SOME NEW RESULTS ON Δ-SPACES

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Abstract. A topological space X is a Δ-space (or $X \in \Delta$) if for any decreasing sequence $\{A_n : n < \omega\}$ of subsets of X with empty intersection there is a (decreasing) sequence $\{U_n : n < \omega\}$ of open sets with empty intersection such that $A_n \subset U_n$ for all $n < \omega$. In this note we prove the following results concerning Δ-spaces.

- Every T_3 countably compact Δ-space is compact.
- If there is a T_1 crowded Baire Δ-space then there is an inner model with a measurable cardinal.
- If $X \in \Delta$ and $cf\big(o(X)\big) > \omega$ then $|X| < o(X)$. (Here $o(X)$ is the number of open subsets of X.)

The first two of these provide full and/or partial solutions to problems raised in the literature, while the third improves a known result.

1. Introduction

There has recently been quite an interest in the study of Δ-spaces, i.e. spaces X such that for any decreasing sequence $\{A_n : n < \omega\}$ of subsets of X with empty intersection there is a (decreasing) sequence $\{U_n : n < \omega\}$ of open sets with empty intersection such that $A_n \subset U_n$ for all $n < \omega$. Also, the class of these spaces is denoted by Δ.

This interest is perhaps mainly motivated by the fact that for a Tykhonov space X the locally convex space $C_p(X)$ is distinguished if and only if X is a Δ-space, see e.g. [6].

On the other hand the simple and attractive definition of this class, moreover, the large number of number of other interesting features of the class of Δ-spaces justifies their study. The aim of our present paper is to add a few new results to this study.

We emphasize that in this work we do not assume that the Δ-spaces in question, unless otherwise stated, satisfy any separation axiom.

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The notation and terminology we use in this paper are pretty standard, so we do not think it is necessary to waste any space to them.

2. Countably compact Δ-spaces

As is mentioned in [7], the Proper Forcing Axiom (PFA), or rather a very deep result in [1] derived from PFA, implies that every T_3 and countably compact Δ-space is actually compact. The aim of this section is to show that, in fact, this is already provable in ZFC.

We start with a preliminary definition.

Definition 2.1. (i) If X is any *non-compact* topological space, we denote by $\mathcal{C}(X)$ the family of all non-compact closed (i.e. closed unbounded) subsets of X.
(ii) We say that $S \subset X$ is *stationary in* X iff $S \cap C \neq \emptyset$ for all $C \in \mathcal{C}(X)$. We shall denote by $St(X)$ the family of all stationary subsets of X.

Now, the following simple lemma will play a key role in our proof.

Lemma 2.2. *Assume that X is a non-compact Δ-space and there is a decreasing sequence $\{S_n : n < \omega\} \subset St(X)$ with empty intersection. Then X is σ-compact.*

Proof. Indeed, as $X \in \Delta$, there are open sets $\{U_n : n < \omega\}$ such that $S_n \subset U_n$ for each $n < \omega$ and $\bigcap\{U_n : n < \omega\} = \emptyset$. But then, as S_n stationary in X, the closed set $K_n = X \setminus U_n$ must be compact for every $n < \omega$, hence $X = \bigcup\{K_n : n < \omega\}$ is indeed σ-compact. □

Note that if $\{T_n : n < \omega\} \subset St(X)$ are pairwise disjoint then putting $S_n = \bigcup\{T_m : n \leq m < \omega\}$ the sequence $\{S_n : n < \omega\} \subset St(X)$ is decreasing with empty intersection, hence we may apply Lemma 2.2 to conclude that X is σ-compact.

We are now ready to formulate and prove the main result of this section.

Theorem 2.3. *Every T_3 and countably compact Δ-space is compact.*

Proof. Our proof of this is indirect: Assume, on the contrary, that

$$\mathcal{E} = \{X \in \Delta \cap T_3 : X \text{ is countably compact and non-compact}\} \neq \emptyset.$$

We shall show that this assumption leads to a contradiction. Our first step towards this contradiction is the following claim.

Claim 1. $\mathcal{E} \neq \emptyset$ implies that there is $X \in \mathcal{E}$ which is *locally compact.*

To see this, we first note that every T_3 and countably compact Δ-space is scattered. In fact, crowded countably compact T_3-spaces are ω-resolvable by Theorem 6.9 of [2], moreover Proposition 6.6 of [11] says that no Δ-space can be both Baire and ω-resolvable. We also refer about this to the first paragraph of the next section.

We shall actually show that if $X \in \mathcal{E}$ is of minimum possible scattered height $ht(X) = \alpha$ then X is locally compact. To see this, we first show that α has to be a limit ordinal. Indeed, if we had $\alpha = \beta + 1$ then the non-empty top level X_β of X has to be finite because X is countably compact. So, if $\mathcal{U}$ is any open cover of X then there is a finite $\mathcal{V} \subset \mathcal{U}$ that covers X_β. But then $Y = X \setminus \cup\mathcal{V}$ is also a T_3 and countably compact Δ-space with $ht(Y) \le \beta < \alpha$, hence Y is compact by the minimality of α. So, Y is also covered by finitely many members of $\mathcal{U}$, showing that X is also compact, a contradiction.

So, we know that $ht(X) = \alpha$ is a limit ordinal. But then, using that X is T_3, every point $x \in X$ has a closed, hence countably compact, neighbourhood W_x such that $ht(W_x) = ht(x, X) < \alpha$. But the minimality of α then implies that W_x is actually compact, i.e. X is indeed locally compact.

Now consider the Alexandrov one-point compactification $A(X) = X \cup \{p\}$ of X. Since any one-point extension of a Δ-space is trivially also a Δ-space, we have $A(X) \in \Delta$. So, $A(X)$ is a compact Δ-space, hence by Theorem 4.2 of [7] it has countable tightness. Consequently, there is a countable subset H of X such that $p \in \overline{H}$, the closure taken in $A(X)$. Hence $X \cap \overline{H}$ is a locally compact and separable member of $\mathcal{E}$. This argument clearly also yields the following.

Claim 2. Every locally compact $X \in \mathcal{E}$ includes a separable locally compact member of $\mathcal{E}$.

As is well-known and easy to prove, see e.g. [4], separable scattered T_3-spaces have cardinality at most $\mathfrak{c}$, the size of the continuum. So, it is immediate from the above that from $\mathcal{E} \ne \emptyset$ it follows that

$$\mathcal{E}_0 = \{X \in \mathcal{E} : X \text{ is locally compact and } |X| \le \mathfrak{c}\} \ne \emptyset$$

as well. Note that for every $X \in \mathcal{E}_0$ we obviously have $\mathcal{C}(X) \subset \mathcal{E}_0$.

Now we distinguish two cases.

Case 1. There is $X \in \mathcal{E}_0$ such that for any two $C_0, C_1 \in \mathcal{C}(X)$ we have $C_0 \cap C_1 \ne \emptyset$.

Note that in this case we even have $C_0 \cap C_1 \in \mathcal{C}(X)$ as well. Indeed, if $K = C_0 \cap C_1$ would be compact then, as X is locally compact, we could find an open neighborhood U of K such that its closure in X is compact, and then $C_0 \setminus U$ and $C_1 \setminus U$ would be two disjoint members of $\mathcal{C}(X)$, contradicting that we are in Case 1.

We also claim that $\cap\mathcal{C}(X) = \emptyset$. This is obvious because every point $x \in X$ has an open neighborhood U_x with compact closure, hence $x \notin X \setminus U_x \in \mathcal{C}(X)$.

Thus, $\mathcal{C}(X)$ is a collection closed under finite intersections and with empty intersection, hence any ultrafilter u on X extending $\mathcal{C}(X)$ is free. But $|X| \le \mathfrak{c}$ is less than the first measurable cardinal, if it exists, hence the ultrafilter u is not σ-complete. Thus there is a decreasing sequence $\{S_n : n < \omega\} \subset u$ with empty intersection. Now, $\mathcal{C}(X) \subset u$ clearly implies that every member of u is stationary in X, hence by Lemma 2.2 we have that X is σ-compact. But countably compact and σ-compact spaces are compact, contradicting that X is not compact.

Thus we are left with the following.

Case 2. For every $X \in \mathcal{E}_0$ there are two $C_0, C_1 \in \mathcal{C}(X)$ such that $C_0 \cap C_1 = \emptyset$.

Claim 3. In this case we have $|X| = \mathfrak{c}$ for all $X \in \mathcal{E}_0$.

Indeed, let us consider any $X \in \mathcal{E}_0$. Clearly, we only have to show that $|X| \ge \mathfrak{c}$. To see this, we use the fact that we are in Case 2. and a standard procedure to define a family of sets

$$\{C_s : s \in 2^{<\omega}\} \subset \mathcal{C}(X) \subset \mathcal{E}_0$$

such that for every $s \in 2^{<\omega}$ we have

$$C_{s^\frown 0} \cup C_{s^\frown 1} \subset C_s \text{ and } C_{s^\frown 0} \cap C_{s^\frown 1} = \emptyset.$$

Then for any ω-sequence $t \in 2^\omega$, we have the decreasing ω-sequence $\{C_{t \restriction n} : n \in \omega\}$ of closed sets in X whose intersection $C_t \ne \emptyset$ because X is countably compact. So, the family $\{C_t : t \in 2^\omega\}$ clearly consists of pairwise disjoint non-empty sets, hence we indeed get $|X| = \mathfrak{c}$.

For every $X \in \mathcal{E}_0$ let us now denote by $\mathcal{C}_\omega(X)$ the family of separable members of $\mathcal{C}(X)$. Clearly, then $|\mathcal{C}_\omega(X)| \le \mathfrak{c}$, moreover every member of $\mathcal{C}_\omega(X)$ has cardinality $\mathfrak{c}$. We can thus apply Kuratowski's disjoint refinement lemma to find for every $C \in \mathcal{C}_\omega(X)$ a $\mathfrak{c}$-sized subset $A_C \subset C$ in such a way that the family $\{A_C : C \in \mathcal{C}_\omega(X)\}$ is disjoint.

Note that it follows from Claim 2. that if $S \subset X$ intersects every A_C then S is stationary in X. But we clearly can find infinitely many, even $\mathfrak{c}$-many disjoint such sets. Using Lemma 2.2, we may thus conclude that X is σ-compact and hence compact. So, we ran into a contradiction in this case as well, completing our proof. $\square$

In [11] the authors write the following: "Whether any countably compact Δ-space has countable tightness in ZFC is an open problem." Since compact Δ-space do have countable tightness, Theorem 2.3 yields an immediate affirmative answer to this question as well.

3. Baire Δ-spaces

It is obvious that if $X \in \Delta$ is Baire then the intersection of any decreasing sequence $\{S_n : n < \omega\}$ of *dense* subsets of X is non-empty. In particular, this clearly implies that X is not ω-resolvable.

Note that if X has an isolated point then, trivially, the intersection of any family of dense subsets of X is non-empty. This lead the authors to raise the following natural question in [8], Problem 4.1: If $X \in \Delta$ is Baire, must X have an isolated point? In this section we give a partial affirmative answer to this question by showing that the existence of a counterexample implies that there is an inner model with a measurable cardinal. By the above observation that any Baire space $X \in \Delta$ is not ω-resolvable, this will be an immediate consequence of our next result.

Theorem 3.1. *If the crowded Baire T_1-space X is not ω-resolvable then X has a crowded Baire irresolvable subspace.*

Proof. Illanes proved in [3] that if a space X is not ω-resolvable then there is $0 < n < \omega$ such that X is n-resolvable but not $(n+1)$-resolvable. Thus we may fix a partition $\{S_i : i < n\}$ of X into n irresolvable dense subsets.

Let $\mathcal{V}$ be a maximal disjoint family of $(n+1)$-resolvable open subsets of X. Then $W = \bigcup \mathcal{V}$ cannot be dense in X because it is not $(n+1)$-resolvable. Thus we have $U = X \setminus \overline{W} \neq \emptyset$.

We claim that then $U \cap S_i$ is *open hereditary irresolvable* (in short OHI) for every $i < n$. Indeed, if we had a non-empty open $V \subset U$ and $i < n$ such that $V \cap S_i$ is resolvable, then clearly V would be $(n+1)$-resolvable, contradicting the maximal disjointness of $\mathcal{V}$. Note also that $V \cap S_i$ is crowded for any open set V because X is T_1. We claim that there are $V \subset U$ non-empty open and $i < n$ for which the crowded and irresolvable subspace $V \cap S_i$ is also Baire.

Indeed, otherwise we could define by induction a decreasing sequence of non-empty open sets $\{U_i : i < n\}$ as follows. Since $U \cap S_0$ is not Baire, there is a non-empty open $U_0 \subset U$ such that $U_0 \cap S_0$ is of first category. If U_i has been defined for some $i < n-1$ so that $U_i \cap S_i$ is of first category then, by our indirect assumption, we may find a non-empty open $U_{i+1} \subset U_i$ such that $U_{i+1} \cap S_{i+1}$ is of first category. But then at the end we have

$$U_{n-1} = \bigcup \{U_{n-1} \cap S_i : i < n\},$$

hence the non-empty open $U_{n-1} \subset X$, as a finite union of first category sets, would be of first category in X, contradicting that X is Baire. $\square$

Now, as it had been established in [9] and [10], the existence of a crowded Baire irresolvable space implies that there is an inner model M of the set-theoretic universe V that contains a measurable cardinal. In other words, we have the following result that yields a partial solution of problems 4.1 and 4.2 from [8].

Corollary 3.2. *If there is no inner model of* V *containing a measurable cardinal, for instance if Gödel's axiom of constructibility* $V = L$ *holds, then every* T_1 *crowded Baire* Δ*-space has an isolated point; consequently, any* T_1 *hereditary Baire* Δ*-space is scattered.*

As is well-known, pseudocompact spaces are Baire, hence we may also ask: is there a crowded pseudocompact Δ-space? Of course, in view of Theorem 3.1, the existence of such a space implies the consistency of having a measurable cardinal. On the other hand, as far as we know, it is open if ZFC implies that all crowded pseudocompact spaces are ω-resolvable. we actually conjecture that this is true, and hence there are no crowded and Baire pseudocompact Δ-spaces.

In any case, it follows from the results in [5] that the existence of a non-ω-resolvable crowded pseudocompact space would imply much stronger large cardinals than measurable.

4. Connection with $o(X)$

We start with a general, somewhat technical result that gives a condition for $X \notin \Delta$.

Theorem 4.1. *Let* X *be a topological space with* $|X| = \kappa$*, where* $cf(\kappa) > \omega$*. Assume also that there is a family* $\mathcal{A} \subset [X]^\kappa$ *such that* $|\mathcal{A}| \le \kappa$ *and for for every closed set* $F \in [X]^\kappa$ *there is* $A \in \mathcal{A}$ *with* $A \subset F$*. Then* $X \notin \Delta$*.*

Proof. By our assumptions we can apply Kuratowski's disjoint refinement lemma to $\mathcal{A}$ to obtain a disjoint collection

$$\{B_A : A \in \mathcal{A}\} \subset [X]^\kappa$$

such that $B_A \subset A$ for all $A \in \mathcal{A}$. Now, it is easy to fix ω-many disjoint sets $\{S_n : n < \omega\}$ such that $S_n \cap B_A \neq \emptyset$ for any $n < \omega$ and $A \in \mathcal{A}$.

It is clear from our assumptions that then we also have $S_n \cap F \neq \emptyset$ for every $n < \omega$ and for every closed set $F \in [X]^\kappa$. In other words, then for any open set $U \supset S_n$ we have $|X \setminus U| < \kappa$. Consequently, if we take open sets $U_n \supset S_n$ for al $n < \omega$, then $cf(\kappa) > \omega$ implies

$$|X \setminus \bigcap\{U_n : n < \omega\}| < \kappa$$

as well. But this clearly implies $X \notin \Delta$. $\square$

Theorem 4.1 trivially applies if we have

$$|\{F \in [X]^\kappa : F \text{ is closed}\}| \leq \kappa.$$

In particular, we trivially have this if $o(X) \leq \kappa$, where $o(X)$ denotes the number of all open, or equivalently closed, subsets of X. Thus we have the following corollary of Theorem 4.1.

Corollary 4.2. *If $X \in \Delta$ and $cf\big(o(X)\big) > \omega$ then $|X| < o(X)$.*

Since $\kappa^\omega = \kappa$ implies $cf(\kappa) > \omega$, Corollary 4.2 improves Theorem 6.1 of [11] which says that $X \in \Delta$ implies $|X| < o(X)^\omega$.

Clearly, if X is hereditary Lindelöf of weight $\leq \mathfrak{c}$ or if $|X| = \mathfrak{c}$ and X is hereditary separable then $o(X) \leq \mathfrak{c}$. So, if either X is hereditary Lindelöf of weight $\leq \mathfrak{c}$ or hereditary separable then $X \in \Delta$ implies $|X| < \mathfrak{c}$.

Thus we are lead to the following problem.

Problem 4.3. *Does every (T_2 or T_3) hereditary Lindelöf Δ-space have cardinality $< \mathfrak{c}$?*

Provided that the answer to this is affirmative, we may ask the following.

Problem 4.4. *Does every (T_2 or T_3) Δ-space of countable spread have cardinality $< \mathfrak{c}$?*

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