Some results on spectra and certain norms

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Abstract

Given the norms of powers $(\|x^n\|)_{n\geq 0}$ of a Banach algebra element x, the largest possible value of the minimum modulus on the spectrum of x is determined. It is also shown that, given a Banach algebra element x and a compact set $K \subset \mathbb{C}$ with maximum modulus no more than the spectral radius of x, there exists a Banach algebra element y with $\|y^n\| = \|x^n\|$ for all $n \geq 0$ and spectrum equal to the union of the spectrum of x and K. These results, along with the spectral radius formula, are generalized to the joint spectrum of several commutative Banach algebra elements. The generalization of the spectral radius formula presented gives the maximum possible joint spectrum for commutative Banach algebra elements $x_1, \dots, x_n$, given the norms $(\|x_1^{i_1} \cdots x_n^{i_n}\|)_{i_1,\dots,i_n \geq 0}$.

1 Introduction

This paper contains results regarding subsets of $\mathbb{C}$ that are possible spectra for a Banach algebra element x, given the norms $(\|x^n\|)_{n\geq 0}$, and subsets of $\mathbb{C}^n$ that are possible joint spectra for n elements $x_1, \dots, x_n$ in a commutative Banach algebra, given the norms $(\|x_1^{i_1} \cdots x_n^{i_n}\|)_{i_1,\dots,i_n \geq 0}$. For a Banach algebra element x, we denote by $\sigma(x)$ the spectrum of x, i.e. the set of all complex numbers λ such that $\lambda \cdot 1 - x$ is not invertible. Before defining the joint spectrum, we introduce some notation. Call a linear functional ϕ on a complex algebra A multiplicative if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$.

Definition 1.1. Let A be a commutative Banach algebra. Denote by $\mathcal{M}_A$ the set of all nonzero multiplicative linear functionals on A.

With the Gelfand topology, the set $\mathcal{M}_A$ is the maximal ideal space of A (see [8, pp. 265, 268]).

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Definition 1.2. Let A be a commutative Banach algebra, and let $x_1, \ldots, x_n \in A$. The *joint spectrum* $\sigma(x_1, \ldots, x_n)$ of $x_1, \ldots, x_n$ is given by

$$\sigma(x_1, \ldots, x_n) = \{(\phi(x_1), \ldots, \phi(x_n)) \in \mathbb{C}^n : \phi \in \mathcal{M}_A\}.$$

It is well known that the spectrum of an element in a Banach algebra is a nonempty compact subset of $\mathbb{C}$ and that the joint spectrum of n elements in a commutative Banach algebra is a nonempty compact subset of $\mathbb{C}^n$. When $n = 1$, the joint spectrum is equal to the spectrum (see [2, Corollary 1.2.14]). Two necessary conditions for a sequence $(a_n)_{n \geq 0}$ of nonnegative numbers to be the sequence of norms of powers of an element in a Banach algebra are that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$. A fact, due to L. J. Wallen and found in [5, pp. 50, 234], is that these necessary conditions are also sufficient.

Lemma 1.3. *Let $(a_n)_{n \geq 0}$ be a sequence of nonnegative numbers. Then there exists an element x in some Banach algebra with $\|x^n\| = a_n$ for all $n \geq 0$ if and only if $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$.*

An analogue to Lemma 1.3 holds in several variables and is a special case of Lemma 5.2, to be proved in Section 5.

Lemma 1.4. *Let $(a_{i_1 \ldots i_n})_{i_1, \ldots, i_n \geq 0}$ be an indexed collection of nonnegative numbers. Then there exist elements $x_1, \ldots, x_n$ in some commutative Banach algebra with $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_{i_1 \ldots i_n}$ for all $i_1, \ldots, i_n \geq 0$ if and only if $a_{0 \ldots 0} = 1$ and $a_{i_1 + j_1 \ldots i_n + j_n} \leq a_{i_1 \ldots i_n} a_{j_1 \ldots j_n}$ for all $i_1, \ldots, i_n, j_1, \ldots, j_n \geq 0$.*

In saying that a certain compact set $K \subset \mathbb{C}$ is a possible spectrum for a sequence of norms of powers $(a_n)_{n \geq 0}$, we shall mean that $(a_n)_{n \geq 0}$ is a sequence of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$ and that there exists an element x in some Banach algebra such that $\|x^n\| = a_n$ for all $n \geq 0$ and $\sigma(x) = K$. An analogous expression will be used in several variables.

We denote by $r(x)$ the spectral radius of a Banach algebra element x, i.e. $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$. Recall the spectral radius formula, that $r(x) = \lim\limits_{n \to \infty} \|x^n\|^{1/n} = \inf\limits_{n > 0} \|x^n\|^{1/n}$ (see [8, Theorem 10.13]). Prior work has shown that certain sequences of norms of powers give more information about the spectrum than just the spectral radius. Letting Γ denote the unit circle, a theorem of Y. Katznelson and L. Tzafriri in [6] can be stated as follows. If x is an element in a Banach algebra with $\sup\limits_{n > 0} \|x^n\| < \infty$, then $\|x^n - x^{n+1}\| \to 0$ $(n \to \infty)$ if and only if $\sigma(x) \cap \Gamma \subset \{1\}$. (The theorem in [6] is stated for an operator on a Banach space, but is valid for any Banach algebra element. This follows from the fact that every Banach algebra is isometrically isomorphic to an algebra of operators on a Banach space (see [8, p. 230]).) It follows from the triangle inequality of normed vector spaces that if x is a Banach algebra element, then $\|x^n - x^{n+1}\| \geq \|x^n\| - \|x^{n+1}\|$ for all $n \geq 0$. It is then not hard to use the theorem to get examples of sequences of norms of powers giving a positive spectral radius that do not permit a spectrum consisting of a single point. (For one example, let $(a_n)_{n\geq 0}$ be the sequence given by $a_{2n} = 1$ and $a_{2n+1} = 2$ for all $n \geq 0$.) A generalization of this theorem was proved by G. Allan and T. Ransford in [1]. These results give relationships between the sequence of norms of powers of a Banach algebra element and the peripheral spectrum, i.e. the intersection of the spectrum with the circle centered at the origin of radius the spectral radius.

In this paper, rather than focus on the peripheral spectrum, we give results related to the minimum modulus on the spectrum. We begin by establishing the existence of a sequence of norms of powers giving positive spectral radius that forces the spectrum to contain the origin. In Theorem 2.13, we determine the smallest annulus centered at the origin that is a possible spectrum for a given sequence of norms of powers. As a special case, we provide a characterization of the sequences of norms of powers that permit spectrum the circle centered at the origin of radius the spectral radius. It is also shown that if a certain compact set is a possible spectrum for a sequence of norms of powers $(a_n)_{n\geq 0}$, then any larger compact set contained in $\overline{D}(0, \lim\limits_{n\to\infty} a_n^{1/n})$ is also a possible spectrum for $(a_n)_{n\geq 0}$, where $\overline{D}(0, r)$ denotes the closed disc centered at the origin of radius $r \geq 0$. We provide several variable analogues of the main results. The several variable results begin with a generalization of the spectral radius formula to joint spectra, which gives the maximum possible joint spectrum for commutative Banach algebra elements $x_1, \ldots, x_n$, given the norms $(\|x_1^{i_1} \cdots x_n^{i_n}\|)_{i_1,\ldots,i_n\geq 0}$. In particular, the smallest polynomially convex circled set containing the joint spectrum is determined. As Theorem 5.15, we generalize Theorem 2.13 to joint spectra, giving the intersection of all rationally convex connected circled sets that are possible joint spectra for a given collection of norms of products $(a_{i_1,\ldots,i_n})_{i_1,\ldots,i_n\geq 0}$.

2 One Variable Results

This section contains results on the spectrum of a single Banach algebra element. The main theorem is Theorem 2.13, which gives the largest possible value of the minimum modulus on $\sigma(x)$, given the norms of powers $(\|x^n\|)_{n\geq 0}$ of a Banach algebra element x. As Theorem 2.4, we show that there exists a sequence of norms of powers giving positive spectral radius that forces the spectrum to contain the origin. We begin with a lemma mentioned in the introduction, which shows that if a certain compact set is a possible spectrum for a sequence of norms of powers $(a_n)_{n\geq 0}$, then any larger compact set contained in $\overline{D}(0, \lim\limits_{n\to\infty} a_n^{1/n})$ is also a possible spectrum for $(a_n)_{n\geq 0}$. This is presented as Lemma 2.3.

Before presenting Lemma 2.3, we consider a few facts. If A and B are Banach algebras, then the direct sum of algebras $A \oplus B$ is a Banach algebra with the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$. The following elementary fact is not hard to verify.

Lemma 2.1. *If A and B are Banach algebras, $x \in A$, and $y \in B$, then $\sigma((x,y)) = \sigma(x) \cup \sigma(y)$, where (x,y) is considered as an element of $A \oplus B$.*

For a Banach algebra element x, we note the following alternate description of the disc $\overline{D}(0, r(x))$.

Proposition 2.2. *If x is an element in a Banach algebra, then*

$$\overline{D}(0, r(x)) = \{\lambda \in \mathbb{C} : |\lambda^n| \leq \|x^n\| \text{ for all } n \geq 0\}.$$

For a compact set $K \subset \mathbb{C}^n$, we denote by $C(K)$ the algebra of all continuous complex-valued functions on K with the supremum norm, $\|f\| = \sup\limits_{\lambda \in K} |f(\lambda)|$.

Lemma 2.3. *If x is an element in a Banach algebra and K is a compact subset of $\overline{D}(0, r(x))$, then there exist a Banach algebra B and an element y of B such that $\|y^n\| = \|x^n\|$ for all $n \geq 0$ and $\sigma(y) = \sigma(x) \cup K$.*

Proof. We assume that K is nonempty, since otherwise there is nothing to prove. Let A be the Banach algebra containing x. Let $B = A \oplus C(K)$, and let y be the element (x, z) of B, where z is the inclusion of K into $\mathbb{C}$. It follows from Proposition 2.2 that $\|z^n\| \leq \|x^n\|$ for all $n \geq 0$. Therefore $\|y^n\| = \|x^n\|$ for all $n \geq 0$. It is easy to see that $\sigma(z) = K$. Lemma 2.3 then follows from Lemma 2.1. $\square$

We now turn to Theorem 2.4.

Theorem 2.4. *There exists a sequence $(a_n)_{n \geq 0}$ of positive numbers with $a_0 = 1$, $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$, and $\lim\limits_{n \to \infty} a_n^{1/n} = 1$ such that $\sigma(x)$ must contain the origin whenever x is an element in a Banach algebra with $\|x^n\| = a_n$ for all $n \geq 0$.*

It is immediate from the definition of the spectrum that the spectrum of a Banach algebra element contains the origin if and only if the element is noninvertible. When x is an invertible Banach algebra element, the supremum $\sup\limits_{n \geq 1} \frac{\|x^{n-1}\|}{\|x^n\|}$ is no more than $\|x^{-1}\|$. Thus, if a sequence of positive norms of powers $(a_n)_{n \geq 0}$ has $\sup\limits_{n \geq 1} \frac{a_{n-1}}{a_n} = \infty$, then the spectrum of a Banach algebra element x with $\|x^n\| = a_n$ for all $n \geq 0$ must contain the origin. Theorem 2.4 follows from the next lemma, which establishes existence of such a sequence of norms of powers giving spectral radius 1.

Lemma 2.5. *There exists a sequence $(a_n)_{n \geq 0}$ of positive numbers with $a_0 = 1$, $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$, and $\lim\limits_{n \to \infty} a_n^{1/n} = 1$ such that $\sup\limits_{n \geq 1} \frac{a_{n-1}}{a_n} = \infty$.*

Proof. We define $(a_n)_{n \geq 0}$ recursively, as follows. Let $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$. For $n \geq 3$, given $a_1, \ldots, a_{n-1}$, define

$$a_n = \begin{cases} \min\{a_i a_j : i, j \geq 1 \text{ and } i + j = n\}, & \text{if } a_{n-1} \leq \max\{a_1, \ldots, a_{n-2}\} \\ 2, & \text{otherwise.} \end{cases}$$

We first note that

$$a_n \text{ is a positive power of 2 for all } n \geq 1. \tag{1}$$

It is easy to see that $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$. Before turning to the remaining assertions of the lemma, we show that (a_n) is unbounded. Assume, to get a contradiction, that (a_n) is bounded. Then by (1), the sequence takes on only finitely many values. We show that

$$\text{for each } k \geq 1, \text{ we have } a_n = 2^k \text{ for only finitely many } n, \tag{2}$$

which will give a contradiction. Since (a_n) takes on only finitely many values, we have $a_{n-1} > \max\{a_1, \ldots, a_{n-2}\}$ for only finitely many n. Consequently, only finitely many terms of (a_n) are equal to 2. Given $k \geq 1$, if $a_n \leq 2^k$ for only finitely many n, then $a_n \leq 2^{k+1}$ for only finitely many n, for if for some N we have $a_n > 2^k$ for all $n > N$, then $a_n > 2^{k+1}$ for all $n > 2N$. Statement (2) follows, and we have arrived at a contradiction. Therefore (a_n) is unbounded.

Since (a_n) is unbounded, we have $a_n = 2$ for infinitely many n. Therefore $\lim\limits_{n\to\infty} a_n^{1/n} = 1$. It remains to see that $\sup\limits_{n\geq 1} \frac{a_{n-1}}{a_n} = \infty$. Noting that $a_{n+1} \leq 2a_n$ for all $n \geq 0$, every nonnegative power of 2 is achieved in the sequence. Given $j \geq 1$, if a_k is the first instance of 2^{j+1}, then $a_{k+1} = 2$, and we have

$$\sup_{n\geq 1} \frac{a_{n-1}}{a_n} \geq \frac{a_k}{a_{k+1}} = 2^j. \qquad \square$$

Remark 2.6. The proof of Lemma 2.5 remains valid if powers of 2 are replaced by powers of R, for any $R > 1$.

The rest of this section is dedicated to proving Theorem 2.13. Several relevant results are presented first. Theorem 2.13 concerns the smallest annulus centered at the origin containing the spectrum. This annulus is necessarily closed due to the compactness of the spectrum. We use an extended definition of an annulus and regard a closed disc as a closed annulus with inner radius 0. We first state the form of the smallest annulus centered at the origin containing the spectrum when the norms of all powers, including negative powers, if present, are known. When x is an invertible Banach algebra element, this annulus has inner radius $\frac{1}{r(x^{-1})}$ by the spectral mapping theorem (see [8, Theorem 10.28]).

Lemma 2.7. *If x is an element in a Banach algebra, then the smallest annulus centered at the origin containing $\sigma(x)$ is*

$$\begin{cases} \{\lambda \in \mathbb{C} : |\lambda| \leq \inf\limits_{k>0} \|x^k\|^{1/k}\}, & \text{if } x \text{ is noninvertible} \\ \{\lambda \in \mathbb{C} : \frac{1}{\inf\limits_{k>0} \|x^{-k}\|^{1/k}} \leq |\lambda| \leq \inf\limits_{k>0} \|x^k\|^{1/k}\}, & \text{if } x \text{ is invertible.} \end{cases}$$

A theorem in [9] asserts that if T is a bounded unilateral weighted shift or an invertible bounded bilateral weighted shift, then T has spectrum equal to the annulus from Lemma 2.7. We use this to prove the next lemma. Let $\mathbb{N}_0$ denote the set of all nonnegative integers. By a unilateral weighted shift, we mean an operator T on $\ell^2(\mathbb{N}_0)$ of the form $T(e_n) = \omega_n e_{n+1}$ for $n \geq 0$, where $(e_n)_{n\geq 0}$ is the standard basis for $\ell^2(\mathbb{N}_0)$ and $(\omega_n)_{n\geq 0}$ is a sequence of complex numbers. A unilateral weighted shift is necessarily noninvertible. By a bilateral weighted shift, we mean an operator T on $\ell^2(\mathbb{Z})$ of the form $T(e_n) = \omega_n e_{n+1}$ for $n \in \mathbb{Z}$, where $(e_n)_{n\in\mathbb{Z}}$ is the standard basis for $\ell^2(\mathbb{Z})$ and $(\omega_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers. Part *(i)* of the next lemma is Wallen's result (Lemma 1.3) with an addendum.

Lemma 2.8. *(i) Let $(a_n)_{n\geq 0}$ be a sequence of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$. Then there exists an element x in some Banach algebra such that $\|x^n\| = a_n$ for all $n \geq 0$ and $\sigma(x) = \{\lambda \in \mathbb{C} : |\lambda| \leq \inf\limits_{k>0} a_k^{1/k}\}$.*

(ii) Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of positive numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \in \mathbb{Z}$. Then there exists an invertible element x in some Banach algebra such that $\|x^n\| = a_n$ for all $n \in \mathbb{Z}$ and $\sigma(x) = \{\lambda \in \mathbb{C} : \frac{1}{\inf_{k>0} a_{-k}^{1/k}} \leq |\lambda| \leq \inf\limits_{k>0} a_k^{1/k}\}$.

Implicit in *(ii)* is that $\inf\limits_{k>0} a_{-k}^{1/k} \neq 0$. The proof of Lemma 1.3 given in [5, pp. 50, 234] uses a bounded unilateral weighted shift, and thus provides a proof of *(i)*. The proof we provide of *(ii)* is a natural modification of this proof using a bilateral weighted shift.

Proof. To see *(ii)*, suppose $(a_n)_{n\in\mathbb{Z}}$ is a sequence of positive numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \in \mathbb{Z}$. Let T be the bilateral weighted shift (weighted forward shift on $\ell^2(\mathbb{Z})$) given by

$$T(e_n) = \frac{a_{n+1}}{a_n} e_{n+1} \text{ for all } n \in \mathbb{Z}.$$

The inequality $\frac{a_{n+1}}{a_n} \leq a_1$ holds for all $n \in \mathbb{Z}$, and it follows that T is bounded, with $\|T\| \leq a_1$. (In fact, $\|T\| = a_1$ since $T(e_0) = a_1 e_1$.) The operator T has inverse the weighted backward shift on $\ell^2(\mathbb{Z})$ given by

$$T^{-1}(e_n) = \frac{a_{n-1}}{a_n} e_{n-1} \text{ for all } n \in \mathbb{Z}.$$

We have

$$T^k(e_n) = \frac{a_{n+k}}{a_n} e_{n+k} \text{ for all } k, n \in \mathbb{Z},$$

and it follows that $\|T^n\| = a_n$ for all $n \in \mathbb{Z}$. By the theorem in [9] mentioned in the paragraph preceding the statement of the current lemma, we have $\sigma(T) = \{\lambda \in \mathbb{C} : \frac{1}{\inf\limits_{k>0} a_{-k}^{1/k}} \le |\lambda| \le \inf\limits_{k>0} a_k^{1/k}\}$. Therefore, T satisfies the properties in *(ii)*. $\square$

We now give a definition to be used in the rest of this section. Given an indexed collection of positive numbers $(a_n)_{n\ge0}$ such that $a_0 = 1$ and $a_{i+j} \le a_i a_j$ for all $i, j \ge 0$, define, for each $i \in \mathbb{Z}$,

$$C_i = \sup_{\substack{n\ge0 \\ n+i\ge0}} \frac{a_{n+i}}{a_n}.$$

Each C_i lies in $(0, \infty]$. Here are two elementary properties of $(C_i)_{i\in\mathbb{Z}}$. Proposition 2.10 is used in the remainder of this section without mention.

Proposition 2.9. *The equality $C_i = a_i$ holds for all $i \ge 0$.*

Proof. Let $i \ge 0$. Since the inequality $a_{n+i} \le a_n a_i$ holds for all $n \ge 0$, we have $\frac{a_{n+i}}{a_n} \le a_i$ for all $n \ge 0$. It follows that $C_i \le a_i$. But also, we have $a_i = \frac{a_{0+i}}{a_0} \le C_i$. $\square$

Proposition 2.10. *If C_{-1} is finite, then C_i is finite for all $i \in \mathbb{Z}$.*

Proof. If $i < 0$, then for all $n \ge 0$ such that $n + i \ge 0$, we have

$$\frac{a_{n+i}}{a_n} = \frac{a_{n+i}}{a_{n+i+1}} \cdots \frac{a_{n-1}}{a_n} \le C_{-1}^{-i}.$$ $\square$

The next two lemmas are used to prove Theorem 2.13. The first is a simple consequence of the definition of $(C_i)_{i\in\mathbb{Z}}$.

Lemma 2.11. *Let $(a_n)_{n\ge0}$ be a sequence of positive numbers such that $a_0 = 1$ and $a_{i+j} \le a_i a_j$ for all $i, j \ge 0$. Suppose x is an invertible Banach algebra element with $\|x^n\| = a_n$ for all $n \ge 0$. Then $C_k \le \|x^k\|$ for all $k \in \mathbb{Z}$.*

Lemma 2.12. *Let $(a_n)_{n\ge0}$ be a sequence of positive numbers such that $a_0 = 1$ and $a_{i+j} \le a_i a_j$ for all $i, j \ge 0$. Then $C_0 = 1$, and if C_{-1} is finite, then $C_{i+j} \le C_i C_j$ for all $i, j \in \mathbb{Z}$.*

Proof. That $C_0 = 1$ follows from Proposition 2.9. Suppose C_{-1} is finite. Let $i, j \in \mathbb{Z}$, and let $n \ge 0$ such that $n + i + j \ge 0$. If $i \ge 0$, then $n + i \ge 0$, and we have

$$\frac{a_{n+i+j}}{a_n} = \frac{a_{n+i+j}}{a_{n+i}} \cdot \frac{a_{n+i}}{a_n} \le C_j C_i.$$

If $i < 0$, then $n + j > 0$, and we have

$$\frac{a_{n+i+j}}{a_n} = \frac{a_{n+i+j}}{a_{n+j}} \cdot \frac{a_{n+j}}{a_n} \le C_i C_j.$$ $\square$

We now present the main result of this section.

Theorem 2.13. *Let $(a_n)_{n\geq 0}$ be a sequence of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$.*

(i) Suppose $a_k = 0$ for some $k > 0$. If x is an element in a Banach algebra with $\|x^n\| = a_n$ for all $n \geq 0$, then $\sigma(x) = \{0\}$. Furthermore, there exists an element y in some Banach algebra such that $\|y^n\| = a_n$ for all $n \geq 0$.

(ii) Suppose each a_n is positive. Let K be the annulus given by

$$K = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| \leq \inf\limits_{k>0} C_k^{1/k}\}, \text{ if } C_{-1} = \infty \\ \{\lambda \in \mathbb{C} : \frac{1}{\inf\limits_{k>0} C_{-k}^{1/k}} \leq |\lambda| \leq \inf\limits_{k>0} C_k^{1/k}\}, \text{ if } C_{-1} \text{ is finite.} \end{cases}$$

If x is an element in a Banach algebra with $\|x^n\| = a_n$ for all $n \geq 0$, then the smallest annulus centered at the origin containing $\sigma(x)$ contains K. Furthermore, there exists an element y in some Banach algebra such that $\|y^n\| = a_n$ for all $n \geq 0$ and $\sigma(y) = K$.

Proof. To see *(i)*, suppose $a_k = 0$ for some $k > 0$. Then $a_n = 0$ for all $n \geq k$, and thus $r(x) = 0$ whenever x is an element in a Banach algebra with $\|x^n\| = a_n$ for all $n \geq 0$. Existence of an element y in some Banach algebra such that $\|y^n\| = a_n$ for all $n \geq 0$ is given by *(i)* of Lemma 2.8.

To see *(ii)*, suppose each a_n is positive. We first show that there exists an element y in some Banach algebra with $\|y^n\| = a_n$ for all $n \geq 0$ and $\sigma(y) = K$. If $C_{-1} = \infty$, then existence of an element y in some Banach algebra such that $\|y^n\| = a_n$ for all $n \geq 0$ and $\sigma(y) = K$ is given by *(i)* of Lemma 2.8 and Proposition 2.9. If C_{-1} is finite, then it follows from Lemma 2.12 and *(ii)* of Lemma 2.8 that there exists an invertible element y in some Banach algebra such that $\|y^n\| = C_n$ for all $n \in \mathbb{Z}$ and $\sigma(y) = K$. For this y, we have $\|y^n\| = a_n$ for all $n \geq 0$ by Proposition 2.9. To see the remaining assertion of *(ii)*, suppose x is an element in a Banach algebra with $\|x^n\| = a_n$ for all $n \geq 0$. If x is noninvertible, then by Lemma 2.7 and Proposition 2.9, the smallest annulus centered at the origin containing $\sigma(x)$ is $\{\lambda \in \mathbb{C} : |\lambda| \leq \inf\limits_{k>0} C_k^{1/k}\}$. It is clear that this annulus contains K. Suppose x is invertible. By Lemma 2.11, we have $C_{-1} \leq \|x^{-1}\|$, and thus $K = \{\lambda \in \mathbb{C} : \frac{1}{\inf\limits_{k>0} C_{-k}^{1/k}} \leq |\lambda| \leq \inf\limits_{k>0} C_k^{1/k}\}$. By Proposition 2.9, we have $\|x^k\| = C_k$ for all $k > 0$, and it follows from Lemma 2.11 that $\frac{1}{\inf\limits_{k>0} \|x^{-k}\|^{1/k}} \leq \frac{1}{\inf\limits_{k>0} C_{-k}^{1/k}}$. It then follows from Lemma 2.7 that the smallest annulus centered at the origin containing $\sigma(x)$ contains K. $\square$

Corollary 2.14. *Let $(a_n)_{n\geq 0}$ be a sequence of positive numbers such that $a_0 = 1$, $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$, and $\lim\limits_{n\to\infty} a_n^{1/n} = r > 0$. Then there exists an element*

x in some Banach algebra with $\|x^n\| = a_n$ for all $n \geq 0$ and spectrum equal to the circle $\{\lambda \in \mathbb{C} : |\lambda| = r\}$ if and only if C_{-1} is finite and $\inf\limits_{k>0} C_{-k}^{1/k} = \frac{1}{r}$.

Lemma 2.5 gives existence of a sequence of norms of powers giving spectral radius 1 such that $C_{-1} = \infty$. The next lemma gives, for each $0 < r \leq 1$, existence of a sequence of norms of powers giving spectral radius 1 such that C_{-1} is finite and $\frac{1}{\inf\limits_{k>0} C_{-k}^{1/k}} = r$. In conjunction with Lemma 2.5 (and noting Proposition 2.9), this shows that for each $0 \leq r \leq 1$, the annulus $\{\lambda \in \mathbb{C} : r \leq |\lambda| \leq 1\}$ is a possible annulus K for *(ii)* of Theorem 2.13. It follows that every closed annulus centered at the origin is a possible annulus K for *(ii)* of Theorem 2.13. Indeed, if $r \geq 0$ and $R > 0$, and if $K = \{\lambda \in \mathbb{C} : r \leq |\lambda| \leq 1\}$ for a sequence $(a_n)_{n\geq 0}$, then we have $K = \{\lambda \in \mathbb{C} : Rr \leq |\lambda| \leq R\}$ for the sequence $(R^n a_n)_{n\geq 0}$.

Lemma 2.15. *For each $0 < r \leq 1$, there exists a sequence $(a_n)_{n\geq 0}$ of positive numbers with $a_0 = 1$, $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$, and $\lim\limits_{n\to\infty} a_n^{1/n} = 1$ such that C_{-1} is finite and $\frac{1}{\inf\limits_{k>0} C_{-k}^{1/k}} = r$.*

Proof. For $r = 1$, the constant sequence given by $a_n = 1$ for all $n \geq 0$ satisfies the conclusion of the lemma, for this sequence has $C_k = 1$ for all $k \in \mathbb{Z}$. Suppose $0 < r < 1$, and let $R = \frac{1}{r}$. We define $(a_n)_{n\geq 0}$ recursively, as follows. Let $a_0 = 1$, $a_1 = R$, and $a_2 = R^2$. For $n \geq 3$, given $a_1, \dots, a_{n-1}$, define

$$a_n = \begin{cases} R^{m-k} \text{ , if } a_{n-k} = R^m > \max\{a_1, \dots, a_{n-k-1}\} \text{ for some } m > 1 \text{ and } \\ \hfill 1 \leq k \leq m-1 \\ \min\{a_i a_j : i, j \geq 1 \text{ and } i + j = n\} \text{ , otherwise.} \end{cases}$$

We note that a_n is a positive power of R for all $n \geq 1$. A simple modification of the proof that the sequence in Lemma 2.5 is unbounded shows that (a_n) is unbounded. We have $a_{n+1} \leq Ra_n$ for all $n \geq 0$, and thus, every power of R is achieved in the sequence. For each $m > 1$, let n_m be the index of the first instance of R^m. We can then write, for $n \geq 3$,

$$a_n = \begin{cases} R^{m-k} \text{ , if } n = n_m + k \text{ for some } m > 1 \text{ and } 1 \leq k \leq m-1 \\ \min\{a_i a_j : i, j \geq 1 \text{ and } i + j = n\} \text{ , otherwise.} \end{cases}$$

To see that $a_{i+j} \leq a_i a_j$ for all $i, j \geq 0$ and that C_{-1} is finite, we prove by induction on $n \geq 1$ that

$$a_n \leq a_i a_j \text{ for all } n \geq 1 \text{ and } i, j \geq 0 \text{ such that } i + j = n, \text{ and} \tag{3}$$

$$\frac{a_{n-1}}{a_n} \leq R \text{ for all } n \geq 1. \tag{4}$$

It is clear that (3) and (4) are true when $n = 1$ or $n = 2$. Suppose $N \geq 3$ and that (3) and (4) hold when $1 \leq n \leq N-1$. We show that (3) and (4) hold when $n = N$. If $N = n_m + k$ for some $m > 1$ and $1 \leq k \leq m-1$, then by definition of (a_n), we have $\frac{a_{N-1}}{a_N} = R$. In this case, we also have, by the inductive assumptions, that if $i, j \geq 1$ such that $i + j = N$, then

$$a_N = \frac{1}{R} a_{i+j-1} \leq \frac{1}{R} a_{i-1} a_j \leq \frac{1}{R} \cdot R a_i a_j = a_i a_j.$$

If $a_N = \min\{a_i a_j : i, j \geq 1 \text{ and } i + j = N\}$, then it is clear that $a_N \leq a_i a_j$ for all $i, j \geq 0$ such that $i + j = N$. In this case, we have $a_N = a_k a_l$ for some $k, l \geq 1$ such that $k + l = N$. Using the inductive assumptions, we then have

$$a_{N-1} \leq a_{k-1} a_l \leq R a_k a_l = R a_N.$$

Therefore (3) and (4) are proved.

That $\lim\limits_{n\to\infty} a_n^{1/n} = 1$ follows from noting that $a_{n_m+m-1} = R$ for all $m > 1$. To see that $\frac{1}{\inf\limits_{k>0} C_{-k}^{1/k}} = r$, it suffices to show that $C_{-k} = R^k$ for all $k > 0$. It follows from (4) that $C_{-k} \leq R^k$ for all $k > 0$. But also, for all $k > 0$, we have

$$C_{-k} \geq \frac{a_{n_{k+1}}}{a_{n_{k+1}+k}} = \frac{R^{k+1}}{R} = R^k. \qquad \square$$

3 A Generalization of the Spectral Radius Formula

The remaining sections concern the joint spectrum of several commutative Banach algebra elements. For the remainder of the paper, when $x_1, \ldots, x_n$ are not explicitly defined, it is understood that $x_1, \ldots, x_n$ are elements in an arbitrary commutative Banach algebra. In this section, we present a generalization of the spectral radius formula to joint spectra. In [7], V. Müller determined, for $1 \leq p \leq \infty$, the radius of the smallest closed ℓ^p ball centered at the origin containing the joint spectrum of $x_1, \ldots, x_n$ from the norms $(\|x_1^{i_1} \cdots x_n^{i_n}\|)_{i_1,\ldots,i_n \geq 0}$. In this section, rather than find a radius, we determine the smallest polynomially convex circled set containing the joint spectrum. This is presented as Theorem 3.6. In Theorem 3.9, we show that this is the maximum possible joint spectrum for given norms of products. In Lemma 3.8, we generalize Lemma 2.3.

Before defining polynomial convexity and circled sets, we introduce some notation used in the remaining sections. For an element s of $\mathbb{C}^n$, we denote by s_k the k^{th} coordinate of s, for $1 \leq k \leq n$. Similarly, for an element i of $\mathbb{Z}^n$, we denote by i_k the k^{th} coordinate of i, for $1 \leq k \leq n$. We denote by S_0 the subset of $\mathbb{Z}^n$ of all elements with nonnegative coordinates, that is, $S_0 = \{i \in \mathbb{Z}^n : i_1, \ldots, i_n \geq 0\}$. For $1 \leq k \leq n$, we denote by ϵ_k the element of $\mathbb{Z}^n$ with k^{th} coordinate equal to 1 and all other coordinates equal to 0. Addition in $\mathbb{Z}^n$ is the usual coordinatewise addition.

Definition 3.1. Let K be a compact subset of $\mathbb{C}^n$. The *polynomial hull* of K is

$$\widehat{K} = \{s \in \mathbb{C}^n : |p(s)| \leq \|p\|_K \text{ for every polynomial } p\}.$$

A compact subset K of $\mathbb{C}^n$ is *polynomially convex* if $\widehat{K} = K$.

The polynomials in Definition 3.1 are polynomials in the complex coordinate functions $z_k : \mathbb{C}^n \to \mathbb{C}$, $1 \leq k \leq n$, where z_k is the projection on the k^{th} coordinate. The norm $\|p\|_K$ is the supremum norm of p over K, $\|p\|_K = \sup\limits_{s \in K} |p(s)|$.

Definition 3.2. A subset K of $\mathbb{C}^n$ is *circled* if, given $s \in K$ and complex numbers $\alpha_1, \ldots, \alpha_n$ of modulus 1, the element $(\alpha_1 s_1, \ldots, \alpha_n s_n)$ of $\mathbb{C}^n$ also lies in K.

It follows from the maximum modulus principle that the polynomially convex circled sets in $\mathbb{C}$ are the closed discs centered at the origin. Before presenting Theorem 3.6, we state a relevant theorem of K. deLeeuw. In general, when an element s of $\mathbb{C}^n$ does not lie in the polynomial hull of a compact set $K \subset \mathbb{C}^n$, there is a polynomial p such that $|p(s)| > \|p\|_K$. The next theorem, which can be found in [3], says that in the particular case when K is circled, this polynomial can be chosen to be a monomial.

Theorem 3.3. *Let $K \subset \mathbb{C}^n$ be a compact circled set. If $s \in \mathbb{C}^n \setminus \widehat{K}$, then there exist nonnegative integers $i_1, \ldots, i_n$ such that $|s_1^{i_1} \cdots s_n^{i_n}| > \|z_1^{i_1} \cdots z_n^{i_n}\|_K$.*

Corollary 3.4. *If K is a compact subset of $\mathbb{C}^n$, then the smallest polynomially convex circled subset of $\mathbb{C}^n$ containing K is*

$$\{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq \|z_1^{i_1} \cdots z_n^{i_n}\|_K \text{ for all } i \in S_0\}.$$

Proof. Let $K' = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq \|z_1^{i_1} \cdots z_n^{i_n}\|_K \text{ for all } i \in S_0\}$. It is clear that K' is a polynomially convex circled set containing K. Suppose $K'' \subset \mathbb{C}^n$ is a polynomially convex circled set containing K. If $s \in \mathbb{C}^n \setminus K''$, then by Theorem 3.3, for some $i \in S_0$, we have

$$|s_1^{i_1} \cdots s_n^{i_n}| > \|z_1^{i_1} \cdots z_n^{i_n}\|_{K''} \geq \|z_1^{i_1} \cdots z_n^{i_n}\|_K,$$

so that $s \notin K'$. Therefore $K' \subset K''$. $\square$

The next lemma will be used to prove Theorem 3.6.

Lemma 3.5. *If $x_1, \ldots, x_n$ are elements in a commutative Banach algebra, then $\|z_1^{i_1} \cdots z_n^{i_n}\|_{\sigma(x_1,\ldots,x_n)} = r(x_1^{i_1} \cdots x_n^{i_n})$ for all $i \in S_0$.*

Proof. Let A be the commutative Banach algebra containing $x_1, \ldots, x_n$, and let $i \in S_0$. Then we have

$$\begin{aligned}
\|z_1^{i_1} \cdots z_n^{i_n}\|_{\sigma(x_1,\ldots,x_n)} &= \sup\{|z_1^{i_1} \cdots z_n^{i_n}(\phi(x_1), \ldots, \phi(x_n))| : \phi \in \mathcal{M}_A\} \\
&= \sup\{|\phi(x_1)^{i_1} \cdots \phi(x_n)^{i_n}| : \phi \in \mathcal{M}_A\} \\
&= \sup\{|\phi(x_1^{i_1} \cdots x_n^{i_n})| : \phi \in \mathcal{M}_A\} \\
&= r(x_1^{i_1} \cdots x_n^{i_n}).
\end{aligned}$$

$\square$

We now present the main result of this section.

Theorem 3.6. *If $x_1, \ldots, x_n$ are elements in a commutative Banach algebra, then the smallest polynomially convex circled subset of $\mathbb{C}^n$ containing $\sigma(x_1, \ldots, x_n)$ is*

$$\{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq \|x_1^{i_1} \cdots x_n^{i_n}\| \text{ for all } i \in S_0\}.$$

Proof. Let $K = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq \|x_1^{i_1} \cdots x_n^{i_n}\| \text{ for all } i \in S_0\}$, and let $K' = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq r(x_1^{i_1} \cdots x_n^{i_n}) \text{ for all } i \in S_0\}$. It follows from Corollary 3.4 and Lemma 3.5 that the smallest polynomially convex circled subset of $\mathbb{C}^n$ containing $\sigma(x_1, \ldots, x_n)$ is equal to K'. To complete the proof, we show that $K = K'$. Since the spectral radius of any Banach algebra element is no more than its norm, we have $K' \subset K$. To see that $K \subset K'$, suppose $s \in K$. Given $i \in S_0$, we have $|s_1^{i_1k} \cdots s_n^{i_nk}| \leq \|x_1^{i_1k} \cdots x_n^{i_nk}\|$ for all $k > 0$, which implies that $|s_1^{i_1} \cdots s_n^{i_n}| \leq \|(x_1^{i_1} \cdots x_n^{i_n})^k\|^{1/k}$ for all $k > 0$. Thus, we have

$$|s_1^{i_1} \cdots s_n^{i_n}| \leq \inf_{k>0} \|(x_1^{i_1} \cdots x_n^{i_n})^k\|^{1/k} = r(x_1^{i_1} \cdots x_n^{i_n}) \text{ for all } i \in S_0,$$

so that $s \in K'$. Therefore $K \subset K'$. $\square$

Before turning to Lemma 3.8, we present a lemma concerning the joint spectrum and the direct sum of Banach algebras, which is a generalization of Lemma 2.1. The joint spectrum of $x_1, \ldots, x_n$ can be written

$$\sigma(x_1, \ldots, x_n) = \{s \in \mathbb{C}^n : 1 \notin (s_1 \cdot 1 - x_1)A + \cdots + (s_n \cdot 1 - x_n)A\}.$$

This follows from the one-to-one correspondence between the members of $\mathcal{M}_A$ and the maximal ideals in A that is given in [8, p. 265]. The next lemma can be verified using this alternate formulation of the joint spectrum.

Lemma 3.7. *Let A and B be commutative Banach algebras, let $x_1, \ldots, x_n \in A$, and let $y_1, \ldots, y_n \in B$. If $(x_1, y_1), \ldots, (x_n, y_n)$ are considered as elements of $A \oplus B$, then $\sigma((x_1, y_1), \ldots, (x_n, y_n)) = \sigma(x_1, \ldots, x_n) \cup \sigma(y_1, \ldots, y_n)$.*

If K is a nonempty compact subset of $\mathbb{C}^n$ and $z_1, \ldots, z_n$ are regarded as elements of $C(K)$, then $\sigma(z_1, \ldots, z_n) = K$. This follows from the fact that the members of $\mathcal{M}_{C(K)}$ are exactly the evaluation functionals $\phi_s(f) = f(s)$, where s ranges over K (see [8, p. 271]).

Lemma 3.8. *Let $x_1, \ldots, x_n$ be elements in a commutative Banach algebra, and let K be a compact subset of $\{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq \|x_1^{i_1} \cdots x_n^{i_n}\| \text{ for all } i \in S_0\}$. Then there exist a commutative Banach algebra B and elements $y_1, \ldots, y_n$ in B such that $\|y_1^{i_1} \cdots y_n^{i_n}\| = \|x_1^{i_1} \cdots x_n^{i_n}\|$ for all $i \in S_0$ and $\sigma(y_1, \ldots, y_n) = \sigma(x_1, \ldots, x_n) \cup K$.*

Proof. We assume that K is nonempty, since otherwise there is nothing to prove. Let A be the commutative Banach algebra containing $x_1, \ldots, x_n$. Let $B = A \oplus C(K)$, and for $1 \leq k \leq n$, let y_k be the element (x_k, z_k) of B. It is clear that $\|z_1^{i_1} \cdots z_n^{i_n}\| \leq \|x_1^{i_1} \cdots x_n^{i_n}\|$ for all $i \in S_0$, and it follows that $\|y_1^{i_1} \cdots y_n^{i_n}\| = \|x_1^{i_1} \cdots x_n^{i_n}\|$ for all $i \in S_0$. Since $\sigma(z_1, \ldots, z_n) = K$, the lemma then follows from Lemma 3.7. □

We now present Theorem 3.9.

Theorem 3.9. *Let $(a_i)_{i \in S_0}$ be an indexed collection of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \in S_0$. Then there exist elements $x_1, \ldots, x_n$ in some commutative Banach algebra such that $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in S_0$ and $\sigma(x_1, \ldots, x_n) = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq a_i \text{ for all } i \in S_0\}$.*

Proof. Apply Lemma 1.4 and apply Lemma 3.8 with $K = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq a_i \text{ for all } i \in S_0\}$. □

4 The Joint Spectrum and Rationally Convex Connected Circled Sets, Part I

The purpose of the remaining sections is to generalize Theorem 2.13 to joint spectra. In this section, we provide a several variable generalization of Lemma 2.7. In particular, as Theorem 4.6, we determine the smallest rationally convex connected circled set containing the joint spectrum of $x_1, \ldots, x_n$ from the norms of all defined monomials $x_1^{i_1} \cdots x_n^{i_n}$, where $i \in \mathbb{Z}^n$. All compact subsets of $\mathbb{C}$ are rationally convex, so in $\mathbb{C}$, the closed annuli centered at the origin are exactly the rationally convex connected circled sets. With the exception of Lemma 4.2, all results in this section are analogues of results in Section 3.

Definition 4.1. Let K be a compact subset of $\mathbb{C}^n$. The *rational hull* of K is

$$\widehat{K}_R = \{s \in \mathbb{C}^n : |f(s)| \leq \|f\|_K \text{ for every rational function } f \text{ analytic on } K\}.$$

A compact subset K of $\mathbb{C}^n$ is *rationally convex* if $\widehat{K}_R = K$.

By a rational function, we mean a quotient of polynomials in the complex coordinate functions. To say that a rational function f is analytic on a compact set $K \subset \mathbb{C}^n$ means that f can be written $f = \frac{p}{q}$, where p and q are polynomials and q has no zeroes on K. The norm $\|f\|_K$ is the supremum norm of f over K, $\|f\|_K = \sup\limits_{s \in K} |f(s)|$.

Before presenting Theorem 4.6, we present some relevant results. Theorem 4.3 and Corollary 4.4 are analogues of Theorem 3.3 and Corollary 3.4 for rational convexity. The next lemma is used to prove Corollary 4.4.

Lemma 4.2. *Let M be a subset of $\mathbb{Z}^n$ containing ϵ_k, for $1 \leq k \leq n$, and let $(a_i)_{i \in M}$ be an indexed collection of nonnegative numbers. Then the set $K = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq a_i \text{ for all } i \in M\}$ is a rationally convex connected circled set.*

Requiring M to contain ϵ_k, for $1 \le k \le n$, ensures that K is bounded. As in Section 1, we denote the unit circle in $\mathbb{C}$ by Γ.

Proof. It is clear from Definition 4.1 that $K' \subset \widehat{K'}_R$ for any compact set $K' \subset \mathbb{C}^n$. To see that K is rationally convex, we show that $\widehat{K}_R \subset K$. It is clear from the definition of K that for each $i \in M$, the rational monomial $z_1^{i_1} \cdots z_n^{i_n}$ is analytic on K and satisfies $\|z_1^{i_1} \cdots z_n^{i_n}\|_K \le a_i$. Thus, if $s \in \widehat{K}_R$, then we have

$$|s_1^{i_1} \cdots s_n^{i_n}| \le \|z_1^{i_1} \cdots z_n^{i_n}\|_K \le a_i \text{ for all } i \in M,$$

so that $s \in K$. To see that K is circled, we note that if $s \in K$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with $|\alpha_1| = \cdots = |\alpha_n| = 1$, then

$$|(\alpha_1 s_1)^{i_1} \cdots (\alpha_n s_n)^{i_n}| = |s_1^{i_1} \cdots s_n^{i_n}| \le a_i \text{ for all } i \in M,$$

so that $(\alpha_1 s_1, \ldots, \alpha_n s_n) \in K$. It remains to prove that K is connected. First, define

$$B = \{(r_1, \ldots, r_n) \in \mathbb{R}^n : r_k \ge 0 \text{ for each } k \text{ and } r_1^{i_1} \cdots r_n^{i_n} \le a_i \text{ for all } i \in M\}.$$

The set K is a continuous image of the product of B and an n-torus, for if $F : B \times \Gamma^n \to K$ is given by $F((r_1, \ldots, r_n), (\alpha_1, \ldots, \alpha_n)) = (r_1\alpha_1, \ldots, r_n\alpha_n)$ for all $(r_1, \ldots, r_n) \in B$ and $(\alpha_1, \ldots, \alpha_n) \in \Gamma^n$, then the image of $B \times \Gamma^n$ under F is equal to K. Indeed, given $s \in K$ and writing, for each $1 \le k \le n$, $s_k = |s_k|\alpha_k$ for some $\alpha_k \in \mathbb{C}$ with $|\alpha_k| = 1$, we have $F((|s_1|, \ldots, |s_n|), (\alpha_1, \ldots, \alpha_n)) = s$. To prove that K is connected, we show that B is connected. Given $(r_1, \ldots, r_n), (s_1, \ldots, s_n) \in B$, the map $f : [0,1] \to \mathbb{R}^n$ given by $f(t) = (r_1^{1-t} s_1^t, \ldots, r_n^{1-t} s_n^t)$ is a path in $\mathbb{R}^n$ from $(r_1, \ldots, r_n)$ to $(s_1, \ldots, s_n)$. The image of $[0,1]$ under f lies in B, for if $i \in M$ and $t \in [0,1]$, we have

$$(r_1^{1-t} s_1^t)^{i_1} \cdots (r_n^{1-t} s_n^t)^{i_n} = (r_1^{i_1} \cdots r_n^{i_n})^{1-t} (s_1^{i_1} \cdots s_n^{i_n})^t \le a_i^{1-t} a_i^t = a_i.$$

We have therefore shown that B is path connected, hence connected. $\square$

The next theorem can be found in [4, p.76].

Theorem 4.3. *Let $K \subset \mathbb{C}^n$ be a connected compact circled set. If $s \in \mathbb{C}^n \setminus \widehat{K}_R$, then there exist integers $i_1, \ldots, i_n$ such that the rational monomial $f(z) = z_1^{i_1} \cdots z_n^{i_n}$ is analytic on K and satisfies $|f(s)| > \|f\|_K$.*

Corollary 4.4. *If K is a compact subset of $\mathbb{C}^n$, then the smallest rationally convex connected circled subset of $\mathbb{C}^n$ containing K is*

$$\{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \le \|z_1^{i_1} \cdots z_n^{i_n}\|_K \text{ for all } i \in \mathbb{Z}^n \text{ such that } z_1^{i_1} \cdots z_n^{i_n} \text{ is analytic on } K\}.$$

Proof. Let $K' = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq \|z_1^{i_1} \cdots z_n^{i_n}\|_K$ for all $i \in \mathbb{Z}^n$ such that $z_1^{i_1} \cdots z_n^{i_n}$ is analytic on $K\}$. It is clear that K' contains K, and by Lemma 4.2, K' is a rationally convex connected circled set. The rest of the proof is similar to the proof of Corollary 3.4. We suppose $K'' \subset \mathbb{C}^n$ is a rationally convex connected circled set containing K and apply Theorem 4.3 to show that $K' \subset K''$. $\square$

The next lemma is a generalization of Lemma 3.5.

Lemma 4.5. *Suppose $x_1, \ldots, x_n$ are elements in a commutative Banach algebra.*

(i) For each $i \in \mathbb{Z}^n$, the monomial $x_1^{i_1} \cdots x_n^{i_n}$ is defined if and only if $z_1^{i_1} \cdots z_n^{i_n}$ is analytic on $\sigma(x_1, \ldots, x_n)$.

(ii) If $i \in \mathbb{Z}^n$ such that $x_1^{i_1} \cdots x_n^{i_n}$ is defined, then $\|z_1^{i_1} \cdots z_n^{i_n}\|_{\sigma(x_1,\ldots,x_n)} = r(x_1^{i_1} \cdots x_n^{i_n})$.

Proof. The projection of $\sigma(x_1, \ldots, x_n)$ onto the k^{th} coordinate is equal to $\sigma(x_k)$. Consequently, x_k is invertible if and only if no element of $\sigma(x_1, \ldots, x_n)$ has k^{th} coordinate equal to 0. Statement *(i)* follows. The proof of Lemma 3.5 is valid for all $i \in \mathbb{Z}^n$ such that $x_1^{i_1} \cdots x_n^{i_n}$ is defined. This yields *(ii)*. $\square$

We now present the main result of this section.

Theorem 4.6. *If $x_1, \ldots, x_n$ are elements in a commutative Banach algebra, then the smallest rationally convex connected circled subset of $\mathbb{C}^n$ containing $\sigma(x_1, \ldots, x_n)$ is*

$$\{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq \|x_1^{i_1} \cdots x_n^{i_n}\| \text{ for all } i \in \mathbb{Z}^n \text{ such that } x_1^{i_1} \cdots x_n^{i_n} \text{ is defined}\}.$$

Proof. Let $K = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq \|x_1^{i_1} \cdots x_n^{i_n}\|$ for all $i \in \mathbb{Z}^n$ such that $x_1^{i_1} \cdots x_n^{i_n}$ is defined$\}$, and let $K' = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq r(x_1^{i_1} \cdots x_n^{i_n})$ for all $i \in \mathbb{Z}^n$ such that $x_1^{i_1} \cdots x_n^{i_n}$ is defined$\}$. We conclude from Corollary 4.4 and Lemma 4.5 that the smallest rationally convex connected circled subset of $\mathbb{C}^n$ containing $\sigma(x_1, \ldots, x_n)$ is equal to K'. To complete the proof, we show that $K = K'$. Since the spectral radius of any Banach algebra element is no more than its norm, we have $K' \subset K$. To see that $K \subset K'$, suppose $s \in K$. Given $i \in \mathbb{Z}^n$ such that $x_1^{i_1} \cdots x_n^{i_n}$ is defined, we have $|s_1^{i_1 k} \cdots s_n^{i_n k}| \leq \|x_1^{i_1 k} \cdots x_n^{i_n k}\|$ for all $k > 0$, which implies that $|s_1^{i_1} \cdots s_n^{i_n}| \leq \|(x_1^{i_1} \cdots x_n^{i_n})^k\|^{1/k}$ for all $k > 0$. Thus, for all $i \in \mathbb{Z}^n$ such that $x_1^{i_1} \cdots x_n^{i_n}$ is defined, we have

$$|s_1^{i_1} \cdots s_n^{i_n}| \leq \inf_{k>0} \|(x_1^{i_1} \cdots x_n^{i_n})^k\|^{1/k} = r(x_1^{i_1} \cdots x_n^{i_n}).$$

Thus, $s \in K'$. Therefore $K \subset K'$. $\square$

The next lemma is an analogue of Lemma 3.8 and will be used in the next section.

Lemma 4.7. *Let $x_1, \ldots, x_n$ be elements in a commutative Banach algebra. Let $M = \{i \in \mathbb{Z}^n : x_1^{i_1} \cdots x_n^{i_n} \text{ is defined}\}$, and let K be a compact subset of $\{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq \|x_1^{i_1} \cdots x_n^{i_n}\| \text{ for all } i \in M\}$. Then there exist a commutative Banach algebra B and elements $y_1, \ldots, y_n$ in B such that $\{i \in \mathbb{Z}^n : y_1^{i_1} \cdots y_n^{i_n} \text{ is defined}\} = M$, $\|y_1^{i_1} \cdots y_n^{i_n}\| = \|x_1^{i_1} \cdots x_n^{i_n}\|$ for all $i \in M$, and $\sigma(y_1, \ldots, y_n) = \sigma(x_1, \ldots, x_n) \cup K$.*

Proof. We assume that K is nonempty, since otherwise there is nothing to prove. Let A be the commutative Banach algebra containing $x_1, \ldots, x_n$. Let $B = A \oplus C(K)$, and for $1 \leq k \leq n$, let y_k be the element (x_k, z_k) of B. If x_k is invertible, then z_k is invertible, for if x_k is invertible, then no element of K has k^{th} coordinate equal to 0. Hence, y_k is invertible if and only if x_k is invertible. Therefore $\{i \in \mathbb{Z}^n : y_1^{i_1} \cdots y_n^{i_n} \text{ is defined}\} = M$. It is clear that $\|z_1^{i_1} \cdots z_n^{i_n}\| \leq \|x_1^{i_1} \cdots x_n^{i_n}\|$ for all $i \in M$, and it follows that $\|y_1^{i_1} \cdots y_n^{i_n}\| = \|x_1^{i_1} \cdots x_n^{i_n}\|$ for all $i \in M$. Recalling that $\sigma(z_1, \ldots, z_n) = K$, the lemma then follows from Lemma 3.7. □

5 The Joint Spectrum and Rationally Convex Connected Circled Sets, Part II

In this section, we generalize Theorem 2.13 to joint spectra. In particular, we determine the largest rationally convex connected circled set that must be contained in the smallest rationally convex connected circled set containing the joint spectrum of $x_1, \ldots, x_n$, given the norms $(\|x_1^{i_1} \cdots x_n^{i_n}\|)_{i \in S_0}$. This is presented as Theorem 5.15. We remark after the proof of this result that this set is the intersection of all rationally convex connected circled sets that are possible joint spectra for the given norms of products. As Question 5.16, we leave as an open question whether this set is a possible joint spectrum for the given norms of products. A related result is presented in Theorem 5.17, in which we find the smallest possible value of $\|z_1^{j_1} \cdots z_n^{j_n}\|_{\sigma(x_1, \ldots, x_n)}$ when $j \in \mathbb{Z}^n$ and $z_1^{j_1} \cdots z_n^{j_n}$ is analytic on $\sigma(x_1, \ldots, x_n)$, given the norms $(\|x_1^{i_1} \cdots x_n^{i_n}\|)_{i \in S_0}$.

We present two main lemmas, Lemma 5.2 and Lemma 5.12, before presenting Theorem 5.15. The first of these is a several variable generalization of Wallen's result (Lemma 1.3). The next lemma contains two facts that are trivial, but will be useful for the proof of Lemma 5.2.

Lemma 5.1. *Let S be a (possibly empty) subset of $\{1, \ldots, n\}$, and let $M = \{i \in \mathbb{Z}^n : i_k \geq 0 \text{ for all } k \in \{1, \ldots, n\} \setminus S\}$. Let $(a_i)_{i \in M}$ be an indexed collection of nonnegative numbers such that $a_{i+j} \leq a_i a_j$ for all $i, j \in M$.*

(i) If $i, j \in M$ and $a_i = 0$, then $a_{i+j} = 0$.

(ii) If $i, j \in M$ and $a_{i+j} \neq 0$, then $a_i \neq 0$.

Lemma 5.2. *Let S be a (possibly empty) subset of $\{1,\dots,n\}$, and let $M = \{i \in \mathbb{Z}^n : i_k \geq 0 \text{ for all } k \in \{1,\dots,n\} \setminus S\}$. Suppose $(a_i)_{i\in M}$ is an indexed collection of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \in M$. Then there exist elements $x_1,\dots,x_n$ in some commutative Banach algebra such that $\{i \in \mathbb{Z}^n : x_1^{i_1}\cdots x_n^{i_n} \text{ is defined}\} = M$ and $\|x_1^{i_1}\cdots x_n^{i_n}\| = a_i$ for all $i \in M$.*

For a Banach space X, we denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators on X.

Proof. Given $i \in M$, let $j(i)$ be the element of M given by

$$j(i)_k = \begin{cases} i_k \text{ , if } k \in S \\ 0 \text{ , if } k \in \{1,\dots,n\} \setminus S. \end{cases}$$

Noting that $-j(i) \in M$ for all $i \in M$, by Lemma 5.1 we have $a_{j(i)} \neq 0$ for all $i \in M$. We also note that

$$j(i+m) = j(i) + m \text{ whenever } i, m \in M \text{ and } m_k = 0 \text{ for all } k \in \{1,\dots,n\} \setminus S. \tag{5}$$

We now define bounded operators $T_1,\dots,T_n$ on $\ell^2(M)$ and show that they satisfy the conclusion of the lemma when considered as elements of a certain commutative subalgebra of $\mathcal{L}(\ell^2(M))$. Let $(e_i)_{i\in M}$ be the standard basis for $\ell^2(M)$. For each $k \in \{1,\dots,n\} \setminus S$, let T_k be the weighted forward shift on $\ell^2(M)$ given by

$$T_k(e_i) = \begin{cases} \frac{a_{i+\epsilon_k}}{a_i} e_{i+\epsilon_k} \text{ , if } a_i \neq 0 \\ 0 \text{ , if } a_i = 0 \end{cases} \qquad \text{for all } i \in M.$$

For each $k \in S$, let T_k be the weighted forward shift on $\ell^2(M)$ given by

$$T_k(e_i) = \begin{cases} \frac{a_{i+\epsilon_k}}{a_i} e_{i+\epsilon_k} \text{ , if } a_i \neq 0 \\ \frac{a_{j(i)+\epsilon_k}}{a_{j(i)}} e_{i+\epsilon_k} \text{ , if } a_i = 0 \end{cases} \qquad \text{for all } i \in M.$$

Each T_k is bounded, with $\|T_k\| = a_{\epsilon_k}$. We show that

$$T_1,\dots,T_n \text{ commute,} \tag{6}$$

$$T_k \text{ is invertible if and only if } k \in S \text{, and} \tag{7}$$

$$\|T_1^{i_1}\cdots T_n^{i_n}\| = a_i \text{ for all } i \in M. \tag{8}$$

We first prove (7). If $k \in \{1,\dots,n\} \setminus S$, then the image of T_k is contained in $\operatorname{span}\{e_i : i \in M \text{ and } i_k > 0\}$. Thus, T_k is not surjective in this case. If $k \in S$, then, with some applications of Lemma 5.1 and (5), it can be seen that T_k has inverse the weighted backward shift on $\ell^2(M)$ given by

$$T_k^{-1}(e_i) = \begin{cases} \frac{a_{i-\epsilon_k}}{a_i} e_{i-\epsilon_k} \text{ , if } a_i \neq 0 \\ \frac{a_{j(i)-\epsilon_k}}{a_{j(i)}} e_{i-\epsilon_k} \text{ , if } a_i = 0 \end{cases} \qquad \text{for all } i \in M.$$

To prove (6), we show that for all $1 \leq k, l \leq n$ and $i \in M$, we have

$$T_k T_l(e_i) = \begin{cases} \frac{a_{i+\epsilon_k+\epsilon_l}}{a_i} e_{i+\epsilon_k+\epsilon_l} \text{ , if } a_{i+\epsilon_k+\epsilon_l} \neq 0 \\ \frac{a_{j(i)+\epsilon_k+\epsilon_l}}{a_{j(i)}} e_{i+\epsilon_k+\epsilon_l} \text{ , if } a_{i+\epsilon_k+\epsilon_l} = 0 \text{ and } k, l \in S \\ 0 \text{ , if } a_{i+\epsilon_k+\epsilon_l} = 0 \text{ and either } k \notin S \text{ or } l \notin S. \end{cases} \tag{9}$$

The first and second cases of (9) can be shown with some applications of Lemma 5.1. To see the second case, also apply (5). To see the third case of (9), suppose $a_{i+\epsilon_k+\epsilon_l} = 0$ and either $k \notin S$ or $l \notin S$. If $k \notin S$, then $T_k(e_{i+\epsilon_l}) = 0$, and it follows that $T_k T_l(e_i) = 0$. If $k \in S$ and $l \notin S$, then an application of Lemma 5.1 shows that $a_{i+\epsilon_l} = 0$. In this case, we then have $T_l(e_i) = 0$, and therefore $T_k T_l(e_i) = 0$. Thus, (9) is proved. Property (6) follows from noting that the right side of (9) remains the same if the roles of k and l are reversed.

A similar proof shows that, more generally, for all $m, i \in M$, we have

$$T_1^{m_1} \cdots T_n^{m_n}(e_i) = \begin{cases} \frac{a_{i+m}}{a_i} e_{i+m} \text{ , if } a_{i+m} \neq 0 \\ \frac{a_{j(i)+m}}{a_{j(i)}} e_{i+m} \text{ , if } a_{i+m} = 0 \text{ and } k \in S \text{ whenever } m_k \neq 0 \\ 0 \text{ , if } a_{i+m} = 0 \text{ and there exists } k \in \{1, \ldots, n\} \setminus S \\ \qquad\qquad \text{such that } m_k \neq 0. \end{cases} \tag{10}$$

Property (8) follows.

The closure of the subalgebra of $\mathcal{L}(\ell^2(M))$ generated by $T_1, \ldots, T_n$ and the inverses T_k^{-1}, where k ranges over S, is a commutative Banach algebra. The weighted shifts $T_1, \ldots, T_n$ satisfy the conclusion of the lemma when considered as elements of this algebra. □

Corollary 5.3. *Let S be a (possibly empty) subset of $\{1, \ldots, n\}$, and let $M = \{i \in \mathbb{Z}^n : i_k \geq 0 \text{ for all } k \in \{1, \ldots, n\} \setminus S\}$. Suppose $(a_i)_{i \in M}$ is an indexed collection of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \in M$. Then there exist elements $x_1, \ldots, x_n$ in some commutative Banach algebra such that $\{i \in \mathbb{Z}^n : x_1^{i_1} \cdots x_n^{i_n} \text{ is defined}\} = M$, $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in M$, and $\sigma(x_1, \ldots, x_n) = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq a_i \text{ for all } i \in M\}$.*

Proof. Apply Lemma 5.2 and apply Lemma 4.7 with $K = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq a_i \text{ for all } i \in M\}$, keeping in mind Theorem 4.6. □

We now give some definitions to be used in the rest of this section. Given an indexed collection $(a_i)_{i \in S_0}$ of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \in S_0$, we define the following. For each $i \in \mathbb{Z}^n$, let

$$A_i = \{m \in S_0 : m + i \in S_0 \text{ and } a_m, a_{m+i} \text{ are not both } 0\},$$

and define

$$C_i = \begin{cases} \sup\limits_{m \in A_i} \frac{a_{m+i}}{a_m}, \text{ if } A_i \text{ is nonempty} \\ \infty, \text{ otherwise.} \end{cases}$$

Let $S = \{k : 1 \leq k \leq n \text{ and } C_{-\epsilon_k} \text{ is finite}\}$, and let $M = \{i \in \mathbb{Z}^n : i_k \geq 0 \text{ for all } k \in \{1, \dots, n\} \setminus S\}$.

Each C_i lies in $[0, \infty]$. It will be proved below as *(i)* of Lemma 5.5 that C_i is finite for all $i \in M$. We note the following property of $(C_i)_{i \in \mathbb{Z}^n}$.

Proposition 5.4. *The equality $C_i = a_i$ holds for all $i \in S_0$.*

Proof. Let $i \in S_0$. We have $0 \in A_i$, so that A_i is nonempty. If $m \in A_i$, then the inequality $a_{m+i} \leq a_m a_i$ implies that $a_m \neq 0$ and $\frac{a_{m+i}}{a_m} \leq a_i$. Therefore $C_i \leq a_i$. But also, we have $a_i = \frac{a_{0+i}}{a_0} \leq C_i$. □

In Theorem 5.15, we show that the set $K = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \leq C_i \text{ for all } i \in M\}$ is the largest rationally convex connected circled set that must be contained in the smallest rationally convex connected circled set containing the joint spectrum of $x_1, \dots, x_n$ when $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in S_0$. When the inequality $C_{i+j} \leq C_i C_j$ holds for all $i, j \in M$, we have, from Corollary 5.3 and Proposition 5.4, existence of elements $x_1, \dots, x_n$ in some commutative Banach algebra such that $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in S_0$ and $\sigma(x_1, \dots, x_n) = K$. In this case, Theorem 5.15 is not hard to prove. However, this inequality is not in general satisfied for all $i, j \in M$. An example can be seen with $n = 2$, setting $a_{(1,0)} = 2$ and $a_i = 1$ for all other $i \in S_0$. We then have $M = \mathbb{Z}^2$ and $C_{(2,-1)} = C_{(-1,1)} = 1$, while $C_{(1,0)} = 2$. Lemma 5.12 establishes a weaker property of the collection $(C_i)_{i \in M}$ that will be used to prove Theorem 5.15.

Before presenting Lemma 5.12, we present several results used in its proof. The next lemma and its corollary give relevant properties of $(C_i)_{i \in M}$. For each integer l, define

$$\operatorname{sign}(l) = \begin{cases} 1 \text{ , if } l > 0 \\ 0 \text{ , if } l = 0 \\ -1 \text{ , if } l < 0. \end{cases}$$

Lemma 5.5. *Let $(a_i)_{i \in S_0}$ be an indexed collection of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \in S_0$. Then*

(i) C_i is finite for all $i \in M$.

(ii) Suppose $i, j \in M$ satisfy

$$m + i \in S_0 \text{ or } m + j \in S_0 \text{ whenever } m \in S_0 \text{ and } m + i + j \in S_0. \quad (11)$$

Then $C_{i+j} \leq C_i C_j$.

Proof. We first show that

$$A_i \text{ is nonempty for all } i \in M \text{, and} \quad (12)$$

$$\text{if } i, j \in M \text{ satisfy (11) and } C_i \text{ and } C_j \text{ are finite, then } C_{i+j} \leq C_i C_j. \quad (13)$$

To see (12), we make the following observations. By definition of S, we have that $C_{-\epsilon_k}$ is finite for each $k \in S$. Thus, if $k \in S$, $m \in S_0$, and $a_m \neq 0$, then $a_{m+\epsilon_k} \neq 0$. Since $a_0 = 1$, it follows that

$$\text{if } m \in S_0 \text{ and } m_k \neq 0 \text{ only if } k \in S, \text{ then } a_m \neq 0. \tag{14}$$

Now, let $i \in M$, and let $m = (\max\{-i_1, 0\}, \ldots, \max\{-i_n, 0\})$. Then m and $m+i$ lie in S_0, and $m_k \neq 0$ only if $k \in S$. By (14), we then have $m \in A_i$. Thus, (12) is proved.

To see (13), suppose $i, j \in M$ satisfy (11) and that C_i and C_j are finite. The set A_{i+j} is nonempty by (12). We show that

$$\frac{a_{m+i+j}}{a_m} \leq C_i C_j \text{ for all } m \in A_{i+j}.$$

Let $m \in A_{i+j}$. Applying (11), we assume, without loss of generality, that $m + i \in S_0$. Suppose $a_{m+i+j} \neq 0$. Then $m + i \in A_j$, and since C_j and C_i are finite, we have $a_{m+i} \neq 0$, $m \in A_i$, and $a_m \neq 0$. We can then write

$$\frac{a_{m+i+j}}{a_m} = \frac{a_{m+i+j}}{a_{m+i}} \cdot \frac{a_{m+i}}{a_m} \leq C_j C_i.$$

Therefore (13) is proved.

To see *(i)*, let $i \in M$. By definition of M and Proposition 5.4, we have that $C_{\text{sign}(i_k)\epsilon_k}$ is finite for all $1 \leq k \leq n$. Also, for all $j \in M$ and $1 \leq k \leq n$, we have that $j, \text{sign}(i_k)\epsilon_k$ satisfy (11). Therefore, successive applications of (13) shows that

$$C_i \leq C_{\text{sign}(i_1)\epsilon_1}^{|i_1|} \cdots C_{\text{sign}(i_n)\epsilon_n}^{|i_n|}.$$

Thus, *(i)* is proved. Statement *(ii)* follows from *(i)* and (13). $\square$

Corollary 5.6. *The collection $(C_i)_{i \in M}$ satisfies*

$$C_{m+i} \leq C_m C_{\mathit{sign}(i_1)\epsilon_1}^{|i_1|} \cdots C_{\mathit{sign}(i_n)\epsilon_n}^{|i_n|} \textit{ for all } m, i \in M.$$

We now give some definitions to be used in the proof of Lemma 5.12. Given an indexed collection $(a_i)_{i \in S_0}$ of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \in S_0$, we make the following definitions. Let p be the number of elements of S. Supposing $p > 0$, and given a permutation $k_1, \ldots, k_p$ of the elements of S, we define certain subsets R_l of M and indexed collections $(B_i^l)_{i \in M}$ of nonnegative numbers, for $0 \leq l \leq p$. Define

$$R_l = \{i \in M : i_{k_{l+1}}, \ldots, i_{k_p} \geq 0\}, \text{ for } 0 \leq l \leq p-1, \text{ and}$$

$$R_p = M.$$

We note that $R_0 = S_0$ and that $R_0 \subset R_1 \subset \ldots \subset R_p$. Given $1 \leq l \leq p$ and an indexed collection $(B_i^{l-1})_{i \in M}$ of nonnegative numbers satisfying

(a) $B_i^{l-1} = a_i$ for all $i \in S_0$,

(b) $B_{i+j}^{l-1} \leq B_i^{l-1} B_j^{l-1}$ for all $i, j \in R_{l-1}$, and

(c) $B_{m+i}^{l-1} \leq B_m^{l-1} (B_{\text{sign}(i_1)\epsilon_1}^{l-1})^{|i_1|} \cdots (B_{\text{sign}(i_n)\epsilon_n}^{l-1})^{|i_n|}$ for all $m, i \in M$,

define

$$B_i^l = \sup_{m \in A_i^l} \frac{B_{m+i}^{l-1}}{B_m^{l-1}} \text{ for all } i \in M, \tag{15}$$

where

$$A_i^l = \{m \in R_{l-1} : m + i \in R_{l-1} \text{ and } B_m^{l-1}, B_{m+i}^{l-1} \text{ are not both } 0\}.$$

By (a), the set A_i^l contains A_i and is therefore nonempty by *(i)* of Lemma 5.5. It follows from (c) that each B_i^l is finite.

As analogues to Proposition 5.4, *(ii)* of Lemma 5.5, and Corollary 5.6, we have the following properties of $(B_i^l)_{i \in M}$. These have the same proofs as the mentioned results, with the obvious substitutions.

Lemma 5.7. *The collection* $(B_i^l)_{i \in M}$ *satisfies*

(i) $B_i^l = B_i^{l-1}$ *for all* $i \in R_{l-1}$,

(ii) if $i, j \in M$ *satisfy*

$$m + i \in R_{l-1} \textit{ or } m + j \in R_{l-1} \textit{ whenever } m \in R_{l-1} \textit{ and } m + i + j \in R_{l-1}, \tag{16}$$

then $B_{i+j}^l \leq B_i^l B_j^l$, *and*

(iii) $B_{m+i}^l \leq B_m^l (B_{\textit{sign}(i_1)\epsilon_1}^l)^{|i_1|} \cdots (B_{\textit{sign}(i_n)\epsilon_n}^l)^{|i_n|}$ *for all* $m, i \in M$.

Corollary 5.8. *The collection* $(B_i^l)_{i \in M}$ *satisfies*

(i) $B_i^l = a_i$ *for all* $i \in S_0$, *and*

(ii) $B_{i+j}^l \leq B_i^l B_j^l$ *for all* $i, j \in R_l$.

Proof. Property *(i)* follows from (a) and *(i)* of Lemma 5.7. To see *(ii)*, let $i, j \in R_l$. For the sake of applying *(ii)* of Lemma 5.7, suppose $m \in R_{l-1}$ such that $m+i+j \in R_{l-1}$. It is not hard to see that if $i_{k_l} \geq 0$, then $m + i \in R_{l-1}$, and if $i_{k_l} < 0$, then $m + j \in R_{l-1}$. Therefore i, j satisfy (16), and by *(ii)* of Lemma 5.7, we have $B_{i+j}^l \leq B_i^l B_j^l$. □

We now set $B_i^0 = C_i$ for all $i \in M$ and define indexed collections $(B_i^l)_{i \in M}$ of nonnegative numbers, for $0 \leq l \leq p$, recursively using (15). By Proposition 5.4 and Corollary 5.6, the collection $(B_i^0)_{i \in M}$ satisfies (a), (b), and (c). This recursive definition is valid by Corollary 5.8 and *(iii)* of Lemma 5.7. Noting Proposition 5.4, we have the following remark direct from the definition of $(B_i^1)_{i \in M}$.

Remark 5.9. The equality $B_i^1 = C_i$ holds for all $i \in M$.

The next two lemmas give relevant properties of the collections $(B_i^l)_{i\in M}$, $0 \leq l \leq p$.

Lemma 5.10. *The inequality $B_i^{l-1} \leq B_i^l$ holds for all $1 \leq l \leq p$ and $i \in M$.*

Proof. When $l = 1$, the lemma holds by Remark 5.9. Suppose $2 \leq l \leq p$, and let $i \in M$. Applying *(i)* of Lemma 5.7 and noting that $A_i^{l-1} \subset A_i^l$, we have

$$B_i^{l-1} = \sup_{m\in A_i^{l-1}} \frac{B_{m+i}^{l-2}}{B_m^{l-2}} = \sup_{m\in A_i^{l-1}} \frac{B_{m+i}^{l-1}}{B_m^{l-1}} \leq \sup_{m\in A_i^l} \frac{B_{m+i}^{l-1}}{B_m^{l-1}} = B_i^l. \qquad \square$$

Lemma 5.11. *Suppose $2 \leq l \leq p$. If $i \in M$ such that $i_{k_{l-1}} \leq 0$, then $B_i^l = B_i^{l-1}$.*

Proof. Let $2 \leq l \leq p$, and suppose $i \in M$ such that $i_{k_{l-1}} \leq 0$. By Lemma 5.10, it suffices to show that $B_i^l \leq B_i^{l-1}$. For all $j \in R_{l-1}$, we have that i, j satisfy (16), with $l-1$ in place of l. Indeed, if $j \in R_{l-1}$, $m \in R_{l-2}$, and $m+i+j \in R_{l-2}$, it is not hard to see that since $i_{k_{l-1}} \leq 0$, we have $m + j \in R_{l-2}$. We can then apply *(ii)* of Lemma 5.7, with $l-1$ in place of l, to see that $B_{j+i}^{l-1} \leq B_j^{l-1} B_i^{l-1}$ for all $j \in A_i^l$. It follows that $B_i^l \leq B_i^{l-1}$. $\square$

We now present Lemma 5.12.

Lemma 5.12. *Let $(a_i)_{i\in S_0}$ be an indexed collection of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \leq a_i a_j$ for all $i, j \in S_0$. Then for each $m' \in M$, there exists an indexed collection $(B_i)_{i\in M}$ of nonnegative numbers such that $B_{i+j} \leq B_i B_j$ for all $i, j \in M$, $B_i = a_i$ for all $i \in S_0$, $C_i \leq B_i$ for all $i \in M$, and $B_{m'} = C_{m'}$.*

Proof. If S is empty, then $M = S_0$, and, noting Proposition 5.4, the lemma is trivial. Suppose S is nonempty, and let $m' \in M$. Let N be the number of negative coordinates of m'. We note that $N \leq p$. If $N \geq 1$, we let $k_1, \ldots, k_p$ be a permutation of the elements of S such that $k_1, \ldots, k_N$ are the distinct elements k of $\{1, \ldots, n\}$ such that $m'_k < 0$. If $N = 0$, we let $k_1, \ldots, k_p$ be any permutation of the elements of S. Define $(B_i^l)_{i\in M}$, for $0 \leq l \leq p$, as in the paragraph following Corollary 5.8, and let

$$B_i = B_i^p \text{ for all } i \in M.$$

We show that $(B_i)_{i\in M}$ satisfies the properties of the lemma.

By Corollary 5.8, we have that $B_i = a_i$ for all $i \in S_0$ and $B_{i+j} \leq B_i B_j$ for all $i, j \in M$. It follows from Lemma 5.10 that $C_i \leq B_i$ for all $i \in M$. It remains to show that $B_{m'} = C_{m'}$. First, define

$$S_l = \{i \in M : i_{k_1} \leq 0, \ldots, i_{k_l} \leq 0, i_{k_{l+1}} \geq 0, \ldots, i_{k_p} \geq 0\}, \text{ for } 1 \leq l \leq p-1, \text{ and}$$

$$S_p = \{i \in M : i_{k_1}, \ldots, i_{k_p} \leq 0\}.$$

We note that $S_l \subset R_l$ for all $0 \le l \le p$. By the choice of $k_1, \dots, k_p$, we have that

$$m' \in S_N. \tag{17}$$

By Remark 5.9 and Lemma 5.11, we have

$$B_i^q = B_i^{q-1} \text{ whenever } 1 \le q \le l \le p \text{ and } i \in S_l.$$

It follows that

$$B_i^l = C_i \text{ for all } 0 \le l \le p \text{ and } i \in S_l.$$

It then follows from (17) and *(i)* of Lemma 5.7 that $B_{m'} = C_{m'}$. □

Remark 5.13. The proof of Lemma 5.12 actually shows that $B_i = C_i$ for all $i \in \bigcup_{l=0}^{p} S_l$.

The next lemma will also be used to prove Theorem 5.15.

Lemma 5.14. *Let $(a_i)_{i \in S_0}$ be an indexed collection of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \le a_i a_j$ for all $i, j \in S_0$. Suppose $x_1, \dots, x_n$ are elements in a commutative Banach algebra with $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in S_0$. If $i \in \mathbb{Z}^n$ such that $x_1^{i_1} \cdots x_n^{i_n}$ is defined, then*

- *(i)* $i \in M$, *and*
- *(ii)* $C_i \le \|x_1^{i_1} \cdots x_n^{i_n}\|$.

Proof. Let $i \in \mathbb{Z}^n$ such that $x_1^{i_1} \cdots x_n^{i_n}$ is defined. To prove *(i)*, we show that $C_{-\epsilon_k}$ is finite if $i_k < 0$. For each $1 \le k \le n$, the set $A_{-\epsilon_k}$ is nonempty, since $\epsilon_k \in A_{-\epsilon_k}$. If $i_k < 0$, then x_k is invertible, and it is easy to see that $C_{-\epsilon_k} \le \|x_k^{-1}\|$. Thus, *(i)* is proved. It then follows from *(i)* of Lemma 5.5 that A_i is nonempty, and it is then easy to see *(ii)*. □

We now present the main result of this section.

Theorem 5.15. *Let $(a_i)_{i \in S_0}$ be an indexed collection of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \le a_i a_j$ for all $i, j \in S_0$. Let*

$$K = \{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \le C_i \text{ for all } i \in M\}.$$

Then

- *(i)* *K is a rationally convex connected circled set, and*
- *(ii)* *whenever $x_1, \dots, x_n$ are elements in a commutative Banach algebra with $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in S_0$, the smallest rationally convex connected circled subset of $\mathbb{C}^n$ containing $\sigma(x_1, \dots, x_n)$ contains K.*

Furthermore, K is the largest subset of $\mathbb{C}^n$ for which (ii) holds.

To say that K is the largest subset of $\mathbb{C}^n$ for which *(ii)* holds means that if *(ii)* holds when K is replaced by a set $K' \subset \mathbb{C}^n$, then $K' \subset K$.

Proof. Statement *(i)* follows from Lemma 4.2. Statement *(ii)* follows from Lemma 5.14 and Theorem 4.6. It remains to see that K is the largest subset of $\mathbb{C}^n$ for which *(ii)* holds. Let $\mathscr{E}$ be the collection of all sets $E \subset \mathbb{C}^n$ such that E is the smallest rationally convex connected circled subset of $\mathbb{C}^n$ containing $\sigma(x_1, \dots, x_n)$ for some commutative Banach algebra elements $x_1, \dots, x_n$ with $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in S_0$. We show that K is equal to the intersection $\bigcap\limits_{E \in \mathscr{E}} E$. That $K \subset \bigcap\limits_{E \in \mathscr{E}} E$ follows from *(ii)*. To see that $\bigcap\limits_{E \in \mathscr{E}} E \subset K$, let $\lambda \in \bigcap\limits_{E \in \mathscr{E}} E$. By Lemma 5.12, Lemma 5.2, and Theorem 4.6, for each $m \in M$, there is a member of $\mathscr{E}$ of the form $\{s \in \mathbb{C}^n : |s_1^{i_1} \cdots s_n^{i_n}| \le B_i \text{ for all } i \in M\}$ for some indexed collection $(B_i)_{i \in M}$ of nonnegative numbers such that $B_m = C_m$. It follows that $|\lambda_1^{i_1} \cdots \lambda_n^{i_n}| \le C_i$ for all $i \in M$, so that $\lambda \in K$. Thus, $\bigcap\limits_{E \in \mathscr{E}} E \subset K$. □

By Theorem 4.6 and Corollary 5.3, each member of the collection $\mathscr{E}$ in the proof of Theorem 5.15 is the joint spectrum of some commutative Banach algebra elements $x_1, \dots, x_n$ with $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in S_0$. Thus, the proof of Theorem 5.15 shows that K is the intersection of all rationally convex connected circled sets that are possible joint spectra for the norms of products $(a_i)_{i \in S_0}$. Since the inequality $C_{i+j} \le C_i C_j$ is not in general satisfied for all $i, j \in M$, it is unclear if K is a possible joint spectrum for $(a_i)_{i \in S_0}$. This is left as an open question.

Question 5.16. If $(a_i)_{i \in S_0}$ is an indexed collection of nonnegative numbers with $a_0 = 1$ and $a_{i+j} \le a_i a_j$ for all $i, j \in S_0$ and K is defined as in Theorem 5.15, do there exist elements $x_1, \dots, x_n$ in some commutative Banach algebra such that $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in S_0$ and $\sigma(x_1, \dots, x_n) = K$?

We now present Theorem 5.17. Recall that for $i \in \mathbb{Z}^n$, the monomial $x_1^{i_1} \cdots x_n^{i_n}$ is defined if and only if $z_1^{i_1} \cdots z_n^{i_n}$ is analytic on $\sigma(x_1, \dots, x_n)$. This is *(i)* of Lemma 4.5. For $k > 0$ and $i \in \mathbb{Z}^n$, we denote by ki the element $(ki_1, \dots, ki_n)$ of $\mathbb{Z}^n$.

Theorem 5.17. *Let $(a_i)_{i \in S_0}$ be an indexed collection of nonnegative numbers such that $a_0 = 1$ and $a_{i+j} \le a_i a_j$ for all $i, j \in S_0$.*

(i) Suppose $x_1, \dots, x_n$ are elements in a commutative Banach algebra with $\|x_1^{i_1} \cdots x_n^{i_n}\| = a_i$ for all $i \in S_0$. If $i \in \mathbb{Z}^n$ and $x_1^{i_1} \cdots x_n^{i_n}$ is defined, then $i \in M$ and $\|z_1^{i_1} \cdots z_n^{i_n}\|_{\sigma(x_1, \dots, x_n)} \ge \inf\limits_{k > 0} C_{ki}^{1/k}$.

(ii) *For each $j \in M$, there exist elements $y_1, \ldots, y_n$ in some commutative Banach algebra such that $\|y_1^{i_1} \cdots y_n^{i_n}\| = a_i$ for all $i \in S_0$, $y_1^{j_1} \cdots y_n^{j_n}$ is defined, and $\|z_1^{j_1} \cdots z_n^{j_n}\|_{\sigma(y_1,\ldots,y_n)} = \inf_{k>0} C_{kj}^{1/k}$.*

Proof. Statement *(i)* follows from Lemma 4.5 and Lemma 5.14. Statement *(ii)* follows from Lemma 4.5, Lemma 5.12 with Remark 5.13, and Lemma 5.2. □

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