OPTIMAL FRAMES FOR PHASE RETRIEVAL FROM EDGE VECTORS OF OPTIMAL POLYGONS

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Abstract. This paper aims to characterize the optimal frame for phase retrieval, defined as the frame whose condition number for phase retrieval attains its minimal value. In the context of the two-dimensional real case, we reveal the connection between optimal frames for phase retrieval and the perimeter-maximizing isodiametric problem, originally proposed by Reinhardt in 1922. Our work establishes that every optimal solution to the perimeter-maximizing isodiametric problem inherently leads to an optimal frame in $\mathbb{R}^2$. By recasting the optimal polygons problem as one concerning the discrepancy of roots of unity, we characterize all optimal polygons. Building upon this connection, we then characterize all optimal frames with m vectors in $\mathbb{R}^2$ for phase retrieval when $m \geq 3$ has an odd factor. As a key corollary, we show that the harmonic frame E_m is *not* optimal for any even integer $m \geq 4$. This finding disproves a conjecture proposed by Xia, Xu, and Xu (Math. Comp., 94(356): 2931-2960). Previous work has established that the harmonic frame $E_m \subset \mathbb{R}^2$ is indeed optimal when m is an odd integer.

Exploring the connection between phase retrieval and discrete geometry, this paper aims to illuminate advancements in phase retrieval and offer new perspectives on the perimeter-maximizing isodiametric problem.

1. Introduction

1.1. **Stability of phase retrieval.** Let $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]^T \in \mathbb{H}^{m\times d}$ be a given measurement matrix, where each $\boldsymbol{a}_j \in \mathbb{H}^d$ and $\mathbb{H} \in \{\mathbb{R}, \mathbb{C}\}$. The phase retrieval problem seeks to recover an unknown signal $\boldsymbol{x} \in \mathbb{H}^d$ from its phaseless measurements $|\langle \boldsymbol{a}_j, \boldsymbol{x}\rangle|$, $j = 1, \ldots, m$. Define the nonlinear measurement map $\Phi_{\boldsymbol{A}} : \mathbb{H}^d \to \mathbb{R}_+^m$ by

$$\Phi_{\boldsymbol{A}}(\boldsymbol{x}) = |\boldsymbol{A}\boldsymbol{x}| := (|\langle \boldsymbol{a}_1, \boldsymbol{x}\rangle|, |\langle \boldsymbol{a}_2, \boldsymbol{x}\rangle|, \ldots, |\langle \boldsymbol{a}_m, \boldsymbol{x}\rangle|)^T \in \mathbb{R}_+^m \tag{1.1}$$

A matrix $\boldsymbol{A}$ is said to have the phase retrieval property if $|\boldsymbol{A}\boldsymbol{x}| = |\boldsymbol{A}\boldsymbol{y}|$ implies $\boldsymbol{x} = c \cdot \boldsymbol{y}$ for some $c \in \mathbb{H}$ with $|c| = 1$. It is known that $m \geq 2d - 1$ (when $\mathbb{H} = \mathbb{R}$) or $m \geq 4d - 4$ (when $\mathbb{H} = \mathbb{C}$) generic measurements suffice for phase retrieval property [5, 8, 13, 31].

A key metric for evaluating the robustness of the phase retrieval is the condition number. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{H}^d$, we define the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$ as

$$\mathrm{dist}_{\mathbb{H}}(\boldsymbol{x}, \boldsymbol{y}) := \min\{\|\boldsymbol{x} - c \cdot \boldsymbol{y}\|_2 : c \in \mathbb{H}, |c| = 1\},$$

where $\|\cdot\|_2$ represents the Euclidean norm. The condition number $\beta_{\boldsymbol{A}}^{\mathbb{H}}$ of a given measurement matrix $\boldsymbol{A} \in \mathbb{H}^{m\times d}$ is defined as

$$\beta_{\boldsymbol{A}}^{\mathbb{H}} := \frac{U_{\boldsymbol{A}}^{\mathbb{H}}}{L_{\boldsymbol{A}}^{\mathbb{H}}},$$

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where $L_{\boldsymbol{A}}^{\mathbb{H}}$ and $U_{\boldsymbol{A}}^{\mathbb{H}}$ represent the optimal lower and upper Lipschitz constants of the map $\Phi_{\boldsymbol{A}}$, i.e.,

$$L_{\boldsymbol{A}}^{\mathbb{H}} := \inf_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d \\ \mathrm{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y})\neq 0}} \frac{\||\boldsymbol{A}\boldsymbol{x}|-|\boldsymbol{A}\boldsymbol{y}|\|_2}{\mathrm{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y})} \quad \text{and} \quad U_{\boldsymbol{A}}^{\mathbb{H}} := \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d \\ \mathrm{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y})\neq 0}} \frac{\||\boldsymbol{A}\boldsymbol{x}|-|\boldsymbol{A}\boldsymbol{y}|\|_2}{\mathrm{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y})}.$$

A smaller value of $\beta_{\boldsymbol{A}}^{\mathbb{H}}$ entails that $\Phi_{\boldsymbol{A}}$ behaves more like a near-isometry, indicating greater stability against the measurement noise. If $\boldsymbol{A}$ lacks the phase retrieval property, then $L_{\boldsymbol{A}}^{\mathbb{H}}=0$ and $\beta_{\boldsymbol{A}}^{\mathbb{H}}=+\infty$. We say that a matrix $\boldsymbol{A}\in\mathbb{H}^{m\times d}$ has *the minimal condition number* if $\beta_{\boldsymbol{A}}^{\mathbb{H}}=\min_{\boldsymbol{M}\in\mathbb{H}^{m\times d}}\beta_{\boldsymbol{M}}^{\mathbb{H}}$. To simplify the notation, we often omit the superscript in $\beta_{\boldsymbol{A}}^{\mathbb{H}}$, determining whether $\beta_{\boldsymbol{A}}^{\mathbb{H}}$ is defined over the real or complex field based on whether the matrix $\boldsymbol{A}$ is real or complex. In the same way, we drop the superscripts in $L_{\boldsymbol{A}}^{\mathbb{H}}, U_{\boldsymbol{A}}^{\mathbb{H}}$, and we omit the subscript in $\mathrm{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y})$ when the underlying field is clear from context.

Recently, the authors in [32] derive the first universal constant lower bound on the condition number $\beta_{\boldsymbol{A}}^{\mathbb{H}}$ for all $\boldsymbol{A}\in\mathbb{H}^{m\times d}$, namely,

$$\beta_{\boldsymbol{A}}^{\mathbb{H}} \geq \beta_0^{\mathbb{H}} := \begin{cases} \sqrt{\frac{\pi}{\pi-2}} \approx 1.659 & \text{if } \mathbb{H}=\mathbb{R}, \\ \sqrt{\frac{4}{4-\pi}} \approx 2.159 & \text{if } \mathbb{H}=\mathbb{C}. \end{cases} \tag{1.2}$$

The condition number of a standard Gaussian matrix in $\mathbb{H}^{m\times d}$ asymptotically matches the lower bound $\beta_0^{\mathbb{H}}$ as $m\to\infty$ [32]. Furthermore, in the real case $\mathbb{H}=\mathbb{R}$, they show that for all $\boldsymbol{A}\in\mathbb{R}^{m\times d}$,

$$\beta_{\boldsymbol{A}}^{\mathbb{R}} \geq \frac{1}{\sqrt{1-\frac{1}{m\cdot\sin\frac{\pi}{2m}}}}. \tag{1.3}$$

According to Theorem 3.3 in [32], we have

$$\beta_{\boldsymbol{E}_m}^{\mathbb{R}} = \begin{cases} \frac{1}{\sqrt{1-\frac{2}{m\cdot\sin\frac{\pi}{m}}}} & \text{if } m\geq 3 \text{ is even}, \\ \frac{1}{\sqrt{1-\frac{1}{m\cdot\sin\frac{\pi}{2m}}}} & \text{if } m\geq 3 \text{ is odd}, \end{cases} \tag{1.4}$$

where $\boldsymbol{E}_m$ is the harmonic frame in $\mathbb{R}^2$ which is defined by

$$\boldsymbol{E}_m := \begin{pmatrix} 1 & \cos\frac{1}{m}\pi & \cdots & \cos\frac{m-1}{m}\pi \\ 0 & \sin\frac{1}{m}\pi & \cdots & \sin\frac{m-1}{m}\pi \end{pmatrix}^T \in \mathbb{R}^{m\times 2}. \tag{1.5}$$

Combining (1.3) and (1.4), we have $\beta_{\boldsymbol{E}_m}^{\mathbb{R}}=\min_{\boldsymbol{A}\in\mathbb{R}^{m\times 2}}\beta_{\boldsymbol{A}}^{\mathbb{R}}$ for each *odd* integer $m\geq 3$. The following conjecture is proposed in [32] for each even integer $m\geq 4$.

Conjecture 1.1. *If $m\geq 4$ is even, then*

$$\beta_{\boldsymbol{E}_m}^{\mathbb{R}} = \min_{\boldsymbol{A}\in\mathbb{R}^{m\times 2}} \beta_{\boldsymbol{A}}^{\mathbb{R}}.$$

In this paper, we will show that minimizing the condition number of $\boldsymbol{A}\in\mathbb{R}^{m\times 2}$ can be transformed into solving the perimeter-maximizing isodiametric problem proposed by Reinhardt in 1922 [25]. We prove that each optimal solution to the perimeter-maximizing isodiametric problem gives rise to a matrix with the minimal condition number. Based on that, we can show that Conjecture 1.1 is false. For instance, when $m=4$, we will demonstrate in Remark 1.3 that

$$\beta_{\boldsymbol{E}_4}^{\mathbb{R}} = \sqrt{2+\sqrt{2}} \approx 1.84 > \beta_{\boldsymbol{E}_4'}^{\mathbb{R}} = \min_{\boldsymbol{M}\in\mathbb{R}^{4\times 2}} \beta_{\boldsymbol{M}}^{\mathbb{R}} = \sqrt{1+\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}} \approx 1.71,$$

where

$$\boldsymbol{E}_4' = \begin{pmatrix} -1 & \cos\frac{2\pi}{3} & \sqrt{2\sin\frac{\pi}{12}}\cos\frac{3\pi}{8} & \sqrt{2\sin\frac{\pi}{12}}\cos\frac{7\pi}{24} \\ 0 & \sin\frac{2\pi}{3} & \sqrt{2\sin\frac{\pi}{12}}\sin\frac{3\pi}{8} & \sqrt{2\sin\frac{\pi}{12}}\sin\frac{7\pi}{24} \end{pmatrix}^T. \tag{1.6}$$

1.2. **The perimeter-maximizing isodiametric problem for convex polygons.** The perimeter-maximizing isodiametric problem has a long and rich history. It was first studied by Reinhardt in 1922 [25], and has been furthered investigated by Vincze [30] in 1950 and Datta [14] in 1997 (see also [15, 16, 24, 9]). The convex polygon in $\mathbb{R}^2$ with m edges is called the convex m-gon. The diameter of a polygon P, denoted by $\operatorname{diam}(P)$, is the largest distance between any pair of its vertices, i.e., $\operatorname{diam}(P) := \max_{\boldsymbol{x},\boldsymbol{y}\in P}\|\boldsymbol{x}-\boldsymbol{y}\|_2$. The perimeter of a polygon P, denoted by $\operatorname{perim}(P)$, is the sum of the length of its edges. The perimeter-maximizing isodiametric problem asks:

Problem 1.1. *Among all convex m-gons with fixed diameter, which ones has the maximal perimeter?*

For clarity, we introduce the following definitions regarding polygons, which are used consistently throughout this paper.

Definition 1.1.
(i) *Optimal polygon: A convex m-gon is defined as optimal if it is a solution to Problem 1.1. We use $\mathcal{E}_m$ to denote the collection of all optimal convex m-gons in $\mathbb{R}^2$.*
(ii) *Equilateral polygon: A polygon is called equilateral if all its edges have equal length.*
(iii) *Regular m-gon: A regular m-gon, denoted as P_m, is an equilateral polygon with equal interior angles.*
(iv) *Strictly convex polygon: A convex m-gon is called strictly convex if all its interior angles are strictly less than π. We use $\mathcal{P}_m$ to denote the set of all strictly convex m-gons in $\mathbb{R}^2$.*
(v) *Edge vector: Assume the vertices of a convex m-gon are ordered counterclockwise. For each edge, the edge vector is defined by subtracting the coordinates of its starting vertex from those of its ending vertex. The edge set of the polygon is the collection of all such edge vectors.*

It is known that the optimal convex m-gon must be strictly convex [25]. Denote $r(P)$ as the diameter-to-perimeter ratio of a polygon P, i.e.,

$$r(P) := \frac{\operatorname{diam}(P)}{\operatorname{perim}(P)}. \tag{1.7}$$

Then solving Problem 1.1 is equivalent to solving

$$\min_{P\in\mathcal{P}_m} r(P). \tag{1.8}$$

We say that the solution to (1.8) is unique if it is unique up to scaling, translations, rotations, flips, or any combination of these transformations.

Theorem 1.1 summarizes the existing results for the perimeter-maximizing isodiametric problem.

Theorem 1.1.
(i) *([25, 14, 16, 15, 30, 24]) Let $m \geq 3$ be a positive integer that has an odd factor. Then*

$$\min_{P\in\mathcal{P}_m} r(P) = \frac{1}{2m\cdot\sin\frac{\pi}{2m}},$$

where the equality is achieved by finitely many equilateral strictly convex m-gons .
(ii) *([25, 14, 16, 15]) Let $m = 2^s$, where s is an integer and $s \geq 2$. Then*

$$\min_{P\in\mathcal{P}_m} r(P) > \frac{1}{2m\cdot\sin\frac{\pi}{2m}}.$$

(iii) *([25, 14, 16, 15, 30]) The regular m-gon P_m is optimal if and only if m is odd. Furthermore, P_m is uniquely optimal if and only if m is a prime .*

When $m \geq 3$ has an odd factor, the optimal convex m-gons can be nicely characterized [25, 30, 14, 15, 16, 24]. When $m \geq 3$ has no odd factors, i.e., $m = 2^s, s \geq 2$, the problem remains largely unresolved. The characterization is known only for $m = 4$ [22, 29, 28] and $m = 8$ [17, 4, 9].

1.3. **Our contribution.** In this paper, we focus on estimating the minimal condition number for $m \times 2$ real matrices and characterizing all $m \times 2$ real matrices that has the minimal condition number. For convenience, we introduce the following definitions.

Definition 1.2. *A finite set in $\mathbb{R}^d$ is called a* frame *if the vectors in this set can span $\mathbb{R}^d$. We say that a frame $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subset \mathbb{R}^d$ is* optimal *if its associated matrix $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]^T \in \mathbb{R}^{m \times d}$ has the minimal condition number, i.e., $\beta_{\boldsymbol{A}}^{\mathbb{R}} = \min_{\boldsymbol{M} \in \mathbb{R}^{m\times d}} \beta_{\boldsymbol{M}}^{\mathbb{R}}$. A frame $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subset \mathbb{R}^d$ is called a tight frame if $\sum_{i=1}^m \boldsymbol{a}_i \boldsymbol{a}_i^T = C \cdot \boldsymbol{I}_d$ for some constant $C > 0$.*

Definition 1.3. *Denote by $\mathcal{T}$ the upper half-plane in $\mathbb{R}^2$ excluding the positive x-axis, i.e.,*

$$\mathcal{T} := \{(x, y)^T \in \mathbb{R}^2 : x = t \cdot \cos\phi, y = t \cdot \sin\phi,\ t \geq 0, \phi \in (0, \pi]\}. \tag{1.9}$$

Let $f : \mathcal{T} \to \mathbb{R}^2$ be a map defined by

$$f(\boldsymbol{a}) \ = \ t^2 \cdot (\cos 2\phi, \sin 2\phi)^T \tag{1.10}$$

for any $\boldsymbol{a} = t \cdot (\cos\phi, \sin\phi)^T \in \mathcal{T}$.

Remark 1.1. *We will now show that the map $f : \mathcal{T} \to \mathbb{R}^2$ is a bijection. The bijectivity can be established by considering two cases. Firstly, for the zero vector, it holds that $\boldsymbol{a} = \boldsymbol{0}$ if and only if $f(\boldsymbol{a}) = \boldsymbol{0}$. Secondly, for any nonzero vector $\boldsymbol{e}_0 \in \mathbb{R}^2$, it possesses a unique representation in the form $\boldsymbol{e}_0 = t_0 \cdot (\cos 2\phi_0, \sin 2\phi_0)^T$, where $t_0 > 0$ and $\phi_0 \in (0, \pi]$. Given this, the unique pre-image $\boldsymbol{a}_0 \in \mathcal{T}$ such that $f(\boldsymbol{a}_0) = \boldsymbol{e}_0$ is given by $\boldsymbol{a}_0 = \sqrt{t_0} \cdot (\cos\phi_0, \sin\phi_0)^T$. Therefore, f is both injective and surjective, implying it is a bijection. This ensures that the inverse map $f^{-1} : \mathbb{R}^2 \to \mathcal{T}$ is well-defined, and is explicitly given by $f^{-1}(\boldsymbol{e}_0) = \boldsymbol{a}_0$.*

1.3.1. *Estimating the minimal condition number.* We first estimate the minimal condition number of real $m \times 2$ matrices. Note that the condition number of a matrix $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]^T \in \mathbb{R}^{m\times 2}$ does not change if the row vector $\boldsymbol{a}_i$ is replaced by $-\boldsymbol{a}_i$ for any $i \in [m] := \{1, \ldots, m\}$. Hence, to estimate the minimal condition number, it is enough to consider the case when each $\boldsymbol{a}_i \in \mathcal{T}$.

We begin by establishing a bijective correspondence between optimal tight frames with m vectors in $\mathbb{R}^2$ and optimal convex m-gons.

Theorem 1.2. *Assume that $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subset \mathcal{T} \subset \mathbb{R}^2$, where $\mathcal{T}$ is defined in (1.9). The set $\mathcal{A}$ is an optimal tight frame if and only if $\{f(\boldsymbol{a}_1), \ldots, f(\boldsymbol{a}_m)\}$ is the edge vectors set of an optimal m-gon $P \in \mathcal{P}_m$, where $f(\cdot)$ is defined in (1.10). In either case, we have*

$$\beta_{\boldsymbol{A}} = \frac{1}{\sqrt{1 - 2 \cdot r(P)}},$$

where $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]^T$ and $r(P)$ is defined in (1.7).

Combining Theorem 1.2 with Theorem 1.1, we can derive Corollary 1.1, which presents a more precise estimate on the minimal condition number for phase retrieval.

Corollary 1.1. (i) *For any integers $m \geq 3$ which has an odd factor, we have*

$$\min_{\boldsymbol{A} \in \mathbb{R}^{m\times 2}} \beta_{\boldsymbol{A}} = \frac{1}{\sqrt{1 - \frac{1}{m \cdot \sin\frac{\pi}{2m}}}}.$$

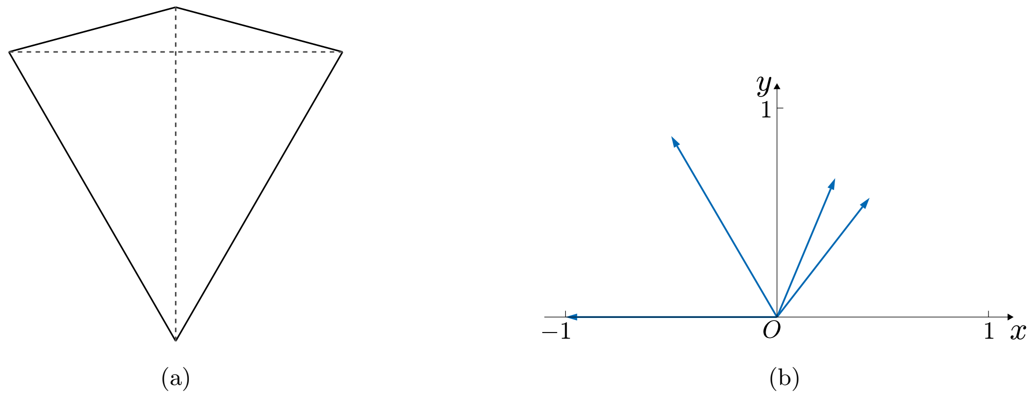

FIGURE 1. (a): Optimal quadrilateral. The dashed lines connect pairs of vertices at maximal distance; (b): Optimal frame consisting of 4 vectors in $\mathcal{T}$.

(ii) *For $m = 2^s$, where s is an integer with $s \geq 2$, we have*

$$\min_{\boldsymbol{A} \in \mathbb{R}^{m\times 2}} \beta_{\boldsymbol{A}} > \frac{1}{\sqrt{1 - \frac{1}{m\cdot \sin\frac{\pi}{2m}}}}.$$

(iii) *The harmonic frame $\boldsymbol{E}_m$ in (1.5) has the minimal condition number if and only if m is odd. Moreover, $\boldsymbol{E}_m$ is the unique matrix with minimal condition number if and only if m is a odd prime.*

Remark 1.2. *The authors in* [32] *proved that the lower bound (1.3) is tight if $m \geq 3$ is an odd integer. Corollary 1.1* (i) *generalizes this result by showing that the lower bound* (1.3) *is tight as long as $m \geq 3$ has an odd factor. Moreover, Corollary 1.1* (iii) *shows that Conjecture 1.1 is false, i.e., the condition number of harmonic frame $\boldsymbol{E}_m$ is not minimal for each even integer $m \geq 4$.*

Remark 1.3. *Take $m = 4$ as an example. Let $P \subset \mathbb{R}^2$ be the convex 4-gon with the edge set*

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos\frac{4\pi}{3} \\ \sin\frac{4\pi}{3} \end{pmatrix}, 2\sin\frac{\pi}{12} \begin{pmatrix} \cos\frac{3\pi}{4} \\ \sin\frac{3\pi}{4} \end{pmatrix}, 2\sin\frac{\pi}{12} \begin{pmatrix} \cos\frac{7\pi}{12} \\ \sin\frac{7\pi}{12} \end{pmatrix} \right\}.$$

It is known that P is the unique optimal 4-gon, and we have $r(P) = \frac{1}{2+\sqrt{6}-\sqrt{2}}$ [29, 28]*. Let $\boldsymbol{A} \in \mathbb{R}^{4\times 2}$ be defined as in* (1.6)*. By Theorem 1.2 we see that the matrix $\boldsymbol{A}$ has the minimal condition number $\beta_{\boldsymbol{A}}^{\mathbb{R}} = \sqrt{1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}} \approx 1.71$. According to (1.4), we have $\beta_{\boldsymbol{E}_4}^{\mathbb{R}} = \sqrt{2+\sqrt{2}} \approx 1.84$, which is strictly greater than $\beta_{\boldsymbol{A}}^{\mathbb{R}}$. This is aligned with Corollary 1.1* (iii)*. The optimal convex 4-gon P and the rows of $\boldsymbol{A}$ are both illustrated in Figure 1.*

1.3.2. *Characterization of all optimal polygons.* Recall that $\mathcal{E}_m$ denotes the set of all optimal convex m-gons in $\mathbb{R}^2$. We then proceed to characterize these optimal polygons $P \in \mathcal{E}_m$. This characterization begins with the introduction of the following problem concerning the discrepancy of roots of unity. Subsequently, we will demonstrate the connection between this discrepancy problem and the optimal polygons.

Problem 1.2. *Let $m \geq 3$ be an integer. Define $\zeta_m := e^{i\pi/m}$. Determine all vectors $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)^T \in \{\pm 1\}^m$ such that*

$$g(\boldsymbol{\varepsilon}) := \sum_{j=1}^{m} \varepsilon_j \zeta_m^j = 0. \tag{1.11}$$

Theorem 1.3. *Let $m \geq 3$ be an integer with an odd factor. Set $\boldsymbol{\mu}_j := (\cos \frac{j\pi}{m}, \sin \frac{j\pi}{m})^T$ for each $j \in [m]$. Then we have*

- (i) *If $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)^T \in \{\pm 1\}^m$ satisfies (1.11), then there exists an optimal polygon $P \in \mathcal{E}_m$ with the edge vectors $\{\varepsilon_1 \boldsymbol{\mu}_1, \ldots, \varepsilon_m \boldsymbol{\mu}_m\}$. Such a polygon P is unique up to translation.*
- (ii) *Conversely, for any optimal polygon $P \in \mathcal{E}_m$, there exists a vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)^T \in \{\pm 1\}^m$ satisfying (1.11), such that the edge vectors of P are $\{\varepsilon_1 \boldsymbol{\mu}_1, \ldots, \varepsilon_m \boldsymbol{\mu}_m\}$, after a proper scaling and rotation.*

Remark 1.4. *For a vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)^T \in \{\pm 1\}^m$, we define the shifting operator $\mathcal{S}$ and the flip operator $\mathcal{R}$ as*

$$\mathcal{S}(\boldsymbol{\varepsilon}) = (\varepsilon_2, \ldots, \varepsilon_m, -\varepsilon_1)^T \quad \text{and} \quad \mathcal{R}(\boldsymbol{\varepsilon}) = (\varepsilon_m, \varepsilon_{m-1}, \ldots, \varepsilon_1)^T. \tag{1.12}$$

For each $i \in \{1, 2\}$, let $\boldsymbol{\varepsilon}_i \in \{\pm 1\}^m$ be a vector satisfying (1.11), and let $P_i \in \mathcal{E}_m$ be the corresponding optimal polygon with the edge set as described in Theorem 1.3 (i). A simple calculation shows that

- (i) *if P_2 is obtained from P_1 through a proper translation and scaling, then $\boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_1$;*
- (ii) *if P_2 is obtained from P_1 via a clockwise rotation of $\frac{k\pi}{m}$ for some integer $k \in [2m]$, then $\boldsymbol{\varepsilon}_2 = \mathcal{S}^k(\boldsymbol{\varepsilon}_1)$;*
- (iii) *if P_2 is obtained from P_1 by a flip with respect to the y-axis, then $\boldsymbol{\varepsilon}_2 = \mathcal{S}(\mathcal{R}(\boldsymbol{\varepsilon}_1))$.*

Since we treat polygons as equivalent under scaling, translations, rotations, flips, or any combination of these transformations, we define two vectors $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \{\pm 1\}^m$ to be equivalent if one can be obtained from the other by applying a finite sequence of the shifting operator $\mathcal{S}$ and the flip operator $\mathcal{R}$. A simple calculation shows that $\mathcal{R}^2 = \mathcal{S}^{2m} = \mathrm{Id}$, where Id denotes the identity operator. Furthermore, we have $\mathcal{S}^k \mathcal{R} = \mathcal{R} \mathcal{S}^{2m-k}$ for any integer $k \in [1, 2m]$. Hence, $\boldsymbol{\varepsilon}_1$ is equivalent to $\boldsymbol{\varepsilon}_2$ if and only if $\mathcal{R}^{k_1} \mathcal{S}^{k_2}(\boldsymbol{\varepsilon}_1) = \boldsymbol{\varepsilon}_2$ for some $k_1 \in \{0, 1\}$ and $k_2 \in \{0, 1, \ldots, 2m-1\}$. Under this convention, the number of distinct solutions to Problem 1.1 equals that of Problem 1.2.

1.3.3. *Characterization of all optimal frames.* We next characterize all optimal frames with m vectors when $m \geq 3$ has an odd factor. For convenience, for any vector $\boldsymbol{\nu} = t \cdot (\cos \phi, \sin \phi)^T \in \mathcal{T}$ with $t \geq 0$ and $\phi \in (0, \frac{\pi}{2}]$, we define $\boldsymbol{\nu}^{\perp} := t \cdot (\cos(\phi + \frac{\pi}{2}), \sin(\phi + \frac{\pi}{2}))^T \in \mathcal{T} \subset \mathbb{R}^2$.

Theorem 1.4. *Assume that $m \geq 3$ is an integer with an odd factor. Let $\mathcal{T}_m$ denote the set of all optimal frames with m vectors in $\mathcal{T}$, where $\mathcal{T}$ is defined in (1.9). Set $\boldsymbol{\nu}_j := (\cos \frac{j\pi}{2m}, \sin \frac{j\pi}{2m})^T$ for $j \in [m]$. Then we have*

- (i) *Assume that $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \in \mathcal{T}_m$. Then $\mathcal{A}$ forms a tight frame, and all vectors in $\mathcal{A}$ have the same norm, i.e.,*
$$\|\boldsymbol{a}_1\|_2 = \|\boldsymbol{a}_2\|_2 = \cdots = \|\boldsymbol{a}_m\|_2.$$
- (ii) *If $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)^T \in \{\pm 1\}^m$ satisfies (1.11), then the frame $\mathcal{A} = \{\boldsymbol{\nu}_j : j \in I\} \cup \{\boldsymbol{\nu}_j^{\perp} : j \in [m] \setminus I\}$ is an optimal frame, i.e., $\mathcal{A} \in \mathcal{T}_m$, where $I = \{j \in [m] : \varepsilon_j = 1\}$.*
- (iii) *For any optimal frame $\mathcal{A} \in \mathcal{S}_m$, there exists a vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)^T \in \{\pm 1\}^m$ satisfying (1.11), such that $\mathcal{A} = \{\boldsymbol{\nu}_j : j \in I\} \cup \{\boldsymbol{\nu}_j^{\perp} : j \in [m] \setminus I\}$ where $I = \{j \in [m] : \varepsilon_j = 1\}$, after a proper scaling and rotation.*

Remark 1.5. *Theorem 1.4 establishes that any optimal frame $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subset \mathbb{R}^2$ forms a tight frame when $m \geq 3$ has an odd factor. Inspired by this result, we propose the following conjecture:*

Any optimal frame $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subset \mathbb{R}^d$ *for phase retrieval in* $\mathbb{R}^d$ *must be a tight frame, provided* $m \geq 2d - 1$.

Remark 1.6. *Let* $m \geq 3$ *be an integer with an odd factor. A brute-force algorithm can be used to generate all optimal polygons in* $\mathcal{E}_m$, *up to scaling, translation, rotation, and flip. It can also generate all optimal frames in* $\mathcal{T}_m$, *up to scaling, rotation, and flip* [1]. *The algorithm begins by examining all vectors* $\boldsymbol{\varepsilon} \in \{\pm 1\}^m$ *that satisfy (1.11). Then Theorem 1.3 and Theorem 1.4 are applied. Take* $m = 12$ *as an example. After checking all vectors* $\boldsymbol{\varepsilon} \in \{\pm 1\}^{12}$ *for which (1.11) holds, by Theorem 1.3, we identify all optimal polygons in* $\mathcal{E}_{12}$ *(up to scaling, translation, rotation and flip), shown in Figure 2 (a1) and (a2). Also, by Theorem 1.4, all the optimal frames in* $\mathcal{T}_{12}$ *(up to scaling, rotation and flip) are obtained, shown in Figure 2 (b1) and (b2). Denote by* $\#\mathcal{T}_m$ *the number of distinct frames in* $\mathcal{T}_m$, *modulo scaling, rotation and flip. Table 1 lists the number* $\#\mathcal{T}_m$ *for* $3 \leq m \leq 15$. *In particular, the cases* $\#\mathcal{T}_4 = \#\mathcal{T}_8 = 1$ *follow from established results in* [29, 28, 4, 9]. *A promising direction for future research is to design an efficient algorithm capable of finding all solutions to Problem 1.2.*

TABLE 1. The number $\#\mathcal{T}_m$ of distinct frames in $\mathcal{T}_m$, modulo scaling, rotation and flip, for $m \in [3, 15]$.

m	3	4	5	6	7	8	9	10	11	12	13	14	15
$\#\mathcal{T}_m$	1	1	1	1	1	1	2	1	1	2	1	1	5

2. Preliminaries

2.1. **Notation.** For a positive integer m, we define $[m] := \{1, \ldots, m\}$. Let $\mathbb{H} \in \{\mathbb{R}, \mathbb{C}\}$. We use $\|\boldsymbol{x}\|_2$ to denote the Euclidean norm of a vector $\boldsymbol{x} \in \mathbb{H}^d$, and we use $\|\boldsymbol{A}\|_2$ to denote the spectral norm of a matrix $\boldsymbol{A} \in \mathbb{H}^{m\times d}$. We denote the unit sphere in $\mathbb{H}^d$ by $\mathbb{S}_{\mathbb{H}}^{d-1}$, i.e.,

$$\mathbb{S}_{\mathbb{H}}^{d-1} = \{\boldsymbol{x} \in \mathbb{H}^d : \|\boldsymbol{x}\|_2 = 1\}.$$

If $\mathbb{H} = \mathbb{R}$ then we simply write $\mathbb{S}_{\mathbb{R}}^{d-1}$ as $\mathbb{S}^{d-1}$.

2.2. **Equivalent forms of Lipschitz constants and condition numbers.** We briefly introduce the existing results on the optimal lower and upper Lipschitz constants.

Theorem 2.1. [8, 6, 1] *Let* $\boldsymbol{A} \in \mathbb{H}^{m\times d}$, *where* $\mathbb{H} = \mathbb{R}$ *or* $\mathbb{C}$. *Then* $U_{\boldsymbol{A}} = \|\boldsymbol{A}\|_2$.

Theorem 2.2. [2] *Let* $\boldsymbol{A} \in \mathbb{H}^{m\times d}$, *where* $\mathbb{H} = \mathbb{R}$ *or* $\mathbb{C}$. *Then*

$$L_{\boldsymbol{A}} = \min_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d \\ \|\boldsymbol{x}\|_2=1, \|\boldsymbol{y}\|_2\leq 1, \langle \boldsymbol{x},\boldsymbol{y}\rangle=0}} \frac{\||\boldsymbol{A}\boldsymbol{x}| - |\boldsymbol{A}\boldsymbol{y}|\|_2}{\operatorname{dist}(\boldsymbol{x},\boldsymbol{y})}.$$

The following lemma yields an equivalent form for the condition number of a tight frame.

Lemma 2.1. *Assume that* $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subset \mathbb{H}^d$ *forms a tight frame, where* $\mathbb{H} = \mathbb{R}$ *or* $\mathbb{C}$. *Then*

$$\beta_{\boldsymbol{A}} = \left(1 - \frac{1}{C} \max_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d, \\ \|\boldsymbol{x}\|_2=\|\boldsymbol{y}\|_2=1, \langle \boldsymbol{x},\boldsymbol{y}\rangle=0}} \sum_{j=1}^m \left|\boldsymbol{x}^*\boldsymbol{a}_j\boldsymbol{a}_j^*\boldsymbol{y}\right|\right)^{-\frac{1}{2}}, \tag{2.1}$$

where $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]^* \in \mathbb{H}^{m\times d}$ *and* $C = \frac{1}{d}\sum_{j=1}^m \|\boldsymbol{a}_j\|_2^2$.

[1]The MATLAB and Maple implementations of the algorithm are available at https://github.com/zxynhy/Optimal-polygon-for-the-perimeter-maximizing-isodiametric-problem.git.

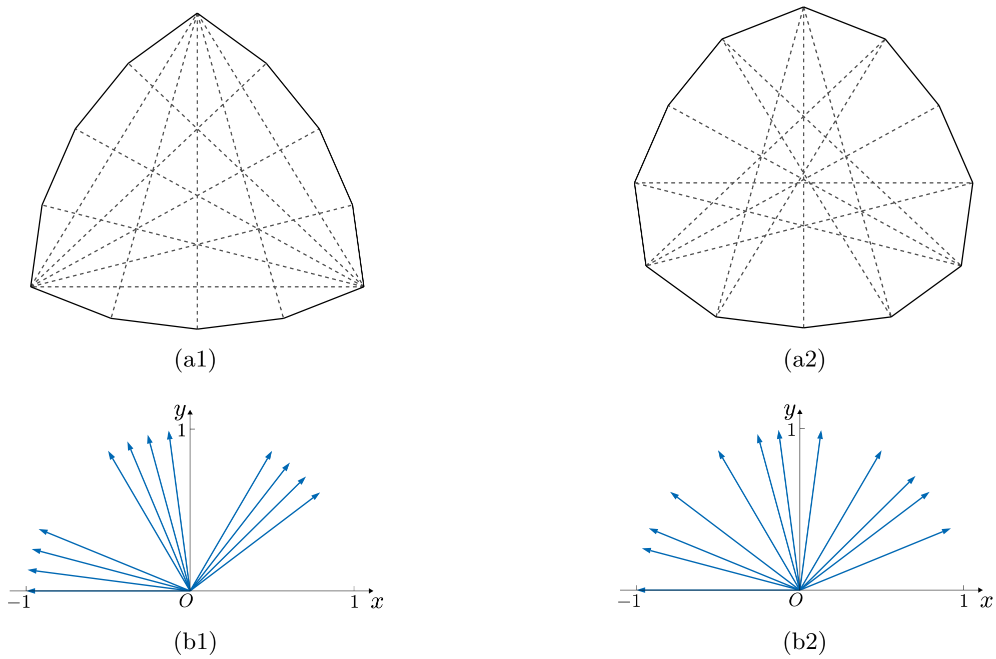

FIGURE 2. (a1), (a2): All optimal dodecagons. The dashed lines connect pairs of vertices at maximal distance; (b1), (b2): All optimal frames consisting of 12 vectors in $\mathcal{T}$, corresponding to (a1) and (a2) respectively.

Proof. Since $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}\subset\mathbb{H}^d$ forms a tight frame, we have $\boldsymbol{A}^*\boldsymbol{A}=C\cdot\boldsymbol{I}_d$, where $C=\frac{1}{d}\sum_{j=1}^m\|\boldsymbol{a}_j\|_2^2$. By Theorem 2.2 we have

$$(L_{\boldsymbol{A}})^2=\min_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d\\ \|\boldsymbol{x}\|_2=1,\|\boldsymbol{y}\|_2\le 1,\langle\boldsymbol{x},\boldsymbol{y}\rangle=0}}\frac{\||\boldsymbol{A}\boldsymbol{x}|-|\boldsymbol{A}\boldsymbol{y}|\|_2^2}{\operatorname{dist}^2(\boldsymbol{x},\boldsymbol{y})}. \tag{2.2}$$

If $\boldsymbol{y}=\boldsymbol{0}$, then we simply have

$$\frac{\||\boldsymbol{A}\boldsymbol{x}|-|\boldsymbol{A}\boldsymbol{y}|\|_2^2}{\operatorname{dist}^2(\boldsymbol{x},\boldsymbol{y})}=\frac{\|\boldsymbol{A}\boldsymbol{x}\|_2^2}{\|\boldsymbol{x}\|_2^2}=C. \tag{2.3}$$

We next consider the case when $\boldsymbol{y}\neq\boldsymbol{0}$. Assume that $\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d$, $\|\boldsymbol{x}\|_2=1$, $0<\|\boldsymbol{y}\|_2\le 1$, and $\langle\boldsymbol{x},\boldsymbol{y}\rangle=0$. A direct calculation shows $\operatorname{dist}^2(\boldsymbol{x},\boldsymbol{y})=\|\boldsymbol{x}-\boldsymbol{y}\|_2^2=1+\|\boldsymbol{y}\|_2^2$ and

$$\begin{aligned}\||\boldsymbol{A}\boldsymbol{x}|-|\boldsymbol{A}\boldsymbol{y}|\|_2^2&=\boldsymbol{x}^*\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{x}+\boldsymbol{y}^*\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{y}-2\sum_{j=1}^m\left|\boldsymbol{x}^*\boldsymbol{a}_j\boldsymbol{a}_j^*\boldsymbol{y}\right|\\&=C\left(1+\|\boldsymbol{y}\|_2^2\right)-2\|\boldsymbol{y}\|_2\sum_{j=1}^m\left|\boldsymbol{x}^*\boldsymbol{a}_j\boldsymbol{a}_j^*\widetilde{\boldsymbol{y}}\right|,\end{aligned}$$

where $\widetilde{\boldsymbol{y}}:=\boldsymbol{y}/\|\boldsymbol{y}\|_2$. It follows that

$$\frac{\||\boldsymbol{A}\boldsymbol{x}|-|\boldsymbol{A}\boldsymbol{y}|\|_2^2}{\operatorname{dist}^2(\boldsymbol{x},\boldsymbol{y})}=C-\frac{2\sum_{j=1}^m\left|\boldsymbol{x}^*\boldsymbol{a}_j\boldsymbol{a}_j^*\widetilde{\boldsymbol{y}}\right|}{\frac{1}{\|\boldsymbol{y}\|_2}+\|\boldsymbol{y}\|_2}\overset{(a)}{\ge}C-\sum_{j=1}^m\left|\boldsymbol{x}^*\boldsymbol{a}_j\boldsymbol{a}_j^*\widetilde{\boldsymbol{y}}\right|. \tag{2.4}$$

Here, we apply the Cauchy-Schwarz inequality in (a), where the equality is achieved if and only if $\|\boldsymbol{y}\|_2 = 1$. Therefore, combining (2.2) with (2.3) and (2.4), we have

$$\begin{aligned}(L_{\boldsymbol{A}})^2 &= \min_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d,\\ \|\boldsymbol{x}\|_2=\|\boldsymbol{y}\|_2=1,\,\langle\boldsymbol{x},\boldsymbol{y}\rangle=0}} \left(C - \sum_{j=1}^m \left|\boldsymbol{x}^*\boldsymbol{a}_j\boldsymbol{a}_j^*\boldsymbol{y}\right|\right) \\ &= C - \max_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d,\\ \|\boldsymbol{x}\|_2=\|\boldsymbol{y}\|_2=1,\,\langle\boldsymbol{x},\boldsymbol{y}\rangle=0}} \sum_{j=1}^m \left|\boldsymbol{x}^*\boldsymbol{a}_j\boldsymbol{a}_j^*\boldsymbol{y}\right|.\end{aligned} \tag{2.5}$$

Combining (2.5) with $\beta_{\boldsymbol{A}} = U_{\boldsymbol{A}}/L_{\boldsymbol{A}}$ and $U_{\boldsymbol{A}} = \|\boldsymbol{A}\|_2 = \sqrt{C}$, we arrive at (2.1). □

The following lemma establishes that, for determining the minimal condition number of $\boldsymbol{A}\in\mathbb{H}^{m\times d}$, it suffices to consider cases where its rows form a tight frame.

Lemma 2.2. *Assume that $m \geq d$. We have*

$$\min_{\boldsymbol{A}\in\mathbb{H}^{m\times d}} \beta_{\boldsymbol{A}} = \min_{\boldsymbol{A}\in\mathbb{H}^{m\times d},\boldsymbol{A}^*\boldsymbol{A}=\boldsymbol{I}_d} \beta_{\boldsymbol{A}}.$$

Proof. If $\operatorname{rank}(\boldsymbol{A}) < d$, then $\beta_{\boldsymbol{A}} = +\infty$. We next consider the case when $\boldsymbol{A}\in\mathbb{H}^{m\times d}$ has rank d. Let $\boldsymbol{M} = \boldsymbol{A}^*\boldsymbol{A}\in\mathbb{H}^{d\times d}$ and $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{M}^{-\frac{1}{2}}\in\mathbb{H}^{m\times d}$. We claim that

$$\beta_{\boldsymbol{A}} \geq \beta_{\boldsymbol{B}}. \tag{2.6}$$

Note that the rows of $\boldsymbol{B}$ form a tight frame. Also note that the condition number $\beta_{\boldsymbol{B}}$ is invariant under the scaling of $\boldsymbol{B}$. Thus, (2.6) implies the conclusion. It suffices to prove (2.6).

Note that for any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d$,

$$\||\boldsymbol{B}\boldsymbol{x}| - |\boldsymbol{B}\boldsymbol{y}|\|_2 = \||\boldsymbol{A}\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{x}| - |\boldsymbol{A}\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{y}|\|_2 = \||\boldsymbol{A}\widehat{\boldsymbol{x}}| - |\boldsymbol{A}\widehat{\boldsymbol{y}}|\|_2,$$

where $\widehat{\boldsymbol{x}} = \boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{x}$ and $\widehat{\boldsymbol{y}} = \boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{y}$. Also note that for any $c\in\mathbb{H}$,

$$\|\boldsymbol{x} - c\cdot\boldsymbol{y}\|_2 = \|\boldsymbol{M}^{\frac{1}{2}}(\widehat{\boldsymbol{x}} - c\cdot\widehat{\boldsymbol{y}})\|_2 \leq \|\boldsymbol{M}^{\frac{1}{2}}\|_2\cdot\|\widehat{\boldsymbol{x}} - c\cdot\widehat{\boldsymbol{y}}\|_2,$$

which implies that

$$\begin{aligned}\operatorname{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y}) &= \min_{c\in\mathbb{H},|c|=1}\|\boldsymbol{x} - c\cdot\boldsymbol{y}\|_2 \\ &\leq \|\boldsymbol{M}^{\frac{1}{2}}\|_2\cdot\min_{c\in\mathbb{H},|c|=1}\|\widehat{\boldsymbol{x}} - c\cdot\widehat{\boldsymbol{y}}\|_2 = \|\boldsymbol{M}^{\frac{1}{2}}\|_2\cdot\operatorname{dist}_{\mathbb{H}}(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}}).\end{aligned}$$

Therefore,

$$\frac{\||\boldsymbol{B}\boldsymbol{x}| - |\boldsymbol{B}\boldsymbol{y}|\|_2}{\operatorname{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y})} = \frac{\||\boldsymbol{A}\widehat{\boldsymbol{x}}| - |\boldsymbol{A}\widehat{\boldsymbol{y}}|\|_2}{\operatorname{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y})} \geq \frac{1}{\|\boldsymbol{M}^{\frac{1}{2}}\|_2}\cdot\frac{\||\boldsymbol{A}\widehat{\boldsymbol{x}}| - |\boldsymbol{A}\widehat{\boldsymbol{y}}|\|_2}{\operatorname{dist}_{\mathbb{H}}(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}})}.$$

Since $\boldsymbol{M}$ has full rank, we have

$$\begin{aligned}L_{\boldsymbol{B}} &= \inf_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{H}^d\\ \operatorname{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y})\neq 0}} \frac{\||\boldsymbol{B}\boldsymbol{x}| - |\boldsymbol{B}\boldsymbol{y}|\|_2}{\operatorname{dist}_{\mathbb{H}}(\boldsymbol{x},\boldsymbol{y})} \\ &\geq \frac{1}{\|\boldsymbol{M}^{\frac{1}{2}}\|_2}\cdot\inf_{\substack{\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}}\in\mathbb{H}^d\\ \operatorname{dist}_{\mathbb{H}}(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}})\neq 0}} \frac{\||\boldsymbol{A}\widehat{\boldsymbol{x}}| - |\boldsymbol{A}\widehat{\boldsymbol{y}}|\|_2}{\operatorname{dist}_{\mathbb{H}}(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}})} = \frac{L_{\boldsymbol{A}}}{\|\boldsymbol{M}^{\frac{1}{2}}\|_2}.\end{aligned}$$

Note that $\boldsymbol{B}^*\boldsymbol{B} = \boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{M}^{-\frac{1}{2}} = \boldsymbol{I}_d$ and $\|\boldsymbol{M}^{\frac{1}{2}}\|_2 = \|\boldsymbol{A}\|_2$, so we have $U_{\boldsymbol{B}} = \|\boldsymbol{B}\|_2 = 1$ and

$$\beta_{\boldsymbol{B}} = \frac{U_{\boldsymbol{B}}}{L_{\boldsymbol{B}}} \leq \frac{\|\boldsymbol{M}^{\frac{1}{2}}\|_2}{L_{\boldsymbol{A}}} = \frac{\|\boldsymbol{A}\|_2}{L_{\boldsymbol{A}}} = \frac{U_{\boldsymbol{A}}}{L_{\boldsymbol{A}}} = \beta_{\boldsymbol{A}}.$$

We arrive at our conclusion.

□

2.3. **Basic properties of convex polygons.** Here we introduce some basic properties of convex polygons.

Definition 2.1. [26] *The width function of a convex polygon $P \subset \mathbb{R}^2$ in the direction $\boldsymbol{u} \in \mathbb{S}^1$ is defined by $w(P, \boldsymbol{u}) := h(P, \boldsymbol{u}) + h(P, -\boldsymbol{u})$, where $h(P, \boldsymbol{z}) := \max_{\boldsymbol{x} \in P} \langle \boldsymbol{x}, \boldsymbol{z} \rangle$ for any $\boldsymbol{z} \in \mathbb{S}^1$.*

Lemma 2.3. [26] *For any convex polygon $P \subset \mathbb{R}^2$, we have*

$$\mathrm{diam}(P) = \max_{\boldsymbol{u} \in \mathbb{S}^1} w(P, \boldsymbol{u}).$$

Lemma 2.4. *Assume that $E = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\} \subset \mathbb{R}^2$ is the edge set of a convex m-gon $P \subset \mathbb{R}^2$. Then*

$$r(P) = \frac{\max\limits_{\boldsymbol{u} \in \mathbb{S}^1} \sum_{j=1}^m |\langle \boldsymbol{e}_j, \boldsymbol{u} \rangle|}{2 \sum_{j=1}^m \|\boldsymbol{e}_j\|_2}. \tag{2.7}$$

Proof. We first consider the case when P is strongly convex. Since $r(P) = \frac{\mathrm{diam}(P)}{\mathrm{perim}(P)}$ and the perimeter $\mathrm{perim}(P)$ equals $\sum_{j=1}^m \|\boldsymbol{e}_j\|_2$, it is enough to prove

$$\mathrm{diam}(P) = \frac{1}{2} \max_{\boldsymbol{u} \in \mathbb{S}^1} \sum_{j=1}^m |\langle \boldsymbol{e}_j, \boldsymbol{u} \rangle|. \tag{2.8}$$

Without loss of generality, we assume that $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ are listed in counterclockwise order. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be the vertices of P such that $\boldsymbol{e}_j = \boldsymbol{v}_{j+1} - \boldsymbol{v}_j$ for each $j \in [m]$, where we denote $\boldsymbol{v}_{m+1} = \boldsymbol{v}_1$. For any $\boldsymbol{u} \in \mathbb{S}^1$, since P is convex, there exist $j_1, j_2 \in [m]$ depending on $\boldsymbol{u}$ such that

$$\boldsymbol{v}_{j_1} \in \operatorname*{argmin}_{\boldsymbol{x} \in P} \langle \boldsymbol{x}, \boldsymbol{u} \rangle \quad \text{and} \quad \boldsymbol{v}_{j_2} \in \operatorname*{argmax}_{\boldsymbol{x} \in P} \langle \boldsymbol{x}, \boldsymbol{u} \rangle. \tag{2.9}$$

Assume without loss of generality that $j_1 < j_2$. By Lemma 2.3, we have

$$\begin{aligned} \mathrm{diam}(P) = \max_{\boldsymbol{u} \in \mathbb{S}^1} w(P, \boldsymbol{u}) &= \max_{\boldsymbol{u} \in \mathbb{S}^1} (h(P, \boldsymbol{u}) + h(P, -\boldsymbol{u})) \\ &= \max_{\boldsymbol{u} \in \mathbb{S}^1} (\max_{\boldsymbol{x} \in P} \langle \boldsymbol{x}, \boldsymbol{u} \rangle + \max_{\boldsymbol{x} \in P} \langle \boldsymbol{x}, -\boldsymbol{u} \rangle) \\ &= \max_{\boldsymbol{u} \in \mathbb{S}^1} (\max_{\boldsymbol{x} \in P} \langle \boldsymbol{x}, \boldsymbol{u} \rangle - \min_{\boldsymbol{x} \in P} \langle \boldsymbol{x}, \boldsymbol{u} \rangle) \\ &= \max_{\boldsymbol{u} \in \mathbb{S}^1} \langle \boldsymbol{v}_{j_2} - \boldsymbol{v}_{j_1}, \boldsymbol{u} \rangle. \end{aligned} \tag{2.10}$$

We claim that

$$\langle \boldsymbol{e}_j, \boldsymbol{u} \rangle \begin{cases} \geq 0 & \text{if } j_1 \leq j \leq j_2 - 1, \\ \leq 0 & \text{else.} \end{cases} \tag{2.11}$$

Then we have

$$\begin{aligned} \langle \boldsymbol{v}_{j_2} - \boldsymbol{v}_{j_1}, \boldsymbol{u} \rangle = \langle \sum_{j=j_1}^{j_2-1} \boldsymbol{e}_j, \boldsymbol{u} \rangle &\overset{(a)}{=} \langle \frac{1}{2} \sum_{j=j_1}^{j_2-1} \boldsymbol{e}_j - \frac{1}{2} \sum_{j=1}^{j_1-1} \boldsymbol{e}_j - \frac{1}{2} \sum_{j=j_2}^{m} \boldsymbol{e}_j, \boldsymbol{u} \rangle \\ &\overset{(b)}{=} \frac{1}{2} \sum_{j=1}^{m} |\langle \boldsymbol{e}_j, \boldsymbol{u} \rangle|, \end{aligned} \tag{2.12}$$

where (a) follows from $\sum_{j=1}^m \boldsymbol{e}_j = \boldsymbol{0}$ and (b) follows from (2.11). Substituting (2.12) into (2.10), we arrive at (2.8).

It remains to prove (2.11). We first prove $\langle \boldsymbol{e}_j, \boldsymbol{u}\rangle \geq 0$ for each $j_1 \leq j \leq j_2 - 1$. By the definition of j_1 and j_2, we have

$$\langle \boldsymbol{e}_{j_1}, \boldsymbol{u}\rangle = \langle \boldsymbol{v}_{j_1+1}, \boldsymbol{u}\rangle - \langle \boldsymbol{v}_{j_1}, \boldsymbol{u}\rangle \geq 0,\ \ \langle \boldsymbol{e}_{j_2-1}, \boldsymbol{u}\rangle = \langle \boldsymbol{v}_{j_2}, \boldsymbol{u}\rangle - \langle \boldsymbol{v}_{j_2-1}, \boldsymbol{u}\rangle \geq 0. \tag{2.13}$$

Suppose, for contradiction, that there exists an integer s with $j_1+1 \leq s \leq j_2-2$ such that $\langle \boldsymbol{e}_s, \boldsymbol{u}\rangle < 0$. Then there must exist an integer $t \in \{s+1, \ldots, j_2-1\}$ satisfying

$$\langle \boldsymbol{e}_{t-1}, \boldsymbol{u}\rangle < 0, \quad \langle \boldsymbol{e}_t, \boldsymbol{u}\rangle \geq 0, \tag{2.14}$$

since otherwise we have $\langle \boldsymbol{e}_i, \boldsymbol{u}\rangle < 0$ for each $s+1 \leq i \leq j_2 - 1$, which contradicts with (2.13). Since P is strongly convex, for any $\boldsymbol{x} \in P$ there exist $\gamma_1, \gamma_2 \geq 0$ depending on $\boldsymbol{x}$ such that

$$\boldsymbol{x} - \boldsymbol{v}_t = \gamma_1 \cdot (-\boldsymbol{e}_{t-1}) + \gamma_2 \cdot \boldsymbol{e}_t.$$

Taking inner products with $\boldsymbol{u}$ and using (2.14), we have

$$\langle \boldsymbol{x}, \boldsymbol{u}\rangle - \langle \boldsymbol{v}_t, \boldsymbol{u}\rangle = \gamma_1\langle -\boldsymbol{e}_{t-1}, \boldsymbol{u}\rangle + \gamma_2\langle \boldsymbol{e}_t, \boldsymbol{u}\rangle \geq 0, \quad \forall \boldsymbol{x} \in P,$$

implying that

$$\langle \boldsymbol{v}_{j_1}, \boldsymbol{u}\rangle = \langle \boldsymbol{v}_t, \boldsymbol{u}\rangle = \min\nolimits_{\boldsymbol{x}\in P}\langle \boldsymbol{x}, \boldsymbol{u}\rangle. \tag{2.15}$$

It is well known that a linear function on a convex polygon attains its extrema at either a vertex or along an edge of the polygon [27]. Hence, we must have $t = j_1 + 1$. This contradicts with $t \geq s+1 \geq j_1+2$. Therefore, we have $\langle \boldsymbol{e}_j, \boldsymbol{u}\rangle \geq 0$ for each $j_1 \leq j \leq j_2 - 1$. Similarly, we can prove that $\langle \boldsymbol{e}_j, \boldsymbol{u}\rangle \leq 0$ for each $j < j_1$ and $j \geq j_2$. Hence, we arrive at (2.11).

We next consider the case when P is not strictly convex. By replacing all vectors in the same direction with their sum, we obtain a strictly convex k-gon $\widehat{P} \in \mathcal{P}_k$ with the edge set $\widehat{E} = \{\widehat{\boldsymbol{e}}_1, \ldots, \widehat{\boldsymbol{e}}_k\}$, where k is an integer less than m. Note that $\sum_{j=1}^m \|\boldsymbol{e}_j\|_2 = \sum_{j=1}^k \|\widehat{\boldsymbol{e}}_j\|_2$ and $\sum_{j=1}^m |\langle \boldsymbol{e}_j, \boldsymbol{u}\rangle| = \sum_{j=1}^k |\langle \widehat{\boldsymbol{e}}_j, \boldsymbol{u}\rangle|$ for any $\boldsymbol{u} \in \mathbb{S}^1$. Hence, we have

$$r(P) = r(\widehat{P}) = \frac{\max\limits_{\boldsymbol{u}\in\mathbb{S}^1} \sum_{j=1}^m |\langle \widehat{\boldsymbol{e}}_j, \boldsymbol{u}\rangle|}{2\sum_{j=1}^m \|\widehat{\boldsymbol{e}}_j\|_2} = \frac{\max\limits_{\boldsymbol{u}\in\mathbb{S}^1} \sum_{j=1}^m |\langle \boldsymbol{e}_j, \boldsymbol{u}\rangle|}{2\sum_{j=1}^m \|\boldsymbol{e}_j\|_2}.$$

Hence, we arrive at our conclusion. This completes the proof.

□

The following theorem discovered by Minkowski indicates that a zero-sum vector set in $\mathbb{R}^2$ with pairwise distinct directions uniquely determines a strictly convex polygon.

Theorem 2.3. [21] *Suppose $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \in \mathbb{R}^2$ are distinct unit vectors that span $\mathbb{R}^2$, and suppose that $\alpha_1, \ldots, \alpha_m > 0$. Then there exists a strictly convex m-gon P in $\mathbb{R}^2$, having edge unit outer normals $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ and corresponding edge lengths $\alpha_1, \ldots, \alpha_m > 0$, if and only if $\sum_{j=1}^m \alpha_j \boldsymbol{u}_j = \boldsymbol{0}$. Moreover, such a polygon P is unique up to translation.*

Corollary 2.1. *Suppose $E = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\} \subset \mathbb{R}^2$. Then there exists a strictly convex m-gon P in $\mathbb{R}^2$, having the edge set E if and only if $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ are nonzero vectors with pairwise distinct directions and $\sum_{j=1}^m \boldsymbol{e}_j = \boldsymbol{0}$. Moreover, such a polygon P is unique up to translation.*

Proof. Assume that $\sum_{j=1}^m \boldsymbol{e}_j = \boldsymbol{0}$. For all $j \in [m]$, rotating each vector $\frac{\boldsymbol{e}_j}{\|\boldsymbol{e}_j\|_2}$ counterclockwise by $\frac{\pi}{2}$ yields $\boldsymbol{u}_j$, while preserving the zero-sum property $\sum_{j=1}^m \|\boldsymbol{e}_j\|_2\boldsymbol{u}_j = \boldsymbol{0}$. It is easy to verify that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \in \mathbb{R}^2$ are distinct unit vectors that span $\mathbb{R}^2$. Therefore we can utilize Theorem 2.3 to obtain a strictly convex m-gon P with edge unit outer normals $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ and corresponding edge lengths $\|\boldsymbol{e}_1\|_2, \ldots, \|\boldsymbol{e}_m\|_2$, i.e., with edge vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$. Moreover, P is unique, disregarding the translation of polygons.

Conversely, assume that P is a strictly convex m-gon with the edge set E. Assume that $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_m$ are arranged counterclockwise. It is clear that $\sum_{j=1}^m \boldsymbol{e}_j=\boldsymbol{0}$ and each $\boldsymbol{e}_i$ is nonzero. Without loss of generality, it is enough to prove that $\boldsymbol{e}_1$ and $\boldsymbol{e}_i$ have distinct directions for any $i>1$. For each $i\in\{2,\ldots,m\}$, define α_i as the counterclockwise angle from $\boldsymbol{e}_1$ to $\boldsymbol{e}_i$. Since P is strictly convex, we have $\alpha_i\in(0,2\pi)$, and α_i is strictly increasing as i increases. Thus, there does not exist an integer $i\in\{2,\ldots,m\}$ such that $\alpha_i=0$, i.e., $\boldsymbol{e}_1$ and $\boldsymbol{e}_j$ have the same direction. This completes the proof.

□

3. Proofs of Theorem 1.2 and Corollary 1.1

3.1. The relationship between tight frames and convex polygons in $\mathbb{R}^2$.

Definition 3.1. *Let $\mathcal{A}=\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}\subset\mathbb{R}^d$. If there exist two distinct integers $i,j\in[m]$ such that $\boldsymbol{a}_i$ is a scalar multiple of $\boldsymbol{a}_j$, i.e., $\boldsymbol{a}_i=\lambda\boldsymbol{a}_j$ for some $\lambda\in\mathbb{R}$, then $\mathcal{A}$ is called reducible; otherwise, $\mathcal{A}$ is called irreducible.*

Lemma 3.1. *Assume that $\mathcal{A}=\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}\subset\mathcal{T}$, where $\mathcal{T}$ is defined in (1.9). Let $E=\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_m\}$, where each $\boldsymbol{e}_j=f(\boldsymbol{a}_j)$, where f is defined in (1.10). Then we have the following results:*

- (i) *The set $\mathcal{A}$ is irreducible if and only if the vectors in E are nonzero and have pairwise distinct directions.*
- (ii) *The set $\mathcal{A}$ is a tight frame if and only if $\sum_{j=1}^m \boldsymbol{e}_j=\boldsymbol{0}$.*
- (iii) *The set $\mathcal{A}$ is an irreducible tight frame if and only if E is the edge set of a strictly convex m-gon $P\in\mathcal{P}_m$.*
- (iv) *Assume that the set $\mathcal{A}$ forms a tight frame. Let $\boldsymbol{A}=[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m]^T$. If $\mathcal{A}$ is irreducible, then*
$$\beta_{\boldsymbol{A}}=\frac{1}{\sqrt{1-2\cdot r(P)}},$$
where P is the strictly convex m-gon defined in (iii). *If $\mathcal{A}$ is reducible, then the condition number of $\boldsymbol{A}$ is not the minimum value attainable among all $m\times 2$ matrices.*

Proof. For clarity, we write $\boldsymbol{a}_j=t_j(\cos\phi_j,\sin\phi_j)^T$ for each $j\in[m]$, where $t_j\geq 0$ and $\phi_j\in(0,\pi]$.

(i) We prove the equivalence in two directions. Assume that the set $\mathcal{A}$ is irreducible. First, we show that vectors in E are nonzero. It is important to note that $\boldsymbol{e}_j=f(\boldsymbol{a}_j)$ is the zero vector if and only if $\boldsymbol{a}_j$ is the zero vector. Since $\mathcal{A}$ is irreducible, by definition, it does not contain the zero vector. Consequently, given this property of f, the set E also does not contain the zero vector. Next, we show that vectors in E have pairwise distinct directions. Suppose, for contradiction, that there exist distinct vectors $\boldsymbol{e}_i=f(\boldsymbol{a}_i)$ and $\boldsymbol{e}_j=f(\boldsymbol{a}_j)$ (with $i\neq j$) that have the same direction. This implies that their associated directional parameters are equal, i.e., $\phi_j=\phi_i$. This condition is, in turn, equivalent to $\boldsymbol{a}_j=\lambda\boldsymbol{a}_i$ for some nonzero scalar $\lambda\in\mathbb{R}$. However, this directly contradicts the definition of an irreducible set $\mathcal{A}$, which requires that no two distinct vectors are scalar multiples of each other. Hence, we conclude that if $\mathcal{A}$ is irreducible, then the vectors in E are nonzero and have pairwise distinct directions.

Conversely, if the vectors in E are nonzero and have pairwise distinct directions, the irreducibility of $\mathcal{A}$ can be established by an analogous argument.

(ii) We define
$$a=\sum_{j=1}^m t_j^2\cos^2\phi_j,\quad b=\sum_{j=1}^m t_j^2\sin^2\phi_j,\quad\text{and}\quad c=\sum_{j=1}^m t_j^2\cos\phi_j\sin\phi_j.$$

A direct computation shows that $\sum_{j=1}^m \boldsymbol{e}_j = (a-b, 2c)^T$ and

$$\sum_{j=1}^m \boldsymbol{a}_j\boldsymbol{a}_j^T = \begin{pmatrix} \sum_{j=1}^m t_j^2\cos^2\phi_j & \sum_{j=1}^m t_j^2\cos\phi_j\sin\phi_j \\ \sum_{j=1}^m t_j^2\cos\phi_j\sin\phi_j & \sum_{j=1}^m t_j^2\sin^2\phi_j \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}. \tag{3.1}$$

Therefore, $\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_m\}$ is a tight frame if and only if $a=b$ and $c=0$, which is equivalent to $\sum_{j=1}^m \boldsymbol{e}_j = \mathbf{0}$.

(iii) Assume that $\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_m\}$ is an irreducible tight frame. It follows from (i) and (ii) that $\sum_{j=1}^m \boldsymbol{e}_j = \mathbf{0}$ and $\boldsymbol{e}_1,\dots,\boldsymbol{e}_m$ have pairwise distinct directions. Since $\boldsymbol{e}_j \neq \mathbf{0}$ for all $j\in[m]$, according to Corollary 2.1, there exists a strictly convex m-gon P with the edge set E.

Assume that E is the edge set of a strictly convex m-gon $P\in\mathcal{P}_m$. By Corollary 2.1 we have $\sum_{j=1}^m \boldsymbol{e}_j = \mathbf{0}$, and $\boldsymbol{e}_1,\dots,\boldsymbol{e}_m$ are nonzero vectors with pairwise distinct directions. If follows from (i) and (ii) that $\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_m\}$ is an irreducible tight frame.

(iv) Since $\mathcal{A}$ is a tight frame, by Lemma 2.1 we have

$$\beta_{\boldsymbol{A}} = \left(1 - \frac{1}{C}\max_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{S}^1,\langle\boldsymbol{x},\boldsymbol{y}\rangle=0}\sum_{j=1}^m \left|\boldsymbol{x}^T\boldsymbol{a}_j\boldsymbol{a}_j^T\boldsymbol{y}\right|\right)^{-\frac{1}{2}}, \tag{3.2}$$

where $C=\frac{1}{2}\sum_{j=1}^m\|\boldsymbol{a}_j\|_2^2 = \frac{1}{2}\sum_{j=1}^m\|\boldsymbol{e}_j\|_2$. Let

$$\boldsymbol{x}=(\cos\theta,\sin\theta)^T,\quad \boldsymbol{y}=(-\sin\theta,\cos\theta)^T,\quad\text{and}\quad \boldsymbol{u}=(\sin2\theta,\cos2\theta)^T,$$

where $\theta\in[0,2\pi)$. A direct calculation shows that for each $j\in[m]$

$$\left|\boldsymbol{x}^T\boldsymbol{a}_j\boldsymbol{a}_j^T\boldsymbol{y}\right| = \frac{t_j^2}{2}|\sin(2\theta-2\phi_j)| = \frac{1}{2}|\langle\boldsymbol{e}_j,\boldsymbol{u}\rangle|.$$

Therefore, we have

$$\max_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{S}^1,\langle\boldsymbol{x},\boldsymbol{y}\rangle=0}\sum_{j=1}^m\left|\boldsymbol{x}^T\boldsymbol{a}_j\boldsymbol{a}_j^T\boldsymbol{y}\right| = \max_{\theta\in[0,2\pi)}\sum_{j=1}^m\frac{t_j^2}{2}|\sin(2\theta-2\phi_j)| = \frac{1}{2}\cdot\max_{\boldsymbol{u}\in\mathbb{S}^1}\sum_{j=1}^m|\langle\boldsymbol{e}_j,\boldsymbol{u}\rangle|. \tag{3.3}$$

Substituting (3.3) and $C=\frac{1}{2}\sum_{j=1}^m\|\boldsymbol{e}_j\|_2$ into (3.2), we obtain

$$\beta_{\boldsymbol{A}} = \left(1-\frac{\max_{\boldsymbol{u}\in\mathbb{S}^1}\sum_{j=1}^m|\langle\boldsymbol{e}_j,\boldsymbol{u}\rangle|}{\sum_{j=1}^m\|\boldsymbol{e}_j\|_2}\right)^{-\frac{1}{2}}. \tag{3.4}$$

If $\mathcal{A}$ is irreducible, it follows from (iii) that E is the edge set of a strictly convex m-gon $P\in\mathcal{P}_m$. Combining (3.4) with Lemma 2.4, we obtain $\beta_{\boldsymbol{A}} = \frac{1}{\sqrt{1-2\cdot r(P)}}$.

We next prove that the condition number of $\boldsymbol{A}$ is not minimal if $\mathcal{A}$ is reducible. Assume $\mathcal{A}$ is reducible. Then the set E contains zero vectors or vectors that have the same direction. By deleting zero vectors from E and replacing all vectors in the same direction with their sum, we obtain a new set $\widehat{E}=\{\widehat{\boldsymbol{e}}_1,\dots,\widehat{\boldsymbol{e}}_k\}$ with $k<m$. The vectors in $\widehat{E}$ are nonzero and have pairwise distinct directions. Note that $\sum_{j=1}^k\widehat{\boldsymbol{e}}_j = \sum_{j=1}^m\boldsymbol{e}_j = 0$. By Corollary 2.1, there exists a strictly convex k-gon $\widehat{P}\in\mathcal{P}_k$ with the edge set $\widehat{E}$. Note that $\sum_{j=1}^m\|\boldsymbol{e}_j\|_2 = \sum_{j=1}^k\|\widehat{\boldsymbol{e}}_j\|_2$ and $\sum_{j=1}^m|\langle\boldsymbol{e}_j,\boldsymbol{u}\rangle| = \sum_{j=1}^k|\langle\widehat{\boldsymbol{e}}_j,\boldsymbol{u}\rangle|$ for any $\boldsymbol{u}\in\mathbb{S}^1$. Combining with (3.4) and Lemma 2.4, we have

$$\beta_{\boldsymbol{A}} = \frac{1}{\sqrt{1-2\cdot r(\widehat{P})}}. \tag{3.5}$$

Let $Q \in \mathcal{P}_m$ be an optimal strictly convex m-gon. Denote the edge set of Q by $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m\}$. Let $\boldsymbol{B} = [f^{-1}(\boldsymbol{q}_1), \ldots, f^{-1}(\boldsymbol{q}_m)]^T \in \mathbb{R}^{m\times 2}$, where $f(\cdot)$ is defined in (1.10). According to (iii), the set $\{f^{-1}(\boldsymbol{q}_1), \ldots, f^{-1}(\boldsymbol{q}_m)\}$ forms an irreducible tight frame. Then, we have

$$\beta_{\boldsymbol{B}} = \frac{1}{\sqrt{1 - 2 \cdot r(Q)}}. \tag{3.6}$$

Recall that any optimal convex m-gon is strictly convex [25]. Since $\widehat{P} \in \mathcal{P}_k$ is a convex m-gon but not strictly convex (it has a flat angle), it cannot be optimal. Hence, we have $r(\widehat{P}) > r(Q)$.

Combining with (3.5) and (3.6), we have $\beta_{\boldsymbol{A}} > \beta_{\boldsymbol{B}}$. Therefore, the condition number of $\boldsymbol{A}$ is not minimal. This completes the proof.

□

3.2. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. (i) Assume that $\mathcal{A}$ is an optimal tight frame, i.e., the matrix $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]^T$ has the minimal condition number and its rows form a tight frame. It follows from Lemma 3.1 (iv) that $\mathcal{A}$ is irreducible. Then Lemma 3.1 (iii) shows that the set $\{f(\boldsymbol{a}_1), \ldots, f(\boldsymbol{a}_m)\}$ is the edge set of a strictly convex m-gon $P \subset \mathbb{R}^2$, and we have $\beta_{\boldsymbol{A}} = \frac{1}{\sqrt{1-2\cdot r(P)}}$. We prove that P is optimal by contradiction. Suppose, to the contrary, that P is not optimal. Then there exists a strictly convex m-gon $Q \subset \mathbb{R}^2$ such that $r(Q) < r(P)$. Denote the edge vectors set of Q by $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m\}$. Let $\boldsymbol{B} = [f^{-1}(\boldsymbol{q}_1), \ldots, f^{-1}(\boldsymbol{q}_m)]^T \in \mathbb{R}^{m\times 2}$. By Lemma 3.1 (iii) we have $\beta_{\boldsymbol{B}} = \frac{1}{\sqrt{1-2\cdot r(Q)}}$. Since $r(Q) < r(P)$, we have $\beta_{\boldsymbol{B}} < \beta_{\boldsymbol{A}}$. This contradicts with the fact that $\boldsymbol{A}$ has the minimal condition number. Hence, P is optimal.

(ii) Assume that $\{f(\boldsymbol{a}_1), \ldots, f(\boldsymbol{a}_m)\}$ is the edge set of an optimal m-gon $P \in \mathcal{P}_m$. Then P is strictly convex. According to Lemma 3.1 (iii) and (iv), the set $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ is an irreducible tight frame, and we have $\beta_{\boldsymbol{A}} = \frac{1}{\sqrt{1-2\cdot r(P)}}$. We prove that $\boldsymbol{A}$ has the minimal condition number by contradiction. Assume that $\beta_{\boldsymbol{B}} < \beta_{\boldsymbol{A}}$, where $\boldsymbol{B} \in \mathbb{R}^{m\times 2}$ is a matrix with the minimal condition number and its rows form a tight frame. By Lemma 3.1 (iii), the rows of $\boldsymbol{B}$ is irreducible, and there is a strictly convex m-gon $Q \subset \mathbb{R}^2$ such that $\beta_{\boldsymbol{B}} = \frac{1}{\sqrt{1-2\cdot r(Q)}}$. Since $\beta_{\boldsymbol{B}} < \beta_{\boldsymbol{A}}$, we have $r(Q) < r(P)$. This contradicts with P being an optimal convex m-gon. Hence, $\boldsymbol{A}$ has the minimal condition number, and its rows form a tight frame.

□

3.3. **Proof of Corollary 1.1.**

Proof of Corollary 1.1. By Lemma 2.2 we have

$$\min_{\boldsymbol{A}\in\mathbb{R}^{m\times 2}} \beta_{\boldsymbol{A}} = \min_{\boldsymbol{A}\in\mathbb{R}^{m\times 2}, \boldsymbol{A}^T\boldsymbol{A}=\boldsymbol{I}_2} \beta_{\boldsymbol{A}}.$$

Let $\boldsymbol{B} \in \mathbb{R}^{m\times 2}$ be such that $\boldsymbol{B}^T\boldsymbol{B} = \boldsymbol{I}_2$ and the condition number of $\boldsymbol{B}$ is minimal. By Theorem 1.2, there is an optimal convex m-gon $P \subset \mathbb{R}^2$ such that

$$\min_{\boldsymbol{A}\in\mathbb{R}^{m\times 2}} \beta_{\boldsymbol{A}} = \beta_{\boldsymbol{B}} = \frac{1}{\sqrt{1 - 2 \cdot r(P)}}. \tag{3.7}$$

(i) For any integers $m \geq 3$ which has an odd factor, by Theorem 1.1 (i) we have $r(P) = \frac{1}{2m\cdot\sin\frac{\pi}{2m}}$. Hence, combining with (3.7), we obtain

$$\min_{\boldsymbol{A}\in\mathbb{R}^{m\times 2}} \beta_{\boldsymbol{A}} = \frac{1}{\sqrt{1-\frac{1}{m\cdot\sin\frac{\pi}{2m}}}}.$$

(ii) For $m = 2^s$ with $s \geq 2$, by Theorem 1.1 (ii) we have $r(P) > \frac{1}{2m\cdot\sin\frac{\pi}{2m}}$. Hence, combining with (3.7), we obtain

$$\min_{\boldsymbol{A}\in\mathbb{R}^{m\times 2}} \beta_{\boldsymbol{A}} > \frac{1}{\sqrt{1-\frac{1}{m\cdot\sin\frac{\pi}{2m}}}}.$$

(iii) Recall that P_m is the regular m-gon. Theorem 1.1 (iii) shows that $r(P_m) = r(P)$ if and only if m is odd, and P_m is uniquely optimal if and only if m is a prime. According to Theorem 1.2, the matrix constructed from P_m corresponds to the harmonic frame $\boldsymbol{E}_m$; in particular, $\boldsymbol{E}_m$ consists of the vectors $f^{-1}(\boldsymbol{e}_j)$ obtained from the edge vectors $\boldsymbol{e}_j$ of $P_m, j = 1, \ldots, m$. Combining Theorem 1.2 and Theorem 1.1, we conclude that $\boldsymbol{E}_m$ has the minimal condition number if and only if m is odd. Moreover, $\boldsymbol{E}_m$ is the unique matrix with minimal condition number if and only if m is a prime. □

4. Proofs of Theorem 1.3 and Theorem 1.4

4.1. **Characterization of optimal convex polygons in $\mathbb{R}^2$ for the perimeter-maximizing isodiametric problem.** We begin by recalling the definition of a Reuleaux polygon, which will be used in the sequel.

Definition 4.1. *A Reuleaux polygon is a constant-width curve constructed from circular arcs, all having the same radius.*

Here, we present several key properties of Reuleaux polygons given by [24, 11, 22, 19, 20].

Theorem 4.1. [24, 19, 20] *Let R be a Reuleaux polygon. The following holds:*

(i) *The boundary of R is composed of r circular arcs, where $r \geq 3$ is an odd integer.*
(ii) *Connecting all pairs of vertices at maximal distance from one another in R generates a star polygon S, which is a closed planar curve composed of r line segments, each intersecting all the others. The total sum of the interior angles at the vertices of S equals π.*

Theorem 4.2. [25] *Suppose $m \geq 3$ is an integer with an odd factor. A convex m-gon P is optimal if and only if P is equilateral and P can be inscribed in a Reuleaux polygon R with* $\operatorname{diam}(R) = \operatorname{diam}(P)$ *such that every vertex of R is also a vertex of P.*

Remark 4.1. *When $m \geq 3$ has an odd factor, one can easily construct a convex m-gon that satisfies the criteria of Theorem 4.2, using the procedure described in* [25, 24, 9].

Remark 4.2. *Let P be an optimal convex m-gon, where $m \geq 3$ is an integer with an odd factor. By Theorem 4.2 and Theorem 4.1 (i), P is inscribed in a Reuleaux polygon R with r boundary arcs, where r is an odd integer with $3 \leq r \leq m$. Label the vertices of R as $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r$ in counterclockwise order, and extend the indexing cyclically by setting $\boldsymbol{w}_{r+i} = \boldsymbol{w}_i$ for each $i \in [r]$. Define $h = \frac{r-1}{2}$. Then for each $i \in [r]$, the vertex $\boldsymbol{w}_i$ is the the center of the arc $\widehat{\boldsymbol{w}_{i+h}\boldsymbol{w}_{i+h+1}}$ on the boundary of R. This can be seen as follows. By Theorem 4.1 (ii), we can start at $\boldsymbol{s}_1 := \boldsymbol{w}_1$ and construct a star polygon S by sequentially connecting pairs of vertices in R at maximal distance in the counterclockwise direction. Let $\boldsymbol{s}_i$ denote the i-th vertex of R visited during this construction. According to* [25, Page 256]*, the*

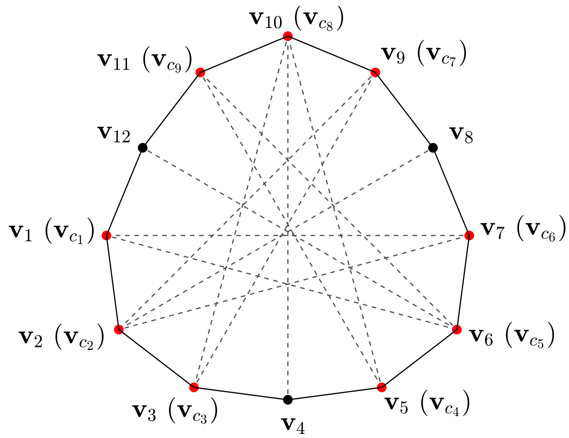

FIGURE 3. One of the optimal dodecagon as in Figure 2 (a2), with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{12}$. The corresponding Reuleaux polygon consists of 9 circular arcs with vertices $\boldsymbol{v}_{c_1}, \ldots, \boldsymbol{v}_{c_9}$ indicated in red dots. The dashed lines connect pairs of vertices at maximal distance.

vertices $\boldsymbol{s}_1, \boldsymbol{s}_3, \ldots, \boldsymbol{s}_r, \boldsymbol{s}_2, \boldsymbol{s}_4, \ldots, \boldsymbol{s}_{r-1}$ are ordered counterclockwise. Comparing this with the labeling $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r$, we have

$$\boldsymbol{w}_1 = \boldsymbol{s}_1, \boldsymbol{w}_2 = \boldsymbol{s}_3, \ldots, \boldsymbol{w}_{\frac{r+1}{2}} = \boldsymbol{s}_r, \boldsymbol{w}_{\frac{r+1}{2}+1} = \boldsymbol{s}_2, \ldots, \boldsymbol{w}_r = \boldsymbol{s}_{r-1}.$$

Set $\boldsymbol{s}_0 := \boldsymbol{s}_r$ and $\boldsymbol{s}_{r+1} := \boldsymbol{s}_1$. Note that $\boldsymbol{s}_i$ is the center of the arc $\widehat{\boldsymbol{s}_{i-1}\boldsymbol{s}_{i+1}}$ for each $i \in [r]$. Translating this to the $\boldsymbol{w}_i$ notation, we conclude that for each $i \in [r]$, $\boldsymbol{w}_i$ is the center of the arc $\widehat{\boldsymbol{w}_{i+h}\boldsymbol{w}_{i+h+1}}$, where $h = \frac{r-1}{2}$.

The following lemma, motivated by [24], establishes a characterization of the directional angles of the edge vectors for optimal convex polygons.

Lemma 4.1. *Assume that $m \geq 3$ is an integer with an odd factor. Let P be a convex m-gon with edge vectors $\{\boldsymbol{e}_1, \ldots \boldsymbol{e}_m\}$. For each $i \in [m]$, define $\alpha_i \in [0, 2\pi)$ as the counterclockwise angle from $\boldsymbol{e}_1$ to $\boldsymbol{e}_i$, and let $\psi_i \in (0, \pi]$ satisfy $\psi_i \equiv \alpha_i \pmod{\pi}$. Let $\mathcal{E}_m$ denote the collection of all optimal strictly convex m-gons in $\mathbb{R}^2$. If $P \in \mathcal{E}_m$ then*

$$\{\psi_1, \psi_2, \ldots, \psi_m\} = \left\{ \frac{j\pi}{m} \ : \ j \in [m] \right\}. \tag{4.1}$$

Proof. Assume that $P \in \mathcal{E}_m$. It follows from Theorem 4.2 that P is equilateral. We now prove that P satisfies (4.1). Assume without loss of generality that $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ are arranged counterclockwise. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be the vertices of P such that $\boldsymbol{e}_j = \boldsymbol{v}_{j+1} - \boldsymbol{v}_j$ for each $j \in [m-1]$ and $\boldsymbol{e}_m = \boldsymbol{v}_1 - \boldsymbol{v}_m$. For each $i \in [m]$, we denote $\boldsymbol{v}_{m+i} = \boldsymbol{v}_i$, and let θ_i be the counterclockwise angle from $\boldsymbol{e}_{i-1}$ to $\boldsymbol{e}_i$, where $\boldsymbol{e}_0 := \boldsymbol{e}_m$. Then we have $\alpha_1 = 0$ and $\alpha_i = \alpha_{i-1} + \theta_i$ for each $i \in \{2, \ldots, m\}$. Moreover, since P is strictly convex, we have $\theta_i \in (0, \pi)$, $\forall i \in [m]$.

We first calculate each θ_i by plane geometry. By Theorem 4.2, P is inscribed in a Reuleaux polygon R with the same diameter, and every vertex of R is also a vertex of P. By Theorem 4.1 (i), R consists of r circular arcs, where $3 \leq r \leq m$ is an odd number. Denote these circular arcs by $\widehat{\boldsymbol{v}_{c_1}\boldsymbol{v}_{c_2}}, \widehat{\boldsymbol{v}_{c_2}\boldsymbol{v}_{c_3}}, \ldots, \widehat{\boldsymbol{v}_{c_r}\boldsymbol{v}_{c_1}}$, where $1 \leq c_1 < c_2 < \cdots < c_{r-1} < c_r \leq m$. Without loss of generality, we may assume that $c_1 = 1$, because the choice of the edge $\boldsymbol{e}_1$ can be arbitrary and does not affect the outcome of the lemma. For convenience, we denote

$$c_{r+i} := c_i + m, \quad \forall i \in [r].$$

By Remark 4.2, for each $i \in [r]$, $\boldsymbol{v}_{c_i}$ is the center of the arc $\widehat{\boldsymbol{v}_{c_{i+h}}\boldsymbol{v}_{c_{i+h+1}}}$, where $h = \frac{r-1}{2}$ is a fixed integer. See Figure 3 for a clear and intuitive illustration. Since P is equilateral and the sum of the interior angles at the vertices of R equals π, the interior angle of the arc $\widehat{\boldsymbol{v}_{c_{i+h}}\boldsymbol{v}_{c_{i+h+1}}}$ centered on $\boldsymbol{v}_{c_i}$ is $(c_{i+h+1} - c_{i+h}) \cdot \frac{\pi}{m}$ for each $i \in [r]$. By plane geometry, if $i \notin \{c_1, \ldots, c_r\}$ then we have

$$\theta_i = \frac{\pi}{m}. \tag{4.2}$$

Moreover, for any integer $i \in [r]$, we have

$$\theta_{c_i} = \pi - (\pi - \frac{\pi}{m}) + (c_{i+h+1} - c_{i+h}) \cdot \frac{\pi}{m} = (c_{i+h+1} - c_{i+h} + 1) \cdot \frac{\pi}{m}. \tag{4.3}$$

We next calculate $\alpha_1, \ldots, \alpha_m$. Recall that $\alpha_i = \alpha_{i-1} + \theta_i$ for each $i \in \{2, \ldots, m\}$. Hence, using (4.2) we have

$$\alpha_{L+c_i} = \alpha_{c_i} + L \cdot \frac{\pi}{m}, \quad \forall\, i \in [r],\ \forall\, 0 \le L \le c_{i+1} - c_i - 1. \tag{4.4}$$

For any integer $2 \le i \le r$, using (4.3) and (4.4) we have

$$\begin{aligned}\alpha_{c_i} &= \alpha_{c_i - 1} + \theta_{c_i} = \alpha_{c_{i-1}} + (c_i - c_{i-1} - 1) \cdot \frac{\pi}{m} + \theta_{c_i} \\ &= \alpha_{c_{i-1}} + (c_i - c_{i-1} + c_{i+h+1} - c_{i+h}) \cdot \frac{\pi}{m}.\end{aligned} \tag{4.5}$$

Since $c_1 = 1$ and $\alpha_1 = 0$, repeatedly using (4.5), we obtain

$$\alpha_{c_i} = \alpha_{c_1} + (c_i - c_1 + c_{i+h+1} - c_{h+2}) \cdot \frac{\pi}{m} = (c_i + c_{i+h+1} - c_{h+2} - 1) \cdot \frac{\pi}{m}. \tag{4.6}$$

Then we can use (4.6) and (4.4) to obtain α_l, $l = 1, \ldots, m$.

Now we turn to prove (4.1). Note that

$$\begin{aligned}\alpha_{c_{h+2}} &\overset{(a)}{=} (c_{h+2} + c_{2h+3} - c_{h+2} - 1) \cdot \frac{\pi}{m} = \pi + (c_2 - 1) \cdot \frac{\pi}{m} > \pi, \\ \alpha_{c_{h+2}-1} &\overset{(b)}{=} \alpha_{c_{h+2}} - (c_{2h+3} - c_{2h+2} + 1) \cdot \frac{\pi}{m} = \alpha_{c_{h+2}} - c_2 \cdot \frac{\pi}{m} = \pi - \frac{\pi}{m},\end{aligned}$$

where (a) follows from (4.6) and (b) follows from (4.5). Hence, by the definition of each α_l, we have

$$0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{c_{h+2}-1} < \pi < \alpha_{c_{h+2}} < \cdots < \alpha_m < 2\pi. \tag{4.7}$$

Since each α_l is a multiple of $\frac{\pi}{m}$, and by the definition $\psi_i \equiv \alpha_i \pmod{\pi}$, to prove (4.1), it is sufficient to show that there are no integers i, j such that $1 \le i < c_{h+2} \le j \le m$ and $\alpha_j = \alpha_i + \pi$. We prove by contradiction. Assume that such integers i and j exist. Then there is an integer t with $h + 2 \le t \le r$ such that $c_t \le j < c_{t+1}$. Using (4.4), (4.6) and $c_{t+h+1} = c_{t-h+r} = c_{t-h} + m$, we obtain

$$\alpha_j = \alpha_{c_t} + (j - c_t) \cdot \frac{\pi}{m} = (c_{t+h+1} + j - c_{h+2} - 1) \cdot \frac{\pi}{m} = \pi + (c_{t-h} + j - c_{h+2} - 1) \cdot \frac{\pi}{m}. \tag{4.8}$$

However, note that

$$\begin{aligned}\alpha_{c_{t-h}} &\overset{(a)}{=} (c_{t-h} + c_{t+1} - c_{h+2} - 1) \cdot \frac{\pi}{m} \\ \alpha_{c_{t-h}-1} &\overset{(b)}{=} \alpha_{c_{t-h}} - (c_{t+1} - c_t + 1) \cdot \frac{\pi}{m} = (c_{t-h} + c_t - c_{h+2} - 2) \cdot \frac{\pi}{m},\end{aligned}$$

where (a) follows from (4.6) and (b) follows from (4.5). Combining with (4.8) and $c_t \le j < c_{t+1}$, we have

$$\alpha_{c_{t-h}-1} < \alpha_i = \alpha_j - \pi < \alpha_{c_{t-h}}.$$

By (4.7) we have $c_{t-h} - 1 < i < c_{t-h}$. Since no integer lies between two consecutive integers, such integer i does not exist, and we have a contradiction. This completes the proof of (4.1). $\square$

Now we can present a proof of Theorem 1.3.

Proof of Theorem 1.3. (i) We first prove that there exists a convex m-gon P with the edge set $\{\varepsilon_1\boldsymbol{\mu}_1, \ldots, \varepsilon_m\boldsymbol{\mu}_m\}$. Since $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)^T$ satisfies (1.11), we have

$$0 = \sum_{j=1}^m \varepsilon_j \zeta_m^j = \sum_{j=1}^m \varepsilon_j \cos\frac{j\pi}{m} + \mathrm{i}\sum_{j=1}^m \varepsilon_j \sin\frac{j\pi}{m},$$

where $\zeta_m = e^{\mathrm{i}\pi/m}$. It follows that

$$\sum_{j=1}^m \varepsilon_j \boldsymbol{\mu}_j = (\sum_{j=1}^m \varepsilon_j \cos\frac{j\pi}{m}, \sum_{j=1}^m \varepsilon_j \sin\frac{j\pi}{m})^T = \mathbf{0}. \tag{4.9}$$

Combining with the fact that $\varepsilon_1\boldsymbol{\mu}_1, \ldots, \varepsilon_m\boldsymbol{\mu}_m$ have pairwise distinct directions, we can utilize Corollary 2.1 to obtain a strictly convex m-gon P with the edge set $\{\varepsilon_1\boldsymbol{\mu}_1, \ldots, \varepsilon_m\boldsymbol{\mu}_m\}$, and P is unique up to translation.

We next prove that $P \in \mathcal{E}_m$. For each $i \in [m]$, define $\alpha_i \in [0, 2\pi)$ as the counterclockwise angle from $\varepsilon_1\boldsymbol{\mu}_1$ to $\varepsilon_i\boldsymbol{\mu}_i$, and let $\psi_i \in (0, \pi]$ satisfy $\psi_i \equiv \alpha_i \pmod{\pi}$. Since each $\boldsymbol{\mu}_j = (\cos\frac{j\pi}{m}, \sin\frac{j\pi}{m})^T$, we have

$$\{\psi_1, \psi_2, \ldots, \psi_m\} = \left\{\frac{j\pi}{m} \ : \ j \in [m]\right\}. \tag{4.10}$$

Let $Q \in \mathcal{E}_m$ be an optimal m-polygon with the edge set $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m\}$ such that $\boldsymbol{q}_1 = \varepsilon_1\boldsymbol{\mu}_1$. By Theorem 4.2 we see that Q is equilateral, so we have $\|\boldsymbol{q}_j\|_2 = 1$ for each $j \in [m]$. For each $i \in [m]$, define $\beta_i \in [0, 2\pi)$ as the counterclockwise angle from $\boldsymbol{q}_1$ to $\boldsymbol{q}_i$, and let $\phi_i \in (0, \pi]$ satisfy $\phi_i \equiv \beta_i \pmod{\pi}$. By Lemma 4.1 we have

$$\{\phi_1, \phi_2, \ldots, \phi_m\} = \left\{\frac{j\pi}{m} \ : \ j \in [m]\right\}.$$

Therefore, there exists a permutation $\{s_1, s_2, \ldots, s_m\}$ of $[m]$ such that $\psi_i = \phi_{s_i}$ for each $i \in [m]$. This means that for each $i \in [m]$ we have $\alpha_i \equiv \beta_{s_i} \pmod{\pi}$. Recall that $\boldsymbol{q}_1 = \varepsilon_1\boldsymbol{\mu}_1 = \varepsilon_1(\cos\frac{\pi}{m}, \sin\frac{\pi}{m})^T$. Let $I := \{i \in [m] \mid \alpha_i = \beta_{s_i}\}$. If $i \in I$, then

$$\boldsymbol{q}_{s_i} = \varepsilon_1(\cos(\frac{\pi}{m} + \beta_{s_i}), \sin(\frac{\pi}{m} + \beta_{s_i}))^T = \varepsilon_1(\cos(\frac{\pi}{m} + \alpha_i), \sin(\frac{\pi}{m} + \alpha_i))^T = \varepsilon_i\boldsymbol{\mu}_i.$$

Otherwise, if $i \in [m] \setminus I$, then $\alpha_i = \beta_{s_i} - \pi$ or $\alpha_i = \beta_{s_i} + \pi$. So we have

$$\boldsymbol{q}_{s_i} = \varepsilon_1(\cos(\frac{\pi}{m} + \beta_{s_i}), \sin(\frac{\pi}{m} + \beta_{s_i}))^T = -\varepsilon_1(\cos(\frac{\pi}{m} + \alpha_i), \sin(\frac{\pi}{m} + \alpha_i))^T = -\varepsilon_i\boldsymbol{\mu}_i.$$

Hence, we have $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m\} = \{\varepsilon_i\boldsymbol{\mu}_i\}_{i\in I} \cup \{-\varepsilon_i\boldsymbol{\mu}_i\}_{i\in[m]\setminus I}$. According to (2.7) in Lemma 2.4, we have

$$r(P) = \frac{\max_{\boldsymbol{u}\in\mathbb{S}^1} \sum_{i=1}^m |\langle\boldsymbol{\mu}_i, \boldsymbol{u}\rangle|}{2\sum_{i=1}^m \|\boldsymbol{\mu}_i\|_2} = \frac{\max_{\boldsymbol{u}\in\mathbb{S}^1} \sum_{i=1}^m |\langle\boldsymbol{q}_i, \boldsymbol{u}\rangle|}{2\sum_{i=1}^m \|\boldsymbol{q}_i\|_2} = r(Q).$$

Since $r(Q)$ is minimal, we see that P is also optimal, i.e., $P \in \mathcal{E}_m$. This completes the proof.

(ii) Let $P \in \mathcal{E}_m$ be an optimal polygon with edge vectors $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$. Without loss of generality, we may assume that $\boldsymbol{e}_1 = (1, 0)^T$ after a proper scaling and rotation. For each $i \in [m]$, define $\alpha_i \in [0, 2\pi)$ as the counterclockwise angle from $\boldsymbol{e}_1$ to $\boldsymbol{e}_i$, and let $\psi_i \in (0, \pi]$ satisfy $\psi_i \equiv \alpha_i \pmod{\pi}$. By Lemma 4.1, we see that P is equilateral and

$$\{\psi_1, \psi_2, \ldots, \psi_m\} = \left\{\frac{j\pi}{m} \ : \ j \in [m]\right\}. \tag{4.11}$$

Hence, there exists a permutation $\{s_1,\ldots,s_m\}$ of $[m]$ such that $\psi_{s_j}=\frac{j\pi}{m}$ for each $j\in[m]$. Let $I:=\{j\in[m]:\alpha_{s_j}=\psi_{s_j}\}$, and let $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_m)^T$, where $\varepsilon_j=1$ if $j\in I$ and $\varepsilon_j=-1$ if $j\notin I$. Recall that $\boldsymbol{e}_1=(1,0)^T$ and α_i is the counterclockwise angle from $\boldsymbol{e}_1$ to $\boldsymbol{e}_i$. Then we have

$$\begin{aligned}\boldsymbol{e}_{s_j}&=(\cos\alpha_{s_j},\sin\alpha_{s_j})^T=(\cos\psi_{s_j},\sin\psi_{s_j})^T=(\cos\frac{j\pi}{m},\sin\frac{j\pi}{m})^T=\boldsymbol{\mu}_j\quad\text{for } j\in I,\\ \boldsymbol{e}_{s_j}&=(\cos\alpha_{s_j},\sin\alpha_{s_j})^T=(\cos(\pi+\psi_{s_j}),\sin(\pi+\psi_{s_j}))^T\\ &\qquad\qquad\qquad\quad=-(\cos\frac{j\pi}{m},\sin\frac{j\pi}{m})^T=-\boldsymbol{\mu}_j\quad\text{for } j\in[m]\setminus I.\end{aligned}$$

Therefore, the edge vectors of P are $\{\varepsilon_1\boldsymbol{\mu}_1,\ldots,\varepsilon_m\boldsymbol{\mu}_m\}$.

It suffices to prove that $\boldsymbol{\varepsilon}$ satisfies (1.11). By Corollary 2.1, we have

$$\mathbf{0}=\sum_{i=1}^m\varepsilon_i\boldsymbol{\mu}_i=(\sum_{j=1}^m\varepsilon_j\cos\frac{j\pi}{m},\sum_{j=1}^m\varepsilon_j\sin\frac{j\pi}{m})^T,$$

which implies that

$$\sum_{j=1}^m\varepsilon_j\zeta_m^j=\sum_{j=1}^m\varepsilon_j\cos\frac{j\pi}{m}+\mathrm{i}\sum_{j=1}^m\varepsilon_j\sin\frac{j\pi}{m}=0.$$

This completes the proof. $\square$

4.2. **Proof of Theorem 1.4.** The following lemma is useful in the proof of Theorem 1.4, which is essentially proved in [32] (see equation (2.2) and (3.16) in [32]).

Lemma 4.2. [32] *Let $\boldsymbol{A}=[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m]^T\in\mathbb{R}^{m\times2}$. Then*

$$(L_{\boldsymbol{A}})^2\le\frac{1}{2}\sum_{j=1}^m\|\boldsymbol{a}_j\|_2^2-\frac{1}{2m\cdot\sin\frac{\pi}{2m}}\sum_{j=1}^m\|\boldsymbol{a}_j\|_2^2,$$

where $L_{\boldsymbol{A}}$ is the optimal lower Lipschitz constant of the map $\Phi_{\boldsymbol{A}}$ defined in (1.1).

Now we can present a proof of Theorem 1.4.

Proof of Theorem 1.4. (i) We first prove that $\mathcal{A}$ forms a tight frame by contradiction. Suppose, for contradiction, that $\mathcal{A}$ is not a tight frame. Set $\boldsymbol{A}=[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m]^T$ with $\boldsymbol{A}\in\mathbb{R}^{m\times2}$. By Theorem 2.2, we have

$$(U_{\boldsymbol{A}})^2=\|\boldsymbol{A}\|_2^2=\|\boldsymbol{A}^T\boldsymbol{A}\|_2>\frac{1}{2}\cdot\mathrm{Tr}(\boldsymbol{A}^T\boldsymbol{A})=\frac{1}{2}\cdot\mathrm{Tr}(\boldsymbol{A}\boldsymbol{A}^T)=\frac{1}{2}\sum_{j=1}^m\|\boldsymbol{a}_j\|_2^2.$$

Combining with Lemma 4.2, we derive that

$$\beta_{\boldsymbol{A}}=\frac{U_{\boldsymbol{A}}}{L_{\boldsymbol{A}}}>\frac{1}{\sqrt{1-\frac{1}{m\cdot\sin\frac{\pi}{2m}}}}.\tag{4.12}$$

Corollary 1.1 (i) shows that $\min_{\boldsymbol{M}\in\mathbb{R}^{m\times2}}\beta_{\boldsymbol{M}}=\frac{1}{\sqrt{1-\frac{1}{m\cdot\sin\frac{\pi}{2m}}}}$. Combining with (4.12), we obtain that $\beta_{\boldsymbol{A}}>\min_{\boldsymbol{M}\in\mathbb{R}^{m\times2}}\beta_{\boldsymbol{M}}$, meaning that $\mathcal{A}$ is not optimal. This contradicts with $\mathcal{A}\in\mathcal{T}_m$. Therefore, $\mathcal{A}$ forms a tight frame.

We next prove that all vectors in $\mathcal{A}$ have the same norm. By Theorem 1.2, there is an optimal strictly convex m-gon P with the edge set $\{f(\boldsymbol{a}_1),\ldots,f(\boldsymbol{a}_m)\}$. Since $m\ge3$ is an integer with an odd factor, Theorem 1.1 (i) implies that P is equilateral, i.e., $\|f(\boldsymbol{a}_1)\|_2=\|f(\boldsymbol{a}_2)\|_2=\cdots=\|f(\boldsymbol{a}_m)\|_2$. Consequently, according to the definition of $f(\cdot)$ in (1.10), all vectors in $\mathcal{A}$ have equal norm.

(ii) If $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_m)^T\in\{\pm1\}^m$ satisfies (1.11), by Theorem 1.3, there exists an optimal polygon $P\in\mathcal{E}_m$ with the edge vectors $\{\varepsilon_j(\cos(\frac{j}{m}\pi),\sin(\frac{j}{m}\pi))^T : j\in[m]\}$. Note that for $j\in[m]$ we have

$$f(\boldsymbol{\nu}_j)=(\cos\frac{j\pi}{m},\sin\frac{j\pi}{m})^T \quad \text{and} \quad f(\boldsymbol{\nu}_j^{\perp})=-(\cos\frac{j\pi}{m},\sin\frac{j\pi}{m})^T. \tag{4.13}$$

Hence, the edge set of P can be rewritten as $\{f(\boldsymbol{\nu}_j): j\in I\}\cup\{f(\boldsymbol{\nu}_j^{\perp}): j\in[m]\setminus I\}$ where $I=\{j\in[m]:\varepsilon_j=1\}$, and such a polygon P is unique up to translation. Then it follows from Theorem 1.2 that the set $\{\boldsymbol{\nu}_j : j\in I\}\cup\{\boldsymbol{\nu}_j^{\perp} : j\in[m]\setminus I\}$ is an optimal tight frame.

(iii) Let $\mathcal{A}=\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}\in\mathcal{T}_m$. As established in the proof of (i), $\mathcal{A}$ is an optimal tight frame. Hence, by Theorem 1.2, there is an optimal strictly convex m-gon P with the edge set $\{f(\boldsymbol{a}_1),\ldots,f(\boldsymbol{a}_m)\}$. According to Theorem 1.3, there exists a vector $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_m)^T\in\{\pm1\}^m$ satisfying (1.11), such that after a proper scaling and rotation, the edge set $\{f(\boldsymbol{a}_1),\ldots,f(\boldsymbol{a}_m)\}$ equals $\{(\cos\frac{j\pi}{m},\sin\frac{j\pi}{m})^T : j\in I\}\cup\{-(\cos\frac{j\pi}{m},\sin\frac{j\pi}{m})^T : j\in[m]\setminus I\}$, where $I=\{j\in[m]:\varepsilon_j=1\}$. By (4.13), this edge set can be rewritten as $\{f(\boldsymbol{\nu}_j): j\in I\}\cup\{f(\boldsymbol{\nu}_j^{\perp}): j\in[m]\setminus I\}$. Note that the function $f(\cdot)$ is invertible on $\mathcal{T}$. It follows that $\mathcal{A}=\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}=\{\boldsymbol{\nu}_j : j\in I\}\cup\{\boldsymbol{\nu}_j^{\perp} : j\in[m]\setminus I\}$, up to scaling and rotation. This completes the proof. □

5. Discussion and Future Work

In this paper, we investigated the optimal frame for phase retrieval. Specifically, in the context of $\mathbb{R}^2$, we established a connection between the optimal frame and the perimeter-maximizing polygon problem, which is a classical research problem in discrete geometry. Leveraging this connection, we were able to characterize all optimal frames with m vectors, provided m possesses an odd factor. Future research will naturally explore the extension of these results to higher-dimensional and complex domains. We anticipate that such investigations will reveal deeper connections between phase retrieval and the fundamental properties of various polytopes, potentially enriching our understanding of both fields.

We conclude this paper by highlighting an intriguing connection between the perimeter-maximizing polygon problem and a fundamental question in discrepancy theory. Assume that $E=\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_m\}$ is the edge set of a convex m-gon $P\subset\mathbb{R}^2$ with fixed perimeter $\mathrm{perim}(P)=m$. Lemma 2.4 shows that

$$r(P)=\frac{1}{2m}\cdot\max_{\boldsymbol{u}\in\mathbb{S}^1}\sum_{j=1}^m|\langle\boldsymbol{e}_j,\boldsymbol{u}\rangle|. \tag{5.1}$$

Hence, solving the perimeter-maximizing polygon problem in (1.8) is equivalent to solving the following optimization problem:

$$\begin{aligned}\min_{\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_m\}\subset\mathbb{R}^2}\ &\max_{\boldsymbol{u}\in\mathbb{S}^1}\sum_{j=1}^m|\langle\boldsymbol{e}_j,\boldsymbol{u}\rangle|\\ \text{s.t.}\ &\sum_{j=1}^m\boldsymbol{e}_j=\boldsymbol{0},\ \sum_{j=1}^m\|\boldsymbol{e}_j\|_2=m.\end{aligned} \tag{5.2}$$

As shown in [3, Proposition 3], for any $\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_m\}\subset\mathbb{S}^1$, we have

$$\max_{\boldsymbol{u}\in\mathbb{S}^1}\sum_{j=1}^m|\langle\boldsymbol{e}_j,\boldsymbol{u}\rangle|=\max_{\boldsymbol{\varepsilon}\in\{\pm1\}^m}\left\|\sum_{j=1}^m\varepsilon_j\cdot\boldsymbol{e}_j\right\|_2.$$

Hence, if we assume that each $\|\boldsymbol{e}_j\| = 1$, then (5.2) can be rewritten as

$$\begin{aligned} \min_{\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_m\}\subset\mathbb{S}^1} \max_{\boldsymbol{\varepsilon}\in\{\pm1\}^m} &\left\| \sum_{j=1}^m \varepsilon_j \cdot \boldsymbol{e}_j \right\|_2 \\ \text{s.t. } &\sum_{j=1}^m \boldsymbol{e}_j = \boldsymbol{0}. \end{aligned} \tag{5.3}$$

This means that the problem of minimizing $r(P)$ among all equilateral convex m-gons can be reformulated as the discrepancy problem in (5.3). Understanding how to effectively solve or approximate the perimeter-maximizing polygon problem using discrepancy-theoretic methods presents an exciting direction for future investigation.

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