.17 (Mori fiber space). Let $\mathfrak{A}/U$ be an adjoint foliated structure with ambient variety X and $\phi : X \dashrightarrow X'$ a birational map/ U . Assume that ϕ is $K_{\mathfrak{A}}$ -negative, $\mathfrak{A}' := \phi_* \mathfrak{A}$, and let $X \rightarrow Z$ be a $K_{\mathfrak{A}'}$ -Mori fiber space/ U . Then we say that $\mathfrak{A}' \rightarrow Z$ is a *Mori fiber space* of $\mathfrak{A}/U$.

We need the following two auxiliary results.

Lemma 2.18 ([Cas+24, Lemma 3.16(2)]). *Let $\mathfrak{A}/U$ be an adjoint foliated structure with ambient variety X and let $\phi : X \dashrightarrow X'$ be a birational map/ U which does not extract any divisor. Let $\mathfrak{A}' := \phi_* \mathfrak{A}$. Assume that $K_{\mathfrak{A}'}$ is nef/ U .*

If $a(D, \mathfrak{A}) < a(D, \mathfrak{A}')$ for any prime divisor D on X that is exceptional/ X' , then $\mathfrak{A}'/U$ is a minimal model of $\mathfrak{A}/U$.

Lemma 2.19 (cf. [LMX24a, Lemma 5.4]). *Let $(X, B)/U$ be an lc pair. Let $H \geq 0$ be an $\mathbb{R}$ -divisor on X such that $(X, B + H)$ is lc and $K_X + B + H$ is nef/ U . Assume that*

for any $\mu \in [0, 1]$, either $(X, B + \mu H)/U$ has a log minimal model, or $K_X + B + \mu H$ is not pseudo-effective/ U . Then there exists a $(K_X + B)$ -MMP/ U with scaling of H which terminates with a minimal model of $(X, B)/U$ after finitely many steps.

2.4. Locally stable families.

Definition 2.20 (Slc). A *semi-pair* (X, B) consists of a demi-normal scheme X and an $\mathbb{R}$ -divisor $B \geq 0$ such that $K_X + B$ is $\mathbb{R}$ -Cartier. Let $\nu : X^\nu \rightarrow X$ be the normalization of X and let $D \subset X$, $D^\nu \subset X^\nu$ be the conductors. We say that (X, B) is *slc* if

- (1) $\text{Supp } B$ does not contain any irreducible component of the conductor D .
- (2) $(X^\nu, B^\nu + D^\nu)$ is lc, where B^ν is the divisorial part of $\nu^{-1}(B)$.

We say that (X, B) is *stable* if (X, B) is projective, slc, and $K_X + B$ is ample.

Definition 2.21 (Mumford divisor). Let d be a positive integer and let $f : X \rightarrow Z$ be a flat morphism over a reduced scheme such that the fibers of f are reduced, connected, S_2 , and of pure dimension d . A *Mumford divisor*/ Z is a divisor D on X satisfying the following.

- (1) (Equidimensionality) Every irreducible component of $\text{Supp } D$ dominates an irreducible component of Z , and all nonempty fibers of the induced morphism $f|_D : \text{Supp } D \rightarrow Z$ are of pure dimension $d - 1$.
- (2) (Mumford) f is smooth near the generic point of $f^{-1}(z) \cap \text{Supp } D$ for any point $z \in Z$.
- (3) (Generic Cartier) D is Cartier near the generic points of $f^{-1}(z) \cap \text{Supp } D$ for any point $z \in Z$.

An $\mathbb{R}$ -divisor B on X is called a *Mumford $\mathbb{R}$ -divisor*/ Z if $B = \sum a_i D_i$, where each $a_i \geq 0$ and each D_i is a Mumford divisor/ Z .

Definition 2.22 (Locally stable family). A *locally stable family* $f : (X, B) \rightarrow Z$ consists of a flat projective morphism $f : X \rightarrow Z$ over a reduced scheme with demi-normal and connected fibers, and a Mumford $\mathbb{R}$ -divisor/ Z B on X , such that

- (1) $K_{X/Z} + B$ is $\mathbb{R}$ -Cartier, and
- (2) (X_z, B_z) is slc for any $z \in Z$, where $X_z = f^{-1}(z)$ and $B_z = B|_{X_z}$ (we refer the reader to [Kol23, Section 4.1] for the definition of such restriction).

Here $K_{X/Z}$ is the relative canonical divisor corresponding to the relative dualizing sheaf $\omega_{X/Z}$ and we refer the reader to [Kol23, 2.68] for more details.

A *stable family* is a locally stable family $f : (X, B) \rightarrow Z$ such that $K_{X/Z} + B$ is ample/ Z .

Definition 2.23 (Variation). Let $f : (X, B) \rightarrow Z$ be a stable family. The *variation* of $f : (X, B) \rightarrow Z$ is $\dim Z - d$ where d is the dimension of a general isomorphism equivalence class of the log fibers. We say that $f : (X, B) \rightarrow Z$ is of *maximal variation* if the variation of $f : (X, B) \rightarrow Z$ equals to $\dim Z$.

Definition 2.24 (Base change). Let $f : (X, B) \rightarrow Z$ be a (locally) stable family. A *base change* of $f : (X, B) \rightarrow Z$ is $f' : (X', B') \rightarrow Z'$ satisfying the following:

- There is a morphism $h_Z : Z' \rightarrow Z$ such that $X' = X \times_Z Z'$, and $h : X' \rightarrow X$ is the induced morphism.
- B' is the unique $\mathbb{R}$ -divisor such that $K_{X'/Z'} + B' = h^*(K_{X/Z} + B)$.

By definition (cf. [Kol23, Theorem 4.8]), $f' : (X', B') \rightarrow Z'$ is also a (locally) stable family. If h_Z is birational, then we say that $f' : (X', B') \rightarrow Z'$ is a *birational base change* of $f : (X, B) \rightarrow Z$.

Definition 2.25 ((d, Γ, v) -stable family). Let d be a positive integer, $\Gamma \subset [0, 1]$ a set such that $\Gamma = \Gamma_+$, v a positive real number, and (X, B) a semi-pair. We say that (X, B) is a (d, Γ, v) -stable semi-pair if $\dim X = d$, (X, B) is stable, the coefficients of B belong to Γ , and $\text{vol}(K_X + B) = v$. A family of (d, Γ, v) -stable semi-pairs is a locally stable family $f : (X, B) \rightarrow Z$, such that

- (1) $B = \sum b_i B_i$ where each $b_i \in \Gamma$ and each B_i is a Mumford divisor/ Z , and
- (2) (X_z, B_z) is a (d, Γ, v) -stable semi-pair for any point $z \in Z$, where $X_z = f^{-1}(z)$ and $B_z = B|_{X_z}$.

We denote by $\mathfrak{S}(d, \Gamma, v)$ the set of all (d, Γ, v) -stable semi-pairs. We denote by $\mathfrak{S}(d, \Gamma, v)$ the moduli functor by setting

$$\mathfrak{S}(d, \Gamma, v)(Z) = \{\text{families of } (d, \Gamma, v)\text{-stable semi-pairs over } Z\}$$

where Z is any reduced scheme.

Definition-Theorem 2.26 (cf. [Kol23, Theorem 4.1], [HJLL24, Definition-Theorem 2.12]). Let d be a positive integer, $\Gamma \subset [0, 1]$ a DCC set such that $\Gamma = \Gamma_+$, and v a positive real number. Then the functor $\mathfrak{S}(d, \Gamma, v)$ has a projective coarse moduli space $M(d, \Gamma, v)$. For any family $f : (X, B) \rightarrow Z$ of (d, Γ, v) -stable semi-pairs, we call the induced morphism $Z \rightarrow M(d, \Gamma, v)$ the corresponding *moduli map*.

Theorem 2.27 ([KP17, Corollary 6.19], [HJLL24, Theorem 2.13]). Let d be a positive integer, $\Gamma \subset [0, 1]$ a DCC set such that $\Gamma = \Gamma_+$, and v a positive real number. Then there exists a reduced scheme Z and $f \in \mathfrak{S}(d, \Gamma, v)(Z)$, such that the corresponding moduli map $Z \rightarrow M(d, \Gamma, v)$ is a finite cover.

2.5. Iitaka dimensions and abundance.

Definition 2.28. Let X be a normal projective variety and D an $\mathbb{R}$ -divisor on X . The *Iitaka dimension* $\kappa(D)$ (resp. *numerical Iitaka dimension* $\nu(D)$) of D is defined in the following way. If $|[mD]| \neq \emptyset$ for some positive integer m (resp. D is pseudo-effective), then

$$\kappa(D) := \max \left\{ k \in \mathbb{N} \left| \limsup_{m \rightarrow +\infty} \frac{\dim H^0(X, [mD])}{m^k} > 0 \right. \right\}$$

$$\left(\text{resp. } \nu(D) := \max \left\{ k \in \mathbb{N} \left| A \text{ is Cartier, } \limsup_{m \rightarrow +\infty} \frac{\dim H^0(X, [mD] + A)}{m^k} > 0 \right. \right\} \right).$$

Otherwise, we let $\kappa(D) := -\infty$ (resp. $\nu(D) := -\infty$). If $|D|_{\mathbb{R}} \neq \emptyset$, then we define $\kappa_\ell(D) := \kappa(D')$ for some $D' \in |D|_{\mathbb{R}}$. Otherwise, we define $\kappa_\ell(D) := -\infty$. κ_ℓ is well-defined by [Cho08, Section 2].

Let $\mathcal{F}$ be a foliation on X . We define $\kappa(X) := \kappa(K_X)$ (resp. $\kappa(\mathcal{F}) := \kappa(K_{\mathcal{F}})$) as the *Kodaira dimension* of X (resp. $\mathcal{F}$), and define $\nu(X) := \nu(K_X)$ (resp. $\nu(\mathcal{F}) := \nu(K_{\mathcal{F}})$) as the *numerical dimension* of X (resp. $\mathcal{F}$). We say that X (resp. $\mathcal{F}$) is *abundant* if $\kappa(X) = \nu(X)$ (resp. $\kappa(\mathcal{F}) = \nu(\mathcal{F})$).

Let $\pi : X \rightarrow U$ be a projective morphism from between normal quasi-projective varieties and D an $\mathbb{R}$ -divisor on X . Let $X \xrightarrow{f} T \rightarrow \pi(U)$ be the Stein factorization and let F be a general fiber of f . We define $\kappa(X/U, D) := \kappa(D|_F)$, $\kappa_\ell(X/U, D) := \kappa_\ell(D|_F)$, and $\nu(X/U, D) := \nu(D|_F)$ as the *Iitaka dimension*, *invariant Iitaka dimension*, and *numerical Iitaka dimension* of D over U respectively. We say that D is *abundant/U* if $\kappa_\ell(X/U, D) = \nu(X/U, D)$. Here by convention, if $\dim F = 0$, then we define $\kappa(D|_F) := \kappa_\ell(D|_F) := \nu(D|_F) := 0$.

We refer the reader to [Nak04, Chapters II,V], [Cho08, Section 2], and [HH20, Section 2] for basic properties of Iitaka dimensions.

We need the following result which characterizes nef and abundant divisors.

Lemma 2.29 ([Hu20, Lemma 2.8]). *Let X be a normal projective variety and D a nef/ U $\mathbb{R}$ -divisor on X . Then the followings are equivalent.*

- (1) D is abundant/ U .
- (2) There exists a birational morphism $g : Y \rightarrow X$ and a contraction/ U $h : Y \rightarrow Z$ such that $g^*D = h^*H$ for some big/ U and nef/ U $\mathbb{R}$ -divisor H on Z .
- (3) For any very general fiber F of the Stein factorization of $X \rightarrow U$, either $\dim F = 0$, or $\dim F > 0$ and for any prime divisor P on F and any $\epsilon > 0$, there exists $0 \leq D_{P,\epsilon} \sim_{\mathbb{R}} D|_F$ such that $\text{mult}_P D_{P,\epsilon} < \epsilon$.

Finally, we recall the following version of the abundance conjecture which is one assumption in many of our results.

Conjecture 2.30 (Abundance). *Let (X, B) be a klt pair. Then $\kappa_\ell(K_X + B) = \nu(K_X + B)$.*

2.6. Nakayama-Zariski decomposition.

Definition 2.31. Let $\pi : X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a quasi-projective variety, D a pseudo-effective/ U $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X , and P a prime divisor on X . We define $\sigma_P(X/U, D)$ as in [LX25a, Definition 3.1] by considering $\sigma_P(X/U, D)$ as a number in $[0, +\infty) \cup \{+\infty\}$. We define $N_\sigma(X/U, D) = \sum_Q \sigma_Q(X/U, D)Q$ where the sum runs through all prime divisors on X and consider it as a formal sum of divisors with coefficients in $[0, +\infty) \cup \{+\infty\}$. We say that D is movable/ U if $N_\sigma(X/U, D) = 0$.

If U is a point then we denote by $\sigma_P(D) := \sigma(X/U, D)$ and $N_\sigma(D) := N_\sigma(X/U, D)$. By [Nak04, III 1.5 Lemma], if U is a point, then $N_\sigma(D)$ is an $\mathbb{R}$ -divisor.

We need the following properties on the negative part of Nakayama-Zariski decompositions.

Lemma 2.32 ([LMX24a, Lemma 2.25]). *Let $X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a quasi-projective variety and $\phi : X \dashrightarrow X'$ a birational map/ U . Let D be an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X such that ϕ is D -negative and $D' := \phi_*D$. Then:*

- (1) The divisors contracted by ϕ are contained in $\text{Supp } N_\sigma(X/U, D)$.
- (2) If D' is movable/ U , then $\text{Supp } N_\sigma(X/U, D)$ is the set of all ϕ -exceptional divisors.

Lemma 2.33 (cf. [Cas+25a, Lemma 3.21]). *Let $X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a quasi-projective variety and $\phi : X \dashrightarrow X'$ a birational map/ U which does not extract any divisor. Let D be an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X such that $D' := \phi_*D$ is $\mathbb{R}$ -Cartier and ϕ contracts all divisors that are contained in $\text{Supp } N_\sigma(X/U, D)$. Then D' is movable/ U .*

Lemma 2.34 (cf. [Cas+25a, Lemma 3.22]). *Let $X \rightarrow U$ be a projective morphism between normal quasi-projective varieties and let C, D be two pseudo-effective/ U $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisors on X . Assume that $N_\sigma(X/U, D)$ is an $\mathbb{R}$ -divisor, i.e. it does not have $+\infty$ as a coefficient. Then there exists a positive real number s_0 and a reduced divisor E satisfying the following.*

- (1) $\text{Supp } N_\sigma(X/U, C + sD) = E$ for any $0 < s \leq s_0$.
- (2) $\text{Supp } N_\sigma(X/U, C) \subset E$.

3. ABUNDANCE OF FOLIATIONS

The goal of this section is to prove Theorem 1.1.

3.1. Abundant and semi-ampleness. The following theorem is the real coefficient version of the recently proven Prokhorov-Shokurov-Bakker-Filipazzi-Mauri-Tsimmerman's $\mathbf{b}$ -semi-ampleness theorem [BFMT25, Theorem 1.5].

Theorem 3.1. *Let (X, B) be a projective lc pair and let $f : X \rightarrow Z$ be a contraction such that $K_X + B \sim_{\mathbb{R}, Z} 0$. Let $(Z, B_Z, \mathbf{M})$ be the generalized pair induced by the canonical bundle formula of $f : (X, B) \rightarrow Z$,*

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + \mathbf{M}_Z).$$

Then $\mathbf{M}$ is $\mathbf{b}$ -semi-ample.

Proof. Write $B = \sum_{i=1}^m v_i^0 B_i$ where $B_i \geq 0$ are Weil divisors. Let $\mathbf{v}_0 := (v_1^0, \dots, v_m^0)$, and let $V \ni \mathbf{v}_0$ be the rational envelope of $\mathbf{v}$ in $\mathbb{R}^m$. Let $B(\mathbf{v}) := \sum_{i=1}^m v_i B_i$ for any $\mathbf{v} \in \mathbb{R}^m$. By [HLX23, Theorem 3.3] and [HLS24, Theorem 5.6], there exists an open subset $U \ni \mathbf{v}_0$ of V , such that for any $\mathbf{v} \in U$,

- $f : (X, B(\mathbf{v})) \rightarrow Z$ is an lc-trivial fibration and $(X, B(\mathbf{v}))$ is lc, and
- Let $(Z, B_Z(\mathbf{v}), \mathbf{M}(\mathbf{v}))$ be the generalized pair induced by the canonical bundle formula of $f : (X, B(\mathbf{v})) \rightarrow Z$,

$$K_X + B(\mathbf{v}) \sim_{\mathbb{R}} f^*(K_Z + B_Z(\mathbf{v}) + \mathbf{M}(\mathbf{v})_Z),$$

then for any vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in U$ and positive real numbers $a_1, \dots, a_k$ such that $\sum_{i=1}^k a_i = 1$, we have

$$\mathbf{M} \left(\sum_{i=1}^k a_i \mathbf{v}_i \right) = \sum_{i=1}^k a_i \mathbf{M}(\mathbf{v}_i).$$

Pick $\mathbf{v}_i^0 \in \mathbb{Q}^m \cap U$ and a_i^0 such that $\sum_{i=1}^k a_i^0 \mathbf{v}_i^0 = \mathbf{v}_0$ and $\sum_{i=1}^k a_i^0 = 1$. By [BFMT25, Theorem 1.5], $\mathbf{M}(\mathbf{v}_i^0)$ is $\mathbf{b}$ -semi-ample for any i , hence

$$\mathbf{M} = \mathbf{M} \left(\sum_{i=1}^k a_i^0 \mathbf{v}_i^0 \right) = \sum_{i=1}^k a_i^0 \mathbf{M}(\mathbf{v}_i^0)$$

is $\mathbf{b}$ -semi-ample. □

3.2. Abundance for stable families. The following proposition is the real coefficient version of [KP17, Corollary 6.19].

Proposition 3.2. *Let $f : (X, B) \rightarrow Z$ be a stable family such that Z is normal. Then there exist two stable families $f' : (X', B') \rightarrow Z'$, $f'' : (X'', B'') \rightarrow Z''$, a finite surjective morphism $g : Z'' \rightarrow Z$, and a surjective morphism $g' : Z'' \rightarrow Z'$, such that*

- (1) $f'' : (X'', B'') \rightarrow Z''$ is the base change of both $f : (X, B) \rightarrow Z$ and $f' : (X', B') \rightarrow Z'$,
- (2) Z' and Z'' are normal,
- (3) $f' : (X', B') \rightarrow Z'$ is of maximal variation, and
- (4) if the general fiber of f is normal, then the general fibers of f' and f'' are normal.

Proof. Let F be a general fiber of f and let $B_F := B|_F$. Let $d := \dim F$, Γ' the set of coefficients of B_F , $\Gamma := \Gamma'_+$, and $v = \text{vol}(K_F + B_F)$. Then there exists an induced moduli map $Z \rightarrow M(d, \Gamma, v)$. By Theorem 2.27, there exists a reduced scheme $\mathcal{Z}$ and a family $\mathfrak{f} : (\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{Z}$ such that $\mathfrak{f} \in \mathfrak{S}(d, \Gamma, v)(\mathcal{Z})$ and the induced moduli map $\mathcal{Z} \rightarrow M(d, \Gamma, v)$ is a finite cover.

Then map $Z \times_{M(d, \Gamma, v)} \mathcal{Z} \rightarrow Z$ is finite and surjective. Let Z'' be the normalization of an irreducible component of $Z \times_{M(d, \Gamma, v)} \mathcal{Z}$ which dominant Z with induced finite surjective morphism $g : Z'' \rightarrow Z$, and let Z' be the normalization of the image of Z'' in $\mathcal{Z}$ with induced surjective morphism $g' : Z'' \rightarrow Z'$. The base change of $\mathfrak{f} : (\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{Z}$ induces two stable families $f' : (X', B') \rightarrow Z'$, $f'' : (X'', B'') \rightarrow Z''$ and they satisfy our requirements. $\square$

Lemma 3.3. *Let $f : (X, B) \rightarrow Z$ be a stable family such that Z is projective and the general fiber of f is normal. Then $K_{X/Z} + B$ is abundant.*

Proof. By Proposition 3.2, there exists a stable family of maximal variation $f' : (X', B') \rightarrow Z'$, and two projective surjective morphisms $h : X'' \rightarrow X$, $h' : X'' \rightarrow X'$, such that the general fiber of f' is normal and

$$h^*(K_{X/Z} + B) = h'^*(K_{X'/Z'} + B').$$

Since the general fiber of f' is normal, the generic log fiber (X'_η, B'_η) of $f' : (X', B') \rightarrow Z'$ is lc. Since Z is projective, Z' is projective. By [HJLL24, Proposition 2.18] (rational coefficient case [PX17, Proposition 2.15]), $K_{X'/Z'} + B'$ is big. Thus

$$\kappa_\iota(K_{X/Z} + B) = \kappa_\iota(K_{X'/Z'} + B') = \nu(K_{X'/Z'} + B') = \nu(K_{X/Z} + B).$$

$\square$

3.3. Transformation of generalized foliated quadruples. The following proposition is the generalized foliated quadruple version of [HJLL24, Proposition 3.16] which characterizes generalized foliated quadruples under finite covers.

Proposition 3.4. *Let $\mathfrak{A} := (X, \mathcal{F}, B, \mathbf{M})$ be a generalized foliated quadruple and let $h : X' \rightarrow X$ be a finite cover between normal varieties. Let $\mathcal{F}' := h^{-1}\mathcal{F}$ and $\mathbf{M}' := h^*\mathbf{M}$. For any prime divisor D on X , let r_D be the ramification index of h along D . Suppose that any codimension one component of the branch locus of h is $\mathcal{F}$ -invariant. Let $K_{\mathcal{F}'} + B' + \mathbf{M}'_{X'} := h^*(K_{\mathcal{F}} + B + \mathbf{M}_X)$ and $\mathfrak{A}' := (X', \mathcal{F}', B', \mathbf{M}')$. Then:*

- (1) *If $\mathfrak{A}$ is lc (resp. klt, canonical), then $\mathfrak{A}'$ is also lc (resp. klt, canonical).*
- (2) *If $\mathfrak{A}'$ is lc (resp. klt), then $\mathfrak{A}$ is lc (resp. klt).*

Proof. Let $g : Y \rightarrow X$ be a birational morphism and E a prime divisor on Y . Let Y' be the normalization of the main component of $Y \times_X X'$ associated with $g' : Y' \rightarrow X'$ and $h' : Y' \rightarrow Y$. Let E' be any prime divisor on Y' that dominates E . Such prime divisor exists as h' is finite. Let $\mathcal{F}_Y := g^{-1}\mathcal{F}$ and $\mathcal{F}_{Y'} := g'^{-1}\mathcal{F}'$.

Let $r_{E'}$ be the ramification index of h' along E' . By [HJLL24, Proposition 3.16(1)], near the generic point of E' , we have

$$\begin{aligned} K_{\mathcal{F}_{Y'}} &= g'^*K_{\mathfrak{A}'} + a(E', \mathfrak{A}')E' = g'^*h^*K_{\mathfrak{A}} + a(E', \mathfrak{A}')E' \\ &= h'^*g^*K_{\mathfrak{A}} + a(E', \mathfrak{A}')E' = h'^*(K_{\mathcal{F}_Y} - a(E, \mathfrak{A})E) + a(E', \mathfrak{A}')E' \\ &= K_{\mathcal{F}_{Y'}} - \epsilon_{\mathcal{F}'}(E')(r_{E'} - 1)E' - a(E, \mathfrak{A})r_{E'}E' + a(E', \mathfrak{A}')E'. \end{aligned}$$

So

$$a(E', \mathfrak{A}') + \epsilon_{\mathcal{F}'}(E') = r_{E'}(a(E, \mathfrak{A}) + \epsilon_{\mathcal{F}}(E)) = r_{E'}(a(E, \mathfrak{A}) + \epsilon_{\mathcal{F}}(E)).$$

Therefore, $a(E, \mathfrak{A}) \geq -\epsilon_{\mathcal{F}}(E)$ (resp. $> -\epsilon_{\mathcal{F}}(E)$) if and only if $a(E', \mathfrak{A}') \geq -\epsilon_{\mathcal{F}'}(E')$ (resp. $> -\epsilon_{\mathcal{F}'}(E')$), and if $a(E, \mathfrak{A}) \geq 0$ (resp. > 0), then $a(E, \mathfrak{A}) \geq 0$ (resp. > 0). Moreover, E is exceptional/ X if and only if E' is exceptional/ X' . By [HJLL24, Lemma 3.6], we get (2). By [Kol13, Lemma 2.22], for any prime divisor E' over X' , we may find Y', Y, g, g', h' and E as above. (1) follows. $\square$

The following proposition is the generalized foliated quadruple version of [HJLL24, Proposition 3.16] which concerns the behavior of generalized foliated quadruples under birational base changes.

Lemma 3.5. *Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M})/U$ be an lc generalized foliated quadruple. Then*

- (1) $\mathfrak{A}$ has a $\mathbb{Q}$ -factorial ACSS modification $h : (\mathfrak{A}'; G)/Z \rightarrow \mathfrak{A}$.
- (2) Let X' be the ambient variety of $\mathfrak{A}'$. Then there exists a core modification $\bar{h} : (\bar{\mathfrak{A}}; \bar{G})/Z \rightarrow \mathfrak{A}$ associated with $\bar{f} : \bar{X} \rightarrow Z$ such that $\bar{X}$ is the core model of $(h, f : X' \rightarrow Z)$ associated with $(\bar{h}, \bar{f})$ and $\bar{f}$ is equidimensional.

Proof. (1) follows from [CHLX23, Theorem 8.2.2]. (2) By [LMX24a, Lemma A.22], the core model of $(h, f : X' \rightarrow Z)$ associated with $(\bar{h}, \bar{f})$ induces a core modification $(\bar{\mathfrak{A}}; \bar{G})/Z \rightarrow \mathfrak{A}$. Since $f : X' \rightarrow Z$ is equidimensional, the induced contraction $\bar{f} : \bar{X} \rightarrow Z$ is also equidimensional. (2) follows. $\square$

3.4. Proof of Theorem 1.1. The following proposition is the key of the proof of Theorem 1.1. Some ideas of the proof of Proposition 3.6 originated from [FS22, Proof of Theorem 4.6] although the details have many differences.

Proposition 3.6. *Let $(X, \mathcal{F}, B)$ be a projective lc algebraically integrable foliated triple. Assume that $(X, \mathcal{F}, B; G)/Z$ satisfies Property (*) associated with an equidimensional contraction $f : X \rightarrow Z$ and $K_{\mathcal{F}} + B$ is semi-ample/ Z . Then $K_{\mathcal{F}} + B$ is abundant.*

Proof. By [HJLL24, Proposition 3.16] (rational coefficient case [FS22, Proposition 3.5]), we may assume that $f : (X, B) \rightarrow Z$ is a locally stable family. We let $g : X \rightarrow Y$ be the ample model/ Z of $K_{X/Z} + B$, then we have

$$K_X + B \sim_{\mathbb{R}, Y} K_{X/Z} + B \sim_{\mathbb{R}, Y} 0.$$

We let $(Y, B_Y, \mathbf{M})$ be the lc generalized pair induced by the canonical bundle formula of $g : (X, B) \rightarrow Y$,

$$K_X + B \sim_{\mathbb{R}} g^*(K_Y + B_Y + \mathbf{M}_Y).$$

Let $h : Y \rightarrow Z$ be the induced contraction and let $\mathcal{F}_Y$ be the foliation induced by h .

For any closed point $z \in Z$, let $H_1, \dots, H_{\dim Z}$ be general hyperplane sections on Z such that $z \in H_i$ for each i and let $H_z := \sum_{i=1}^{\dim Z} H_i$. Since $f : (X, B) \rightarrow Z$ is a locally stable family, X is normal, and Z is smooth, $(X, B + f^*H_z)$ is lc. Since $(Y, B_Y + h^*H_z, \mathbf{M})$ is the generalized pair induced by the canonical bundle formula

$$K_X + B + f^*H_z \sim_{\mathbb{R}} g^*(K_Y + B_Y + h^*H_z + \mathbf{M}_Y),$$

$(Y, B_Y + h^*H_z, \mathbf{M})$ is lc. By Theorem 3.1, $\mathbf{M}$ is $\mathbf{b}$ -semi-ample. Let $\alpha : W \rightarrow Y$ be a birational morphism such that $\mathbf{M}$ descends to W . Then there exists $0 \leq L_{z, W} \sim_{\mathbb{R}} \mathbf{M}_W$ such that such that $(Y, B_Y + h^*H_z + L_z)$ is lc, where $L_z := \alpha_* L_{z, W}$. Let $Y_z := h^{-1}(z)$, $B_{Y_z} := B|_{Y_z}$, and $L_{Y_z} := L_z|_{Y_z}$. By adjunction, $(Y_z, B_{Y_z} + L_{Y_z})$ is slc. By [Kol23, Corollary

4.45], there exists an open subset $U_z \ni z$ with $V_z := h^{-1}(U_z)$, such that $f|_{V_z} : (Y, B_Y + L_z)|_{V_z} \rightarrow U_z$ is a locally stable family, hence a stable family.

By Noetherian property, there exist finitely many closed points $z_1, \dots, z_k \in Z$ such that $\{U_{z_i}\}$ is a cover of Z . We fix a prime divisor P on Y and let η_P be the generic point of P . Then there exists i such that $V := V_{z_i}$ contains η_P . Let $U := U_{z_i}$, $L := L_{z_i}$, $L_V := L|_V$, and $B_V := B_Y|_V$. Then $h_V := h|_V : (V, B_V + L_V) \rightarrow U$ is stable family and the general fiber of f_V is normal.

Let $d := \dim Y$, Γ the set of coefficients of $B + L$, $\Gamma := \Gamma'_+$, and let $v := \text{vol}((K_Y + B_Y + \mathbf{M}_Y)|_F)$ where F is a general fiber of h . Then $f_V : (V, B_V + L_V) \rightarrow U$ is a family of (d, Γ, v) -stable semi-pairs. Thus there exists a moduli map $\phi_0 : U \rightarrow M(d, \Gamma, v)$ which naturally extends to a rational map $\phi : Z \dashrightarrow M(d, \Gamma, v)$. By Theorem 2.27, there exists a reduced scheme $\mathcal{Z}$ and a family $\mathfrak{f} : (\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{Z}$ such that $\mathfrak{f} \in \mathfrak{S}(d, \Gamma, v)(\mathcal{Z})$ and the induced moduli map $\mathcal{Z} \rightarrow M(d, \Gamma, v)$ is a finite cover. Let U' be a component of $U \times_{M(d, \Gamma, v)} \mathcal{Z}$ dominating U . Then $U' \rightarrow U$ is finite and surjective. Thus we can compactify U' to obtain a normal projective variety Z' , such that the maps $U' \rightarrow \mathcal{Z}$ and $U' \rightarrow U$ extend to morphisms $Z' \rightarrow \mathcal{Z}$ and $Z' \rightarrow Z$, and we may let $h' : (Y', D') \rightarrow Z'$ be the stable family induced by base change of $\mathfrak{f} : (\mathcal{X}, \mathcal{D}) \rightarrow \mathcal{Z}$.

Let $\pi_Z : Z' \rightarrow Z$ be the associated morphism. Then π_Z is generically finite, and we may let $Z' \rightarrow Z'' \rightarrow Z$ be the Stein factorization of π_Z . Let $Y'' := Y \times_Z Z''$, $\tilde{Y} = Y \times_Z Z'$, and $\tau : Y'' \rightarrow Y$, $\xi : \tilde{Y} \rightarrow Y''$, $f'' : Y'' \rightarrow Z''$, and $\tilde{f} : \tilde{Y} \rightarrow Z'$ the associated morphisms. Write $\mathbf{M}' := \tau^* \mathbf{M}$. Let $\mu_Z : \hat{Z} \rightarrow Z'$ be a resolution, $\hat{Y} := \tilde{Y} \times_{Z'} \hat{Z}$, and $\mu : \hat{Y} \rightarrow \tilde{Y}$ and $\hat{f} : \hat{Y} \rightarrow \hat{Z}$ the associated morphisms. Let $\mathcal{F}'', \tilde{\mathcal{F}}, \hat{\mathcal{F}}$ be the foliations induced by the contraction $f'', \tilde{f}, \hat{f}$ respectively. Write

$$K_{\mathcal{F}''} + B'' + \mathbf{M}'_{Y''} = \tau^*(K_{\mathcal{F}_Y} + B_Y + \mathbf{M}_Y), K_{\tilde{\mathcal{F}}} + \tilde{B} + \mathbf{M}'_{\tilde{Y}} = \xi^*(K_{\mathcal{F}''} + B'' + \mathbf{M}'_{Y''}),$$

and

$$K_{\hat{\mathcal{F}}} + \hat{B} + \mathbf{M}'_{\hat{Y}} = \mu^*(K_{\tilde{\mathcal{F}}} + \tilde{B} + \mathbf{M}'_{\tilde{Y}}).$$

By Proposition 3.4, $(Y'', \mathcal{F}'', B'', \mathbf{M}')$ is lc and $B'' \geq 0$. By Lemma 3.5 and [HJLL24, Lemma 3.14], $\xi \circ \mu : (\hat{Y}, \hat{\mathcal{F}}, \hat{B}, \mathbf{M}'; G)/\hat{Z} \rightarrow (Y'', \mathcal{F}'', B'', \mathbf{M}')$ is a core modification. In particular, $\hat{B} \geq 0$, so $\tilde{B} \geq 0$.

Let $\pi : \tilde{Y} \rightarrow Y$ be the induced morphism, then π is generically finite. Let $\tilde{W}$ the main component of $W \times_Y \tilde{Y}$ and $\pi_W : \tilde{W} \rightarrow W$, $\tilde{\alpha} : \tilde{W} \rightarrow \tilde{Y}$ the associated morphism, and let $\tilde{L} := \tilde{\alpha}_* \pi_W^* L_{z_i, W}$. Then $\tilde{L} \geq 0$. Let $\psi : \tilde{Y} \dashrightarrow Y'$ be the induced map, then ψ is birational and $\psi : (\tilde{Y}, \tilde{D} := \tilde{B} + \tilde{L}) \dashrightarrow (Y', D')$ is an isomorphism over U' , i.e. ψ is an isomorphism over U' , $\psi_* \tilde{D} = D'$, and $\psi_*^{-1} D' = \tilde{D}$.

We let $p : W \rightarrow \tilde{Y}$ and $q : W \rightarrow Y'$ be a resolution of indeterminacy of ψ and let $\mathcal{F}'$ be the foliation induced by h' . Then $\psi_* \tilde{F} = \mathcal{F}'$. Write

$$p^*(K_{\tilde{\mathcal{F}}} + \tilde{D}) = q^*(K_{\mathcal{F}'} + D') + E.$$

For any prime divisor Q on $\tilde{Y}$, if Q is not $\tilde{\mathcal{F}}$ -invariant, then

$$a(Q, \tilde{\mathcal{F}}, \tilde{D}) = a(Q, \mathcal{F}', D')$$

as ψ is an isomorphism over the generic point of Q . If Q is $\tilde{\mathcal{F}}$ -invariant, then

$$a(Q, \tilde{\mathcal{F}}, \tilde{D}) = -\text{mult}_Q \tilde{D} \leq 0 \leq a(Q, \mathcal{F}', D')$$

as $(Y', \mathcal{F}', D')$ is lc, which in turn is because $h' : (Y', D') \rightarrow Z'$ is stable family. Therefore,

$$p_*E = \sum_{Q \subset \tilde{Y}} \left(a(Q, \mathcal{F}', D') - a(Q, \tilde{\mathcal{F}}, \tilde{D}) \right) \geq 0.$$

Since $E \sim_{\mathbb{R}, \tilde{Y}} -q^*(K_{\mathcal{F}'} + D') = -q^*(K_{Y'/Z'} + D')$ is anti-nef, by the negativity lemma, $E \geq 0$. Moreover, p_*E is vertical/ Z' .

By Lemma 3.3, $K_{Y'/Z'} + D'$ is abundant, so $q^*(K_{Y'/Z'} + D')$ is abundant. By Lemma 2.29, possibly replacing W by a higher model, we may assume that $q^*(K_{Y'/Z'} + D') = A_n + \frac{1}{n}C$ for any integer $n \geq 1$, where A_n are semi-ample $\mathbb{R}$ -divisors and $C \geq 0$. Therefore, for any integer $n \geq 1$, we may pick $H_n \in |A_n|_{\mathbb{R}}$, such that η_P is not contained in $\pi_*p_*H_n$. We have

$$\text{mult}_P \left(\pi_*p_* \left(A_n + \frac{1}{n}C + E \right) \right) = \frac{1}{n} \text{mult}_P \pi_*p_*C.$$

Since

$$\pi_*p_* \left(A_n + \frac{1}{n}C + E \right) \sim_{\mathbb{R}} K_{Y/Z} + B_Y + \mathbf{M}_Y,$$

for any $\epsilon > 0$, there exists $0 \leq N_{P,\epsilon} \sim_{\mathbb{R}} K_{Y/Z} + B_Y + \mathbf{M}_Y$ such that $\text{mult}_P N_{P,\epsilon} < \epsilon$. Since P can be any prime divisor on Y and $K_{Y/Z} + B_Y + \mathbf{M}_Y$ is nef, by Lemma 2.29, $K_{Y/Z} + B_Y + \mathbf{M}_Y$ is abundant. Since

$$K_{\mathcal{F}} + B \sim_{\mathbb{R}} g^*(K_{Y/Z} + B_Y + \mathbf{M}_Y),$$

$K_{\mathcal{F}} + B$ is abundant. $\square$

Theorem 3.7. *Assume Conjecture 2.30 in dimension d . Then for any projective lc algebraically integrable foliated triple $(X, \mathcal{F}, B)$ such that $\text{rank } \mathcal{F} \leq d$, $K_{\mathcal{F}} + B$ is abundant. Moreover, if X is potentially klt, then:*

- (1) $(X, \mathcal{F}, B)$ has a $\mathbb{Q}$ -factorial minimal model or a Mori fiber space.
- (2) We may run a $(K_{\mathcal{F}} + B)$ -MMP with scaling of an ample $\mathbb{R}$ -divisor and any such MMP terminates with either a minimal model or a Mori fiber space of $(X, \mathcal{F}, B)$.

Proof. We may assume that $K_{\mathcal{F}} + B$ is pseudo-effective. Let $h : (X', \mathcal{F}', B'; G)/Z \rightarrow (X, \mathcal{F}, B)$ be a $\mathbb{Q}$ -factorial ACSS modification of $(X, \mathcal{F}, B)$ associated with contraction $f : X' \rightarrow Z$. By [HH20, Theorem 1.4], $(X, B + G)/Z$ has a good minimal model. We run a $(K_{\mathcal{F}'} + B')$ -MMP with scaling of an ample divisor, which is also a sequence of steps of a $(K_X + B + G)$ -MMP/ Z . By [Bir12, Theorem 1.9], this MMP terminates with a good minimal model $(X', B' + G')$ of $(X, B + G)/Z$ where B', G' are the images of B, G on X' respectively, and by [CHLX23, Lemma 9.1.4], $(X', \mathcal{F}', B'; G')/Z$ is $\mathbb{Q}$ -factorial ACSS, where $\mathcal{F}'$ is the foliation induced by $X' \rightarrow Z$. By [ACSS21, Proposition 3.6], $K_{\mathcal{F}'} + B'$ is semi-ample/ Z . By Proposition 3.6, $K_{\mathcal{F}'} + B'$ is abundant. Thus $K_{\mathcal{F}} + B$ is abundant.

To prove the moreover part (1), note that $(X', \mathcal{F}', B')$ is a bs-minimal model of $(X, \mathcal{F}, B)$. (2) follows from [Cas+25a, Theorem 2.3.1] applied to $(X, \mathcal{F}, B)$ and (1) follows from [Cas+25a, Theorem 2.3.1] applied to a small $\mathbb{Q}$ -factorialization of X . $\square$

Proof of Theorem 1.1. It is a special case of Theorem 3.7. $\square$

Proof of Corollary 1.2. By [KMM94, 1.1 Theorem], Conjecture 2.30 holds in dimension ≤ 3 . (Note that [KMM94, 1.1 Theorem] is stated for pairs with rational coefficients but the general case follows easily, cf. [LX25b, Theorem 2.20]). The corollary follows from Theorem 3.7. $\square$

4. ABUNDANCE FOR ALGEBRAICALLY INTEGRABLE ADJOINT FOLIATED STRUCTURES

The goal of this section is to prove Theorem 1.3.

4.1. Abundant divisors. We first state two useful lemmas on klt varieties with polarization of abundant $\mathbb{R}$ -divisors.

Lemma 4.1. *Let $(X, B)/U$ be a klt pair such that $K_X + B$ is abundant/ U .*

- (1) *If $K_X + B$ is not pseudo-effective/ U , then we may run a $(K_X + B)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor and any such MMP terminates with a Mori fiber space of $(X, B)/U$.*
- (2) *If $K_X + B$ is pseudo-effective/ U , then we may run a $(K_X + B)$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor and any such MMP terminates with a good minimal model of $(X, B)/U$. Moreover, $(X, B)/U$ has a $\mathbb{Q}$ -factorial good minimal model.*

Proof. Let $h : X' \rightarrow X$ be a small $\mathbb{Q}$ -factorialization of X and write $K_{X'} + B' := h^*(K_X + B)$. Let A be an ample/ U $\mathbb{R}$ -divisor on X and let $A' := h^*A$. By [Bir12, Remark 2.9], any sequence of steps of a $(K_X + B)$ -MMP/ U with scaling of A induces a $(K_{X'} + B')$ -MMP/ U with scaling of A' . Let λ_i be the scaling numbers of the $(K_{X'} + B')$ -MMP/ U with scaling of A' .

Since A' is big, if $K_X + B$ is not pseudo-effective/ U , then $K_{X'} + B'$ is not pseudo-effective/ U . By [BCHM10, Corollary 1.4.2], this MMP terminates, so that $(K_X + B)$ -MMP/ U with scaling of A terminates with a Mori fiber space of $(X, B)/U$. Thus we may assume that $K_X + B$ is pseudo-effective/ U . By [HH20, Lemma 2.13] (rational coefficient case [GL13, Theorem 4.3]), (X, B) has a good minimal model $(X_{\min}, B_{\min})/U$. Since h is small, $(X_{\min}, B_{\min})/U$ is also a good minimal model of $(X', B')/U$. By [Bir12, Theorem 1.9] and [BCHM10, Corollary 1.4.2], any $(K_{X'} + B')$ -MMP/ U with scaling of A' terminates with a $\mathbb{Q}$ -factorial minimal model $(X'', B'')/U$ of $(X', B')/U$. Thus the $(K_X + B)$ -MMP/ U terminates with a minimal model $(\bar{X}, \bar{B})/U$ of $(X, B)/U$. By [Bir12, Remark 2.7] and since h is small, $(\bar{X}, \bar{B})/U$ is a good minimal model of $(X, B)/U$ and $(X'', B'')/U$ is a $\mathbb{Q}$ -factorial good minimal model of $(X, B)/U$. $\square$

Lemma 4.2. *Let $(X, B)/U$ be a klt pair and let D be a nef/ U and abundant/ U $\mathbb{R}$ -divisor on X . Then there exists a klt pair (X, Δ) such that*

$$K_X + \Delta \sim_{\mathbb{R}, U} K_X + B + D.$$

Proof. There exist a birational morphism/ U $h : X' \rightarrow X$ and a contraction/ U $g : X' \rightarrow Z$ such that $h^*D = g^*L$ for some big/ U and nef/ U $\mathbb{R}$ -divisor L on Z . We may write $L = A_n + \frac{1}{n}E$ for some $\mathbb{R}$ -divisor $E \geq 0$ and ample/ U $\mathbb{R}$ -divisors A_n for any positive integer n . Let $K_{X'} + B' := h^*(K_X + B)$, then there exists $n \gg 0$ such that $(X', B' + \frac{1}{n}g^*E)$ is sub-klt, so we may pick $H_n \in |A_n|_{\mathbb{R}/U}$ such that $(X', B' + g^*H_n + \frac{1}{n}E)$ is sub-klt. We may let $\Delta := h_*(B' + g^*H_n + \frac{1}{n}E)$. $\square$

4.2. Proof of the Theorem 1.3. The following result is the main theorem of this section which is crucial to the proof of Theorem 1.3. Its proof is similar to the proof of [Cas+25a, Theorem 5.2]. The major difference is that we use the abundance conjecture for klt pairs in dimension d instead of the artificial polarization of an ample $\mathbb{R}$ -divisor to deduce the existence of certain good minimal models.

Theorem 4.3. *Assume Conjecture 2.30 in dimension d . Let $\mathfrak{A} := (X, \mathcal{F}, B^{\text{inv}} + (1 - t)B^{\text{inv}}, t)$ be a projective foliated log smooth klt algebraically integrable adjoint foliated*

structure of dimension $\leq d$ such that $K_{\mathfrak{A}}$ is pseudo-effective. Then $\mathfrak{A}$ has a $\mathbb{Q}$ -factorial good minimal model.

Proof. Let $\mathfrak{A}_s := (X, \mathcal{F}, B^{\text{iniv}} + (1-s)B^{\text{inv}}, s)$ for any $s \in [0, 1]$. By [Cas+25a, Theorem 5.1], $K_{\mathfrak{A}_s}$ is pseudo-effective for any $s \in [t, 1]$. We consider the sets

$$\mathcal{P} := \{s \in [t, 1] \mid \mathfrak{A}_s \text{ has a } \mathbb{Q}\text{-factorial good minimal model}\}$$

and

$$\mathcal{Q} := \{s \in [t, 1] \mid \mathfrak{A}_s \text{ has a } \mathbb{Q}\text{-factorial minimal model and } K_{\mathfrak{A}_s} \text{ is abundant}\}.$$

It is obvious that $\mathcal{P} \subset \mathcal{Q}$. By Theorem 3.7, $1 \in \mathcal{Q}$.

Step 1. In this step we prove that, if $\mu \in \mathcal{Q}$ and $\mu > t$, then $\mu - \epsilon \in \mathcal{P}$ for any $0 < \epsilon \ll 1$.

Let $\mathfrak{A}'_\mu$ be a $\mathbb{Q}$ -factorial minimal model of $\mathfrak{A}_\mu$. Let $\phi: X \dashrightarrow X'$ be the induced birational map and let $\mathfrak{A}'_s := \phi_* \mathfrak{A}_s$ for any $s \in [0, 1]$. Since ϕ is $K_{\mathfrak{A}_\mu}$ -negative, for any $0 < \epsilon \ll \mu - t$ and any prime divisor E on X , we have

$$a(E, \mathfrak{A}_{\mu-\epsilon}) \leq a(E, \mathfrak{A}'_{\mu-\epsilon})$$

and strict inequality holds if E is exceptional/ X' . Since $\mathfrak{A}'_\mu$ is klt, by [Cas+24, Proposition 3.3], $\mathfrak{A}'_0$ is a klt. Since $K_{\mathfrak{A}'_\mu}$ is nef and abundant, by Lemma 4.2, there exists a klt pair (X', Δ) such that

$$K_{X'} + \Delta \sim_{\mathbb{R}} K_{\mathfrak{A}'_0} + \frac{\mu - \epsilon}{\epsilon} K_{\mathfrak{A}'_\mu} \sim_{\mathbb{R}, U} \frac{\mu}{\epsilon} K_{\mathfrak{A}'_{\mu-\epsilon}}.$$

By Lemma 4.1, (X', Δ) has a $\mathbb{Q}$ -factorial good minimal model (X'', Δ'') associated with birational map $\psi: X' \dashrightarrow X''$. Then $\psi_* \mathfrak{A}'_{\mu-\epsilon}$ is a $\mathbb{Q}$ -factorial good minimal model of $\mathfrak{A}'_{\mu-\epsilon}$. Since ϕ is $K_{\mathfrak{A}_{\mu-\epsilon}}$ -negative,

$$a(E, \mathfrak{A}'_{\mu-\epsilon}) \leq a(E, \psi_* \mathfrak{A}'_{\mu-\epsilon})$$

for any prime divisor E on X' , and strict inequality holds if E is exceptional/ X'' . Therefore, $a(E, \mathfrak{A}_{\mu-\epsilon}) < a(E, \psi_* \mathfrak{A}'_{\mu-\epsilon})$ for any prime divisor E on X that is exceptional/ X'' . By Lemma 2.18, $\psi \circ \phi$ is $K_{\mathfrak{A}_{\mu-\epsilon}}$ -negative, hence $\psi_* \mathfrak{A}'_{\mu-\epsilon}$ is a $\mathbb{Q}$ -factorial good minimal model of $\mathfrak{A}_{\mu-\epsilon}$. Therefore, $\mathfrak{A}_{\mu-\epsilon}$ has a $\mathbb{Q}$ -factorial good minimal model for any $0 < \epsilon \ll \mu - t$.

Step 2. In this step we prove that, if $\mu \in [t, 1)$ and $\mu + \epsilon \in \mathcal{P}$ for any $0 < \epsilon \ll 1 - \mu$, then $\mu \in \mathcal{P}$.

Let $N_s := N_\sigma(\mathfrak{A}_s)$ for any $s \in [\mu, 1]$. By Lemma 2.34, there exists $1 > s_0 > \mu$ and a reduced divisor E on X such that $s_0 - \mu \ll 1$, and $\text{Supp } N_\mu \subset E$ and $\text{Supp } N_s = E$ for any $s \in (\mu, s_0]$.

We let $\phi: X \dashrightarrow X'$ be a $\mathbb{Q}$ -factorial good minimal model of $\mathfrak{A}_{s_0}$ and let $\mathfrak{A}'_s := \phi_* \mathfrak{A}_s$ for any $s \in [0, 1]$. By Lemma 2.32, $E = \text{Exc}(\phi)$. By Lemma 2.33, $K_{\mathfrak{A}'_s}$ is movable for any $s \in [\mu, s_0]$.

Since $K_{\mathfrak{A}'_{s_0}}$ is semi-ample, we may write $K_{\mathfrak{A}'_{s_0}} = \sum a_i N_i$ where each $a_i > 0$ and each N_i is a base-point-free divisor. Let $0 < \delta_0 \ll s_0 - \mu$ be a real number, such that $a_i \frac{s_0 - \delta_0}{\delta_0} > 2d$ for any i .

We run a $K_{\mathfrak{A}'_{s_0 - \delta_0}}$ -MMP with scaling of an ample divisor. Since $\mathfrak{A}'_{s_0}$ is klt, by [Cas+24, Proposition 3.3], $\mathfrak{A}'_0$ is a klt. Since $K_{\mathfrak{A}'_{s_0}}$ is nef and abundant, by Lemma 4.2, there exists a klt pair (X', Δ) such that

$$K_{X'} + \Delta \sim_{\mathbb{R}} K_{\mathfrak{A}'_0} + \frac{s_0 - \delta_0}{\delta_0} K_{\mathfrak{A}'_{s_0}} \sim_{\mathbb{R}} \frac{s_0}{\delta_0} K_{\mathfrak{A}'_{s_0 - \delta_0}}.$$

This MMP is also a $(K_{\mathfrak{A}'_0} + \frac{s_0 - \delta_0}{\delta_0} K_{\mathfrak{A}'_{s_0}})$ -MMP with scaling of an ample divisor, and is $K_{\mathfrak{A}'_{s_0}}$ -trivial by [BZ16, Lemma 4.4(3)]. By Lemma 4.1, this MMP terminates with a good minimal model $\mathfrak{A}''_{s_0 - \delta_0}$ of $\mathfrak{A}'_{s_0 - \delta_0}$. Let $\psi : X' \dashrightarrow X''$ be the induced birational map and let $\mathfrak{A}''_s := \psi_* \mathfrak{A}'_s$ for any $s \in [0, 1]$. Since $K_{\mathfrak{A}'_{s_0 - \delta_0}}$ is movable, by Lemma 2.32, ψ does not contract any divisor, and $K_{\mathfrak{A}''_s}$ is movable for any $s \in [\mu, s_0]$. Since ψ is $K_{\mathfrak{A}'_{s_0}}$ -trivial, $K_{\mathfrak{A}''_{s_0}}$ is semi-ample and $\mathfrak{A}''_{s_0}$ is klt. By [Cas+24, Proposition 3.3], $\mathfrak{A}''_0$ is klt.

Let $B'' := (\psi \circ \phi)_* B$. Then $\mathfrak{A}''_0 = (X'', B'')$. Since

$$K_{\mathfrak{A}''_0} + \frac{s}{s_0 - s} K_{\mathfrak{A}''_{s_0}} \sim_{\mathbb{R}} \frac{s_0}{s_0 - s} K_{\mathfrak{A}''_s}$$

for any $s \in [0, s_0)$ and $K_{\mathfrak{A}''_0}$ is semi-ample, we may choose $0 \leq H \sim_{\mathbb{R}} K_{\mathfrak{A}''_{s_0}}$ such that

$$\left(X'', B'' + \frac{s_0 - \delta_0}{\delta_0} H \right)$$

is a klt pair. Since

$$K_{X''} + B'' + \frac{s_0 - \delta_0}{\delta_0} H \sim_{\mathbb{R}} \frac{s_0}{\delta_0} K_{\mathfrak{A}_{s_0 - \delta_0}}$$

is nef, by Lemma 2.19, we may run a $(K_{X''} + B'')$ -MMP with scaling of H which terminates. This MMP is also a $K_{\mathfrak{A}''_0}$ -MMP with scaling of $K_{\mathfrak{A}''_{s_0}}$. Let $f_i : X_i \dashrightarrow X_{i+1}$ be the steps of this MMP where $X_1 := X''$ and with scaling numbers λ_i .

Let n be the minimal positive integer such that $\lambda_n \leq \frac{\mu}{s_0 - \mu}$. Such n exists because the MMP terminates. Then there exists $\mu' > \mu$ such that the induced birational map $\tau : X'' \dashrightarrow \bar{X} := X_n$ is a $K_{\mathfrak{A}''_s}$ -MMP with scaling of $K_{\mathfrak{A}''_{s_0}}$ for any $s \in [\mu, \mu']$. Let $\bar{\mathfrak{A}}_s := \tau_* \mathfrak{A}''_s$ for any $s \in [0, 1]$. Since

$$K_{X''} + B'' + \frac{s}{s_0 - s} H \sim_{\mathbb{R}} K_{\mathfrak{A}''_0} + \frac{s}{s_0 - s} K_{\mathfrak{A}''_{s_0}} \sim_{\mathbb{R}, U} \frac{s_0}{s_0 - s} K_{\mathfrak{A}''_s},$$

by our assumption, $K_{\bar{\mathfrak{A}}_s}$ is semi-ample for any $s \in [\mu, \mu']$. Since $K_{\mathfrak{A}''_s}$ is movable for any $s \in [\mu, s_0]$, by Lemma 2.32, τ does not contract any divisor.

For any $s \in (\mu, \mu')$, let $\mathfrak{A}_s^s$ be a good minimal model of $\mathfrak{A}_s$ whose existence is guaranteed by our assumption. Let $\phi_s : X \dashrightarrow X^s$ be the induced birational map. By Lemma 2.32, $\text{Exc}(\phi_s) = N_s = E$. Thus X_s and $\bar{X}$ are isomorphic in codimension 1. Let $p_s : W_s \rightarrow X^s$ and $q_s : W_s \rightarrow \bar{X}$ be a common resolution and let

$$p_s^* K_{\mathfrak{A}_s^s} = q_s^* K_{\bar{\mathfrak{A}}_s} + E_s,$$

then E_s is exceptional/ X^s and exceptional/ $\bar{X}$. By applying the negativity lemma twice, we have that $E_s = 0$. Thus $\bar{\mathfrak{A}}_s$ is a weak lc model of $\mathfrak{A}_s$ for any $s \in (\mu, \mu')$. Let $p : W \rightarrow X$ and $q : W \rightarrow \bar{X}$ be a common resolution, then for any $s \in (\mu, \mu')$,

$$p^* K_{\mathfrak{A}_s} = q^* K_{\bar{\mathfrak{A}}_s} + F_s$$

for some $F_s \geq 0$. Therefore,

$$p^* K_{\mathfrak{A}_\mu} = \lim_{s \rightarrow 0^+} p^* K_{\mathfrak{A}_s} = \lim_{s \rightarrow 0^+} (q^* K_{\bar{\mathfrak{A}}_s} + F_s) = q^* K_{\bar{\mathfrak{A}}_\mu} + \lim_{s \rightarrow 0^+} F_s$$

and $\lim_{s \rightarrow 0^+} F_s \geq 0$.

Therefore, $\mathfrak{A}_\mu$ is a semi-ample model of $\mathfrak{A}_\mu$. By [Cas+25a, Proposition 4.2], $\mathfrak{A}_\mu$ has a $\mathbb{Q}$ -factorial good minimal model.

Step 3. Since $1 \in \mathcal{Q}$, by **Step 1** and **Step 2**, we have $t \in \mathcal{P}$ and we are done. $\square$

Theorem 4.4. *Assume Conjecture 2.30 in dimension d . Let $\mathfrak{A} := (X, \mathcal{F}, B, t)$ be a projective klt algebraically integrable adjoint foliated structure of dimension $\leq d$. Then we may run a $K_{\mathfrak{A}}$ -MMP which scaling of an ample $\mathbb{R}$ -divisor and any such MMP terminates with either a good minimal model or a Mori fiber space.*

Proof. By [Cas+24, Theorem 1.10(1)], X is potentially klt. If $K_{\mathfrak{A}}$ is not pseudo-effective, then we are done by [Cas+25a, Theorem 2.1.5]. Thus we may assume that $K_{\mathfrak{A}}$ is pseudo-effective.

If $\mathcal{F} = T_X$, then we are done by Theorem 3.7, so we may assume that $\mathcal{F} \neq T_X$. In this case, $t < 1$. Let $h : X' \rightarrow X$ be a foliated log resolution of $\mathfrak{A}$ and $\mathcal{F}' := h^{-1}\mathcal{F}$. Then there exists $0 < \epsilon \ll 1$ such that

$$tK_{\mathcal{F}'} + (1-t)K_{X'} + h_*^{-1}B + (1-\epsilon)(\text{Exc}(h)^{\text{nin}} + (1-t)\text{Exc}(h)^{\text{inv}}) = h^*K_{\mathfrak{A}} + E$$

for some $E \geq 0$ that is exceptional/ X and $\text{Supp } E = \text{Exc}(h)$. Let $B' := h_*^{-1}B + (1-\epsilon)(\text{Exc}(h)^{\text{nin}} + (1-t)\text{Exc}(h)^{\text{inv}})$, then $\mathfrak{A}' := (X', \mathcal{F}', B', t)$ is foliated log smooth and klt, and $K_{\mathfrak{A}'}$ is pseudo-effective. By Theorem 4.3, $\mathfrak{A}'$ has a good minimal model. Moreover, $\mathfrak{A}'$ is a foliated log smooth model of $\mathfrak{A}$. By [Cas+25b, Lemma 3.8], $\mathfrak{A}$ has a bs-good minimal model. The theorem follows from [Cas+25b, Theorem 4.5]. $\square$

Proof of Theorem 1.3. This is a special case of Theorem 4.4. $\square$

5. SURFACE ABUNDANCE

The goal of this section is to prove Theorem 1.4. The idea is to apply the same lines of the arguments as the proof of Theorem 4.3. First, we need to prove the surface non-algebraically integrable version of [Cas+25a, Proposition 4.2, Theorem 5.1] and [Cas+24, Proposition 3.3] which were applied in the proof of Theorem 4.3.

Remark 5.1. The definition of “foliated log smooth” for algebraically integrable foliations and for surface foliations do not coincide in literature (for algebraically integrable foliation on surfaces). To avoid confusion In this section, and in this section only, the term “foliated log smooth” (and related definitions, e.g. F-dlt, foliated log resolution) for rank 1 foliations on surfaces align with the one as in [LLM23, Definition 4.4] and do not align with the definition for algebraically integrable foliations (e.g. [ACSS21]).

5.1. Surface adjoint foliated structures. The following proposition can be seen as the surface non-algebraically integrable case of [Cas+24, Proposition 3.3].

Proposition 5.2. *Let $\mathfrak{A}/U := (X, \mathcal{F}, B^{\text{nin}} + (1-t)B^{\text{inv}}, t)/U$ be an lc (resp. klt) adjoint foliated structure such that $\dim X = 2$ and $t < 1$. Then (X, B) is lc (resp. klt).*

Proof. We may assume that $\text{rank } \mathcal{F} = 1$, otherwise the proposition is trivial. Let $h : X' \rightarrow X$ be a foliated log resolution of $(X, \mathcal{F}, \text{Supp } B)$ (in the sense of cf. [LLM23, Definition 4.4]). Let $\mathcal{F}' := h^{-1}\mathcal{F}$, $B' := h_*^{-1}B + \text{Exc}(h)$, and $\mathfrak{A}' := (X', \mathcal{F}', B', t)$. By [LLM23, Theorem 5.9], we may run a $(K_{\mathcal{F}'} + B'^{\text{nin}})$ -MMP/ X which terminates with a minimal model $(Y, \mathcal{F}_Y, B_Y^{\text{nin}})/X$ of $(X', \mathcal{F}', B'^{\text{nin}})/X$ as X' , and $(Y, \mathcal{F}_Y, B_Y^{\text{nin}})$ is F-dlt (cf. [CS21, Definition 3.1]). Let $g : Y \rightarrow X$ be the induced birational morphism and let B_Y be the image of B' on Y . Then $B_Y = g_*^{-1}B + \text{Exc}(g)$. Moreover, since $K_{\mathcal{F}_Y} + B_Y^{\text{nin}}$ is nef/ X , by the negativity lemma, for any prime divisor D on Y that is exceptional/ X ,

$$a(D, \mathcal{F}_Y, B_Y^{\text{nin}}) \leq -\epsilon_{\mathcal{F}}(D).$$

Here we consider $(Y, \mathcal{F}_Y, B_Y^{\text{nin}})$ as a surface foliated numerical triple (cf. [LMX24b, Definition 3.10]) hence

$$\begin{aligned} -t\epsilon_{\mathcal{F}}(D) - (1-t) &\leq (\text{resp. } <) a(D, X, \mathcal{F}, B^{\text{nin}} + (1-t)B^{\text{inv}}, t) \\ &= ta(D, \mathcal{F}, B^{\text{nin}}) + (1-t)a(D, X, B) \leq -t\epsilon_{\mathcal{F}}(D) + (1-t)a(D, X, B), \end{aligned}$$

and so $a(D, X, B) \geq -1$ (resp. > -1). Therefore,

$$K_Y + B_Y = K_Y + g_*^{-1}B + \text{Exc}(g) \geq g^*(K_X + B).$$

(resp. $K_Y + B_Y - \epsilon \text{Exc}(g) = K_Y + g_*^{-1}B + (1-\epsilon) \text{Exc}(g) \geq g^*(K_X + B)$ for some $\epsilon > 0$). Since $(Y, \mathcal{F}_Y, B_Y^{\text{nin}})$ is dlt, (Y, B_Y) is lc (resp. $(Y, B_Y - \epsilon \text{Exc}(g))$ is klt) by [Spi20, Lemma 8.14] (note that [Spi20, Lemma 8.14] is stated for threefolds, but the same arguments, together with the proof of [Spi20, Lemma 8.9] referred in the proof of [Spi20, Lemma 8.14] work for surfaces). Therefore, (X, B) is numerically lc (resp. klt), hence lc (resp. klt). $\square$

The following proposition is an analogue of [Cas+25a, Theorem 5.1].

Lemma 5.3. *Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a $\mathbb{Q}$ -factorial generalized foliated quadruple such that $\dim X = 2$, and let $\mathfrak{A}_s := (X, \mathcal{F}, B^{\text{nin}} + (1-s)B^{\text{inv}}, \mathbf{M}, s)$ for any $s \in [0, 1]$. Assume that $\mathfrak{A}_t$ is lc and $K_{\mathfrak{A}_t}$ is pseudo-effective/ U for some $t \in [0, 1]$. Then $K_{\mathfrak{A}_s}$ is pseudo-effective/ U for any $s \in [t, 1]$.*

Proof. We may assume that $\text{rank } \mathcal{F} = 1$, otherwise the lemma is trivial. Assume that $K_{\mathfrak{A}_1}$ is not pseudo-effective/ U . By [CP19, Theorem 1.1], [LLM23, Theorem 3.1], $\mathcal{F}$ is algebraically integrable, and we get a contradiction to [Cas+25a, Theorem 5.1]. $\square$

The following lemma is an analogue of [Cas+25a, Theorem 4.1].

Lemma 5.4. *Let $\mathfrak{A}/U := (X, \mathcal{F}, B^{\text{nin}} + (1-t)B^{\text{inv}}, t)/U$ be a klt adjoint foliated structure such that $\dim X = 2, t < 1$, and let $\mathcal{S}$ be a finite set of prime divisors over X such that $a(D, \mathfrak{A}) \leq 0$ for any $D \in \mathcal{S}$. Then there exists a projective birational morphism $f : Y \rightarrow X$ such that Y is $\mathbb{Q}$ -factorial, and the divisors contracted by f are exactly the divisors in $\mathcal{S}$.*

Proof. We may assume that $\text{rank } \mathcal{F} = 1$, otherwise $a(D, X, B) \leq 0$ and we are done by [BCHM10, Corollary 1.4.3]. By Proposition 5.2, (X, B) is klt, hence X is $\mathbb{Q}$ -factorial. By induction on the number of divisors in $\mathcal{S}$, we may assume that $\mathcal{S}$ contains exactly one prime divisor E . If $a(E, X, B) \leq 0$, then we are done by [BCHM10, Corollary 1.4.3], so we may assume that $a(E, X, B) > 0$. Since $a(E, \mathfrak{A}) \leq 0$, by linearity of discrepancies, $a(E, \mathcal{F}, B^{\text{nin}}) < 0$. Since $t < 1$ and $\mathfrak{A}$ is klt, possibly replacing t with $t + \delta_1$ for some $0 < \delta_1 \ll 1$, we may assume that $a(E, \mathfrak{A}) < 0$. Let $b := -a(E, \mathfrak{A})$, then $b > 0$.

Let $h : W \rightarrow X$ be a foliated log resolution of $\mathfrak{A}$ such that E is on W . Since $\mathfrak{A}$ is klt, there exists $\delta_2 > 0$ such that $a(D, \mathfrak{A}) > -(t\epsilon_{\mathcal{F}}(D) + (1-t))(1-2\delta_2)$ for any prime divisor D on W that is exceptional/ X .

We let B_W, F_W be the unique $\mathbb{R}$ -divisors on W such that

- $B_W - h_*^{-1}B := F_W$ is exceptional/ X ,
- $\text{mult}_D F_W = 1 - \delta_2$ if $D \neq E$ and D is exceptional/ X , and
- $(t\epsilon_{\mathcal{F}}(E) + (1-t)) \text{mult}_E F_W = b$.

Let

$$\mathfrak{A}_{W,s} := (W, \mathcal{F}_W, B_W^{\text{nin}} + (1-s)B_W^{\text{inv}}, s)$$

for any $s \in [0, 1]$. Then each $\mathfrak{A}_{W,s}$ is foliated log smooth, and $\mathfrak{A}_{W,s}$ is klt for any $s < 1$.

By [LLM23, Theorem 5.9], we may run a $K_{\mathfrak{A}_{W,1}}$ -MMP/ X which terminates with a $\mathbb{Q}$ -factorial minimal model $\mathfrak{A}_{V,1}/X$ of $\mathfrak{A}_{W,1}/X$ such that $\mathfrak{A}_{V,0}$ is klt, with induced birational morphism $\phi : W \rightarrow V$ and $\mathfrak{A}_{V,s} := \phi_* \mathfrak{A}_{W,s}$ for any $s \in [0, 1]$. Let $g : V \rightarrow X$ be the induced birational morphism. By our construction, we have

$$K_{\mathfrak{A}_{V,t}} = g^* K_{\mathfrak{A}} + \sum_D ((t\epsilon_{\mathcal{F}}(D) + (1-t))(1-\delta_2) + a(D, \mathfrak{A}))D =: g^* K_{\mathfrak{A}} + G$$

where the sum runs through all g -exceptional prime divisors that are not $\phi_* E$. By our construction of δ_2 , $G \geq 0$, $\text{Supp } G = \text{Exc}(g)$ if E is contracted by ϕ , and $\text{Supp } G$ is the closure of $\text{Exc}(g) \setminus \{\phi_* E\}$ if E is not contracted by ϕ .

Let $\mathbf{P} := \overline{K_{\mathfrak{A}_{V,1}}}$. Then $\mathbf{P}$ is nef/ X . Since $\mathfrak{A}_{V,0}$ is klt, $\mathcal{B}_V/X := (\mathfrak{A}_{V,0}, \frac{t}{1-t}\mathbf{P})/X$ is a $\mathbb{Q}$ -factorial generalized pair, and $K_{\mathfrak{A}_{V,t}} = (1-t)K_{\mathcal{B}_V}$. By [BZ16, Lemma 4.4] and [Bir12, Lemma 3.3], we may run a $K_{\mathcal{B}_V}$ -MMP/ U with scaling of an ample divisor that is also a $K_{\mathfrak{A}_{V,t}}$ -MMP/ U with scaling of an ample divisor, which terminates with a model T such that the divisors contracted by the induced birational morphism $\psi : V \rightarrow T$ are exactly the divisors contained in $\text{Supp } G$.

If E is not contracted by ϕ , then since $\phi_* E$ is not a component of $\text{Supp } G$, E is also not contracted by ψ , hence we may let $T =: Y$ and the induced birational morphism $f : Y \rightarrow X$ satisfies our requirements. Therefore, we may assume that E is contracted by ϕ . In this case, since X is $\mathbb{Q}$ -factorial and the induced birational map $T \rightarrow X$ does not contract any divisor, we have $T = X$ hence $\psi : V \rightarrow X$ is a morphism. Let

$$\mathcal{B} := \left(X, B, \frac{t}{1-t}\mathbf{P} \right),$$

then since $H_W \subset \text{Exc}(h)$, $\mathcal{B} = \psi_* \mathcal{B}_V$, hence $\mathcal{B}$ is a klt generalized pair. We have $\mathfrak{A} = \psi_* \mathfrak{A}_{V,t}$.

We set $b_1 := -a(E, \mathfrak{A})$, $b_2 := -a(E, \mathcal{B})$, $c_1 := -a(E, \mathfrak{A}_{V,t})$, and $c_2 := -a(E, \mathcal{B}_V)$. Since $\psi_* K_{\mathcal{B}_V} = K_{\mathcal{B}}$, $\psi_* K_{\mathfrak{A}_{V,t}} = K_{\mathfrak{A}}$, and $K_{\mathfrak{A}_{V,t}} = (1-t)K_{\mathcal{B}_V}$, we have $b_1 - c_1 = (1-t)(b_2 - c_2)$.

Let $h : Z \rightarrow V$ be a projective birational morphism such that E is on Z . Let $\mathfrak{A}_{Z,s} := h_*^{-1} \mathfrak{A}_{V,s}$ for any $s \in [0, 1]$ and $\mathcal{B}_Z := h_*^{-1} \mathcal{B}_V$. Then we have $\text{mult}_E(h^* K_{\mathfrak{A}_{V,t}} - K_{\mathfrak{A}_{Z,t}}) = c_1$ and $\text{mult}_E(h^* K_{\mathcal{B}_V} - K_{\mathcal{B}_Z}) = c_2$. Since E is contracted by ϕ , we have

$$\begin{aligned} s &:= \text{mult}_E(h^* \mathbf{P}_V - h_*^{-1} \mathbf{P}_V) = \text{mult}_E(h^* K_{\mathfrak{A}_{V,1}} - K_{\mathfrak{A}_{Z,1}}) \\ &= -a(E, \mathfrak{A}_{V,1}) < -a(E, \mathfrak{A}_{W,1}) = \text{mult}_E(B_W^{\text{inv}}). \end{aligned}$$

By linearity of discrepancies, we have

$$c_2 = -a(E, \mathfrak{A}_{V,0}) = \frac{1}{1-t}(-a(E, \mathfrak{A}_{V,t}) + ta(E, \mathfrak{A}_{V,1})) = \frac{1}{1-t}(c_1 - ts),$$

hence, $c_1 = (1-t)c_2 + ts$. This implies that

$$b_2 = \frac{1}{1-t}(b_1 - c_1) + c_2 = \frac{1}{1-t}(b_1 - ts).$$

By our construction of B_W , we have

$$\begin{aligned} b_1 &= -a(E, X, \mathcal{F}, B^{\text{inv}} + (1-t)B^{\text{inv}}, t) \\ &= -a(E, \mathfrak{A}) = \text{mult}_E(B_W^{\text{inv}} + (1-t)B_W^{\text{inv}}) \geq \text{mult}_E B_W^{\text{inv}} = s > ts. \end{aligned}$$

Therefore, $b_2 > 0$. Since $\mathcal{B}$ is a klt generalized pair, the existence of f follows from [BZ16, Lemma 4.6]. $\square$

The following proposition is an analogue of [Cas+25a, Proposition 4.2].

Proposition 5.5. *Let $\mathfrak{A} = (X, \mathcal{F}, B, t)/U$ be a klt adjoint foliated structure such that $\dim X = 2$, $\text{rank } \mathcal{F} = 1$, and $t < 1$. If $\mathfrak{A}/U$ has a weak lc model (resp. a semi-ample model) then $\mathfrak{A}/U$ has a $\mathbb{Q}$ -factorial minimal model (resp. $\mathbb{Q}$ -factorial good minimal model).*

Proof. Let $\mathfrak{A}'/U$ be a weak lc model (resp. semi-ample model) of $\mathfrak{A}/U$ and let X, X' be the ambient varieties of $\mathfrak{A}, \mathfrak{A}'$ respectively. Let

$$\mathcal{S} := \{E \mid E \text{ is an exceptional}/X' \text{ prime divisor on } X, a(E, \mathfrak{A}) = a(E, \mathfrak{A}')\}.$$

Then $\mathcal{S}$ is a finite set. Moreover, for any $E \in \mathcal{S}$, $a(E, \mathfrak{A}') \leq 0$. By Lemma 5.4, there exists a birational morphism $f : Y \rightarrow X'$ such that Y is $\mathbb{Q}$ -factorial and the divisors contracted by f are exactly the divisors in $\mathcal{S}$. Let $\phi : X \dashrightarrow Y$ be the induced birational map. By Lemma 2.18, $\phi_*\mathfrak{A}/U$ is a $\mathbb{Q}$ -factorial minimal model (resp. $\mathbb{Q}$ -factorial good minimal model) of $\mathfrak{A}/U$. $\square$

5.2. Proof of Theorem 1.4.

Proof of Theorem 1.4. Let $\mathfrak{A}_s := (X, \mathcal{F}, s)$ for any $s \in [0, 1]$. By Lemma 5.3, $K_{\mathfrak{A}_s}$ is pseudo-effective for any $s \in [t, 1]$. In particular, $\nu := \nu(K_{\mathfrak{A}_s})$ is a constant for any $s \in (t, 1)$ and $\nu \geq \nu(K_{\mathfrak{A}_1})$. We consider the sets

$$\mathcal{P} := \{s \in [t, 1] \mid \mathfrak{A}_s \text{ has a } \mathbb{Q}\text{-factorial good minimal model}\}$$

and

$$\mathcal{Q} := \{s \in [t, 1] \mid \mathfrak{A}_s \text{ has a } \mathbb{Q}\text{-factorial minimal model and } K_{\mathfrak{A}_s} \text{ is abundant}\}.$$

It is obvious that $\mathcal{P} \subset \mathcal{Q}$. By following the same lines of the proof of Theorem 4.3 verbatimly except that the following places:

- We apply Proposition 5.2 instead of [Cas+24, Proposition 3.3].
- We apply Proposition 5.5 instead of [Cas+25a, Proposition 4.2].

We deduce the following:

- If $\mu \in \mathcal{Q}$ and $\mu > t$, then there exists $\epsilon > 0$ such that $[\mu - \epsilon, \mu] \subset \mathcal{P}$.
- If $\mu < 1$ and $\mu + \epsilon \in \mathcal{P}$ for any $0 < \epsilon \ll 1$, then $\mu \in \mathcal{P}$.

Therefore, we only need to show that $\mu \in \mathcal{Q}$ for some $\mu \in (t, 1]$. By [SS23, Theorem 3.1] and [LLM23, Theorem 5.9], there exists $0 < \epsilon_0 < 1 - t$ such that $\mathfrak{A}_s$ has a $\mathbb{Q}$ -factorial minimal model for any $s \in [1 - \epsilon, 1]$. Therefore, we only need to show that $K_{\mathfrak{A}_\mu}$ is abundant for some $\mu \in [1 - \epsilon, 1]$. In particular, we may assume that $\nu \leq 1$, and $\nu(K_{\mathcal{F}}) > \kappa(K_{\mathcal{F}})$. In particular, $K_{\mathcal{F}}$ is not big. We run a $K_{\mathcal{F}}$ -MMP $\phi : X \rightarrow X'$ which terminates with a minimal model $\mathcal{F}'$ of $\mathcal{F}$. Then $\mathcal{F}'$ is canonical and $\nu(K_{\mathcal{F}'}) > \kappa(K_{\mathcal{F}'})$. By [McQ08, Lemma IV.3.1], we have $\nu(K_{\mathcal{F}}) = 1$. By [McQ08, Theorem IV.5.11], $\mathcal{F}'$ is either one of the natural foliations on the Baily-Borel compactification of a Hilbert modular surface. Thus $K_{X'}$ is ample, hence $sK_{\mathcal{F}'} + (1 - s)K_{X'}$ is ample for any $s \in [0, 1]$. Since ϕ is also a sequence of steps of a $K_{\mathfrak{A}_s}$ -MMP for any $0 < 1 - s \ll 1$, we have that $K_{\mathfrak{A}_s}$ is big for any $0 < 1 - s \ll 1$. Thus $K_{\mathfrak{A}_s}$ is big hence abundant for any $0 < 1 - s \ll 1$, and we are done. $\square$

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