

Robust Optimality of Bundling Goods Beyond Finite Variance

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October 7, 2025

Abstract

When selling many goods with independent valuations, we develop a distributionally robust framework, consisting of a two-player game between seller and nature. The seller has only limited knowledge about the value distribution. The seller selects a revenue-maximizing mechanism, after which nature chooses a revenue-minimizing distribution from all distributions that comply with the limited knowledge. When the seller knows the mean and variance of valuations, bundling is known to be an asymptotically optimal deterministic mechanism, achieving a normalized revenue close to the mean. Moving beyond this variance assumption, we assume knowledge of the mean absolute deviation (MAD), accommodating more dispersion and heavy-tailed valuations with infinite variance. We show for a large range of MAD values that bundling remains optimal, but the seller can only guarantee a revenue strictly smaller than the mean. Another noteworthy finding is indifference to the order of play, as both the max-min and min-max versions of the problem yield identical values. This contrasts with deterministic mechanisms and the separate sale of goods, where the order of play significantly impacts outcomes. We further underscore the universality of the optimal bundling price by demonstrating its efficacy in optimizing not only absolute revenue but also the absolute regret and ratio objective among all bundling prices.

1 Introduction

We study a single buyer, multi-good Bayesian mechanism design problem in which the seller wants to maximize expected revenue, i.e., the profit obtained from selling the goods. The expectation here is taken with respect known prior distributions for the values that the buyer has for the goods. Over the last decades, an extensive and beautiful theory has been developed for such Bayesian mechanism design problems, dating back to the work of Myerson [37] on auctions.

In the case of selling $m = 1$ good with non-negative value distribution $\mathbb{P}$, the optimal truthful mechanism [37, 41, 40] maximizing the revenue is characterized as follows: Set the posted “take-it-or-leave-it” price $p^* = \operatorname{argmax}_p p \cdot \mathbb{P}(X \geq p)$ and sell the good to the buyer if and only if the buyer’s (random) value X for the good exceeds p^* . Truthfulness refers to the fact that the buyer should have no incentive to lie about the valuation they have for the good. When selling $m > 1$ goods, the optimal mechanism may exhibit a much more complex structure [22, 12, 34, 11, 27].

The assumption that we have *full knowledge* about the prior value distributions of the buyer for the goods is often deemed unrealistic in practice. Adding to this is what is commonly known as Wilson’s doctrine [46, 36]: We should aim at designing mechanisms that only require minimal assumptions about the value distribution of the buyer. This leads to the notion of (*distributionally*) *robust revenue maximization*, or robust mechanism design, in which the seller only has *partial knowledge* about the value distributions of the buyer. This partial knowledge is typically modelled as an *ambiguity set* that consists of all distributions satisfying the given partial knowledge. A prominent example here is to assume moment knowledge about the unknown distribution, such as the mean and variance. This dates back to the work of Scarf [43]. The idea is now that we have a two-player game where the seller first has to choose a selling mechanism, after which nature gets to choose a distribution from the ambiguity set (satisfying the partial knowledge) for which the seller’s revenue is minimized under the chosen mechanism. Research on robust selling mechanisms and auctions has seen a surge in recent years in the mechanism design literature (see, e.g., Section 1.2). The goal is to either obtain robust counterparts of classical Bayesian results, or to derive tractable robust counterparts of multi-good (and/or multi-buyer) problems for which the Bayesian problem itself has a complex, intractable structure.

Recently there has been an increasing interest in designing robust selling mechanism for the single buyer, multi-good setting [23, 16, 28, 18] (these works are discussed in Section 1.2). In this paper we focus on the setting of robustly selling $m \rightarrow \infty$ *independently distributed* goods, whose distributions come from a common ambiguity set, to an additive buyer using a *deterministic* mechanism. This is an important special case of selling multiple goods whose classical Bayesian counterpart, the independent and identical distribution (i.i.d) setting, has received considerable attention in the literature: Bakos and Brynjolfsson [6] show, using the law of large numbers, that if the known distribution is independent of m , then bundling¹ is asymptotically (i.e., when $m \rightarrow \infty$) optimal. This can be achieved by setting a price just below $m \cdot \mu$, where μ is the expectation of the distribution. When the distribution is allowed to depend on m , bundling is no longer guaranteed to be optimal [30]. Hart and Nisan [30] showed that, instead, bundling achieves a $c/\log(m)$ -fraction of the revenue of the optimal mechanism for some absolute constant $c > 0$. Later, Li and Yao [33] improved this result to show that bundling guarantees a constant fraction of the revenue of the (unknown) optimal mechanism.

In the robust setting, if one assumes to know the mean and variance of the unknown distributions, it follows by an application of Chebyshev’s bound that bundling with a price just below $m\mu$ is asymptotically optimal (with a revenue of roughly $m\mu$). This observation can also be seen as a

¹Setting one price for the grand bundle of goods and selling the bundle as a whole if the sum of the valuations of the buyer exceeds the price.

special case of results on robustly selling multiple goods in [18, 28]. These works in particular rely on the assumption of fixing (or upper bounding) the variance of the unknown distributions. In this work we go beyond this assumption and look at a more general ambiguity set that replaces the variance by the mean absolute deviation, as will be explained next. As our main result, we show that (deterministic) bundling is still asymptotically optimal, but in this case it is no longer possible to achieve a revenue close to $m\mu$, creating a stark contrast with the finite-variance case.

The joint distribution $\mathbb{P}^m = \prod_{i=1}^m \mathbb{P}_i \in \mathcal{P}(\mu, d)^m$ is the product of the marginal distributions of all goods where each $\mathbb{P}_i$ is from the ambiguity set $\mathcal{P}(\mu, d)$ that assumes to know the mean μ and mean absolute deviation (MAD) d of the distribution (referred to as *mean-MAD ambiguity*), i.e.,

$$\mathcal{P}(\mu, d) = \{\mathbb{P} : \mathbb{E}_{\mathbb{P}}[X] = \mu, \mathbb{E}_{\mathbb{P}}[|X - \mu|] = d, \text{ and } X \in [0, \infty)\}. \quad (1.1)$$

Note that while all marginal distributions are contained in the same ambiguity set, they are not assumed to be identical. Hence, the joint distribution $\mathbb{P}^m = \prod_{i=1}^m \mathbb{P}_i$ may consist of different (marginal) $\mathbb{P}_i \in \mathcal{P}(\mu, d)$. The set $\mathcal{P}(\mu, d)$ is non-empty if and only if $0 \leq d < 2\mu$. The MAD is a natural dispersion measure that can substitute for the variance (or standard deviation) and is less sensitive to outliers than the variance. Furthermore, defining the ambiguity set through the MAD allows distributions with heavy tails that have infinite variance. Ambiguity sets based on the MAD have recently gained more interest in the robust revenue maximization literature, see, e.g., [42, 24, 19].

There is ample empirical evidence for heavy-tailed valuations and demand, also spurred by “the long-tail phenomenon” seen in online retail where niche products often determine a large share of the total sales [21]. Examples of reported heavy tails include demand for books [26], movies [9], spare parts [38], and many more [20], and in the majority of cases the tail exponent is such that the mean is finite while the variance is infinite. The ambiguity set (1.1) we consider in this paper conditions on MAD, and hence allows for such heavy-tailed distributions.

The robust revenue maximization problem we want to solve can be seen as a two-player game, in which the seller (first player) needs to choose a truthful mechanism $A \in \mathcal{A}_m$ from the set $\mathcal{A}_m$ of allowed truthful mechanisms, after which nature (the second player) chooses a distribution $\mathbb{P}^m = \prod_{i=1}^m \mathbb{P}_i \in \mathcal{P}^m$ with all (independent) marginal distributions $\mathbb{P}_i \in \mathcal{P}$, and with the goal of minimizing the expected revenue $\text{REV}(A, \mathbb{P})$ that the seller obtains when using mechanism A to sell m independent goods with random values $X_i \sim \mathbb{P}_i$ for $i = 1, \dots, m$. Because in this work we want to let m grow large, we will in fact maximize the revenue scaled by m . That is, we want to solve the maximin problem

$$\lim_{m \rightarrow \infty} \sup_{A \in \mathcal{A}_m} \inf_{\mathbb{P}^m \in \mathcal{P}^m} \frac{\text{REV}(A, \mathbb{P}^m)}{m}. \quad (1.2)$$

Closely related to (1.2) is the minimax problem

$$\lim_{m \rightarrow \infty} \inf_{\mathbb{P}^m \in \mathcal{P}^m} \sup_{A \in \mathcal{A}_m} \frac{\text{REV}(A, \mathbb{P}^m)}{m} \quad (1.3)$$

in which the roles of the players are reversed: First nature gets to choose a distribution, after which the seller has to choose an optimal selling mechanism for that distribution.

Roughly speaking, the difference between the two problems lies in the fact that in the maximin problem, the seller needs to hedge against distributional uncertainty, whereas in the minimax problem the seller has to hedge against a (known) worst-case distribution.

1.1 Our contributions

As our main result we show that, when $0 < d \leq \mu$, both problems (1.2) and (1.3) have the common value $\mu - d/2$. This is summarized in Theorem 1.1. The mechanism solving the maximization problems is a bundling mechanism with a bundling price just below $m(\mu - d/2)$. It is possible to extract a convergence rate in terms of m from our analysis, but leave this to the interested reader.

Theorem 1.1. *Let $0 < d < \mu$, and $\mathcal{D}_m$ the set of all deterministic truthful mechanisms for selling m goods. Then*

$$\lim_{m \rightarrow \infty} \sup_{D \in \mathcal{D}_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{REV}(D, \mathbb{P}^m)}{m} = \lim_{m \rightarrow \infty} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \sup_{D \in \mathcal{D}_m} \frac{\text{REV}(D, \mathbb{P}^m)}{m} = \mu - d/2. \quad (1.4)$$

We next summarize the main insights regarding Theorem 1.1.

- Our result creates a surprising contrast with the mean-variance ambiguity set, where a straightforward concentration argument yields a common value of μ .
- For deterministic mechanisms and fixed m , the maximin and minimax problem typically do not have the same value; compare, e.g., [42, 24] for the case $m = 1$. One has to resort to randomization to establish such results.² As m grows large, we show that the value gap disappears for deterministic mechanisms and the problems are equivalent in terms of revenue.
- Solving the maximin and minimax problem under mean-MAD ambiguity with independent good distributions is arguably much harder than its correlated counterpart, where one can resort to well-known primal-dual semi-infinite linear programming techniques or computing a saddle point of a zero-sum game. Computing optimal independent distributions, from an optimization point of view, results in non-linear (and non-convex) problems. A fitting analogy is comparing the difference in computational complexity between computing a mixed Nash equilibrium (difficult) and a correlated equilibrium in general finite games (tractable via a linear program).

In our technical analysis the subset $\mathcal{P}_2(\mu, d) \subseteq \mathcal{P}(\mu, d)$ of all distributions supported on two points plays an important role. In fact, Theorem 1.1 is also true for this ambiguity set:

Corollary 1.2. *The statement of Theorem 1.1 remains true if we replace $\mathcal{P}(\mu, d)$ by $\mathcal{P}_2(\mu, d)$: The set of all non-negative distributions with mean μ and MAD d whose support consists of two points.*

For the case $\mu \leq d < 2\mu$, which turns out to be more complex, we have the partial result showing that the bundling mechanism achieves a value of $\mu - d/2$ and that no deterministic mechanism can do better than $d/2$. We use $\text{BUND}(p_m, \mathbb{P}^m)$ to denote the revenue of the bundling mechanism that sets price p_m for selling m independent goods with joint distribution $\mathbb{P}^m$ (see Section 2.2 for the formal definition). The first and second inequality in Theorem 1.3 trivially hold.

Theorem 1.3. *For $0 < \mu \leq d < 2\mu$, it holds that*

$$\begin{aligned} \mu - d/2 &= \lim_{m \rightarrow \infty} \sup_{p_m \geq 0} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{m} \leq \lim_{m \rightarrow \infty} \sup_{D \in \mathcal{D}_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{REV}(D, \mathbb{P}^m)}{m} \\ &\leq \lim_{m \rightarrow \infty} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \sup_{D \in \mathcal{D}_m} \frac{\text{REV}(D, \mathbb{P}^m)}{m} \leq d/2. \end{aligned} \quad (1.5)$$

²For the case $m = 1$, the fact that the maximin and minimax problem have the same value when randomization is allowed, follows from [19, Lemma 1]. To the best of our knowledge no such statement is known for the independent multi-good setting.

To prove the above theorems, we first provide results for the maximin and minimax problem separately in Section 3 and conclude with a summary of how Theorems 1.1 and 1.3 follow from them. We will show that for any d the bundling mechanism yields a revenue of $\mu - d/2$ in the maximin problem and that no algorithm can guarantee more than $\max\{\mu - d/2, d/2\}$ for the minimax problem.

One question that arises is what the true answer is for the range $\mu \leq d < 2\mu$. Is there a gap between the revenue guarantee of the maximin and the minimax problem? We show in Proposition E.1 in Appendix E that for the set of two-point distributions $\mathcal{P}_2(\mu, d)$ and i.i.d. goods, the minimax problem for the bundling mechanism gives a value strictly greater than $\mu - d/2$, which indicates that the second part of Theorem 1.1 (i.e., Corollary 1.2) is not true for the range $\mu \leq d < 2\mu$. We therefore conjecture that $\mu - d/2$ is not the true value of the minimax problem for $\mu \leq d < 2\mu$ for the ambiguity set $\mathcal{P}(\mu, d)$. All together, our results and proofs show an intriguing dependence on the dispersion parameter d that deserves further investigation.

Inspired by the optimality of the bundling mechanism, we further investigate this mechanism for two other revenue objectives from the literature: the *ratio objective* (e.g., [28]) and the *absolute regret objective* (e.g., [18]), defined, respectively, by

$$\lim_{m \rightarrow \infty} \sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \quad (1.6)$$

and

$$\lim_{m \rightarrow \infty} \inf_{p_m} \sup_{\mathbb{P}^m \in \mathcal{P}^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m}. \quad (1.7)$$

Here $\text{OPT}(m, \mathbb{P}^m)$ is the expected revenue of the (unknown) optimal deterministic mechanism for selling m independent goods with known joint distribution $\mathbb{P}^m$. When $m = 1$, the absolute revenue, ratio, and absolute regret objectives all have a different optimal posted price for the problems in (1.2), (1.6), and (1.7) under mean-MAD ambiguity [42, 31, 45]. We show that, when $m \rightarrow \infty$, a bundling price $p_m \approx m(\mu - d/2)$ is optimal for both the ratio (Theorem 4.1) and absolute regret objective (Theorem 4.2) among all bundling prices, just as it was for the absolute revenue objective (Theorems 1.1 and 1.3). This shows that all three objectives that we consider in fact have the same optimal bundling price as m grows large, which stands in stark contrast with the case $m = 1$. Roughly speaking, our results give rise to a universality of the bundling price $p_m \approx m(\mu - d/2)$ as an optimal robust solution among various revenue objectives.

An interesting direction for future research is to understand how allowing randomization in the seller's mechanism affects our results. In particular, can one improve over the value of $\mu - d/2$ for the absolute revenue objective, and the corresponding values of the other objectives? Secondly, (dis)proving that the bundling mechanism is optimal among all deterministic mechanisms for the ratio and absolute regret objective is also a very intriguing question.

Techniques. One of the main technical tools that we use to establish our results is a one-sided concentration inequality for distributions with known mean and mean absolute deviation (Proposition 3.1), which might be of independent interest. To establish our tight result in Theorem 1.1 we lower bound the maximin problem with a value of $\mu - d/2$, by analysing the bundling mechanism, and upper bound the minimax problem by $\max\{d/2, \mu - d/2\}$. The latter is the technically most challenging part. We establish this bound by analysing the minimax problem for two-point distributions, and argue that in this case only the smallest two support points of $\sum_i X_i$, the sum of the valuations of the buyer, are relevant for the analysis. Using the definition of a truthful mechanism, we then argue that an optimal mechanism is one that chooses a bundling price to be the better choice of those two support points (the first giving a revenue of $\mu - d/2$ and the second one $d/2$).

1.2 Further related work

In this section we discuss related works, starting with the classical Bayesian setting. We then continue with robust mechanisms for selling multiple goods in other settings, as well as the literature that addresses the case when there are multiple buyers. Finally, we discuss related works for robustly selling a single good. These works demonstrate the growing interest in obtaining distributionally robust solutions to classical mechanism design problems, to which our work contributes.

Mechanisms for selling multiple goods in the classical Bayesian setting. Whereas the optimal mechanism for selling a single good is known, the multi-good setting is more complex. Natural choices like bundling all goods together or selling them separately are in general not optimal for any distribution. Furthermore, the optimal mechanism need not be deterministic, as opposed to the single-good case. These observations are nicely summarized by Hart and Nisan [30, Examples 1-4]. Daskalakis, Deckelbaum and Tzamos [22] provide a primal-dual framework that allows them to characterize the optimal mechanism for selling a fixed number of $m \geq 2$ goods, thereby also generalizing earlier works such as [34, 11, 28]; see also [12] for a more general framework. Daskalakis et al. [22] also provide a characterization for when offering the grand bundle is optimal. As a concrete application of this characterization, they consider i.i.d. goods with uniform distribution on $[c, c + 1]$ for some scalar c . They show that if m is kept fixed, then for c large enough the bundling mechanism is optimal (this was shown by Pavlov [39] for $m = 2$). However, for c fixed, it is shown that for m large enough, the bundling mechanism is *not* optimal for selling m i.i.d. goods.³ In general, there is not a simple economic interpretation of the characterization in [22]. There have also been other works concerned with deriving conditions on the known independently distributed goods for when bundling is the optimal mechanism, such as [29, 39, 22, 35, 34]. See [16, Footnote 1] for an overview.

Robust mechanisms for selling multiple goods. Carroll [14] introduces a robust perspective for selling multiple goods. It is shown that when correlations between the goods are allowed, the optimal robust mechanism is to sell every good separately. See also [15] for an interesting survey. Che and Zhong [16] generalize Carroll’s work by introducing a partition structure. They then assume to know the mean of every good, and some dispersion information about every bundle in the partition, but not the possible correlations between the goods in different parts of the partition. Deb and Roesler [23] consider a robust setting related to ours, but instead of moment information assume to have only support information for the unknown distributions, i.e., they only assume that the distributions have positive probability mass on a given interval. They prove that randomized bundling is the optimal mechanism for selling m goods under an exchangeability assumption (that also applies to independent distributions); see also Che and Zhong [16] for related results in this direction. Giannakopoulos, Pocas and Tsigonias-Dimitriadis [28] consider the robust ratio objective and provide various tight, up to constants, results for selling multiple goods under mean-variance ambiguity for both correlated and independent value distributions. See also the work of Chen, Hu and Perakis [18] for results in this direction.

Robust mechanisms for multiple buyers. There are also various papers that study the setting with multiple buyers where there are unknown distributions coming from (moment-based) ambiguity sets. Anunrojwong, Balseiro and Besbes [3] study the problem of selling one good to n potential buyers, where the unknown (joint) distribution is only assumed to have bounded support, meaning an upper bound on the valuations is known. Bachrach and Talgam-Cohen [5] consider a similar model for selling a single good to n i.i.d. (potential) buyers, but in addition assume to know the mean of the common distribution. In both these works (in [5] only for $n = 2$), a robust version of the

³We suspect that in our minimax setting bundling is also not optimal for fixed m , but leave this for future work.

second price auction with (randomized) reserve price arises as the optimal robust solution, which is proved by using a saddle-point argument for the corresponding robust zero-sum game between the seller and nature. For other works in the multi-buyer setting, see, e.g., Kocyigit et al. [32], Allouah and Besbes [2], or the survey of Carroll [15].

Robust mechanisms for single good. Maximin analysis for mean-variance ambiguity was pioneered in Azar and Micali [4], generalized to higher moments in [13]; see also [18, 44]. Other forms of knowledge were also studied, such as percentiles [25], mean absolute deviation [42, 24, 19], or knowing that the valuation distribution is within the proximity of a given reference distribution [7, 19]. Instead of maximin expected revenue, alternative objectives studied in the literature include minimax regret [8, 7, 24, 19] and the competitive or approximation ratio [25, 28, 24]. We also refer to the work of Allouah, Bahamou and Besbes [1] who assume to have prior knowledge about the probability of sale of a good. In some works in the literature, randomized algorithms are also considered and shown to be optimal in certain cases. For selling one good in the robust setting, Chen, Hu and Wang [19] show, based on a functional version of Von Neumann’s minimax theorem [10], that if randomized mechanisms are allowed in the maximin problem, the values of the maximin and minimax problems are equal to each other. We are not aware of a similar result for the problem of selling multiple goods, in particular not with the additional assumption of independent goods. This relation fails to hold if only (simple) deterministic mechanisms, like posted prices, are allowed.

1.3 Outline

We give formal definitions of all the relevant notions in Section 2. In Section 3 we give the proofs of Theorems 1.1 and 1.3. In Section C we show that $p_m \approx m(\mu - d/2)$ is in fact the optimal bundling price for the ratio (Theorem 4.1) and absolute regret (Theorem 4.2) objective.

2 Preliminaries

We will start by formally defining deterministic truthful mechanisms and the bundling mechanism. We then introduce the various robust revenue objectives using probability theory.

2.1 Mechanism design

A direct revelation, deterministic mechanism D in the setting of a single buyer that bids on m goods is defined by a pair (z, π) . Here $z : \mathbb{R}_{\geq 0}^m \rightarrow \{0, 1\}^m$ is the *allocation rule*, and $\pi : \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}_{\geq 0}$ the *payment rule*. For a given valuation vector $v = (v_1, \dots, v_m)$ with v_i the value the buyer has for good i , they receive the goods i for which $z_i(v) = 1$, and get charged $\pi(v)$ in total for all the goods they receive. The utility of the buyer under mechanism D with valuation vector v is then given by $u(D, v) = \langle z(v), v \rangle - \pi(v)$ with $\langle x, y \rangle = \sum_i x_i y_i$ the inner product for two vectors $x, y \in \mathbb{R}^m$.

A mechanism $D = (z, \pi)$ is *truthful* if the following conditions are satisfied:

1. $\langle z(v), v \rangle - \pi(v) \geq \langle z(w), v \rangle - \pi(w)$ for all $v, w \in \mathbb{R}_{\geq 0}^m$,
2. $\langle z(v), v \rangle - \pi(v) \geq 0$ for all $v \in \mathbb{R}_{\geq 0}^m$.

The first condition ensures that the buyer has no incentive to misreport the true values v that they have for the goods; that is, bidding truthfully is a *dominant strategy* that maximizes their utility. We refer to this property as *incentive compatibility*. The second condition guarantees that if a buyer truthfully reports their values, then their utility (or surplus) is non-negative, i.e., they don’t lose

anything by participating in the selling mechanism. This property is called *individual rationality*. We use $\mathcal{D}_m$ to denote the set of all truthful deterministic mechanisms.

Bundling mechanism. A (deterministic) *bundling mechanism* sells either all goods together at a *bundling price* if the sum of the valuations meets or exceeds this price; otherwise, it sells none of the goods. To be precise, it sets a price $p =: \pi(v)$ and defines

$$z(v) = \begin{cases} (1, 1, \dots, 1) & \text{if } \sum_i v_i \geq p \\ (0, 0, \dots, 0) & \text{if } \sum_i v_i < p \end{cases}.$$

2.2 Distributionally robust revenue maximization

In this work, we assume that the valuations of the buyer are random, denoted by the random vector $X = (X_1, \dots, X_m)$ over which we have a joint probability distribution $\mathbb{P}^m$. We assume that the X_i are *independently distributed* and indicate their marginal distribution by $\mathbb{P}_i$, i.e., $\mathbb{P}^m$ is the product distribution $\mathbb{P}_1 \times \mathbb{P}_2 \times \dots \times \mathbb{P}_m$. The (expected) revenue of a mechanism D used for selling m independent goods $(X_1, \dots, X_m) \sim \mathbb{P}^m$ is given by

$$\text{REV}(D, \mathbb{P}^m) = \mathbb{E}_{\mathbb{P}^m}[\pi(X_1, \dots, X_m)]. \quad (2.1)$$

For the special case of the deterministic bundling mechanism that sells all goods if the sum of their values exceeds p_m , the revenue is denoted by

$$\text{BUND}(p_m, \mathbb{P}^m) = p_m \cdot \mathbb{P}^m \left(\sum_{i=1}^m X_i \geq p_m \right). \quad (2.2)$$

In our robust framework, while the joint distribution $\mathbb{P}^m$ is unknown, it is assumed to be the product of marginal distributions $\mathbb{P}_1, \dots, \mathbb{P}_m$, each contained in a known common ambiguity set $\mathcal{P}$ of distributions. Although all marginals come from the same ambiguity set, they are not required to be the same. We focus on the mean-MAD ambiguity set (1.1), but describe the revenue maximization problems for the absolute revenue, ratio and regret objective in terms of a general ambiguity set.

The goal is to solve the following maximin and minimax problems. In the maximin problem, the seller first chooses a deterministic mechanism $D \in \mathcal{D}_m$, after which an adversary (nature) chooses a distribution $\mathbb{P}^m \in \mathcal{P}^m$ with the goal of minimizing an objective involving the revenue $\text{REV}(D, \mathbb{P}^m)$. We consider various objectives, which in this work are sometimes considered for the set of all bundling mechanisms instead of general deterministic mechanisms (but for which the upcoming definitions are the same). In the minimax version of the problem, the setup is the same but this time first nature chooses a distribution, after which the seller chooses a mechanism. We will next give a precise definition of these problems for the objectives of interest.

In the *robust (absolute) revenue maximization* problem, we want to solve the *maximin* problem

$$\sup_{D \in \mathcal{D}_m} \inf_{\mathbb{P}^m \in \mathcal{P}^m} \frac{\text{REV}(D, \mathbb{P}^m)}{m}. \quad (2.3)$$

Intuitively, the goal will be to maximize the average contribution of every good to the revenue. We use this scaling as we will be interested in what happens when m grows large. The *minimax* problem is given by

$$\inf_{\mathbb{P}^m \in \mathcal{P}^m} \sup_{D \in \mathcal{D}_m} \frac{\text{REV}(D, \mathbb{P}^m)}{m}. \quad (2.4)$$

To define the ratio and absolute regret objective, we need the definition of an optimal deterministic mechanism (i.e., the mechanism that knows the distribution),

$$\text{OPT}(m, \mathbb{P}^m) = \sup_{D' \in \mathcal{D}_m} \text{REV}(D', \mathbb{P}^m). \quad (2.5)$$

This optimal mechanism for selling m independent goods with joint distribution $\mathbb{P}^m$ is generally not known. Because of individual rationality, we always have $\text{OPT}(m, \mathbb{P}^m) \leq m\mu$.

In the *robust (absolute) regret minimization* problem, we want to solve

$$\inf_{D \in \mathcal{D}_m} \sup_{\mathbb{P}^m \in \mathcal{P}^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{REV}(D, \mathbb{P}^m)}{m}. \quad (2.6)$$

That is, we want to bound the absolute difference between the performance of the optimal mechanism that knows the distribution $\mathbb{P}^m$ up front, compared to our chosen robust mechanism.

In the *robust ratio maximization* problem, we want to solve

$$\sup_{D \in \mathcal{D}_m} \inf_{\mathbb{P}^m \in \mathcal{P}^m} \frac{\text{REV}(D, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)}, \quad (2.7)$$

which is the ratio between the optimal mechanism that knows the distribution $\mathbb{P}^m$ and the chosen robust mechanism. This notion is equivalent to the relative regret objective in, e.g., [18]. Note that in (2.7) we do not scale with m , because of the multiplicative nature of this objective.

Remark 2.1. The minimax version

$$\inf_{\mathbb{P}^m \in \mathcal{P}^m} \sup_{D \in \mathcal{D}_m} \frac{\text{REV}(D, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)}$$

of the robust ratio maximization problem is trivial, since for any given $\mathbb{P}^m$,

$$\sup_{D \in \mathcal{D}_m} \frac{\text{REV}(D, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} = 1.$$

by definition of $\text{OPT}(m, \mathbb{P}^m)$ in (2.5). The same argument applies for the regret objective. $\triangle$

In this work, we will often consider a joint distribution of multiple random variables, where each random variable is independently distributed according to a two-point distribution with a given mean μ and mean absolute deviation (MAD) d . The corresponding set of two-point distributions $\mathcal{P}_2(\mu, d) \subset \mathcal{P}(\mu, d)$ can be parameterized by the probability mass α on the left support point. The distribution of such a random variable $X(\alpha) \sim \mathbb{P}_\alpha$ is given by

$$X(\alpha) = \begin{cases} x(\alpha) = \mu - \frac{d}{2\alpha} & \text{w.p. } \alpha \\ y(\alpha) = \mu + \frac{d}{2(1-\alpha)} & \text{w.p. } 1 - \alpha \end{cases} \quad (2.8)$$

for $\alpha \in [d/2\mu, 1)$. When $X_1, \dots, X_m$ are i.i.d. according to $\mathbb{P}_\alpha$, then $Y = \sum_{i=1}^m X_i \sim \mathbb{P}_\alpha^m$ with

$$\mathbb{P}_\alpha^m(Y = (m-k) \cdot x(\alpha) + k \cdot y(\alpha)) = \alpha^{m-k} (1-\alpha)^k \binom{m}{k} \quad \text{for } k = 0, \dots, m. \quad (2.9)$$

3 Absolute revenue maximization

In this section we present the proofs of Theorem 1.1 and 1.3. One of the cornerstones to achieve our main results is Proposition 3.1 which might be of independent interest. Roughly speaking, it states that the sum of m independent random variables with mean μ and mean absolute deviation (MAD) d will be greater than or equal to $\approx m(\mu - d/2)$ with probability close to one as m grows large. The proof of Proposition 3.1 is given in Appendix A.

Proposition 3.1 (One-sided concentration bound). *Let $\mu > 0$ and $0 < d < 2\mu$ be given, and let $0 < \epsilon < 1 - \frac{d}{2\mu}$. Then, for every $m \in \mathbb{N}$, it holds that*

$$\inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \mathbb{P}^m \left(\sum_{i=1}^m X_i \geq (1 - \epsilon)^2 m \left(\mu - \frac{d}{2(1 - \epsilon)} \right) \right) \geq 1 - \frac{f(\mu, d, \epsilon)}{m}, \quad (3.1)$$

where $f(\mu, d, \epsilon)$ is a function independent of m .

3.1 Maximin analysis

In Lemma 3.2, we will show that the bundling mechanism yields a robust absolute revenue of $\mu - d/2$ for the maximin problem.

Lemma 3.2. *For $\mu > 0$ and $0 < d < 2\mu$, it holds that*

$$\lim_{m \rightarrow \infty} \sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{m} = \mu - \frac{d}{2}, \quad (3.2)$$

where the supremum is, roughly speaking, asymptotically attained by the bundling price

$$p_m^*(\epsilon) := (1 - \epsilon)^2 m \left(\mu - \frac{d}{2(1 - \epsilon)} \right), \quad (3.3)$$

for $\epsilon > 0$ sufficiently small.

Proof. Recall from (2.2) that $\text{BUND}(p_m, \mathbb{P}^m) = p_m \mathbb{P}^m(\sum_{i=1}^m X_i \geq p_m)$. Using Proposition 3.1, it follows that for every $0 < \epsilon < 1 - d/2\mu$, we have

$$\begin{aligned} \sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{p_m \cdot \mathbb{P}^m(\sum_{i=1}^m X_i \geq p_m)}{m} &\geq \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{p_m^*(\epsilon) \cdot \mathbb{P}^m(\sum_{i=1}^m X_i \geq p_m^*(\epsilon))}{m} \\ &\geq \frac{p_m^*(\epsilon)}{m} \left(1 - \frac{f(\mu, d, \epsilon)}{m} \right). \end{aligned} \quad (3.4)$$

The expression on the right approaches $p_m^*(\epsilon)/m$ as $m \rightarrow \infty$ for every fixed $0 < \epsilon < 1 - d/2\mu$.

On the other hand, if $p_m \geq m(\mu - d/2)$, then nature can asymptotically put all probability mass of $\sum_{i=1}^m X_i$ below p_m by choosing the two-point distribution $\mathbb{P}_\alpha$ with $\alpha \rightarrow 1$ in (2.8) for all $i \in \{1, \dots, m\}$. That is, the leftmost point in the support of $\sum_{i=1}^m X_i$, $m \cdot x(\alpha)$ approaches $m(\mu - d/2)$ from the left, and the mass on this point approaches 1, as follows from (2.9). This implies that $\inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} p_m \cdot \mathbb{P}^m(\sum_{i=1}^m X_i \geq p_m)/m = 0$, when $p_m \geq m(\mu - d/2)$. Therefore

$$\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{p_m \cdot \mathbb{P}^m(\sum_{i=1}^m X_i \geq p_m)}{m} = \sup_{p_m < m(\mu - d/2)} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{p_m \cdot \mathbb{P}^m(\sum_{i=1}^m X_i \geq p_m)}{m} \leq \mu - \frac{d}{2} \quad (3.5)$$

using that $\mathbb{P}^m(\sum_{i=1}^m X_i \geq p_m) \leq 1$. Combining the lower and upper bounds in (3.4) and (3.5), respectively, it follows that for every fixed $\epsilon \in (0, 1)$, we have

$$(1 - \epsilon)^2 \left(\mu - \frac{d}{2(1 - \epsilon)} \right) \leq \lim_{m \rightarrow \infty} \sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{m} \leq \mu - \frac{d}{2}.$$

By letting $\epsilon \rightarrow 0$, we obtain the result in (3.2). $\square$

3.2 Minimax analysis

In this section we show that for the minimax problem, and therefore also for the maximin problem, there does not exist a deterministic truthful mechanism that can do better than $\max\{\mu - d/2, d/2\}$.

Theorem 3.3. *Let $\mu > 0$ and $0 < d < 2\mu$. There does not exist a deterministic truthful mechanism that can asymptotically achieve a (normalized with m) revenue greater than $\max\{\mu - d/2, d/2\}$ for the minimax problem. That is,*

$$\lim_{m \rightarrow \infty} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \sup_{D \in \mathcal{D}_m} \frac{\text{REV}(D, \mathbb{P}^m)}{m} \leq \max \left\{ \mu - \frac{d}{2}, \frac{d}{2} \right\}. \quad (3.6)$$

Proof. We always have

$$\inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \sup_{D \in \mathcal{D}_m} \text{REV}(D, \mathbb{P}^m) \leq \sup_{D \in \mathcal{D}_m} \text{REV}(D, \mathbb{P}_\alpha^m),$$

where we choose $\mathbb{P}_\alpha^m = \Pi_{i=1}^m \mathbb{P}_\alpha$ with $\mathbb{P}_\alpha$ the two-point distribution (2.8) with $\{x(\alpha(m)), y(\alpha(m))\}$ support and (parameterizing) probability $\alpha(m) = 1 - m^{-(m+1)} \cdot e^{-m}$. For a two-point distribution, every realized valuation for the goods is either x or y . We will first argue that, based on the choice of $\alpha(m)$ and the individual rationality property, the only valuation vectors that matter for the revenue analysis, are those with at most one (realized) valuation being y . For a vector $v \in \{x, y\}^m$ we define $C_y(v) = |\{v_i : v_i = y \text{ for } i = 1, \dots, m\}|$ to be the number of y 's in v . For any deterministic truthful mechanism D , we can then write the revenue as

$$\begin{aligned} \frac{1}{m} \text{REV}(D, \mathbb{P}_\alpha^m) &= \frac{1}{m} \sum_{v: C_y(v) \leq 1} \pi(v) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) \\ &\quad + \frac{1}{m} \sum_{k=2}^m \sum_{v: C_y(v)=k} \pi(v) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) \\ &\leq \frac{1}{m} \sum_{v: C_y(v) \leq 1} \pi(v) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) \\ &\quad + \frac{1}{m} \sum_{k=2}^m \sum_{v: C_y(v)=k} \left(\sum_{i=1}^m v_i \right) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)). \end{aligned} \quad (3.7)$$

In the final inequality of (3.7) we use that, because of individual rationality, we always have $\pi(v) \leq \langle z(v), v \rangle \leq \sum_{i=1}^m v_i$. We first bound the second summation in (3.7). If $C_y(v) = k \geq 2$, we have, for m large enough, that

$$\begin{aligned} \frac{1}{m} \left(\sum_{i=1}^m v_i \right) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) &= \frac{[(m - k) \cdot x(\alpha) + k \cdot y(\alpha)](\alpha)^{m-k}(1 - \alpha)^k}{m} = \\ &\leq C(\mu, d) \cdot m^{-(m+1)} \cdot e^{-m}, \end{aligned} \quad (3.8)$$

where $C(\mu, d)$ is a constant only depending on μ and d (independent of k and m). The last inequality follows from the fact that the dominant term in the expression above is $(1 - \alpha)^{k-1}$, using the definition of $x(\alpha)$ and $y(\alpha)$ in (2.8). Since $k \geq 2$, we then get the desired result as $(1 - \alpha)^{k-1} \leq (1 - \alpha) = m^{-(m+1)} \cdot e^{-m}$. This implies that, still for $k \geq 2$,

$$\frac{1}{m} \sum_{v: C_y(v)=k} \left(\sum_{i=1}^m v_i \right) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) \leq \binom{m}{k} C(\mu, d) \cdot m^{-(m+1)} \cdot e^{-m} \leq C(\mu, d) \cdot \frac{e^{-m}}{m}$$

using that $|v : C_y(v) = k| = \binom{m}{k} \leq m^k$. Finally, because $k \in \{2, \dots, m\}$ in the outer summation, i.e., there are $m - 1$ terms, we obtain

$$\frac{1}{m} \sum_{k=2}^m \sum_{v: C_y(v)=k} \left(\sum_{i=1}^m v_i \right) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) \leq (m - 1) C(\mu, d) \cdot \frac{e^{-m}}{m} \leq C(\mu, d) \cdot e^{-m}. \quad (3.9)$$

Taking $m \rightarrow \infty$ in the right hand side of (3.9) gives an upper bound of 0 on the contribution of all these valuation vectors v to the revenue in (3.7).

We next continue with bounding the summation

$$\frac{1}{m} \sum_{v: C_y(v) \leq 1} \pi(v) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) = \frac{1}{m} \pi(v^{(0)}) \alpha^m + \frac{1}{m} \sum_{j=1}^m \pi(v^{(j)}) \alpha^{m-1} (1 - \alpha),$$

where $v^{(j)}$ is such that $v_i^{(j)} = \begin{cases} y & \text{for } i = j \\ x & \text{for } i \neq j \end{cases}$, and $v^{(0)} = (x, x, \dots, x)$ the vector only containing x 's. We write $\pi_0 = \pi(v^{(0)})$. We next argue that, without loss of generality, we may assume that $\pi(v^{(j)})$ is the same for every $j = 1, \dots, m$; we will denote their common value by π_1 .⁴

Claim 3.4. *Without loss of generality, we may assume that*

- $\pi(v^{(k)}) = \pi(v^{(\ell)}) =: \pi_1$ for all $k, \ell \in \{1, \dots, m\}$.
- z allocates the item with value y to either all $v^{(j)}$ for $j = 1, \dots, m$, or to none of them.

Proof. Let $\pi_{\max} = \max_{j=1, \dots, m} \pi(v^{(j)})$ and let j^* be the index attaining the maximum. We construct a new truthful mechanism that has expected revenue not smaller than that of D , by adjusting the allocation and payment rules for the vectors $v^{(j)}$ for $j = 1, \dots, m$.

One can adjust the allocation rule $z(v^{(j)})$ to $z'(v^{(j)})$ to mimic $z(v^{(j^*)})$. First, we set

$$z'_j(v^{(j)}) = \begin{cases} 1 & \text{if } z_{j^*}(v^{(j^*)}) = 1 \\ 0 & \text{if } z_{j^*}(v^{(j^*)}) = 0 \end{cases}.$$

Next, let t , with $0 \leq t \leq m - 1$, be the number of goods with value x that $z(v^{(j^*)})$ allocates. For $z(v^{(j)})$ we choose the first t goods to be allocated as well; if one of these is the good with value y , we choose the $t + 1$ -th good as well. It then follows that

$$\langle z(v^{(j^*)}), v^{(j^*)} \rangle = \langle z'(v^{(j)}), v^{(j)} \rangle. \quad (3.10)$$

⁴In a nutshell, this follows from the fact that good identities are interchangeable in the definition of dominant strategy and individual rationality.

We'll give a small example of this transformation. Suppose that $j^* = 4$, $j = 2$, $m = 8$ and that $z(v^{(4)}) = (0, 1, 1, 1, 0, 0, 1, 1)$, so that $t = 4$ (the third 1 corresponds to the good with value y being allocated). Then we obtain $z'(v^{(2)}) = (1, 1, 1, 1, 1, 0, 0, 0)$, that is in z' the first, third, fourth and fifth good, having value x , get allocated, as well as the second good with value y .

We set π_{max} as the charged price for every vector $v^{(j)}$ for $j = 1, \dots, m$. Then the resulting mechanism has revenue not smaller than that of D and it is still truthful because of (3.10). $\square$

Using Claim 3.4, we can write

$$\frac{1}{m} \sum_{v: C_y(v) \leq 1} \pi(v) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) = \frac{\pi_0}{m} \cdot \alpha^m + \pi_1 \cdot \alpha^{m-1} (1 - \alpha). \quad (3.11)$$

By individual rationality, we have that (using (2.8) in the final inequality)

$$\pi_0 \leq \langle z(v^{(0)}), v \rangle \leq m \cdot x(\alpha) \leq m(\mu - d/2). \quad (3.12)$$

The transformation in the proof of Claim 3.4 guarantees that the good with value y either gets allocated for all vectors $v^{(j)}$, for $j = 1, \dots, m$, or for none of them.

Case 1: $z_j(v^{(j)}) = 0$ for all $j = 1, \dots, m$. In this case the good with value y is not allocated for any of the valuation vectors $v^{(j)}$ for $j \geq 1$. By individual rationality this mean that $\pi_1 \leq \langle z(v^{(j)}), v \rangle \leq (m-1) \cdot x(\alpha) \leq (m-1)(\mu - d/2)$. In combination with (3.12) and using $\alpha(m) \leq 1$ for all m , we then have that $\frac{\pi_0}{m} \alpha^m + \pi_1 \alpha^{m-1} (1 - \alpha) \leq (\mu - d/2) + (m-1)(\mu - d/2) m^{-(m+1)} e^{-m} \leq (\mu - d/2)(1 + m^{-m} e^{-m})$. Note that taking $m \rightarrow \infty$ then gives an upper bound of $\mu - d/2$ on (3.11).

Case 2: $z_j(v^{(j)}) = 1$ for all $j = 1, \dots, m$. In this case the good with value y is allocated for every of the valuation vectors $v^{(j)}$ for $j \geq 1$. We next make another case distinction, based on whether or not z allocates no, or at least one, good in $v^{(0)}$.

Subcase 2.A: z allocates no good in $v^{(0)}$. This means that $z_i(v^{(0)}) = 0$ for $i = 1, \dots, m$. Because of individual rationality, we then must have that $\pi_0 = 0$. Also because of individual rationality, we know that $\pi_1 \leq (m-1)x(\alpha(m)) + y(\alpha(m)) = (m-1)(\mu - d/2) + \mu + d/(2(1 - \alpha(m)))$. This implies, again using $\alpha(m) \leq 1$, that $\frac{\pi_0}{m} \alpha^m + \pi_1 \alpha^{m-1} (1 - \alpha) \leq (\mu - d/2) \cdot m^{-m} e^{-m} + \mu \cdot m^{-(m+1)} e^{-m} + d/2$. Taking $m \rightarrow \infty$ in the right-hand side gives an upper bound of $d/2$ on (3.11).

Subcase 2.B: z allocates at least one good in $v^{(0)}$. This means that $z_k(v^{(0)}) = 1$ for some $k \in \{1, \dots, m\}$. Without loss of generality, by renumbering the goods if needed, we may assume that $k = 1$. We also know, by assumption of Case 2, that z allocates good 1, with value y , in the valuation vector $v^{(1)}$, i.e., $z_1(v^{(1)}) = 1$. The dominant strategy condition defining a truthful mechanism, applied with $v = v^{(1)}$ and $w = v^{(0)}$ tells us that $\langle z(v^{(1)}), v^{(1)} \rangle - \pi_1 \geq \langle z(v^{(0)}), v^{(1)} \rangle - \pi_0$. Roughly speaking, the seller knows that the mechanism allocates the first good both in the bid vectors $v^{(0)}$ and $v^{(1)}$, so we have to make sure that the seller has no incentive to misreport the value it has for the first good (which is y in $v^{(1)}$). Since the mechanism allocates the first good in $v^{(0)}$, we know that $\langle z(v^{(0)}), v^{(1)} \rangle \geq y(\alpha)$. Furthermore, we also know that $\langle z(v^{(1)}), v^{(1)} \rangle \leq (m-1)x(\alpha) + y(\alpha)$ (which is the trivial upper bound on $\langle z(v^{(1)}), v^{(1)} \rangle$ in case z allocates all goods in $v^{(1)}$). Applying these estimates, we get

$$(m-1)x(\alpha) + y(\alpha) - \pi_1 \geq \langle z(v^{(1)}), v^{(1)} \rangle - \pi_1 \geq \langle z(v^{(0)}), v^{(1)} \rangle - \pi_0 \geq y(\alpha) - \pi_0,$$

from which it follows that $\pi_1 \leq \pi_0 + (m-1)x(\alpha) \leq \pi_0 + (m-1)(\mu - d/2) \leq (2m-1)(\mu - d/2)$, where the last inequality follows by recalling that $\pi_0 \leq m(\mu - d/2)$ because of individual rationality. Plugging this upper bound on π_1 into (3.11), gives us (again also using $\alpha(m) \leq 1$)

$$\frac{\pi_0}{m} \alpha^m + \pi_1 \alpha^{m-1} (1 - \alpha) \leq (\mu - d/2) + (2m-1)(\mu - d/2) m^{-(m+1)} e^{-m} \leq (\mu - d/2)(1 + (2m-1)m^{-m} e^{-m}).$$

Note that when $m \rightarrow \infty$, we obtain an upper bound of $\mu - d/2$ on the right hand side of (3.11).

To summarize, when $m \rightarrow \infty$, we obtain in all cases an upper bound of $\mu - d/2$ or $d/2$. Hence,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{v: C_y(v) \leq 1} \pi(v) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) \leq \max\{\mu - d/2, d/2\}. \quad (3.13)$$

Also recall that, following from (3.9),

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=2}^m \sum_{v: C_y(v)=k} \pi(v) \mathbb{P}_\alpha^m((X_1, \dots, X_m) = (v_1, \dots, v_m)) = 0. \quad (3.14)$$

We then may conclude that $\lim_{m \rightarrow \infty} \frac{\text{REV}(D, \mathbb{P}_\alpha^m)}{m} \leq \max\{\mu - d/2, d/2\}$, as the revenue is the sum of the expressions in (3.13) and (3.14); see (3.7). $\square$

Theorem 1.1 and 1.3 based on the results in Section 3.1 and 3.2, are proven in Appendix B.

4 Ratio revenue maximization

Inspired by the optimality of the bundling mechanism in the absolute revenue maximization problem, we investigate the asymptotic performance of the optimal deterministic robust bundling mechanism for the ratio and absolute regret objective as well. That is, we determine the optimal bundling mechanism among the set of all bundling mechanisms (instead of all deterministic mechanisms). In order to do this, we use the solution of the absolute revenue maximization problem, combined with some additional arguments. An intuitive way of looking at the result in Theorem 4.1 is that the seller can guarantee a revenue close to $m(\mu - d/2)$, whereas the optimal mechanism can achieve a revenue close to $m\mu$, so that the ratio results in $1 - d/(2\mu)$ and the absolute regret in $d/2$. This intuition fails for finite values of m , but becomes valid as m grows large. Formalizing this, though, requires some care. The two proofs differ in the analysis of the term $\text{OPT}(m, \mathbb{P}) = \sup_{D' \in \mathcal{D}_m} \text{REV}(D', \mathbb{P})$ in the objectives.⁵

Theorem 4.1. *Let $\mu > 0$ and $0 < d < 2\mu$. Then*

$$\lim_{m \rightarrow \infty} \sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} = \lim_{m \rightarrow \infty} \sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} = 1 - \frac{d}{2\mu}, \quad (4.1)$$

where the supremum is, roughly speaking, asymptotically attained by the bundling price as in (3.3).

Theorem 4.2. *Let $\mu > 0$ and $0 < d < 2\mu$. Then*

$$\lim_{m \rightarrow \infty} \left[\inf_{p_m} \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m} \right] = \frac{d}{2}, \quad (4.2)$$

where the infimum is, roughly speaking, asymptotically attained by the bundling price as in (3.3).

⁵In fact, the proofs also work if we allow randomized mechanisms in the OPT term.

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A One-sided concentration under given mean and MAD

In this section, we will give the proof of Proposition 3.1. We will rely on the following version of Chebyshev's inequality [17] in the proof.

Lemma A.1 (following from Chebyshev's inequality [17]). *If $Y = \sum_{i=1}^m X_i$ is the sum of i.i.d. random variables $X_1, \dots, X_m$ that all have mean μ and variance at most σ^2 , then for any $0 < \gamma < 1$ it holds that*

$$\mathbb{P}(Y \leq (1 - \gamma)m\mu) \leq \frac{m\sigma^2}{\gamma^2 m^2 \mu^2} = \frac{\sigma^2}{(\gamma\mu)^2 m}.$$

Proof. The rough outline of the proof is as follows. We show that, if one chooses a fixed $t = t(\mu, d, \epsilon)$ large enough, then the contribution of the values greater than or equal to t to the mean μ of any $\mathbb{P}_i \in \mathcal{P}(\mu, d)$ is roughly at most $g(t) \cdot \mu + d/2$ with $g(t)$ small if t is large enough. If we condition the distribution $\mathbb{P}_i$ on values smaller than t , then the resulting distribution has mean roughly $(1 - g(t))\mu - d/2$ and finite variance $\sigma^2(t)$. We can then use Chebyshev's bound to argue that we have concentration around $m((1 - g(t))\mu - d/2)$. This will give the desired result.

We continue with the formal proof, for which we will use the following proposition.

Lemma A.2. *Let $t \geq \mu + d/2$ be fixed. Then*

$$\sup_{\mathbb{P} \in \mathcal{P}(\mu, d)} \mathbb{E}_{\mathbb{P}}[X \cdot \mathbf{1}_{\geq t}(X)] = \frac{d}{2(t - \mu)} \cdot \mu + \frac{d}{2}, \quad (\text{A.1})$$

where $\mathbf{1}_{\geq t}(x) = 1$ if $x \geq t$ and $\mathbf{1}_{\geq t}(x) = 0$ if $x < t$.

Proof. Let $\mathbb{P} \in \mathcal{P}(\mu, d)$ be a fixed probability distribution. We can merge all the probability mass under t in one point, as well as all the probability mass above t . Formally speaking, we look at the two-point distribution $\mathbb{P}'$ supported on $\{x', y'\}$ with $x' = \mathbb{E}[X|X < t]$, $y' = \mathbb{E}[X|X \geq t]$ and $\alpha' = \mathbb{P}(X < t)$ the probability mass on point x' . It is not hard to see that distribution $\mathbb{P}'$ has mean μ and a MAD $d' \leq d$, because of the convex nature of the function $\phi(x) = |x - \mu|$. Furthermore, it holds that $\mathbb{E}_{\mathbb{P}}[X \cdot \mathbf{1}_{\geq t}(X)] = \mathbb{E}_{\mathbb{P}'}[X \cdot \mathbf{1}_{\geq t}(X)]$ since the function $x \cdot \mathbf{1}_{\geq t}(x)$ is piece-wise linear (and the fact that we reduce to a two-point distribution using the point t).

Now, we have $\mathbb{E}_{\mathbb{P}'}[X \cdot \mathbf{1}_{\geq t}(X)] = y'(\alpha')(1 - \alpha') = (1 - \alpha')\mu + d'/2$ if $y'(\alpha') \geq t$ (note that also $t \geq \mu + d'/2$), and zero otherwise. In order to maximize this quantity, we should choose $1 - \alpha'$ as large as possible, but still small enough so that $y(\alpha) \geq t$ (otherwise the objective equals zero). The latter inequality is equivalent to $1 - \alpha \leq d'/(2(t - \mu))$, so we choose $1 - \alpha' = d'/(2(t - \mu))$. It then follows that

$$\sup_{\mathbb{P} \in \mathcal{P}(\mu, d)} \mathbb{E}_{\mathbb{P}}[X \cdot \mathbf{1}_{\geq t}(X)] \leq \frac{d'}{2(t - \mu)} \cdot \mu + \frac{d'}{2} \leq \frac{d}{2(t - \mu)} \cdot \mu + \frac{d}{2},$$

where the second inequality holds because $d' \leq d$. The bound can be attained by choosing the two-point distribution supported on $\{x(\alpha), y(\alpha)\}$ for which $1 - \alpha = d/(2(t - \mu))$. The proof of this claim is the same as the reasoning we gave above for the distribution supported on $\{x', y'\}$. $\square$

Now fix an arbitrary $\epsilon \in (0, 1 - d/2\mu)$ and choose $t = t(\mu, d, \epsilon)$ such that $d/(2(t - \mu)) \leq \epsilon$. Because $0 < \epsilon < 1 - d/2\mu$, this will always result in $t \geq \mu + d/2$. For any given $\mathbb{P}^m \in \mathcal{P}(\mu, d)^m$, let $\mathbb{P}_t^m(h(X_1, \dots, X_m) \in E) := \mathbb{P}^m(h(X_1, \dots, X_m) \in E | X_1 < t, \dots, X_m < t)$ for any function h and event E . Note that by definition of $\mathbb{P}_t^m$, for a given $c_m \geq 0$ it holds that

$$\mathbb{P}^m \left(\sum_{i=1}^m X_i \geq c_m \right) \geq \mathbb{P}^m \left(\sum_{i=1}^m X_i \geq c_m \mid X_i < t \ \forall i \right) = \mathbb{P}_t^m \left(\sum_{i=1}^m X_i^t \geq c_m \right) \quad (\text{A.2})$$

with $(X_1, \dots, X_m) \sim \mathbb{P}^m$ and $(X_1^t, \dots, X_m^t) \sim \mathbb{P}_t^m$. Because of Lemma A.2, and using $\mathbb{P}_i(X_i < t) \leq 1$, we have

$$\mu_i^t := \mathbb{E}_{\mathbb{P}_i}[X_i | X_i < t] = \frac{\mu - \mathbb{E}_{\mathbb{P}_i}[X_i \cdot \mathbb{1}_{\geq t}(X_i)]}{\mathbb{P}_i(X_i < t)} \geq (1 - \epsilon)\mu - \frac{d}{2}, \quad (\text{A.3})$$

and define $\mu_{\min}^t := \min\{\mu_1^t, \dots, \mu_m^t\}$. The conditional distribution $\mathbb{P}_i(X_i | X_i < t)$ has finite support on $[0, t)$ and therefore has

$$(\sigma_i^t)^2 := \mathbb{E}_{\mathbb{P}_i}((X_i^t - \mu_i^t)^2 | X_i < t) \leq \mu_i^t(t - \mu_i^t), \quad (\text{A.4})$$

which is maximal when $\mu_i^t = \frac{1}{2}t$, as $\mu_i^t \in [0, t)$. Hence, we obtain the upper bound

$$(\sigma_i^t)^2 \leq \frac{1}{2}t(t - \frac{1}{2}t) = \frac{1}{4}t^2. \quad (\text{A.5})$$

Then Chebyshev's bound in Lemma A.1 with (A.5) tells us that

$$\mathbb{P}_t^m \left(\sum_{i=1}^m X_i^t > (1 - \epsilon)m\mu_{\min}^t \right) \geq 1 - \frac{t^2}{4(\epsilon\mu_{\min}^t)^2 m}, \quad (\text{A.6})$$

Combined with (A.2) and (A.3), this then tells us that

$$\mathbb{P}^m \left(\sum_{i=1}^m X_i > (1 - \epsilon)m \left((1 - \epsilon)\mu - \frac{d}{2} \right) \right) \geq 1 - \frac{t^2}{4(\epsilon((1 - \epsilon)\mu - \frac{d}{2}))^2 m}. \quad (\text{A.7})$$

As $\mathbb{P}^m$ was chosen arbitrarily, this inequality also holds for the infimum over $\mathbb{P}^m$. The existence of $f(\mu, d, \epsilon)$ then follows. $\square$

B Proof of Theorems 1.1 and 1.3

The proof of Theorem 1.1 follows by observing that

$$\begin{aligned} \mu - \frac{d}{2} &= \lim_{m \rightarrow \infty} \sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{m} \\ &\leq \lim_{m \rightarrow \infty} \sup_{D \in \mathcal{D}_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{REV}(D, \mathbb{P}^m)}{m} \\ &\leq \lim_{m \rightarrow \infty} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \sup_{D \in \mathcal{D}_m} \frac{\text{REV}(D, \mathbb{P}^m)}{m} \\ &\leq \mu - \frac{d}{2}. \end{aligned}$$

Here the first equality follows from Lemma 3.2. The first and second inequality are standard. The last inequality follows from the fact that $\max\{\mu - d/2, d/2\} = \mu - d/2$ when $0 < d \leq \mu$. The observation in Corollary 1.2 follows from the fact that Lemma 3.2 is also true when $\mathcal{P}(\mu, d)$ is replaced by $\mathcal{P}_2(\mu, d)$, in combination with the fact that the proof of Theorem 3.3 works with two-point distributions. The proof of Theorem 1.3 follows in a similar fashion, where the last inequality in the statement of Theorem 1.3 is obtained by observing that $\max\{\mu - d/2, d/2\} = d/2$ when $\mu \leq d < 2\mu$.

C Proof of Theorem 4.1

We will prove Theorem 4.1 by proving the following inequalities:

$$\begin{aligned} 1 - \frac{d}{2\mu} &\leq \lim_{m \rightarrow \infty} \left[\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \right] \\ &\leq \lim_{m \rightarrow \infty} \left[\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \right] \leq 1 - \frac{d}{2\mu}. \end{aligned} \quad (\text{C.1})$$

The second inequality trivially holds, as we take the infimum over a larger set on the left-hand side compared to the right-hand side. We start by proving the last inequality in (C.1) in Lemma C.1.

Lemma C.1. *It holds that*

$$\lim_{m \rightarrow \infty} \left[\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \right] \leq 1 - \frac{d}{2\mu}$$

for any $\mu > 0$ and $0 < d < 2\mu$.

Proof. For any fixed m , it is not hard to argue that for some function $f(\mu, d) > 0$ independent of m , we have $\text{OPT}(m, \mathbb{P}^m) \geq f(\mu, d) > 0$ for every $\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m$, e.g., by using the mechanism that separately sells every good using the optimal robust price in the single good setting. Consider the following case distinction.

Case 1: $p_m \geq m(\mu - d/2)$. Nature can asymptotically get all probability mass of $Y = \sum_{i=1}^m X_i$ below p_m , by choosing $\mathbb{P}_\alpha^m = \prod_{i=1}^m \mathbb{P}_\alpha$ with $\mathbb{P}_\alpha$ the two-point distribution with $\alpha \rightarrow 1$ in (2.8), for all $i \in \{1, \dots, m\}$. That is, the leftmost point in the support of Y , $m \cdot x(\alpha)$ approaches $m(\mu - d/2)$ from the left and the mass on this point approaches 1, as follows from (2.9). This implies that the revenue of bundling approaches zero as $\alpha \rightarrow 1$ (recall that m is fixed at this point), and so

$$\inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \leq \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{f(\mu, d)} = \frac{\inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \text{BUND}(p_m, \mathbb{P}^m)}{f(\mu, d)} = 0. \quad (\text{C.2})$$

Case 2: $p_m < m(\mu - d/2)$. Note that always

$$\text{BUND}(p_m, \mathbb{P}^m) = p_m \mathbb{P}^m \left(\sum_{i=1}^m X_i \geq p_m \right) \leq p_m.$$

This implies that

$$\inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \leq \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{p_m}{\text{OPT}(m, \mathbb{P}^m)} = \frac{p_m}{\sup_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \text{OPT}(m, \mathbb{P}^m)}, \quad (\text{C.3})$$

where the last equality holds because p_m does not depend on $\mathbb{P}^m$.

Fix any $\gamma > 0$. Using the joint distribution $\mathbb{P}_\alpha^m = \prod_{i=1}^m \mathbb{P}_\alpha$ with $\mathbb{P}_\alpha$ the fixed two-point distribution⁶ with $\alpha = d/(2\mu)$, and using the bundling mechanism that sells at the bundle price $(1 - \gamma)m\mu$, gives

$$\sup_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \text{OPT}(m, \mathbb{P}^m) \geq \text{BUND}((1 - \gamma)m\mu, \mathbb{P}_\alpha^m) \geq (1 - \gamma)m\mu \cdot \left[1 - \frac{\text{Var}(X)}{(\gamma\mu)^2 m} \right] \quad (\text{C.4})$$

⁶Any distribution in $\mathcal{P}(\mu, d)$ with finite variance can be used for this argument.

by using Lemma A.1. Note that, when $X \sim \mathbb{P}_\alpha$,

$$\text{Var}(X) = (d/2\mu) \cdot (0 - \mu)^2 + (1 - d/(2\mu)) \cdot (\mu - \mu - d\mu/(2\mu - d))^2 =: g(\mu, d).$$

Plugging (C.4) into (C.3) gives that for $p_m < m(\mu - d/2)$, we have

$$\begin{aligned} \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} &\leq \frac{p_m}{(1 - \gamma)m\mu \cdot \left[1 - \frac{g(\mu, d)}{(\epsilon\mu)^2 m}\right]} \leq \frac{m(\mu - d/2)}{(1 - \gamma)m\mu \cdot \left[1 - \frac{g(\mu, d)}{(\gamma\mu)^2 m}\right]} \\ &= \frac{1}{1 - \gamma} \cdot \frac{2\mu - d}{2\mu} \cdot \left[1 - \frac{g(\mu, d)}{(\gamma\mu)^2 m}\right]^{-1}. \end{aligned} \quad (\text{C.5})$$

Combining Cases 1 and 2 implies that

$$\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \leq \frac{1}{1 - \gamma} \cdot \frac{2\mu - d}{2\mu} \cdot \left[1 - \frac{g(\mu, d)}{(\gamma\mu)^2 m}\right]^{-1},$$

and so

$$\lim_{m \rightarrow \infty} \left[\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \right] \leq \frac{1}{1 - \gamma} \cdot \frac{2\mu - d}{2\mu}.$$

Since $\gamma > 0$ was chosen arbitrarily, we obtain

$$\lim_{m \rightarrow \infty} \left[\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \right] \leq \frac{2\mu - d}{2\mu}. \quad (\text{C.6})$$

This finishes the proof. $\square$

We continue with the first inequality in (C.1) in Lemma C.2.

Lemma C.2. *It holds that*

$$\lim_{m \rightarrow \infty} \left[\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \right] \geq \frac{2\mu - d}{2\mu}$$

for any $\mu > 0$ and $0 < d < 2\mu$.

Proof. For a fixed $0 < \epsilon < 1 - d/(2\mu)$, set

$$p_m^*(\epsilon) = (1 - \epsilon)^2 m \left(\mu - \frac{d}{2(1 - \epsilon)} \right)$$

for $m \in \mathbb{N}$. We have

$$\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \geq \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m^*(\epsilon), \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \geq \frac{\inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \text{BUND}(p_m^*(\epsilon), \mathbb{P}^m)}{m\mu} \quad (\text{C.7})$$

using that $\text{OPT}(m, \mathbb{P}^m) \leq m\mu$. This inequality holds because any truthful mechanism is individually rational, which implies that $\pi(v) \leq \sum_i v_i$. In expectation, this yields a bound of $m\mu$. For any fixed $\mathbb{P}^m \in \mathcal{P}(\mu, d)^m$, recall that

$$\text{BUND}(p_m^*(\epsilon), \mathbb{P}^m) = p_m^*(\epsilon) \cdot \mathbb{P}^m \left(\sum_{i=1}^m X_i \geq p_m^*(\epsilon) \right) \quad (\text{C.8})$$

with $p_m^*(\epsilon) = (1 - \epsilon)^2 m \left(\mu - \frac{d}{2(1-\epsilon)} \right)$.

We next argue that Lemma C.2 follows from Proposition 3.1 by observing that plugging in the concentration bound (3.1) in (C.7) gives

$$\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \geq \frac{(1 - \epsilon)^2 m \left(\mu - \frac{d}{2(1-\epsilon)} \right) (1 - f(\mu, d, \epsilon)/m)}{m\mu},$$

and then

$$\lim_{m \rightarrow \infty} \left[\sup_{p_m} \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m, \mathbb{P}^m)}{\text{OPT}(m, \mathbb{P}^m)} \right] \geq \frac{(1 - \epsilon)^2 \left(2\mu - \frac{d}{(1-\epsilon)} \right)}{2\mu}.$$

Since ϵ was chosen arbitrarily, the statement of the lemma follows. $\square$

This finishes the proof of Theorem 4.1.

D Proof of Theorem 4.2

First, suppose that the seller chooses $p_m^*(\epsilon)$ as the selling price. Then

$$\begin{aligned} \inf_{p_m} \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m} &\leq \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m^*(\epsilon), \mathbb{P}^m)}{m} \\ &\leq \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m)}{m} \\ &\quad - \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{BUND}(p_m^*(\epsilon), \mathbb{P}^m)}{m} \\ &\leq \mu - \frac{p_m^*(\epsilon)}{m} (1 - f(\mu, d, \epsilon)/m) \\ &= \mu - (1 - \epsilon)^2 \left(\mu - \frac{d}{2(1-\epsilon)} \right) (1 - f(\mu, d, \epsilon)/m). \end{aligned}$$

Here we use the upper bound $\text{OPT}(m, \mathbb{P}^m) \leq m\mu$ and $f(\mu, d, \epsilon)$ is as in Proposition 3.1. Taking the limit of $m \rightarrow \infty$, we then find that

$$\lim_{m \rightarrow \infty} \left[\inf_{p_m} \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m} \right] \leq \mu - (1 - \epsilon)^2 \left(\mu - \frac{d}{2(1-\epsilon)} \right).$$

Because this bound holds for all $0 < \epsilon < 1$, by letting $\epsilon \rightarrow 0$ we find that

$$\lim_{m \rightarrow \infty} \left[\inf_{p_m} \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m} \right] \leq d/2. \quad (\text{D.1})$$

We next fix m and consider a case distinction in order to prove that $d/2$ is also a lower bound.

Case 1: $p_m < m(\mu - d/2)$. Note that $\text{BUND}(p_m, \mathbb{P}^m) \leq p_m < m(\mu - d/2)$ always holds. Also, with a similar argument as in the proof of Lemma C.1, we have for any $0 < \gamma < 1$, it holds that

$$\sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m)}{m} \geq (1 - \gamma)\mu \cdot \left[1 - \frac{g(\mu, d)}{(\gamma\mu)^2 m} \right].$$

Then we have

$$\begin{aligned}
\sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m} &\geq \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - m(\mu - d/2)}{m} \\
&= \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m)}{m} - (\mu - d/2) \\
&\geq (1 - \gamma)\mu \cdot \left[1 - \frac{g(\mu, d)}{(\gamma\mu)^2 m} \right] - (\mu - d/2).
\end{aligned}$$

Case 2: $p_m \geq m(\mu - d/2)$. In this case, we consider the joint distribution $\mathbb{P}_\alpha^m = \Pi_{i=1}^m \mathbb{P}_\alpha$ with $\mathbb{P}_\alpha$ the two-point distribution as in (2.8) supported on $\{x(\alpha), y(\alpha)\}$ with $\alpha \rightarrow 1$ (this happens independent of m , which is fixed). Note that

$$\sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m} \geq \frac{\text{OPT}(m, \mathbb{P}_\alpha^m) - \text{BUND}(p_m, \mathbb{P}_\alpha^m)}{m}.$$

Because m is fixed, it follows that $\mathbb{P}_\alpha^m(\sum_{i=1}^m X_i \geq m(\mu - d/2)) \rightarrow 0$ as $\alpha \rightarrow 1$, since all probability mass cumulates on the smallest support point $mx(\alpha) < m(\mu - d/2)$. This means that $\text{BUND}(p_m, \mathbb{P}_\alpha^m) \rightarrow 0$ as $\alpha \rightarrow 1$.

Furthermore, we can lower bound $\text{OPT}(m, \mathbb{P}_\alpha^m)$ by the better option of two bundling mechanisms. Firstly, the bundling mechanism that sets the bundling price p_m just below $mx(\alpha)$ yields a revenue of $\mu - d/2$ as $\alpha \rightarrow 1$. Secondly, if we choose the bundling price p_m just below the second support point of $Y = \sum_{i=1}^m X_i$, namely $(m-1)x(\alpha) + y(\alpha)$, then the revenue would be

$$\begin{aligned}
[(m-1)x(\alpha) + y(\alpha)]\mathbb{P}_\alpha^m(Y \geq (m-1)x(\alpha) + y(\alpha)) &= [(m-1)x(\alpha) + y(\alpha)](1 - \mathbb{P}_\alpha^m(Y = mx(\alpha))) \\
&= [(m-1)x(\alpha) + y(\alpha)](1 - \alpha^m) \\
&= \mu \frac{1 - \alpha^m}{m} + \frac{d}{2} \left[\frac{1 - \alpha^m}{(1 - \alpha)m} - \frac{(1 - \alpha^m)(m-1)}{m\alpha} \right],
\end{aligned}$$

where the last equality follows by the definitions of $x(\alpha)$ and $y(\alpha)$. Since $\alpha \rightarrow 1$, it holds that

$$\lim_{\alpha \rightarrow 1} \frac{1 - \alpha^m}{m} = 0, \quad \lim_{\alpha \rightarrow 1} \frac{1 - \alpha^m}{(1 - \alpha)m} = 1, \quad \text{and} \quad \lim_{\alpha \rightarrow 1} \frac{(1 - \alpha^m)(m-1)}{m\alpha} = 0,$$

meaning that the revenue will approach $d/2$. From this it follows that $\lim_{\alpha \rightarrow 1} \text{OPT}(m, \mathbb{P}_\alpha^m) \geq \max\{\mu - d/2, d/2\}$, and, hence

$$\lim_{\alpha \rightarrow 1} \frac{\text{OPT}(m, \mathbb{P}_\alpha^m) - \text{BUND}(p_m, \mathbb{P}_\alpha^m)}{m} \geq \max\{\mu - d/2, d/2\}.$$

Combining Cases 1 and 2 yields that, for any $0 < \gamma < 1$,

$$\begin{aligned}
&\inf_{p_m} \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m} \\
&\geq \min \left\{ (1 - \gamma)\mu \cdot \left[1 - \frac{g(\mu, d)}{(\gamma\mu)^2 m} \right] - (\mu - d/2), \max\{\mu - d/2, d/2\} \right\},
\end{aligned}$$

and then

$$\begin{aligned}
&\lim_{m \rightarrow \infty} \left[\inf_{p_m} \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m} \right] \\
&\geq \min \{ (1 - \gamma)\mu - (\mu - d/2), \max\{\mu - d/2, d/2\} \}.
\end{aligned}$$

Because this holds for all $0 < \gamma < 1$, by letting $\gamma \rightarrow 0$ it follows that

$$\lim_{m \rightarrow \infty} \left[\inf_{p_m} \sup_{\mathbb{P}^m \in \mathcal{P}(\mu, d)^m} \frac{\text{OPT}(m, \mathbb{P}^m) - \text{BUND}(p_m, \mathbb{P}^m)}{m} \right] \geq \min \{d/2, \max\{\mu - d/2, d/2\}\} = d/2.$$

This completes the proof.

E Minimax analysis for $\mathcal{P}_2(\mu, d)$ when $\mu \leq d < 2\mu$

Proposition E.1. *For $\mu \leq d < 2\mu$ and $X_1, \dots, X_m$ i.i.d., it holds that*

$$\lim_{m \rightarrow \infty} \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \sup_{D \in \mathcal{D}_m} \frac{\text{REV}(D, \mathbb{P}^m)}{m} \geq \lim_{m \rightarrow \infty} \inf_{\mathbb{P}^m \in \mathcal{P}_2(\mu, d)^m} \sup_{p_m} \frac{p_m \cdot \mathbb{P}^m(\sum_{i=1}^m X_i \geq p_m)}{m} \geq \mu - \frac{d}{2} + \xi \quad (\text{E.1})$$

with $\xi = \xi(\mu, d) > 0$ some small number.

Proof. As we are considering two-point distributions in an i.i.d. setting, the choice of $\mathbb{P}^m$ can be characterized by the choice of $\alpha = \alpha(m)$ in (2.8).⁷ We write $\mathbb{P}^m = \mathbb{P}_{\alpha(m)}^m$ to emphasize this, and focus on bounding

$$\lim_{m \rightarrow \infty} \sup_{p_m} \frac{p_m \cdot \mathbb{P}_{\alpha(m)}^m(\sum_{i=1}^m X_i \geq p_m)}{m}. \quad (\text{E.2})$$

We analyse different regimes of growth of the sequence $(\alpha(m))_{m \in \mathbb{N}}$, that together capture all possibilities. To make notation a bit more convenient, we will write $\alpha(m) = 1 - 1/q(m)$ and make our case distinction in terms of $q(m)$.

We will show that for any growth regime of $q(m)$ chosen by nature, the seller can always guarantee a revenue of either $d/2$ or $\mu - d/2 + \xi$ for some small $\xi > 0$, in case $\mu < d < 2\mu$. This yields the lower bound in (3.2) of $\mu - d/2 + \xi$, as this quantity is smaller than $d/2$. There will be three cases, that together capture all possibilities. We emphasize that at this point we assume $\mu < d < 2\mu$.

- Case 1: $\lim_{m \rightarrow \infty} q(m)/m = \infty$. We show the seller can guarantee a revenue of at least $d/2$.
- Case 2: $\lim_{m \rightarrow \infty} q(m)/m = 0$. We show the seller can guarantee a revenue of at least $d/2$.
- Case 3: $\lim_{m \rightarrow \infty} q(m)/m = \lambda$ for constant $\lambda > 0$.
 - If λ is sufficiently small, we show the seller can guarantee a revenue of at least $d/2$.
 - If λ is not small, we show the seller can guarantee a revenue of at least $\mu - d/2 + \xi$.

The "bottleneck" in the analysis, that leads to only being able to guarantee $\mu - d/2 + \xi$ for the seller is caused by the second subcase of Case 3.

Case 1: $\lim_{m \rightarrow \infty} q(m)/m = \infty$. We use a similar argument as that for the upper bound of $\max\{\mu - d/2, d/2\}$. Setting the price p_m just below the second support point $(m-1)x(\alpha) + y(\alpha)$ gives a revenue of

$$\approx (1 - \alpha^m) \left(\mu - \frac{d}{2} \right) + \frac{(1 - \alpha^m) d}{m(1 - \alpha) 2}.$$

⁷If the infimum is not attained we take a sequence $(\alpha(m))_{m \in \mathbb{N}}$ close enough to attaining the infimum, i.e., for which $\sup_p p \cdot \mathbb{P}_{\alpha(m)}^m(\sum_{i=1}^m X_i \geq p) \leq \delta + \inf_{\mathbb{P}^m \in \mathcal{P}(\mu, d)} \sup_p p \cdot \mathbb{P}^m(\sum_{i=1}^m X_i \geq p)$ for an arbitrary $\delta > 0$.

For any choice of $q(m)$ for which $q(m)/m \rightarrow \infty$ as $m \rightarrow \infty$, it can be shown that

$$(1 - \alpha^m) \rightarrow 0 \quad \text{and} \quad (1 - \alpha^m)/(m(1 - \alpha)) \rightarrow 1.$$

This shows that the seller can guarantee a revenue of $d/2$ as $m \rightarrow \infty$ in this case. Since $d > \mu$, we can find $\xi_0(\mu, d) > 0$ such that $d/2 \geq \mu - d/2 + \xi_0$.

Case 2: $\lim_{m \rightarrow \infty} q(m)/m = 0$. Note that for $X(\alpha)$ as in (2.8), we have

$$\text{Var}(X(\alpha)) = \alpha \left(\mu - \frac{d}{2\alpha} - \mu \right)^2 + (1 - \alpha) \left(\mu - \frac{d}{2(1 - \alpha)} - \mu \right)^2 = \frac{d^2}{4\alpha} + \frac{d^2}{4(1 - \alpha)} \approx \frac{d^2}{4} (1 + q(m)).$$

The Chebyshev bound in Lemma A.1 then tells us that

$$\mathbb{P}_{\alpha(m)}^m \left(\sum_{i=1}^m X_i \leq (1 - \gamma)m\mu \right) \leq \frac{\text{Var}(X(\alpha))}{(\gamma\mu)^2 m} = \frac{d^2(1 + q(m))}{(2\gamma\mu)^2 m} \rightarrow 0,$$

as $m \rightarrow \infty$ by the assumption $\lim_{m \rightarrow \infty} q(m)/m = 0$. This means that if for $\gamma > 0$ the seller sets a price of $p_m = (1 - \gamma)m\mu$, she can guarantee a revenue of roughly p_m (as we sell with probability approaching 1), i.e.,

$$\lim_{m \rightarrow \infty} \sup_p \frac{p \cdot \mathbb{P}_{\alpha(m)}^m(\sum_{i=1}^m X_i \geq p)}{m} \geq (1 - \gamma)\mu$$

for any fixed $\gamma > 0$. If we choose γ close enough to zero, we have $(1 - \gamma)\mu \geq d/2 \geq \mu - d/2 + \xi_0$.

Case 3: $\lim_{m \rightarrow \infty} q(m)/m = \lambda$ for a fixed constant $\lambda > 0$. First, we argue that if λ is sufficiently small, we can still use Chebyshev's bound in order to argue that we can obtain a revenue somewhat close to μ . Let $0 < \gamma < 1$, such that

$$0.99(1 - \gamma)\mu \geq \frac{d}{2}, \tag{E.3}$$

which is possible as $d < 2\mu$, and choose $\tau_0 > 0$ such that

$$1 - \left(\frac{d}{2\gamma\mu} \right)^2 \tau_0 \geq 0.99. \tag{E.4}$$

Then for any $\lambda \in (0, \tau_0]$, Chebyshev's bound tells us that

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{(1 - \gamma)m\mu \cdot \mathbb{P}_{\alpha(m)}^m \left(\sum_{i=1}^m X_i \geq (1 - \gamma)m\mu \right)}{m} &\geq \lim_{m \rightarrow \infty} (1 - \gamma)\mu \left(1 - \frac{d^2(1 + q(m))}{(2\gamma\mu)^2 m} \right) \\ &= (1 - \gamma)\mu \left(1 - \frac{d^2}{(2\gamma\mu)^2} \lambda \right) \\ &\geq (1 - \gamma)\mu \left(1 - \frac{d^2}{(2\gamma\mu)^2} \tau_0 \right) \\ &\geq 0.99(1 - \gamma)\mu \geq \frac{d}{2} > \mu - d/2 + \xi_0, \end{aligned}$$

where the final two inequalities use the choice of γ in (E.3) and τ_0 in (E.4), respectively. This means that for $\lambda \in (0, \tau_0]$, the seller can guarantee a revenue of at least $d/2 \geq \mu - d/2 + \xi_0$.

For $\lambda \in [\tau_0, \infty)$, we will again use a similar revenue analysis as in Case 1. We will argue that if we set a price just below $(m-1)x(\alpha) + y(\alpha)$, then the seller can guarantee a revenue strictly larger than $\mu - d/2$ (independent of λ). We have

$$\frac{[(m-1)x(\alpha) + y(\alpha)] \cdot \mathbb{P}_{\alpha(m)}^m(\sum_{i=1}^m X_i \geq (m-1)x(\alpha) + y(\alpha))}{m} \approx (1 - \alpha^m) \left(\mu - \frac{d}{2} \right) + \frac{(1 - \alpha^m) d}{m(1 - \alpha) 2}. \quad (\text{E.5})$$

Note that

$$1 - \alpha^m = 1 - \left(1 - \frac{1}{q(m)} \right)^m \rightarrow 1 - e^{-1/\lambda},$$

as $m \rightarrow \infty$, using $\lim_{n \rightarrow \infty} (1 + x/n)^n = e^x$, and similarly

$$\frac{(1 - \alpha^m)}{m(1 - \alpha)} \rightarrow \lambda \left(1 - e^{-1/\lambda} \right),$$

as $m \rightarrow \infty$, so that

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{[(m-1)x(\alpha) + y(\alpha)] \cdot \mathbb{P}_{\alpha(m)}^m(\sum_{i=1}^m X_i \geq (m-1)x(\alpha) + y(\alpha))}{m} \\ = \left(1 - e^{-1/\lambda} \right) \left(\mu + (\lambda - 1) \frac{d}{2} \right) =: g_{\mu,d}(\lambda). \end{aligned}$$

The function $g_{\mu,d}(\lambda)$ has the property that

$$g_{\mu,d}(\lambda) \rightarrow \begin{cases} \mu - d/2 & \text{if } \lambda \rightarrow 0 \\ d/2 & \text{if } \lambda \rightarrow \infty \end{cases},$$

and it remains bounded away from $\mu - d/2$ on $[\tau_0, \infty)$. To be precise, we can find a $\xi_1(\mu, d, \tau_0)$ such that

$$\left(1 - e^{-1/\lambda} \right) \left(\mu + (\lambda - 1) \frac{d}{2} \right) \geq \mu - \frac{d}{2} + \xi_1(\mu, d, \tau_0)$$

for all $\lambda \in [\tau_0, \infty)$. We can then define $\xi = \min\{\xi_0, \xi_1\}$. □