

ALGEBRAIC AND TOPOLOGICAL ASPECTS OF THE SINGULAR TWIN GROUP AND ITS REPRESENTATIONS

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ABSTRACT. In this article, we introduce the singular twin monoid and its corresponding group, constructed from both algebraic and topological perspectives. We then classify all complex homogeneous 2-local representations of this constructed group. Moreover, we study the irreducibility of these representations and provide clear conditions under which irreducibility holds. Our results give a structured approach to understanding this new algebraic object and its representations.

1. INTRODUCTION

The braid group on n strings, denoted by B_n , was first presented by E. Artin in 1926 [2]. One way to picture B_n is as n parallel strings hanging in three-dimensional space, which may twist around one another as they descend but never cross. The group B_n is generated by $n - 1$ elements denoted by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$, where each σ_i represents the crossing of the i -th strand over the $(i + 1)$ -st strand, see Figure 1. These generators satisfy the classical braid relations, which encode the notion that different ways of performing local crossings can lead to equivalent overall braids. Beyond its algebraic structure, B_n plays a central role in low-dimensional topology, since Alexander's theorem shows that every knot can be expressed as the closure of a braid [1].

A natural counterpart to the braid group, first presented by G. Shabat and V. Voevodsky, is the twin group on n strings, denoted by T_n [23]. It can be viewed as a flattened version of B_n , where the distinction between over and under crossings is ignored. The generators $s_1, s_2, \dots, s_{n-1}$ of T_n still describe swaps between neighboring strands, but without recording which strand passes on top, reflecting the two-dimensional nature of the group, see Figure 2. Thus, T_n preserves the essence of braiding while fitting into a simpler algebraic framework closely related to Coxeter groups. Furthermore, joining the ends of a twin produces a doodle, connecting twin groups to the study of planar curves without self-intersections [11].

Both groups, B_n and T_n , are deeply linked to topology in different ways. The braid group B_n is well known to be isomorphic to the fundamental group of the configuration space of n distinct points in the plane, which explains its rich connections to both algebraic and geometric topology. This interpretation underlies its role in knot theory, where the passage from braids to knots via closures provides a

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powerful bridge between group theory and low-dimensional topology. On the other hand, the twin group T_n corresponds to a more restricted configuration space in which over/under distinctions have been erased. Its uses are mainly combinatorial, since T_n gives a simplified model of braiding. It captures some of the planar features but loses part of the topological information that B_n still preserves.

One of the important algebraic structures that extends B_n is the singular braid monoid on n strings, denoted by SM_n , which was presented first by J. Birman in [7]. The monoid SM_n is generated by the Artin generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ of B_n together with an additional family of singular generators denoted by $\tau_1, \tau_2, \dots, \tau_{n-1}$. In [10], R. Fenn, E. Keyman, and C. Rourke showed that SM_n embeds into a group, namely SB_n , called the singular braid group, which has the same generators and relations as SM_n . Both the monoid SM_n and the group SB_n enrich the study of braids by taking singularities into consideration, making them essential tools in several areas of mathematics. For more information on SM_n and SB_n , see [9, 12].

Group representations and their properties let us study the structure of a group from both algebraic and geometric perspectives. In this setting, abstract group elements are realized as linear transformations, which makes their structure more concrete. One of the important properties to be studied for a representation is its irreducibility. A representation is said to be irreducible if it has no nontrivial subrepresentations, and otherwise it is called reducible. Irreducible representations are particularly significant since they serve as the basic building blocks of representation theory. More deeply, constructing such representations is fundamental in various fields, including quantum mechanics and particle physics [24].

One famous type of representation is the k -local representation which was presented in [19]. For a group G with generators $g_1, g_2, \dots, g_{n-1}$, a representation of G into $\text{GL}_n(\mathbb{Z}[t^{\pm 1}])$ is said to be k -local if each generator g_i is mapped to a block matrix of the form

$$\begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & M_i & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix},$$

where $M_i \in \text{GL}_k(\mathbb{Z}[t^{\pm 1}])$ and I_r is the $r \times r$ identity matrix. The k -local representations of the braid group, the singular braid monoid, and the twin group have been classified and studied for $k = 2$ and $k = 3$ [18, 16, 17, 22].

The main objective of this paper is to construct and study a new group that extends T_n , in complete analogy with the relationship between the singular braid group SB_n and the braid group B_n . This group, which is called the singular twin group and denoted by ST_n , is introduced and developed from both algebraic and topological perspectives (Section 3). The construction provides a natural framework that captures additional singular structures while preserving the fundamental properties of T_n . Once the group is established, we proceed to classify all complex homogeneous 2-local representations of ST_n (Section 4). In order to gain a deeper understanding of the representation of this group, we further examine the irreducibility of these representations and determine precise conditions under which they become irreducible (Section 5). Lastly, we give some open topics to be studied as future work (Section 6).

2. MAIN DEFINITIONS AND PREVIOUS RESULTS

In this section, we present the main group and monoid structures and presentations relevant to our study. We begin with the braid group B_n and its normal subgroup, the pure braid group P_n .

Definition 1. [2, 3] *The braid group on n strands, denoted by B_n , is a discrete group generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ that satisfy the following relations.*

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \dots, n-2, \quad (1)$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2. \quad (2)$$

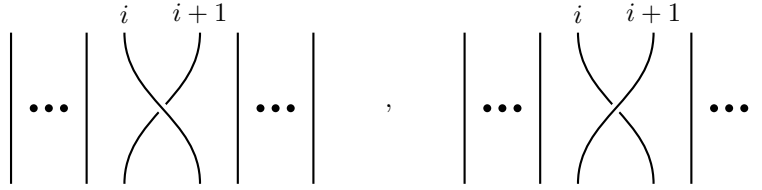


FIGURE 1. The generators σ_i and σ_i^{-1} .

Definition 2. [2, 3] *The pure braid group on n strands, denoted by P_n , is defined as the kernel of the homomorphism $B_n \rightarrow S_n$ defined by $\sigma_i \mapsto (i \ i+1)$, $1 \leq i \leq n-1$, where S_n is the symmetric group of n elements. It admits a presentation with the following generators.*

$$A_{ij} = \sigma_{j-1} \sigma_{j-2} \dots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \dots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}, \quad 1 \leq i < j \leq n.$$

Next, we present the twin group T_n along with its normal subgroup, the pure twin group PT_n .

Definition 3. [23] *The twin group on n strands, denoted by T_n , is a discrete group generated by $s_1, s_2, \dots, s_{n-1}$ that satisfy the following relations.*

$$s_i^2 = 1, \quad i = 1, 2, \dots, n-2, \quad (3)$$

$$s_i s_j = s_j s_i, \quad |i - j| \geq 2. \quad (4)$$

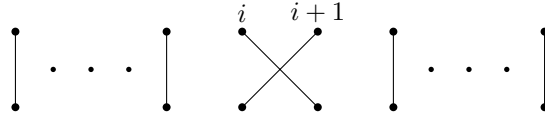


FIGURE 2. The generator s_i .

Definition 4. [13] *The pure twin group on n strands, denoted by PT_n , is defined as the kernel of the homomorphism $T_n \rightarrow S_n$ defined by $s_i \mapsto (i \ i+1)$, $1 \leq i \leq n-1$, where S_n is the symmetric group of n elements.*

In [5], Bardakov et. al found a generating set of PT_n for $n > 2$ using the Reidemeister–Schreier method explained in [15].

We now move through two significant extensions of the braid group B_n : the singular braid monoid SM_n and the singular braid group SB_n . Also, we introduce the singular pure braid group SP_n , which is a normal subgroup of SB_n .

Definition 5. [7] *The singular braid monoid, denoted by SM_n , is the monoid generated by the generators $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$ of B_n and the singular generators $\tau_1, \tau_2, \dots, \tau_{n-1}$. The generators of SM_n satisfy the relations (1) and (2) of B_n in addition to the following relations.*

$$\tau_i \tau_j = \tau_j \tau_i, \quad |i - j| \geq 2, \quad (5)$$

$$\tau_i \sigma_j = \sigma_j \tau_i, \quad |i - j| \geq 2, \quad (6)$$

$$\tau_i \sigma_i = \sigma_i \tau_i, \quad i = 1, 2, \dots, n - 1, \quad (7)$$

$$\sigma_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \dots, n - 2, \quad (8)$$

$$\tau_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \tau_{i+1}, \quad i = 1, 2, \dots, n - 2. \quad (9)$$

By adjoining the inverses of the generators τ_i , $1 \leq i \leq n - 1$, we obtain an extension of B_n , generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and $\tau_1, \tau_2, \dots, \tau_{n-1}$, called the singular braid group and denoted by SB_n .

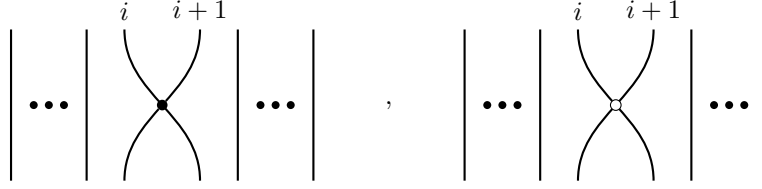


FIGURE 3. The generators τ_i and τ_i^{-1} .

Definition 6. [9] *The singular pure braid group on n strands, denoted by SP_n , is defined as the kernel of the homomorphism $SB_n \rightarrow S_n$ defined by $\sigma_i \mapsto (i \ i + 1)$ and $\tau_i \mapsto (i \ i + 1)$, $1 \leq i \leq n - 1$, where S_n is the symmetric group of n elements.*

Similarly to the case of the pure twin group, Bardakov et al. [6] obtained a generating set of SP_n for $n > 2$ by applying the Reidemeister–Schreier method described in [15].

In the following, we introduce the idea of k -local representations for a group G with finitely many generators.

Definition 7. [19] *Let G be a group generated by $g_1, g_2, \dots, g_{n-1}$. A representation $\theta : G \rightarrow \text{GL}_m(\mathbb{C})$ is called k -local if it takes the form*

$$\theta(g_i) = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & M_i & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix} \quad \text{for } 1 \leq i \leq n - 1,$$

where $M_i \in \text{GL}_k(\mathbb{C})$ with $k = m - n + 2$, and I_r denotes the $r \times r$ identity matrix. The representation is called homogeneous if all the matrices M_i coincide.

In recent years, research on k -local representations has made steady progress. Mikhalechishina first classified the 2-local representations of B_3 and all complex homogeneous 2-local representations of B_n for $n \geq 3$ [18]. Later, Mayassi and Nasser studied the complex homogeneous 3-local representations of B_n for $n \geq 4$ [16]. The following are two famous examples of k -local representations of the braid group B_n with different degrees k .

Definition 8. [8] *Let t be indeterminate. The Burau representation $\rho_B(t) : B_n \rightarrow \text{GL}_n(\mathbb{Z}[t^{\pm 1}])$ is the representation given by*

$$\sigma_i \mapsto \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 1-t & t & 0 \\ & 1 & 0 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array} \right) \text{ for } 1 \leq i \leq n-1.$$

Definition 9. [4] *Let t be indeterminate. The F -representation $\rho_F(t) : B_n \rightarrow \text{GL}_{n+1}(\mathbb{Z}[t^{\pm 1}])$ is the representation given by*

$$\sigma_i \mapsto \left(\begin{array}{c|ccc|c} I_{i-1} & 0 & & & 0 \\ \hline & 1 & 1 & 0 & \\ 0 & 0 & -t & 0 & 0 \\ & 0 & t & 1 & \\ \hline 0 & 0 & & & I_{n-i-1} \end{array} \right) \text{ for } 1 \leq i \leq n-1.$$

Regarding k -local representations of the twin groups, T. Mayassi and M. Nasser classified all 2-local representations of T_n for all $n \geq 2$ [17]. Moreover, M. Nasser determined all 3-local representations of the twin group T_n , the virtual twin group VT_n , and the welded twin group WT_n , for all $n \geq 4$ [22]. On the other hand, in [20], M. Nasser presented two particular representations of the twin group T_n for $n \geq 2$. These representations are referred to as N_1 and N_2 , respectively, and are described explicitly with their main results in the following.

Definition 10. [20] *The N_1 -representation $\eta_1 : T_n \rightarrow \text{GL}_n(\mathbb{Z}[t^{\pm 1}])$, where t is indeterminate, is the representation defined by*

$$s_i \mapsto \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 1-t & t & 0 \\ & 2-t & t-1 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array} \right) \text{ for } 1 \leq i \leq n-1.$$

Theorem 11. [20] *The representation $\eta_1 : T_n \rightarrow \text{GL}_n(\mathbb{Z}[t^{\pm 1}])$ is reducible to the degree $n-1$ for all $n \geq 3$. Moreover, the complex specialization of its $(n-1)$ -composition factor, namely $\eta'_1 : T_n \rightarrow \text{GL}_{n-1}(\mathbb{C})$, is irreducible if and only if $t \neq \frac{2n-2}{n-2}$ and $t \neq 2$.*

Definition 12. [20] *The N_2 -representation $\eta_2 : T_n \rightarrow \text{GL}_n(\mathbb{Z}[t^{\pm 1}])$, where t is indeterminate, is the representation defined by*

$$s_i \mapsto \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 0 & f(t) & 0 \\ & f^{-1}(t) & 0 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array} \right) \text{ for } 1 \leq i \leq n-1,$$

where $f(t) \in \mathbb{Z}[t^{\pm 1}]$ with $f(t) \neq 0$ and $f^{-1}(t) = \frac{1}{f(t)}$.

3. ALGEBRAIC AND TOPOLOGICAL INTERPRETATION OF THE SINGULAR TWIN

In parallel with the singular braid monoid and the singular braid group, we introduce in this section the singular twin monoid and the singular twin group. These constructions provide the twin group counterparts of their braid analogues, serving as a foundation for studying their algebraic and topological structure and representations.

3.1. Algebraic interpretation. In this subsection, we introduce the presentation of the *singular twin monoid* and its corresponding group from algebraic perspective, followed by the definition of the *singular pure twin group*.

Definition 13. *The singular twin monoid, denoted by STM_n , is the monoid generated by the generators $s_1, s_2, \dots, s_{n-1}$ of T_n and the singular generators $\tau_1, \tau_2, \dots, \tau_{n-1}$. The generators of STM_n satisfy the relations (3) and (4) of T_n in addition to the following relations.*

$$\tau_i \tau_j = \tau_j \tau_i, \quad |i - j| \geq 2, \quad (10)$$

$$\tau_i s_j = s_j \tau_i, \quad |i - j| \geq 2, \quad (11)$$

$$\tau_i s_i = s_i \tau_i, \quad i = 1, 2, \dots, n-1, \quad (12)$$

$$s_i s_{i+1} \tau_i = \tau_{i+1} s_i s_{i+1}, \quad i = 1, 2, \dots, n-2, \quad (13)$$

$$\tau_i s_{i+1} s_i = s_{i+1} s_i \tau_{i+1}, \quad i = 1, 2, \dots, n-2. \quad (14)$$

By adjoining the inverses of the generators τ_i , $1 \leq i \leq n-1$, we obtain an extension of T_n , generated by $s_1, s_2, \dots, s_{n-1}$ and $\tau_1, \tau_2, \dots, \tau_{n-1}$, which we call the *singular twin group* and we denote it by ST_n .

Definition 14. [9] *The singular pure twin group on n strands, denoted by SPT_n , is defined as the kernel of the homomorphism $ST_n \rightarrow S_n$ defined by $s_i \mapsto (i \ i+1)$ and $\tau_i \mapsto (i \ i+1)$, $1 \leq i \leq n-1$, where S_n is the symmetric group of n elements.*

Question 15. *Give a presentation of SPT_n for $n > 2$ by generators and relations.*

The Reidemeister–Schreier method provides a way to obtain a presentation of SPT_n by generators and relations. However, the computation in our case becomes quite involved because of the large number of defining relations in the group. Therefore, in what follows we restrict our attention to the case $n = 3$ and show that SPT_3 is generated by the following four elements:

$$a := s_1 \tau_1, \quad b := s_2 \tau_2, \quad c := s_2 a s_2, \quad d := s_1 b s_1.$$

Note that the quotient $ST_3/SPT_3 \cong S_3$. Choose the standard Schreier transversal Λ consisting of reduced words representing each permutation:

$$\Lambda = \{ \lambda_0 = e, \lambda_1 = s_1, \lambda_2 = s_2, \lambda_3 = s_1 s_2, \lambda_4 = s_2 s_1, \lambda_5 = s_1 s_2 s_1 \}.$$

For each coset representative λ_i and each parent generator $x \in \{s_1, s_2, \tau_1, \tau_2\}$ the Schreier generator is

$$a_{\lambda_i, x} := \lambda_i x (\overline{\lambda_i x})^{-1},$$

where $\overline{\lambda_i x} \in \Lambda$ is the chosen representative with the same image in S_3 as $\lambda_i x$. Because $\pi(\tau_j) = \pi(s_j)$ we always have $\overline{\lambda \tau_j} = \overline{\lambda s_j}$, which simplifies many computations.

The nontrivial Schreier generators (after cancelling obvious trivial ones) are computed as follows.

$$\begin{aligned}
s_2 s_1 s_2 (\overline{s_2 s_1 s_2})^{-1} &= s_2 s_1 s_2 s_1 s_2 s_1 = (s_2 s_1)^3, \\
s_1 s_2 s_1 s_2 (\overline{s_1 s_2 s_1 s_2})^{-1} &= s_1 s_2 s_1 s_2 s_2 s_1 = (s_1 s_2)^3, \\
s_1 \tau_1 (\overline{s_1 \tau_1})^{-1} &= s_1 \tau_1 s_1^2 = s_1 \tau_1 = a, \\
s_2 \tau_1 (\overline{s_2 \tau_1})^{-1} &= s_2 \tau_1 (s_2 s_1)^{-1} = s_2 \tau_1 s_1 s_2 = s_2 s_1 \tau_1 s_2 = c, \\
s_1 s_2 \tau_1 (\overline{s_1 s_2 \tau_1})^{-1} &= s_1 s_2 \tau_1 (s_1 s_2 s_1)^{-1} = s_1 s_2 \tau_1 s_1 s_2 s_1 \\
&= \tau_2 s_1 s_2 s_1 s_2 s_1 = \tau_2 s_2 (s_2 s_1)^3 = b (s_2 s_1)^3, \\
s_2 s_1 \tau_1 (\overline{s_2 s_1 \tau_1})^{-1} &= s_2 s_1 \tau_1 (s_2)^{-1} = s_2 s_1 \tau_1 s_2 = c, \\
s_1 s_2 s_1 \tau_1 (\overline{s_1 s_2 s_1 \tau_1})^{-1} &= s_1 s_2 s_1 \tau_1 s_2 s_1 = (s_1 s_2)^3 s_2 \tau_2 = (s_1 s_2)^3 b.
\end{aligned}$$

Similarly, the nontrivial Schreier generators arising from τ_2 are (after cancellation and simplifying by the relations of ST_3)

$$s_1 s_2 \tau_2 s_1 = d, \quad s_2 \tau_2 = b, \quad s_2 s_1 \tau_2 s_1 s_2 s_1 = a, \quad s_1 s_2 s_1 \tau_2 s_1 s_2 = a.$$

By elementary computation using the relations of the singular twin group ST_3 one checks the identities

$$(s_2 s_1)^3 = ((s_1 s_2)^3)^{-1}, \quad (s_1 s_2)^3 = (s_1 s_2 \tau_2 s_1)(s_2 s_1 \tau_1 s_2)^{-1},$$

so the Schreier generators above reduce to the following four elements of SPT_3 :

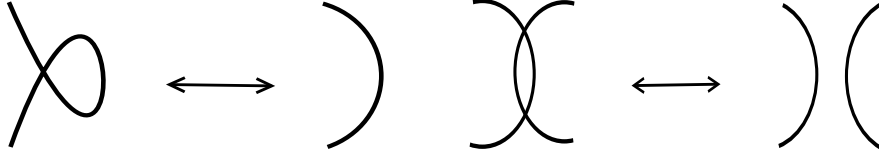
$$a = s_1 \tau_1, \quad b = s_2 \tau_2, \quad c = s_2 a s_2, \quad d = s_1 b s_1.$$

Hence, SPT_3 is generated by a, b, c, d , as required.

3.2. Topological interpretation. Recall that a fundamental theorem in knot theory states that any braid can be closed in a standard manner to yield a knot or a link in the 3-dimensional sphere S^3 [1]. Similarly, the closure of a singular braid yields a singular link, that is, a link which can be represented by a planar diagram that is allowed to have a finite number of transverse double points called singularities. In the same spirit, an element of the twin group can be closed to define a doodle [11]. Formally, a doodle is an immersion of a finite disjoint union of circles into the 2-dimensional sphere S^2 , considered up to homotopy without creating triple points [14]. In other words, two doodles are regarded as equivalent if they can be related by a finite sequence of the following two local moves:

- **Move D_1 :** Creation or elimination of a monogon (a small simple loop with no intersections).
- **Move D_2 :** Creation or elimination of a bigon (two arcs forming a simple lens-shaped region).

These moves play the same role for doodles as the Reidemeister moves do for classical knots and links, providing a combinatorial description of their equivalence classes.

FIGURE 4. The local moves D_1 (left) and D_2 (right).

Likewise, a topological interpretation of the singular twin group can be formulated. Indeed, the closure of a singular twin element can be viewed as a *singular doodle*: an immersion of a 4-valent graph, possibly together with a collection of disjoint circles, into S^2 , see Figure 5. Such singular doodles exhibit two types of singular features:

- (1) **Transverse double points**, corresponding to intersections of edges of the immersed graph, and
- (2) **4-valent vertices**, representing the vertices of the underlying graph.

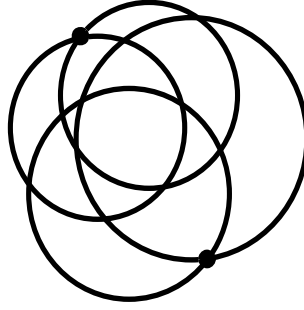
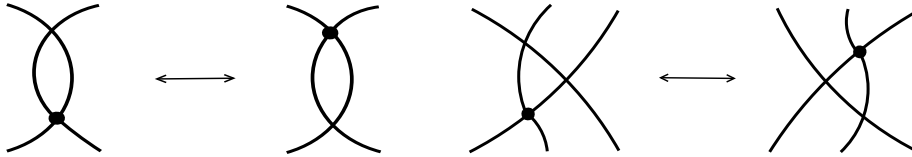


FIGURE 5. A singular doodle obtained as an embedding of a 4-valent graph. The underlying graph is a disjoint union of 2 two-bouquet graphs.

Two singular doodles are said to be equivalent if one can be transformed into the other by a finite sequence of local moves of type D_1 and D_2 (Figure 4), together with moves D_3 and D_4 (Figure 6), which extend the classical doodle moves to configurations involving vertices.

FIGURE 6. The local moves D_3 (left) and D_4 (right).

Now, after introducing the singular twin group from both algebraic and topological perspectives, we aim to construct representations of this group.

Question 16. *What are the possible complex homogeneous k -local representations of ST_n ?*

We answer this question in the following sections for $k = 2$ and $k = 3$.

4. CLASSIFICATION OF HOMOGENEOUS 2-LOCAL REPRESENTATIONS OF ST_n

In this section we classify all complex homogeneous 2-local representations of the singular twin group ST_n for all $n \geq 2$. First of all, we consider the case $n = 2$, which is a special case.

Theorem 17. *Let $\Gamma : ST_2 \rightarrow \text{GL}_2(\mathbb{C})$ be a complex homogeneous 2-local representation of ST_2 . Then, Γ is equivalent to one of the following six representations.*

(1) $\Gamma_1 : ST_2 \rightarrow \text{GL}_2(\mathbb{C})$ such that

$$\Gamma_1(s_1) = \begin{pmatrix} -\sqrt{1-bc} & b \\ c & \sqrt{1-bc} \end{pmatrix} \text{ and } \Gamma_1(\tau_1) = \begin{pmatrix} w & x \\ \frac{cx}{b} & \frac{bw+2x\sqrt{1-bc}}{b} \end{pmatrix},$$

where $b, c, w, x \in \mathbb{C}, b \neq 0$.

(2) $\Gamma_2 : ST_2 \rightarrow \text{GL}_2(\mathbb{C})$ such that

$$\Gamma_2(s_1) = \begin{pmatrix} \sqrt{1-bc} & b \\ c & -\sqrt{1-bc} \end{pmatrix} \text{ and } \Gamma_2(\tau_1) = \begin{pmatrix} w & x \\ \frac{cx}{b} & \frac{bw-2x\sqrt{1-bc}}{b} \end{pmatrix},$$

where $b, c, w, x \in \mathbb{C}, b \neq 0$.

(3) $\Gamma_3 : ST_2 \rightarrow \text{GL}_2(\mathbb{C})$ such that

$$\Gamma_3(s_1) = \begin{pmatrix} -1 & 0 \\ c & 1 \end{pmatrix} \text{ and } \Gamma_3(\tau_1) = \begin{pmatrix} w & 0 \\ y & \frac{cw+2y}{c} \end{pmatrix},$$

where $c, w, y \in \mathbb{C}, c \neq 0$.

(4) $\Gamma_4 : ST_2 \rightarrow \text{GL}_2(\mathbb{C})$ such that

$$\Gamma_4(s_1) = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix} \text{ and } \Gamma_4(\tau_1) = \begin{pmatrix} w & 0 \\ y & \frac{cw-2y}{c} \end{pmatrix},$$

where $c, w, y \in \mathbb{C}, c \neq 0$.

(5) $\Gamma_5 : ST_2 \rightarrow \text{GL}_2(\mathbb{C})$ such that

$$\Gamma_5(s_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \Gamma_5(\tau_1) = \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix},$$

where $w, z \in \mathbb{C}$.

(6) $\Gamma_6 : ST_2 \rightarrow \text{GL}_2(\mathbb{C})$ such that

$$\Gamma_6(s_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \Gamma_6(\tau_1) = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

where $w, x, y, z \in \mathbb{C}$.

Proof. Set

$$\Gamma(s_1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \Gamma(\tau_1) = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

where $a, b, c, d, w, x, y, z \in \mathbb{C}$ such that $ad - bc \neq 0$ and $wz - xy \neq 0$. The defining relations of the group ST_2 are $s_1^2 = 1$ and $s_1\tau_1 = \tau_1s_1$. Consequently, we obtain $\Gamma(s_1)^2 = 1$ and $\Gamma(s_1)\Gamma(\tau_1) = \Gamma(\tau_1)\Gamma(s_1)$. Using these two relations, we derive the following seven equations.

$$a^2 + bc = 1 \tag{15}$$

$$ab + bd = 0 \tag{16}$$

$$ac + cd = 0 \quad (17)$$

$$bc + d^2 = 1 \quad (18)$$

$$-cx + by = 0 \quad (19)$$

$$bw - ax + dx - bz = 0 \quad (20)$$

$$cw - ay + dy - cz = 0 \quad (21)$$

We consider the following two cases.

- (1) The case $b = 0$. From Equations (15) and (18), we get that $a^2 = d^2 = 1$, and so $a = \pm 1$ and $d = \pm 1$. We consider each subcase separately.
 - If $a = d = 1$, then $c = 0$ by Equation (17) and so Γ is equivalent to Γ_6 in this case.
 - If $a = d = -1$, then $c = 0$ by Equation (17) and so Γ is equivalent to Γ_6 in this case.
 - If $a = 1$ and $d = -1$, then $x = 0$ by Equation (20) and so we have Γ is equivalent to Γ_5 if $c = 0$ and Γ is equivalent to Γ_4 if $c \neq 0$.
 - If $a = -1$ and $d = 1$, then $x = 0$ by Equation (20) and so we have Γ is equivalent to Γ_5 if $c = 0$ and Γ is equivalent to Γ_3 if $c \neq 0$.
- (2) The case $b \neq 0$. From Equations (15) and (18), we get that $a^2 = d^2 = 1 - bc$ and, by Equation (16), we get that $a = -d$. So, $a = \pm\sqrt{1 - bc}$ and $d = \mp\sqrt{1 - bc}$. We consider each subcase separately.
 - If $a = -\sqrt{1 - bc}$ and $d = \sqrt{1 - bc}$, then, using Equations (20) and (21), we get that Γ is equivalent to Γ_1 .
 - If $a = \sqrt{1 - bc}$ and $d = -\sqrt{1 - bc}$, then, using Equations (20) and (21), we get that Γ is equivalent to Γ_2 .

□

We now consider the case $n \geq 3$ and we classify all complex homogeneous 2-local representations of ST_n , for all $n \geq 3$.

Theorem 18. *Consider $n \geq 3$ and let $\Theta : ST_n \rightarrow \text{GL}_n(\mathbb{C})$ be a complex homogeneous 2-local representation of ST_n . Then, Θ is equivalent to one of the following five representations.*

- (1) $\Theta_1 : ST_n \rightarrow \text{GL}_n(\mathbb{C})$ such that

$$\Theta_1(s_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 0 & b & 0 \\ & \frac{1}{b} & 0 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array} \right) \text{ and } \Theta_1(\tau_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & w & x & 0 \\ & \frac{x}{b^2} & w & \\ \hline 0 & 0 & & I_{n-i-1} \end{array} \right),$$

where $b, w, x \in \mathbb{C}, b \neq 0, 1 \leq i \leq n - 1$.

- (2) $\Theta_2 : ST_n \rightarrow \text{GL}_n(\mathbb{C})$ such that

$$\Theta_2(s_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & -\sqrt{1 - bc} & b & 0 \\ & c & \sqrt{1 - bc} & \\ \hline 0 & 0 & & I_{n-i-1} \end{array} \right) \text{ and } \Theta_2(\tau_i) = I_n,$$

where $b, c \in \mathbb{C}, 1 \leq i \leq n - 1$.

(3) $\Theta_3 : ST_n \rightarrow \text{GL}_n(\mathbb{C})$ such that

$$\Theta_3(s_i) = \left(\begin{array}{c|cc|c} I_{i-1} & & 0 & 0 \\ \hline & \sqrt{1-bc} & b & \\ 0 & c & -\sqrt{1-bc} & 0 \\ \hline 0 & & 0 & I_{n-i-1} \end{array} \right) \text{ and } \Theta_3(\tau_i) = I_n,$$

where $b, c \in \mathbb{C}$, $1 \leq i \leq n-1$.

(4) $\Theta_4 : ST_n \rightarrow \text{GL}_n(\mathbb{C})$ such that

$$\Theta_4(s_i) = \left(\begin{array}{c|cc|c} I_{i-1} & & 0 & 0 \\ \hline & -1 & 0 & \\ 0 & 0 & -1 & 0 \\ \hline 0 & & 0 & I_{n-i-1} \end{array} \right) \text{ and } \Theta_4(\tau_i) = I_n,$$

where $1 \leq i \leq n-1$.

(5) $\Theta_5 : ST_n \rightarrow \text{GL}_n(\mathbb{C})$ such that

$$\Theta_5(s_i) = I_n \text{ and } \Theta_5(\tau_i) = I_n,$$

where $1 \leq i \leq n-1$.

Proof. The proof follows in a similar manner to that of Theorem 17. $\square$

5. IRREDUCIBILITY OF THE HOMOGENEOUS 2-LOCAL REPRESENTATIONS OF ST_n

In this section, we study the irreducibility of the complex homogeneous 2-local representations of the singular twin group ST_n for all $n \geq 2$. We start by the case $n = 2$, which is a special case.

Theorem 19. *Let $\Gamma : ST_2 \rightarrow \text{GL}_2(\mathbb{C})$ denote a complex homogeneous 2-local representation of ST_2 . Then, Γ is reducible.*

Proof. Theorem 17 yields that Γ is equivalent to one of the six representations Γ_j , $1 \leq j \leq 6$. We consider each case separately.

- (1) If Γ is equivalent to Γ_1 or Γ_2 , then we have the following two subcases.
 - In the case $c = 0$, e_1 is a common eigenvector of both $\Gamma(s_1)$ and $\Gamma(\tau_1)$, and hence Γ is reducible.
 - In the case $c \neq 0$, $\left(-\frac{\sqrt{1-bc}+1}{c}, 1\right)$ is a common eigenvector of both $\Gamma(s_1)$ and $\Gamma(\tau_1)$, and hence Γ is reducible.
- (2) If Γ is equivalent to Γ_3 , Γ_4 or Γ_5 , then e_2 is a common eigenvector of both $\Gamma(s_1)$ and $\Gamma(\tau_1)$, and hence Γ is reducible.
- (3) If Γ is equivalent to Γ_6 , then every eigenvector of $\Gamma(\tau_1)$ is invariant under $\Gamma(s_1)$, and hence Γ is reducible.

$\square$

Theorem 20. *Consider $n \geq 3$ and let $\Theta : ST_n \rightarrow \text{GL}_n(\mathbb{C})$ denote a complex homogeneous 2-local representation of ST_n . By Theorem 18, Θ is equivalent to one of the five representations Θ_i , $1 \leq i \leq 5$. The following hold true.*

- (1) *If Θ is equivalent to Θ_1 , then Θ is irreducible if and only if $w + \frac{x}{b} \neq 1$.*
- (2) *If Θ is equivalent to Θ_2 , then Θ is reducible to the degree $n-1$. Furthermore, by putting $a = -\sqrt{1-bc}$, we have the following cases.*

- If $n = 3$, then the $(n - 1)$ -composition factor, namely Θ' , of Θ is irreducible if and only if $a \notin \{\pm 1, \pm i\sqrt{3}\}$.
- If $n \geq 4$, then the $(n - 1)$ -composition factor, namely Θ' , of Θ is irreducible if and only if $a \notin \{\pm 1\}$ and a is not a root of

$$P(t) = 4(1 + t^2) + \frac{(1 - t)^4}{2t} \left(1 - \left(\frac{1 - t}{1 + t} \right)^{n-4} \right).$$

- (3) If Θ is equivalent to Θ_3 , then Θ is reducible to the degree $n - 1$. Furthermore, by putting $a = \sqrt{1 - bc}$, we have the following cases.
- If $n = 3$, then the $(n - 1)$ -composition factor, namely Θ' , of Θ is irreducible if and only if $a \notin \{\pm 1, \pm i\sqrt{3}\}$.
 - If $n \geq 4$, then the $(n - 1)$ -composition factor, namely Θ' , of Θ is irreducible if and only if $a \notin \{\pm 1\}$ and a is not a root of

$$P(t) = 4(1 + t^2) + \frac{(1 - t)^4}{2t} \left(1 - \left(\frac{1 - t}{1 + t} \right)^{n-4} \right).$$

- (4) If Θ is equivalent to Θ_4 or Θ_5 , then Θ is a direct sum of 1-dimensional representations.

Proof. We examine each case individually in what follows, except for the proofs of (4) and (5), which are straightforward.

- (1) Suppose that Θ is equivalent to Θ_1 . Consider the diagonal matrix defined by $P = \text{diag}(b^{1-n}, b^{2-n}, \dots, b, 1)$, where $\text{diag}(r_1, r_2, \dots, r_n)$ is a diagonal $n \times n$ matrix with $r_{ii} = r_i$. Consider the equivalent representation $\hat{\Theta}$ of Θ given by: $\hat{\Theta}(s_i) = P^{-1}\Theta(s_i)P$ and $\hat{\Theta}(\tau_i) = P^{-1}\Theta(\tau_i)P$ for all $1 \leq i \leq n - 1$. Direct computations give that

$$\hat{\Theta}(s_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & 0 & 1 & 0 \\ & 1 & 0 & \\ \hline 0 & 0 & & I_{n-i-1} \end{array} \right)$$

and

$$\hat{\Theta}(\tau_i) = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & & 0 \\ \hline 0 & w & \frac{x}{b} & 0 \\ & \frac{x}{b} & w & \\ \hline 0 & 0 & & I_{n-i-1} \end{array} \right),$$

where $w, b, x \in \mathbb{C}, b \neq 0$, for $1 \leq i \leq n - 1$. The representation $\hat{\Theta}$ has the same form as the representation ρ_3 obtained in [21, Theorem 30]. Referring to the results in that paper, we obtain that our representation $\hat{\Theta}$ is irreducible if and only if $w + \frac{x}{b} \neq 1$, and consequently the same holds for Θ .

- (2) Suppose that Θ is equivalent to Θ_2 and set $a = -\sqrt{1 - bc}$. The restriction of the representation Θ to T_n in this case has the same form as the representation ξ_1 obtained in [17, Theorem 5]. Referring to the results in that paper, and since $\Theta(\tau_i) = I_n$ for all $1 \leq i \leq n - 1$, we obtain that our representation Θ is reducible to the degree $n - 1$ and the following cases occur.

- If $n = 3$, then the $(n - 1)$ -composition factor, namely Θ' , of Θ is irreducible if and only if $a \notin \{\pm 1, \pm i\sqrt{3}\}$.
- If $n \geq 4$, then the $(n - 1)$ -composition factor, namely Θ' , of Θ is irreducible if and only if $a \notin \{\pm 1\}$ and a is not a root of

$$P(t) = 4(1 + t^2) + \frac{(1 - t)^4}{2t} \left(1 - \left(\frac{1 - t}{1 + t} \right)^{n-4} \right).$$

- (3) In the case Θ is equivalent to Θ_3 , the argument proceeds as in (2), this time taking $a = \sqrt{1 - bc}$.

□

6. FUTURE WORK

In this section, we provide ideas that could be considered as future work.

- (1) One of the important questions that could be addressed for any constructed group is its linearity. A group is said to be linear if it admits a faithful representation. So, the first issue that could be considered for the future is to study the faithfulness of the classified representations.
- (2) In addition to classifying and analyzing k -local representations of the singular twin group, we also encourage the construction of new non-local representations of this group and the investigation of their properties, such as irreducibility and faithfulness.
- (3) Inspired by the relationship between the Burau representation of the braid group and the Alexander polynomial for knots, we propose a future study to investigate whether representations of the twin group and the singular twin group can be used to define analogous invariants for doodles and singular doodles.

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