

Quantum Error Correction with Superpositions of Squeezed Fock States

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Bosonic codes, leveraging infinite-dimensional Hilbert spaces for redundancy, offer great potential for encoding quantum information. However, the realization of a practical continuous-variable bosonic code that can simultaneously correct both single-photon loss and dephasing errors remains elusive, primarily due to the absence of exactly orthogonal codewords and the lack of an experiment-friendly state preparation scheme. Here, we propose a code based on the superposition of squeezed Fock states with an error-correcting capability that scales as $\propto \exp(-7r)$, where r is the squeezing level. The codewords remain orthogonal at all squeezing levels. The Pauli- X operator acts as a rotation in phase space is an error-transparent gate, preventing correctable errors from propagating outside the code space during logical operations. In particular, this code achieves high-precision error correction for both single-photon loss and dephasing, even at moderate squeezing levels. Building on this code, we develop quantum error correction schemes that exceed the break-even threshold, supported by analytical derivations of all necessary quantum gates. Our code offers a competitive alternative to previous encodings for quantum computation using continuous bosonic qubits.

Introduction.—Quantum states are fragile due to their high susceptibility to environmental noise, which poses a significant challenge to realizing quantum computation [1–7]. Quantum error correction (QEC), which restores quantum information degraded by noise channels through syndrome measurements or reservoir engineering, is therefore essential for fault-tolerant quantum computing [8–14]. This process requires encoding information in large Hilbert spaces for redundancy, typically a block of multiple physical qubits or a single higher-dimensional bosonic mode [15–17]. Logical qubits that rely on multiple physical qubits face challenges in QEC and logical operations due to the increased number of error channels, the need for non-local gates, and the growing overhead as error rates accumulate. In contrast, bosonic codes leverage an infinite dimensional Hilbert space within a single quantum mode, enabling efficient higher-dimensional encoding while enhancing stability and scalability [18, 19].

Bosonic codes hold promise for quantum information processing and thus have garnered considerable attention [20–22]. For bosonic codes, single-photon loss and dephasing are the primary noise channels. The more the codewords are distributed across the Fock space (i.e., demanding increased coherence), the more dephasing becomes a significant noise source [23–25]. Various bosonic codes are designed to correct different types of errors and can be broadly classified into continuous and discrete codes based on their distribution in Fock space. Discrete bosonic codes, such as binomial codes, are composed of superpositions over a finite set of Fock states, unlike continuous codes such as cat and Gottesman–Kitaev–Preskill (GKP) codes [26–28], which are continuously distributed over the discrete set of Fock states.

Being a superposition of a finite number of Fock states, discrete bosonic codes exhibit a simple structure especially

if they are tailored to correct a small number of errors. Nevertheless, this apparent simplicity does not translate into trivial schemes to prepare and control these states as addressing a few Fock states requires a fine-tuned control of a quantum system. On the other hand, continuous bosonic codes, such as the squeezed cat or the GKP code, might offer certain advantages for state preparation or control owing to their structure which relies on displacement and squeezing operations available and easy to implement in different quantum technologies, such as superconducting circuits or trapped ions [29–31]. Noteworthy, this class of codes typically relies on codewords which are strictly not orthogonal and therefore, one needs to choose parameter regimes in which orthogonality is approximately satisfied. Despite some possible advantages in their structure as compared to their discrete counterparts, existing continuous bosonic codes perform poorly in correcting errors that involve both single-photon loss and dephasing [32–34]. For example, squeezed cat codes are limited to correcting either single-photon loss or dephasing errors independently, while squeezed Fock codewords are inherently nonorthogonal and exhibit diminished performance, particularly at low squeezing levels [35–39]. Developing orthogonal bosonic codes capable of simultaneously correcting both photon loss and dephasing at experimentally feasible squeezing levels is essential for scalable bosonic quantum computing.

In this Letter, we combine the advantages of continuous and discrete bosonic codes to construct a strictly orthogonal code that *achieves high-precision correction of both single-photon loss and dephasing noise*. The proposed codewords are composed of tailored superpositions of squeezed Fock states, where the specific choice of superposition by construction ensures exact orthogonality of the codewords—at any squeezing level r . Meanwhile,

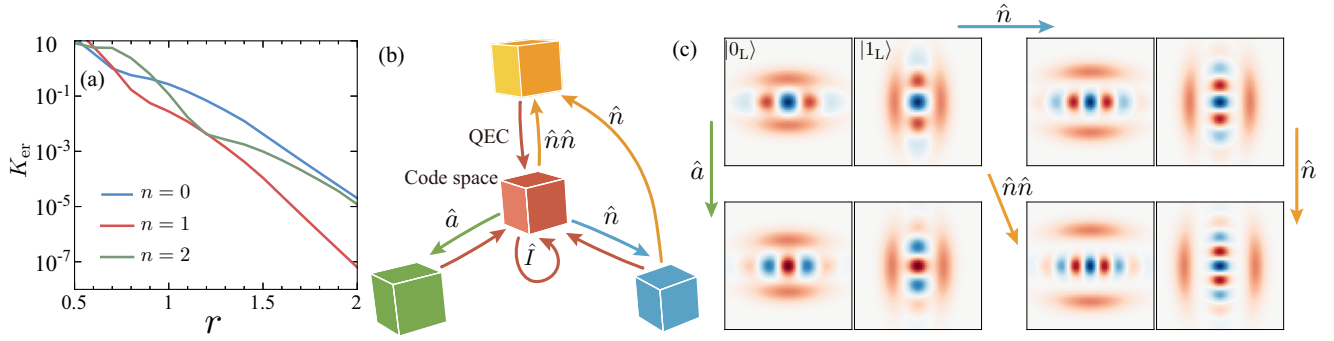


Figure 1. **(a)** Deviation from the KL condition versus the squeezing amplitude r for different values of n . An exponential decay is observed, with $n = 1$ exhibiting the best performance at large r . **(b)** Errors from the set $[\hat{I}, \hat{a}, \hat{n}, \hat{n}^2]$ acting on the logical subspace define the corresponding error spaces. The code and error spaces approximately satisfy the KL condition K_{er} , enabling recovery via QEC. **(c)** Wigner functions of the codewords (related by a $\pi/2$ rotation) and associated error states (also related by a $\pi/2$ rotation) at 8 dB squeezing ($r \approx 0.921$). As the $\hat{n}$ and $\hat{n}^2$ operators do not affect the parity of the codewords (as compared to $\hat{a}$), the resulting error states have nonzero overlap with the codewords and with each other. This, in turn, is the reason for the nonvanishing deviation of the KL condition [cf. Eq. (3)].

the squeezing operation distributes the codewords continuously over the infinite-dimensional Fock space. Our analytical results demonstrate an exponential suppression of the infidelity scaling as $\exp(-7r)$ for combined photon loss and dephasing errors, outperforming all existing continuous bosonic codes. Therefore, the proposed code enables high-precision correction of both single-photon loss and dephasing errors at moderate squeezing levels. In addition, it facilitates straightforward implementation of logical qubit gates with experimentally accessible operations. Specifically, the code supports a simple, error-transparent logical Pauli X gate, allowing for the efficient generation of the codewords. In this respect, our code overcomes the limitations of previous continuous-variable codes. Moreover, we analytically design effective quantum gates to implement both autonomous and parity-measurement-based quantum error correction protocols.

Codewords.—We propose a novel continuous bosonic code that synthesizes discrete and continuous characteristics, specifically designed to correct single-photon loss and dephasing error channels in combination. The logical codewords are defined as

$$\begin{aligned} |0_L\rangle &= \hat{S}(r) (\alpha |n+2\rangle - \beta |n\rangle) \\ |1_L\rangle &= \hat{S}(-r) (\alpha |n+2\rangle + \beta |n\rangle) \end{aligned} \quad (1)$$

where α ($|\beta|^2 + |\alpha|^2 = 1$) is determined by enforcing the orthogonality condition $\langle 0_L | 1_L \rangle = 0$, $\hat{S}(r)$ denotes the squeezing operator with squeezing amplitude r , and $|n\rangle$ are the Fock states [n will be specified below].

To evaluate the QEC capability against single-photon loss and dephasing, we examine the Knill-Laflamme (KL) criterion for the error operator set $E = \{\hat{I}, \hat{a}, \hat{n}, \hat{n}^2\}$ [35], where $\hat{a}$ and $\hat{n}$ are the annihilation and number operator, respectively. The KL criterion reads $\langle u_L | \hat{E}_i^\dagger \hat{E}_j | v_L \rangle = C_{ij} \delta_{uv}$ [40], where the C_{ij} form a Hermitian matrix,

$\hat{E}_i, \hat{E}_j \in E$ denote error operators, and $|u_L\rangle, |v_L\rangle \in \{|0_L\rangle, |1_L\rangle\}$ represent the codewords. Exact satisfaction of the KL condition is a prerequisite for perfect error correction, and the degree of approximate fulfillment determines the maximal achievable fidelity [41–44]. We quantify the deviation from the KL condition using the indicator

$$K_{\text{er}} = \sum_{ij} |M_{ij}^{00} - M_{ij}^{11}|^2 + |M_{ij}^{01}|^2, \quad (2)$$

where $M_{ij}^{\mu\nu} = \langle \mu_L | \hat{E}_i^\dagger \hat{E}_j | \nu_L \rangle$. A smaller value of K_{er} implies a closer approximation to the ideal KL condition and hence stronger error-correcting performance.

The condition $M_{ij}^{00} = M_{ij}^{11}$ is satisfied for the code in Eq. (1), and hence the first term in Eq. (2) vanishes identically. This indicates that the error process does not distinguish between logical basis states—a distinct advantage over the squeezed cat code. Consequently, only the off-diagonal terms quantifying the coherence between logical states affected by errors contribute to the deviation from the KL condition. Although some off-diagonal terms M_{ij}^{01} are nonzero, they remain exponentially small, scaling as $\sim e^{-7r}$ for odd n and $\sim e^{-5r}$ for even n . This difference arises from the distinct probability amplitude distributions of squeezed even and odd Fock states in Fock space. Since increasing n does not improve the scaling behavior but increases the average excitation number—which is approximately proportional to the error occurrence probability—we select $n = 1$ as the optimal choice and focus on it hereafter.

We simulate K_{er} in Fig. 1(a) for different values of n , confirming that $n = 1$ is the optimal choice and that the QEC performance improves with increasing squeezing. The nonzero terms responsible for the deviation from the KL condition can be approximately expanded

in exponentials of the squeezing amplitude r :

$$\begin{aligned} \langle 1_L | \hat{n} | 0_L \rangle &= \pm \frac{32\sqrt{3}e^{-7r}}{5} + \mathcal{O}(e^{-9r}), \\ \langle 1_L | \hat{n}^2 | 0_L \rangle &= -\frac{16\sqrt{2}}{5} \left(5 \pm \sqrt{6} \right) e^{-7r} + \mathcal{O}(e^{-9r}), \\ \langle 1_L | \hat{n}^3 | 0_L \rangle &= \left(24\sqrt{2} \mp \frac{184\sqrt{3}}{5} \right) e^{-7r} + \mathcal{O}(e^{-9r}), \\ \langle 1_L | \hat{n}^4 | 0_L \rangle &= 8\sqrt{2} \left(31 \pm 5\sqrt{6} \right) e^{-7r} + \mathcal{O}(e^{-9r}), \end{aligned} \quad (3)$$

where $\pm$ correspond to different solutions for α . The solutions for the coefficient α are not unique; however, this ambiguity does not affect the exponential scaling of these terms. These analytical results are in excellent agreement with numerical simulations [35]. Note that all other terms of the KL conditions are strictly satisfied. As we will detail next, the scaling of the off-diagonal terms in K_{er} achieved by our proposed code represents a significant improvement over previous continuous bosonic codes.

The value of K_{er} reaches the order of 10^{-6} at $r \approx 0.921$ (approximately 8 dB), a squeezing level readily achievable in current experiments [36]. For the error set E , this corresponds to a reduction in deviation by more than *six* and *four* orders of magnitude compared to the squeezed cat and squeezed Fock codes, respectively. This performance gap stems from the substantial overlap of squeezed cat codewords under the combined action of $\hat{a}$ and $\hat{n}^m$. Consequently, the code performs significantly worse for error sets that simultaneously include both operators, such as $E = \{\hat{I}, \hat{a}, \hat{n}, \hat{n}^2\}$, even though subsets like $\{\hat{I}, \hat{a}\}$ or $\{\hat{I}, \hat{n}, \hat{n}^2\}$ remain approximately correctable. Similarly, the squeezed Fock code suffers from inherent non-orthogonality and off-diagonal terms M_{ij}^{01} that decay no faster than $\sim e^{-3r}$, resulting in markedly inferior error suppression relative to our code.

Short-time quantum dynamics gives rise to three distinct error subspaces, whose relation to the code space is depicted in Fig. 1(b). The corresponding Wigner functions of the codewords and their error-transformed states, related by a $\pi/2$ rotation, are shown in Fig. 1(c). Consequently, the logical Pauli X operator is given by $\hat{X}_L = \exp(-i\frac{\pi}{2}\hat{n})$, which remains effective within the error space; hence, it is an error-transparent gate. Furthermore, the Pauli- Z operator can be implemented as $\hat{Z}_L = \exp[-i\hat{H}_z(\hat{a}, \hat{a}^\dagger)]$, where $\hat{H}_z$ is expressed as a power-series expansion in the operators $\hat{a}$ and $\hat{a}^\dagger$ [35]. Finally, the algebraic properties of the Pauli operators and the structure of the codewords allow for a simplified preparation scheme—once one codeword is prepared, the other can be obtained through a straightforward rotational operation.

Error correction schemes.— The strong alignment of the code with the KL conditions, together with the odd parity structure of the code space, enables the systematic design of QEC schemes, including both autonomous

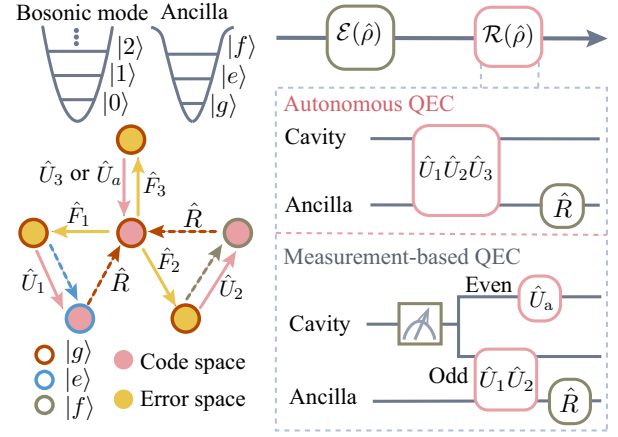


Figure 2. The encoded bosonic mode resides in an infinite-dimensional Hilbert space, while the auxiliary system is a discrete-level system, such as a qutrit. The bosonic mode is initialized in an encoded state $|\psi_L\rangle$, and the auxiliary system in its ground state $|g\rangle$. Noise operators $\hat{F}_i$, derived from the error set E , map the code space into approximately orthogonal error subspaces. In the autonomous QEC protocol, control operations U_i conditionally restore the encoded state while exciting the auxiliary system. A reset operation $\hat{R}$ returns the auxiliary system to $|g\rangle$, completing a measurement-free QEC cycle. In the measurement-based protocol, a parity measurement identifies the error subspace: even parity invokes $\hat{U}_a$ in Eq. (S23) for direct recovery; odd parity triggers a two-step correction via $\hat{U}_1$ and $\hat{U}_2$. Iterating these cycles enables long-term protection of encoded quantum information.

and parity-measurement-based approaches. As shown in Fig. 2, the encoded bosonic mode first undergoes a noise channel $\mathcal{E}(\cdot)$, followed by a recovery operation $\mathcal{R}(\cdot)$. For a short evolution time τ , the combined effect of single-photon loss and dephasing is described by a Kraus expansion, $\mathcal{E}[\hat{\rho}] \approx \sum_{i=1}^3 \hat{A}_i \hat{\rho} \hat{A}_i^\dagger$, where the Kraus operators are given by $\hat{A}_1 \approx \sqrt{\kappa\tau} \hat{a}$, $\hat{A}_2 \approx \sqrt{\kappa_\phi\tau} \hat{n}$, and $\hat{A}_3 \approx \hat{I} - \frac{\kappa\tau}{2} \hat{n} - \frac{\kappa_\phi\tau}{2} \hat{n}^2$ [35]. Here, κ and κ_ϕ are the single-photon loss and dephasing rates, respectively. To construct the QEC channel, we diagonalize the symmetric matrix $\mathbf{J} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^\dagger$, with elements $J_{ij} = \langle u_L | \hat{A}_i^\dagger \hat{A}_j | u_L \rangle$. The noise channel can be recast as $\mathcal{E}[\hat{\rho}] \approx \sum_i \hat{F}_i \hat{\rho} \hat{F}_i^\dagger$, where the transformed Kraus operators are [41]

$$\hat{F}_i = \sum_{k=1}^3 V_{ki} \hat{A}_k. \quad (4)$$

These operators approximately satisfy $\hat{P}_L \hat{F}_i^\dagger \hat{F}_j \hat{P}_L \approx \Lambda_{ij} \delta_{ij} \hat{P}_L$, where $\hat{P}_L$ denotes the projector onto the code space. This ensures that the $\hat{F}_i$ operators map the code space onto orthogonal error subspaces, enabling efficient recovery.

As shown in Fig. 2, we couple the encoded bosonic mode to an ancillary qutrit initialized in its ground state $|g\rangle$ to identify and correct the errors $\hat{F}_i$, resulting in the initial joint state $|\psi_L, g\rangle$. A sequence of unitary

operations $\hat{U}_i$ [$i = 1, 2, 3$ in Eq. (6) and $i = a$ in Eq. (S23)] is then applied to coherently restore the system to the code space. Since only $\hat{a}$ flips the parity of the encoded state, error syndromes can alternatively be extracted via parity measurements, enabling a measurement-based QEC scheme. While conceptually similar to the autonomous scheme, the measurement-based approach requires active readout and feedforward, introducing additional overhead in experimental implementations. We therefore focus on the autonomous scheme in the main text, with the measurement-based protocol detailed in the Supplemental Material [35].

These recovery unitary operations, $\hat{U}_i$, correct errors associated with the operators $\hat{F}_i$ while acting as the identity operator in the other orthogonal error subspaces. As a result, they do not interfere with errors arising from other Kraus operators. Following the recovery process, the encoded state $|\psi_L\rangle$ is approximately restored, while the ancilla qutrit transitions to different states depending on the specific error that occurred [35]. Finally, the ancilla qutrit is rapidly reset to its ground state via a strongly dissipative interaction R with a reservoir, completing the QEC cycle without measurement and thereby enabling fully autonomous QEC. The entire QEC cycle can be described by the following equation

$$\mathcal{R}_a \circ \mathcal{E}[\hat{\rho}_L] = R[\hat{U} \mathcal{E}[\hat{\rho}_L(t)] \otimes |g\rangle\langle g| \hat{U}^\dagger] \approx \hat{\rho}_L \otimes |g\rangle\langle g|, \quad (5)$$

where $\mathcal{R}_a$ is the autonomous QEC channel, $\hat{U} = \hat{U}_3 \hat{U}_2 \hat{U}_1$, $\hat{\rho}_L$ is the encoded state $\hat{\rho}_L = |\psi_L\rangle\langle\psi_L|$, and R is for reset the quantum ancilla qutrit to its ground state.

Here, we propose an analytical approach for the above introduced recovery unitary operations by incorporating an ancilla qutrit system, with the detailed expressions given by

$$\begin{aligned} \hat{U}_1 &= \hat{L}_1 |e\rangle\langle g| + \hat{L}_1^\dagger |g\rangle\langle e| + \hat{U}_{1,\text{re}}, \\ \hat{U}_2 &= \hat{L}_2 |f\rangle\langle g| + \hat{L}_2^\dagger |g\rangle\langle f| + \hat{U}_{2,\text{re}}, \\ \hat{U}_3 &= \left(\hat{L}_3^\dagger + \hat{L}_3 + \hat{I} - \hat{P}_L - \hat{P}_{F_3} \right) |g\rangle\langle g| + \hat{U}_{3,\text{re}}, \end{aligned} \quad (6)$$

where $|e\rangle$, $|f\rangle$ are two excited states used to discriminate between different types of errors, the operator $\hat{L}_i = |0_L\rangle\langle 0_{F_i}| + |1_L\rangle\langle 1_{F_i}|$ is designed to recover information from the error space into the code space ($|u_{F_i}\rangle = \hat{F}_i |u_L\rangle / \|\hat{F}_i |u_L\rangle\|$), $\hat{P}_{F_i}$ is the projector operator of the i th error space, and $\hat{U}_{i,\text{re}}$ supplements $\hat{U}_i$ to ensure it forms a unitary operator (the detailed expression of $\hat{U}_{i,\text{re}}$ is provided in the Supplemental Material [35]). These unitary operations correct the corresponding errors without affecting other parts of the error space or states corrected after other errors occur. The availability of analytic expressions for the recovery unitaries (6) allows for their efficient implementation using well-established quantum control techniques, such as gradient ascent pulse engineering (GRAPE) and machine-learning-based optimization [45–49]. Alternatively, we could use two-qubit or

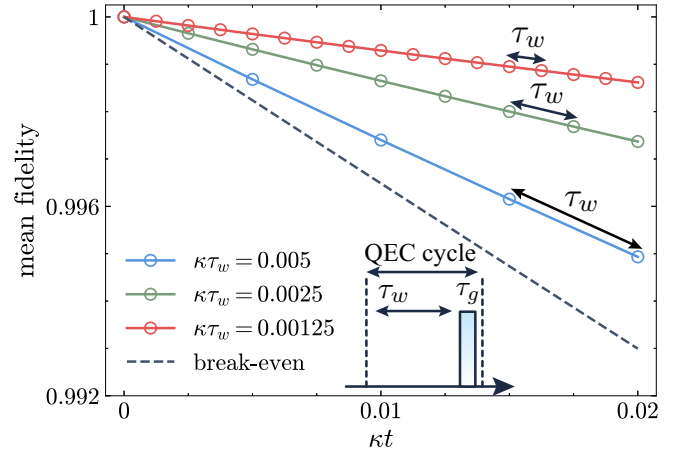


Figure 3. Mean fidelity of the encoded state—averaged over the entire error space—versus time under 8 dB squeezing with a photon loss to dephasing rate ratio $\kappa/\kappa_\phi \approx 8.5$. τ_w represents the waiting time within a single QEC cycle and is much longer than the duration of the recovery unitary operations ($\tau_w \gg \tau_g$).

multilevel ancilla systems to design these unitary operations with the construction methodology similar to that of the qutrit ancilla system and detailed in [35]. These unitary operations can be implemented in platforms such as trapped ions and 3D superconducting microwave cavities, using SNAP gates or GRAPE-based optimal control [50–56]. In principle, arbitrary unitary operations on the system can be achieved through optimal control techniques. As an example, we consider a superconducting cavity coupled to a transmon, where the composite gate $\hat{U}$ is realized by controlling the quadratures $\hat{x}$ and $\hat{p}$ of the bosonic mode, achieving a fidelity exceeding 0.99 [35]. The control fields can be flexibly adjusted to enhance the performance further, but such refinements are beyond the scope of this work.

We assume that the waiting time between successive QEC cycles is much longer than the duration of the recovery operations. Therefore, we effectively consider these operations ideal by neglecting any possible errors induced during their implementation. In Fig. 3, we present the simulated mean fidelities of the autonomous QEC protocol for various waiting times τ_w , enabled by the measurement-free implementation. A squeezing level of approximately 8 dB and a photon loss to dephasing rate ratio $\kappa/\kappa_\phi \approx 8.5$ are used as examples, consistent with current experimental parameters [51, 53, 54, 57–60]. The mean fidelity averaged over the code space surpasses the break-even point—the threshold at which QEC outperforms an unprotected qubit—and improves as the waiting time shortens. This stems from the increased accuracy of the short-time approximation and the reduced probability of multiple errors per cycle. Our scheme is compatible with various experimental platforms, including trapped ions and 3D

superconducting microwave cavities [52, 55, 61–64], and holds strong potential for realizing QEC.

We emphasize that binomial codes can be designed to satisfy the KL conditions for the same error set as our proposed code. The result is a high-order code involving superpositions up to Fock state $|10\rangle$. Whereas these codes offer an error correction performance comparable to ours, they are experimentally challenging due to their reliance on multiple high-Fock components [27]. So far, experiments have only realized the lowest-order binomial code with superpositions up to Fock state $|4\rangle$, which are limited to correcting single-photon loss [65]. In contrast, encodings within low-Fock subspaces such as Fock states $|1\rangle$ and $|3\rangle$ are well established experimentally [66–68]. Our code builds on this subspace, requiring only an additional squeezing operation for codeword preparation. This results in a preparation process that is experimentally more accessible—even simpler than that of the lowest-order binomial codes—while offering enhanced error correction. Consequently, our code combines improved performance with high experimental feasibility.

Conclusion.—We introduced a squeezed bosonic code that robustly corrects for both single-photon loss and dephasing in continuous-variable quantum systems under experimentally feasible conditions. Exploiting the infinite-dimensional Hilbert space of bosonic modes, our code features error-correcting capabilities that scale exponentially with the squeezing amplitude $[\propto \exp(-7r)]$ while maintaining orthogonal codewords at all squeezing levels. As a result, the error-correcting performance improves exponentially with the degree of squeezing, theoretically enabling arbitrarily precise error correction as $r \rightarrow \infty$. Building on this framework, we developed measurement-based and autonomous QEC protocols and provide an analytical description of the required recovery operations. Our analysis establishes that these QEC schemes can be implemented in a bosonic mode coupled to a qutrit. This new code represents a substantial advancement over previous continuous-variable bosonic qubit encodings for quantum computation.

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Supplemental Information for "Quantum Error Correction with Superpositions of Squeezed Fock States"

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I CODEWORD DESIGN

We consider superposition states composed of squeezed Fock states as our codewords, explicitly defined

$$\begin{aligned} |0_L\rangle &= \hat{S}(r)(\alpha|n+2\rangle - \sqrt{1-\alpha^2}|n\rangle) = \alpha|n+2, r\rangle - \sqrt{1-\alpha^2}|n, r\rangle, \\ |1_L\rangle &= \hat{S}(-r)(\alpha|n+2\rangle + \sqrt{1-\alpha^2}|n\rangle) = \alpha|n+2, -r\rangle + \sqrt{1-\alpha^2}|n, -r\rangle. \end{aligned} \quad (S1)$$

where $|n\rangle$ denotes a Fock state, and the squeezed number state is defined as $|n, r\rangle = \hat{S}(r)|n\rangle$, where $\hat{S}(r) = \exp[r(\hat{a}^2 - \hat{a}^{\dagger 2})/2]$ is the squeezing operator with r a real parameter quantifying the squeezing level. Finally, α is a real parameter determined by the orthogonality condition $\langle 0_L | 1_L \rangle = 0$. The Fock-space representation of squeezed Fock states differs markedly between even and odd n . Furthermore, as the squeezing parameter r increases, the population of highly excited Fock states rapidly diminishes. The corresponding occupation probability is given by

$$\langle n | m, r \rangle = \frac{(m!n!)^{\frac{1}{2}}}{\cosh(r)^{\frac{n+m+1}{2}}} \times \sum_k^{\min(m,n)} \left[\frac{\sinh(r)}{2} \right]^{\frac{n+m-2k}{2}} \frac{(-1)^{\frac{n-k}{2}}}{k! \left(\frac{m-k}{2}\right)! \left(\frac{n-k}{2}\right)!}, \quad (S2)$$

where k takes even or odd values in correspondence with the even or odd nature of m and n , respectively [1]. From the above expression, in the limit $r \rightarrow \infty$, it can be shown that the overlap $\langle n | m, r \rangle$ scales as $\cosh(r)^{-\frac{3}{2}}$ for odd n , and as $\cosh(r)^{-\frac{1}{2}}$ for even n .

Owing to the fact that bosonic modes are predominantly susceptible to single-photon loss and dephasing channels, we are going to consider the system dynamics governed by the master equation

$$\frac{d\hat{\rho}}{dt} = \frac{\kappa}{2} \mathcal{D}[\hat{a}]\hat{\rho} + \frac{\kappa_\phi}{2} \mathcal{D}[\hat{n}]\hat{\rho} \quad (S3)$$

where κ (κ_ϕ) are the single photon loss (dephasing) rates and $\mathcal{D}[\hat{x}] = 2\hat{x}\hat{\rho}\hat{x}^\dagger - \hat{x}^\dagger\hat{x}\hat{\rho} - \hat{\rho}\hat{x}^\dagger\hat{x}$ is the Lindblad super-operator. We recast the master equation in terms of Kraus operators at short time scales $\tau \ll 1$

$$\hat{\rho}(t+\tau) = \mathcal{E}[\hat{\rho}(t)] \approx \sum_i \hat{A}_i \hat{\rho}(t) \hat{A}_i^\dagger \quad (S4)$$

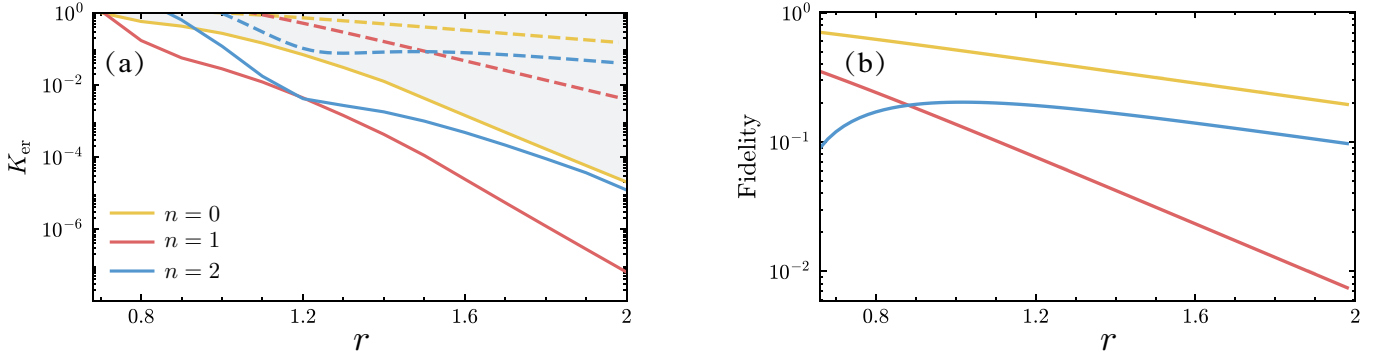


FIG. S1. (a) The deviation K_{er} of our code (solid lines) and the squeezed Fock code (dashed lines) with respect to the squeezing parameter r is presented for different values of n . (b) The fidelity between the codewords of the squeezed Fock code is shown to remain high even under strong squeezing. In contrast, our code is orthogonal by construction.

where $\hat{A}_1 \approx \sqrt{\kappa\tau}\hat{a}$, $\hat{A}_2 \approx \sqrt{\kappa_\phi\tau}\hat{a}^\dagger\hat{a}$, $\hat{A}_3 = \hat{I} - \frac{\kappa\tau}{2}\hat{n} - \frac{\kappa_\phi\tau}{2}\hat{n}^2$. Consequently, the error set accounting for single-photon loss and dephasing can be represented as $E = \{\hat{I}, \hat{a}, \hat{n}, \hat{n}^2\}$. If the elements of this set satisfy the Knill-Laflamme (KL) condition, the associated errors are fully correctable. The KL condition reads $\langle u_L | \hat{A}_i^\dagger \hat{A}_j | v_L \rangle = C_{ij} \delta_{uv}$, where C is a Hermitian matrix. Given the structure of the codewords (S1), it is straightforward to verify that $\langle 0_L | \hat{n}^m | 0_L \rangle = \langle 1_L | \hat{n}^m | 1_L \rangle$. Therefore, the two logical basis states exhibit identical error probabilities under the same noise processes. In addition, the codewords always satisfy $\langle u_L | \hat{a} \hat{n}^m | v_L \rangle = 0$. Furthermore, the KL conditions can be systematically expanded in powers of e^{-r} in the large-squeezing limit $r \rightarrow \infty$. For odd n , we find

$$\langle 1_L | \hat{n}^m | 0_L \rangle \propto e^{-7r} + \mathcal{O}(e^{-9r}), \quad (\text{S5})$$

whereas for even n , the scaling becomes $e^{-5r} + \mathcal{O}(e^{-7r})$.

We evaluate the deviation from the KL condition, denoted as K_{er} in the main text, for both our proposed code and the squeezed Fock code [2]

$$\begin{aligned} |0_L\rangle &= \hat{S}(r) |n\rangle \\ |1_L\rangle &= \hat{S}(-r) |n\rangle, \end{aligned} \quad (\text{S6})$$

at various excitation numbers n , as shown in Fig. S1(a). Notably, the case $n=1$ not only exhibits low excitation levels but also yields superior error correction performance. It is evident that our code exhibits a smaller deviation from the KL condition compared to the squeezed Fock code at all squeezing levels.

Furthermore, as illustrated in Fig. S1(b), the squeezed Fock states are not strictly orthogonal, whereas our codewords are orthogonal by design. In the following, we focus on the minimal nontrivial case $n=1$, which corresponds to the lowest-photon-number encoding within this code family while preserving nontrivial error-correcting capability. In this $n=1$ case, the remaining nonvanishing KL conditions admit the following asymptotic expansions:

$$\begin{aligned} \langle 1_L | \hat{n} | 0_L \rangle &= \pm \frac{32\sqrt{3}e^{-7r}}{5} - \frac{64\sqrt{2}e^{-9r}}{25} + \mathcal{O}(e^{-11r}), \\ \langle 1_L | \hat{n}^2 | 0_L \rangle &= \frac{32\sqrt{2}}{25} \left(2 \pm 35\sqrt{6} \right) e^{-9r} - \frac{16\sqrt{2}}{5} \left(5 \pm \sqrt{6} \right) e^{-7r} + \mathcal{O}(e^{-11r}), \\ \langle 1_L | \hat{n}^3 | 0_L \rangle &= \left(24\sqrt{2} \mp \frac{184\sqrt{3}}{5} \right) e^{-7r} - \frac{16\sqrt{2}}{25} \left(502 \pm 105\sqrt{6} \right) e^{-9r} + \mathcal{O}(e^{-11r}), \\ \langle 1_L | \hat{n}^4 | 0_L \rangle &= \left(640\sqrt{2} \mp \frac{6944\sqrt{3}}{5} \right) e^{-9r} + 8\sqrt{2} \left(31 \pm 5\sqrt{6} \right) e^{-7r} + \mathcal{O}(e^{-11r}). \end{aligned} \quad (\text{S7})$$

Evidently, in the limit $r \rightarrow \infty$, the KL condition is exactly satisfied. Notably, our expansion contains only terms of order equal or higher than e^{-7r} , indicating that the code provides a good approximation to the KL condition for single-photon loss and dephasing even with moderate squeezing. The error-correcting performance further improves with increasing r .

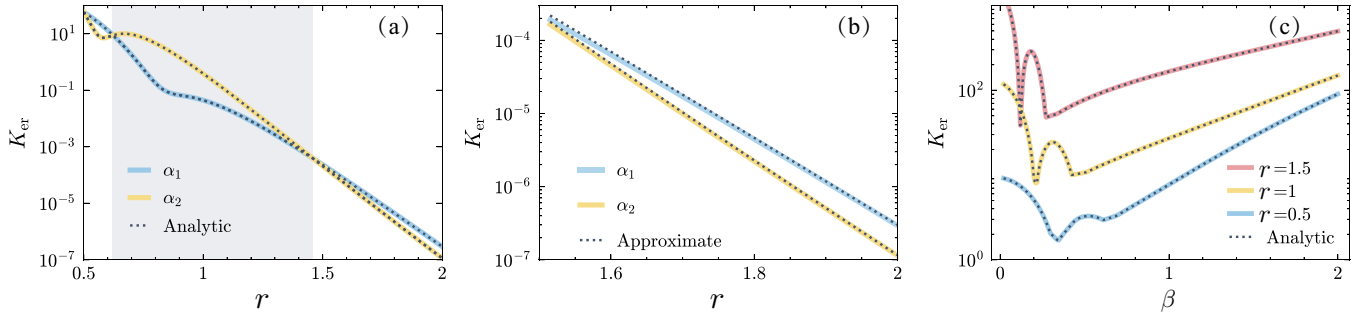


FIG. S2. (a) KL deviation K_{er} between the two codewords corresponding to the two solutions of α , with numerical results shown as solid lines and exact analytical results as dotted lines. (b) Approximate KL deviation K_{er} from Eq. S7, obtained via series expansion (dotted lines) and numerical simulation (solid lines). (c) KL divergence of the squeezed cat code as a function of the amplitude β for various squeezing levels, under the error set $\{\hat{I}, \hat{a}, \hat{n}, \hat{n}^2\}$ that includes single-photon loss and dephasing.

We compare the numerical and analytical results in Fig. S2 (a), and find that they are in good agreement. The two solutions for the coefficient α obtained from the condition $\langle 0_L | 1_L \rangle = 0$ only have a minor effect on the overall QEC performance and do not alter the exponential scaling. This confirms that the choice of α does not affect the fundamental scaling behavior of the code. Furthermore, we evaluate K_{er} using the approximate series expansion given in Eq. S2 (b), which also shows excellent agreement with the exact results, thereby confirming the effectiveness of the approximation solution.

In contrast, the squeezed Fock code (S6) is not inherently orthogonal, with the off-diagonal KL terms scaling as $\langle 0_L | \hat{n}^m | 1_L \rangle \propto e^{-3r}$ [2]. As a result, achieving approximate orthogonality requires significantly larger squeezing. For instance, at 9.5 dB squeezing ($r \approx 1$), the overlap between the codewords of the optimal squeezed Fock code ($n = 1$) is $|\langle 1_L | 0_L \rangle| = \cosh(2r)^{-3/2} \approx 0.137$, whereas our proposed code is exactly orthogonal. This intrinsic orthogonality substantially relaxes the squeezing requirement while ensuring robust and efficient quantum error correction against photon loss and dephasing.

We next evaluate the nonvanishing KL terms for the squeezed cat code.

$$|0_L/1_L\rangle = \frac{1}{\mathcal{N}_{\pm}} (|\beta, r\rangle \pm |-\beta, r\rangle), \quad (\text{S8})$$

and assess its error-correction performance, where $|\beta, r\rangle$ is the squeezed coherent state and the normalization coefficients are $\mathcal{N}_{\pm} = \sqrt{2[1 \pm \exp(-2e^{2r}\beta^2)]}$. These codewords are mutually orthogonal. For $|\beta| > e^{-r}$ and as $r \rightarrow \infty$, the relation $\langle 0_L | \hat{A}_i^\dagger \hat{A}_j | 0_L \rangle = \langle 1_L | \hat{A}_i^\dagger \hat{A}_j | 1_L \rangle$ is approximately satisfied for the error set E . However, the error basis is not orthogonal, i.e., $\langle 1_L | \hat{a} \hat{n}^m | 0_L \rangle$ and $\langle 1_L | \hat{a}^\dagger \hat{n}^m | 0_L \rangle$ do not vanish. The nonvanishing terms $\delta_{\hat{A}, \hat{A}^\dagger} = \langle 1_L | \hat{A} | 0_L \rangle$ and $\langle 1_L | \hat{A}^\dagger | 0_L \rangle$ are listed below.

$$\begin{aligned} \delta_{\hat{a}^\dagger, \hat{a}} &= \frac{\beta (1 \mp e^{-2e^{2r}\beta^2 + 2r})}{\sqrt{1 - e^{-4e^{2r}\beta^2}}}, \quad \delta_{\hat{a}^\dagger \hat{n}, \hat{n} \hat{a}} = \frac{3\beta e^{-2r} + \beta e^{2r} + 4\beta (\beta^2 - 1) \pm \beta e^{-2e^{2r}\beta^2} (4e^{6r}\beta^2 - 1 + 4e^{2r} - 3e^{4r})}{4\sqrt{1 - e^{-4e^{2r}\beta^2}}}, \\ \delta_{\hat{a}^\dagger \hat{n}^2, \hat{n}^2 \hat{a}} &= \frac{1}{16\sqrt{1 - e^{-4e^{2r}\beta^2}}} \left[15\beta e^{-4r} + 3\beta e^{4r} + 8(\beta^2 - 1)\beta e^{2r} + 8(5\beta^2 - 3)\beta e^{-4r} + 2(8\beta^4 - 16\beta^2 + 7)\beta \right. \\ &\quad \left. \pm \beta e^{-2e^{2r}\beta^2} (-16e^{10r}\beta^4 + 40e^{8r}\beta^2 + 8e^{4r}(\beta^2 + 3) - e^{6r}(32\beta^2 + 15) - 3e^{-2r} + 8 - 14e^{2r}) \right], \end{aligned} \quad (\text{S9})$$

where the $\pm$ and $\mp$ subscripts distinguish between $\hat{A}$ and $\hat{A}^\dagger$, respectively. From the above equations, we find that $\delta_{\hat{a}^\dagger, \hat{a}} \rightarrow \beta$ as $r \rightarrow \infty$ and $|\beta| > e^{-r}$, while the other terms cannot simultaneously approach zero. Therefore, the squeezed cat state cannot correct error sets that involve both $\hat{a}$ and $\hat{n}^m$ with $m > 0$. Since the error sets for single-photon loss and dephasing channels are $[\hat{I}, \hat{a}, \hat{n}]$ and $[\hat{I}, \hat{n}, \hat{n}^2]$, respectively, the squeezed cat code can correct dephasing errors but exhibits degraded performance under single-photon loss, as the corresponding terms in Eq. (S9) do not vanish. We also include both the numerical and analytical results for K_{er} of the squeezed cat codes as a function of the parameter β , for squeezing levels $r = 0.5, 1$, and 1.5 , under the error set E , as shown in Fig. S2(c). The excellent agreement between the two confirms the validity of our analytical solutions and further indicates that the squeezed cat code performs poorly under the error set E . We present the error correction performance of these three codes in Table S1.

	Full Orthogonal	Single-photon loss	Dephasing	Single-photon loss & Dephasing
Our code	✓	✓✓ ($\propto e^{-7r}$)	✓✓ ($\propto e^{-7r}$)	✓✓ ($\propto e^{-7r}$)
Squeezed cat code	✗	✗	✓	✗
Squeezed Fock code	✗	✓ ($\propto e^{-3r}$)	✓ ($\propto e^{-3r}$)	✓ ($\propto e^{-3r}$)

TABLE S1. Comparison of different squeezed-bosonic codes for squeezing $r \rightarrow \infty$.

II CONSTRUCTION OF LOGICAL PAULI OPERATORS

Owing to the rotational relation of the encoded logical states $|0_L\rangle$ and $|1_L\rangle$, a logical bit-flip operation can be implemented via a $\pi/2$ rotation generated by $\hat{X}_L = \exp(-i\frac{\pi}{2}\hat{n})$. In contrast, implementing the logical phase-flip operator is less straightforward and is therefore obtained via numerical optimization. Given the continuous nature of the encoding, we adopt an ansatz inspired by the finite-energy Gottesman-Kitaev-Preskill (GKP) logical Pauli operators. Specifically, the logical Pauli- Z operator is realized as

$$\hat{Z}_L = \exp\left[-i\hat{H}_z(\hat{a}, \hat{a}^\dagger)\right], \quad (\text{S10})$$

where $\hat{H}_z$ is an operator parameterized in two alternative forms for numerical construction.

Motivated by the non-Hermitian nature of Pauli operators in the GKP code, we parameterize $\hat{H}_z$ as

$$\hat{H}_z(\hat{a}, \hat{a}^\dagger) = \sum_{k,l=0}^n \alpha_{kl} \frac{\hat{a}^{\dagger k} \hat{a}^l}{\|\hat{a}^{\dagger k} \hat{a}^l\|_2}, \quad (\text{S11})$$

where α_{kl} are complex variational parameters and $\|\cdot\|_2$ denotes the matrix 2-norm. Realizing an effective non-Hermitian logical Pauli- Z operator necessitates incorporating an auxiliary system for its construction.

Alternatively, we consider $\hat{H}_z(\hat{a}, \hat{a}^\dagger)$ in Hermitian form to enable a direct unitary gate acting on the encoded bosonic mode without requiring an auxiliary system. To this end, we employ a symmetrized ansatz:

$$\hat{H}_z(\hat{a}, \hat{a}^\dagger) = \sum_{k,l=0}^n \left(\alpha_{kl} \frac{\hat{a}^{\dagger k} \hat{a}^l}{\|\hat{a}^{\dagger k} \hat{a}^l\|_2} + \alpha_{kl}^* \frac{\hat{a}^k \hat{a}^{\dagger l}}{\|\hat{a}^{\dagger l} \hat{a}^k\|_2} \right), \quad (\text{S12})$$

which guarantees that $\hat{H}_z = \hat{H}_z^\dagger$ by construction.

In both approaches, the variational parameters α_{kl} are optimized numerically to approximate the desired logical Pauli- Z operator. To this end, we minimize the following loss function:

$$E = \sum_{u=0,1} \left(\left| \langle u_L | \hat{Z}_L | u_L \rangle - (-1)^u \right|^2 + \left| \langle u_L | \hat{Z}_L^\dagger | u_L \rangle - (-1)^u \right|^2 + \left| \langle u_L | \hat{Z}_L^\dagger \hat{Z}_L | u_L \rangle - 1 \right|^2 \right), \quad (\text{S13})$$

where the first two terms enforce that $|u_L\rangle$ are eigenstates of the operator $\hat{Z}_L$ with eigenvalues ± 1 , while the last term ensures that $\hat{Z}_L$ acts unitarily on the logical states. This optimization is performed using the Adam algorithm implemented in the PyTorch Python package.

Numerical simulations reveal that constructing the logical Pauli- Z operator $\hat{Z}_L$ using a non-Hermitian generator requires only a small set of operator terms: $\{\hat{I}, \hat{a}^{\dagger 2}, \hat{a}^2, \hat{a}^\dagger \hat{a}, \hat{a}^{\dagger 2} \hat{a}^2\}$. This choice balances expressive power and computational efficiency, involving only five complex variational parameters. Despite its compactness, the ansatz achieves a high-fidelity approximation of the logical gate, with a loss function below 10^{-4} . As a representative example at 8 dB squeezing, the optimized coefficients are found to be

$$\alpha \approx [1.5741 - 0.1206i, \quad 116.2624 - 0.1807i, \quad -53.0023 + 0.1887i, \quad -0.3235 + 19.8160i, \quad 5.9651 - 433.85i]. \quad (\text{S14})$$

In contrast, enforcing Hermiticity to ensure a unitary representation of $\hat{Z}_L$ requires a larger operator basis. For comparable accuracy ($E < 10^{-4}$), we adopt an ansatz with $n = 6$ symmetrized terms. The corresponding optimized parameters for the Hermitian case at 8 dB squeezing are:

$$\alpha \approx \text{Re}(\alpha) + i \text{Im}(\alpha), \quad (\text{S15})$$

where

$$\text{Re}(\alpha) = \begin{bmatrix} 0.0 & 6.7115 & 0.29561 & 240.13 & 0.09506 & 9956.3 \\ 6.6332 & 0.0414 & -360.64 & 0.02024 & 868.27 & 0.20038 \\ -0.2944 & -360.69 & -0.77259 & -11082 & -0.9056 & 118890 \\ 240.35 & 0.02024 & -11082 & 0.19964 & -12945 & 0.0 \\ -0.01497 & 868.11 & -0.9056 & -12944 & 0.0 & 0.0 \\ 9955.9 & 0.20038 & 11889 & 0.0 & 0.0 & 0.0 \end{bmatrix}, \quad (\text{S16})$$

$$\text{Im}(\alpha) = \begin{bmatrix} 0 & 0.22634 & 0.0 & 0.77684 & 0 & 0.43069 \\ 0.22633 & 0.0 & 0.30752 & 0.0 & -0.00143 & 0.0 \\ 0 & 0.31577 & 0 & 0.03513 & 0 & -0.78899 \\ 0.78078 & 0 & 0.03969 & 0.0 & -1.4467 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.3474 & 0.0 & 0.0 \\ 0.26227 & 0.0 & 0.69542 & 0.0 & 0.0 & 0.0 \end{bmatrix}. \quad (\text{S17})$$

III QUANTUM ERROR CORRECTION APPROACHES

We develop error-correction schemes based on the codewords Eq. (S1). These schemes encompass measurement-based error correction, and autonomous error correction techniques. To facilitate the design of recovery unitary operators, we reformulate Eq. (S4) as follows:

$$\hat{\rho}(\tau) = \sum_i \hat{F}_i \hat{\rho} \hat{F}_i^\dagger, \quad \text{where} \quad \hat{F}_i = \sum_{k=1}^2 V_{ki} \hat{A}_k, \quad \text{and} \quad \hat{F}_3 = \hat{a}. \quad (\text{S18})$$

Here, we have diagonalized the symmetric matrix $\mathbf{J} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^\dagger$ with matrix elements $J_{ij} = \langle u_L | \hat{A}_i^\dagger \hat{A}_j | u_L \rangle$. Consequently, we can rewrite the approximate KL condition as

$$\hat{P}_L \hat{F}_i^\dagger \hat{F}_j \hat{P}_L \approx \Lambda_{ij} \delta_{ij} \hat{P}_L. \quad (\text{S19})$$

where $\hat{P}_L = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$ is the projector operator of the code space [3].

Here, we design two physical models to implement quantum error correction based on the encoding method. The first model involves coupling the encoded mode with a three-level atom, while the second model couples the encoded mode to two two-level atoms for error correction. We design the autonomous quantum error correction and the parity-measurement-based quantum error correction as shown in Fig. S3(a). We develop three unitary operations $\hat{U}_i$ to recover the errors $\hat{F}_i$, respectively, where the encoding mode couples to an auxiliary system which exhibits at least three non-degenerate energy levels $|g\rangle$, $|e\rangle$, and $|f\rangle$. This ancilla system is coupled to an auxiliary reservoir, which restores the auxiliary qutrit in its ground state $|g\rangle$. Moreover, we also engineer a parity-based measurement quantum error correction scheme in Fig. S3 (b). It should be noticed that we can realize the quantum gate with different systems.

A Ancilla qutrit system

Here, we consider a bosonic mode coupled with a qutrit auxiliary system. When the system is error-free, the qutrit is in its ground state $|g\rangle$. Upon occurrence of an error represented by the operator $\hat{F}_i$, the qutrit system can be excited to the states $|e\rangle$, $|f\rangle$, or remains in the ground state $|g\rangle$. This allows for a clear distinction between different types of

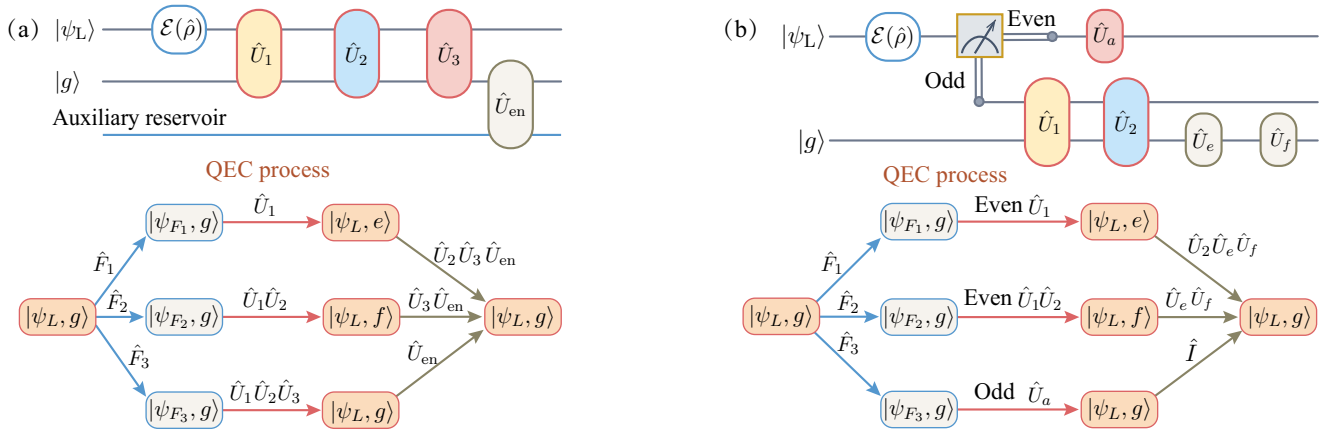


FIG. S3. (a) Schematic of autonomous quantum error correction. The protocol employs a sequence of unitary operations $\hat{U}_i$, each designed to address errors associated with the operators $\hat{F}_i$ ($i = 1, 2, 3$). The errors $\hat{F}_1$ and $\hat{F}_2$ are coherently mapped to the excited states $|e\rangle$ and $|f\rangle$ of an ancilla (respectively), while for a parity error $\hat{F}_3$ the ancilla remains in its ground state $|g\rangle$, thereby enabling selective and efficient correction. An auxiliary reservoir, such as a fast-decaying bosonic mode or a qudit, mediates transitions by transferring the states $|\psi_L, e\rangle$ and $|\psi_L, f\rangle$ to $|\psi_L, g\rangle$ via the unitary $\hat{U}_{\text{en}}$. Each operation $\hat{U}_i$ operates exclusively on its designated error subspace. (b) Quantum error correction based on parity measurement. If an even parity is detected, indicating the occurrence of an error $\hat{a}$, the unitary $\hat{U}_a = \hat{L}_3^\dagger + \hat{L}_3 + \hat{I} - \hat{P}_L - \hat{P}_{F_3}$ is applied to restore the state. For odd parity detection, unitaries $\hat{U}_1$ and $\hat{U}_2$ are employed to recover the encoded states. Additionally, the gates $\hat{U}_e$ and $\hat{U}_f$ facilitate transitions from the excited states $|e\rangle$ and $|f\rangle$ to the ground state $|g\rangle$. Here, $|\psi_L\rangle$ represents the encoded state, while $|\psi_{F_i}\rangle$ denotes the error state in the i -th error space.

errors. We can design the following unitary operators to correct the errors

$$\begin{aligned}\hat{U}_1 &= \hat{L}_1 |e\rangle\langle g| + \hat{L}_1^\dagger |g\rangle\langle e| + (I - \hat{P}_L) |e\rangle\langle e| + (I - \hat{P}_{F_1}) |g\rangle\langle g| + |f\rangle\langle f|, \\ \hat{U}_2 &= \hat{L}_2 |f\rangle\langle g| + \hat{L}_2^\dagger |g\rangle\langle f| + (I - \hat{P}_L) |f\rangle\langle f| + (I - \hat{P}_{F_2}) |g\rangle\langle g| + |e\rangle\langle e|, \\ \hat{U}_3 &= (\hat{L}_3^\dagger + \hat{L}_3 + \hat{I} - \hat{P}_L - \hat{P}_{F_3}) |g\rangle\langle g| + |e\rangle\langle e| + |f\rangle\langle f|,\end{aligned}\tag{S20}$$

where we have introduced the projector operators on the different error spaces $\hat{P}_{F_i} = |0_{F_i}\rangle\langle 0_{F_i}| + |1_{F_i}\rangle\langle 1_{F_i}|$ ($i = 1, 2, 3$) and the error correction operators $\hat{L}_i = |0_L\rangle\langle 0_{F_i}| + |1_L\rangle\langle 1_{F_i}|$. Here, the unitaries $\hat{U}_i$ are applied sequentially, with each of them correcting only the i -th error without influencing other error subspaces. As a result of these operations, errors are transferred to an auxiliary qutrit: after correcting $\hat{F}_1$, $\hat{F}_2$, and $\hat{F}_3$, the auxiliary qutrit is left in the states $|e\rangle$, $|f\rangle$, and $|g\rangle$, respectively. This allows the identification of the error type through a measurement of the auxiliary qutrit's state, followed by resetting the qutrit to the ground state $|g\rangle$.

While this scheme constitutes a semi-autonomous error correction process—owing to the measurement step—a fully autonomous approach can also be implemented. In this case, measurements are avoided, and after error correction, the auxiliary qutrit is coupled to a highly dissipative auxiliary reservoir: a qubit or bosonic mode, as depicted in Fig. S3(a). The quantum gate $\hat{U}_{\text{en}}$ resets the auxiliary qutrit to the ground state $|g\rangle$, with flexibility in its implementation depending on system requirements. For example, we can design the quantum unitary operation $\hat{U}_{\text{en}}$ as

$$\hat{U}_{\text{en}} = \exp \left\{ -i\varsigma \left[(|g\rangle\langle e| + |g\rangle\langle f|) \hat{b}^\dagger + (|e\rangle\langle g| + |f\rangle\langle g|) \hat{b} \right] t \right\},\tag{S21}$$

where $\hat{b}$ describes the auxiliary system with a decay rate κ_b much larger than the coupling strength and the qutrit decay rate $\kappa_b \gg \varsigma \gg \gamma$. Therefore, the entire error correction process can be described as follows

$$\mathcal{R} \circ \mathcal{E}[\hat{\rho}(t)] = \hat{U}_{\text{en}} \hat{U}_3 \hat{U}_2 \hat{U}_1 \mathcal{E}[\hat{\rho}(t)] \otimes |g\rangle\langle g| \hat{U}_1^\dagger \hat{U}_2^\dagger \hat{U}_3^\dagger \hat{U}_{\text{en}}^\dagger \approx \hat{\rho}(t) \otimes |g\rangle\langle g|,\tag{S22}$$

where we have assumed that the initial state of the qutrit is the ground state $|g\rangle\langle g|$.

In the parity-measurement-based QEC scheme, the encoded space lies in the odd-parity Fock subspace, and only the error operator $\hat{a}$ causes a parity change. Therefore, an alternatively parity measurement can be employed: if an

even parity is detected, the corrective operation,

$$\hat{U}_a = |0_L\rangle\langle 0_{F_3}| + |1_L\rangle\langle 1_{F_3}| + |0_{F_3}\rangle\langle 0_L| + |1_{F_3}\rangle\langle 1_L| + \hat{I} - \hat{P}_L - \hat{P}_{F_3}, \quad (\text{S23})$$

is applied; otherwise, the sequence $\hat{U}_1\hat{U}_2\hat{U}_e\hat{U}_f$ is implemented. This process results in the final quantum state $\hat{\rho} \otimes |g\rangle\langle g|$, as shown in Fig. S3(b). Here, $\hat{U}_e$ and $\hat{U}_f$ restore the auxiliary system to its ground state, with their explicit forms defined as follows:

$$\hat{U}_e = |e\rangle\langle g| + |g\rangle\langle e| + |f\rangle\langle f|, \quad \hat{U}_f = |f\rangle\langle g| + |g\rangle\langle f| + |e\rangle\langle e|. \quad (\text{S24})$$

Therefore, we can use the following equation to describe the quantum error correction process

$$\mathcal{R}_m \circ \mathcal{E}[\hat{\rho}(t)] = \hat{U}_a \hat{F}_3 \hat{\rho}(t) \hat{F}_3^\dagger \hat{U}_a^\dagger + \sum_{i=1}^2 \hat{U}_f \hat{U}_e \hat{U}_2 \hat{U}_1 \hat{F}_i \hat{\rho}(t) \otimes |g\rangle\langle g| \hat{F}_i^\dagger \hat{U}_1^\dagger \hat{U}_2^\dagger \hat{U}_e^\dagger \hat{U}_f^\dagger \approx \hat{\rho} \otimes |g\rangle\langle g|. \quad (\text{S25})$$

B Two ancilla qubit systems

The auxiliary system can also be designed by two two-level systems. To extend the scheme to a two-qubit system, it suffices to replace the qutrit states for the joint two-qubit states: $|g\rangle \rightarrow |g_1, g_2\rangle$, $|e\rangle \rightarrow |e_1, g_2\rangle$, and $|f\rangle \rightarrow |g_1, e_2\rangle$, from which the QEC protocol follows directly. Here, we analyze the implementation of autonomous error correction and measurement-based error correction using two auxiliary two-level systems. We only need to replace the quantum gates $\hat{U}_i$ with the following gates to achieve the desired functionality

$$\begin{aligned} \hat{U}_1 &= [\hat{L}_1 |e_1\rangle\langle g_1| + (\hat{I} - \hat{P}_{F_1}) |g_1\rangle\langle g_1| + \hat{L}_1^\dagger |g_1\rangle\langle e_1| + (\hat{I} - \hat{P}_L) |e_1\rangle\langle e_1|] |g_2\rangle\langle g_2| + |e_2\rangle\langle e_2|, \\ \hat{U}_2 &= [\hat{L}_2 |e_2\rangle\langle g_2| + (\hat{I} - \hat{P}_{F_2}) |g_2\rangle\langle g_2| + \hat{L}_2^\dagger |g_2\rangle\langle e_2| + (\hat{I} - \hat{P}_L) |e_2\rangle\langle e_2|] |g_1\rangle\langle g_1| + |e_1\rangle\langle e_1|, \\ \hat{U}_3 &= (\hat{L}_3 + \hat{L}_3^\dagger + \hat{I} - \hat{P}_L - \hat{P}_{F_3}) |e_1 g_2\rangle\langle g_1 g_2| + |e_1 e_2\rangle\langle e_1 e_2| + |g_1 e_2\rangle\langle g_1 e_2| + |g_1 g_2\rangle\langle e_1 g_2|, \end{aligned} \quad (\text{S26})$$

For the measurement-based quantum error correction, we only need to replace the quantum gates $\hat{U}_e$ and $\hat{U}_f$ by the Pauli operators $\hat{\sigma}_x$ of the two ancilla qubits and the design of the quantum gate $\hat{U}_{\text{en}}$ is similar with Eq. (S22). Therefore, we can achieve the quantum error correction with the two ancilla qubit model. By performing a parity measurement, we enable QEC. Specifically, if the even parity is measured, we apply the recovery operation $\hat{U}_a$ in Eq. (S23) to correct errors caused by $\hat{a}$. For odd parity outcomes, the errors are corrected using the operation $\hat{U}_2\hat{U}_1$. After the encoded information is successfully corrected, the ancilla qutrit states are transferred into distinct configurations. Finally, a quantum gate $\hat{U}_r$ is employed to reset the two ancilla qubits to their ground state $|g\rangle$. The entire parity-based QEC process can be formally represented as follows

$$\begin{aligned} \mathcal{R}_m \circ \mathcal{E}[\hat{\rho}_L] &\approx \sum_{i=1}^2 \hat{U}_r \hat{U}_2 \hat{U}_1 \hat{F}_i \hat{\rho}_L \otimes |g\rangle\langle g| \hat{F}_i^\dagger \hat{U}_1^\dagger \hat{U}_2^\dagger \hat{U}_r^\dagger \\ &\quad + \hat{U}_a \hat{F}_3 \hat{\rho}_L \otimes |g\rangle\langle g| \hat{F}_3^\dagger \hat{U}_a^\dagger \\ &\approx \hat{\rho}_L \otimes |g\rangle\langle g|. \end{aligned} \quad (\text{S27})$$

IV EXPERIMENTAL PROPOSAL FOR QUANTUM ERROR CORRECTION

We now propose a physical implementation of our quantum error correction scheme, where quantum gates are designed using optimal quantum control via the gradient ascent pulse engineering (GRAPE) method. Specifically, we consider a superconducting 3D cavity, hosting the bosonic encoding, dispersively coupled to an ancilla transmon qutrit enabling control of the bosonic mode. The Hamiltonian of the cavity-transmon system is expressed as

$$\hat{H} = \omega_s \hat{a}^\dagger \hat{a} + \omega_{ge} |e\rangle\langle e| + \omega_{gf} |f\rangle\langle f| - \chi_e |e\rangle\langle e| \hat{a}^\dagger \hat{a} - \chi_f |f\rangle\langle f| \hat{a}^\dagger \hat{a} + \Omega_d(t) \hat{a}^\dagger e^{-i\omega_d t} + \Omega_d^*(t) \hat{a} e^{i\omega_d t}, \quad (\text{S28})$$

where $|g\rangle, |e\rangle, |f\rangle$ denote the states of the ancilla transmon qutrit, $\hat{a}$ is the photon annihilation operator of the cavity with the frequency ω_s , and $\chi_{e/f}$ represent the dispersive interaction strengths. $\Omega_d(t)$ is the classical complex driving

amplitude of the cavity driving field with frequency ω_d . The transition frequencies for the transmon between the $g - e$ and $g - f$ states are ω_{ge} and ω_{gf} , respectively. This coupled cavity-transmon model has been extensively investigated and reliably demonstrated in recent experiments, supporting significant progress in quantum error correction [4], quantum gate implementation [5], photon blockade [6], and quantum control [7].

In the rotating frame of the drive and transmon transition frequencies, the time-dependent Hamiltonian becomes

$$\hat{H}_I = -\chi_e|e\rangle\langle e|\hat{a}^\dagger\hat{a} - \chi_f|f\rangle\langle f|\hat{a}^\dagger\hat{a} + \Omega_q(t)\hat{q} + \Omega_p(t)\hat{p}, \quad (\text{S29})$$

where $\omega_d \approx \omega_s$ is assumed. The complex classical driving amplitude is expressed as $\sqrt{2}\Omega_d(t) = \Omega_q(t) + i\Omega_p(t)$, with $\Omega_q(t)$ and $\Omega_p(t)$ representing the effective real drives amplitudes along the position and momentum quadratures, respectively. Through a sequence of carefully designed control pulses, all operations on the logical qubit are implemented based on the dispersive interaction between the ancilla and the oscillator.

Note that in our approach, control is applied only to the encoded mode. In principle, however, a variety of control Hamiltonians can be considered, including direct control of the qutrit and its interaction with the bosonic mode, which may further improve the precision. Since the method for optimizing the control fields remains unchanged, we present the encoded-mode control as a representative example. In theory, arbitrary quantum gates for this system can be realized via such optimal control. The dispersive interaction strengths χ_e and χ_f can be freely chosen within the range of current experimental capabilities.

For simplicity, our numerical simulations are performed in units of χ , with $\chi_e \approx \chi_f = 1$; however, this assumption is not essential for implementing the control scheme. During the GRAPE-based simulation, we neglect noise in both the bosonic mode and the qutrit, as the control Hamiltonian $\hat{H}_c$ describes a purely coherent control and not designed to suppress noise. In practical gate implementation, the influence of noise can be accounted for by incorporating it into the system dynamics.

We employ the QuTiP Python library [8–10] to numerically implement the GRAPE algorithm and present the optimized control pulses in Fig. S4. The resulting gate fidelity between the target unitary $\hat{U} = \hat{U}_3\hat{U}_2\hat{U}_1$ and the simulated evolution $\hat{U}_T$ at time T exceeds 0.99. A second transmon qubit can be incorporated in the model in order to realize the protocol described at the end of the previous section. Note that in this case both transmon qubits need to be truncated to their lower two levels. As the procedure closely mirrors that of the two ancilla qubit-based implementation, we omit further details here.

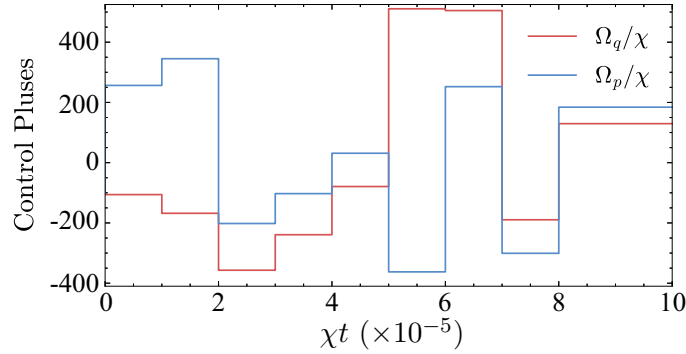


FIG. S4. Control pulses obtained from the GRAPE algorithm. The control field is discretized into 10 segments with a total evolution time of $T\chi = 10^{-4}$. Other choices of control Hamiltonians are also feasible and can achieve comparable gate performance.

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