# QUADRANGULATIONS AND THE LOVÁSZ COMPLEX

CARMEN ARANA AND MATĚJ STEHLÍK

Abstract. The Lovász complex $\mathsf{L}(G)$ of a graph $G$ is a deformation retract of its neighborhood complex, equipped with a canonical $\mathbb{Z}_2$-action. We show that, under mild assumptions, $\mathsf{L}(G)$ is homeomorphic to a surface if and only if $G$ is a non-bipartite quadrangulation of the orbit space $\mathsf{L}(G)/\mathbb{Z}_2$ in which every 4-cycle is facial. This yields a classification of the Lovász complexes of all such quadrangulations. As an application, we contextualize a result of Archdeacon *et al.* and Mohar and Seymour on the chromatic number of quadrangulations, obtaining a stronger statement about the $\mathbb{Z}_2$-index.

## 1. Introduction

In his proof of Kneser's conjecture, Lovász [11] introduced the neighborhood complex $\mathsf{N}(G)$ of a graph $G$ and showed that its homotopic connectivity gives a lower bound on the chromatic number of $G$. In the same paper he also defined another, lesser-known simplicial complex, now called the Lovász complex $\mathsf{L}(G)$ (see, e.g., [6, 12]). The two complexes are homotopy equivalent—indeed, $\mathsf{L}(G)$ is a deformation retract of $\mathsf{N}(G)$ [5]. Although $\mathsf{L}(G)$ typically has more vertices than $\mathsf{N}(G)$, it has two advantages: it admits a canonical free $\mathbb{Z}_2$-action, essential for applications of the Borsuk–Ulam theorem, and can sometimes have lower dimension, which is crucial for our purposes.

From a graph-theoretic perspective, these complexes may appear somewhat mysterious. Indeed, Toft [18] remarked that Lovász's topological bound's "graph theoretic meaning is not well understood." Our goal is to shed light on the Lovász complex in what is arguably the simplest nontrivial case—when it forms a surface. We show that the Lovász complex is closely related to *quadrangulations*—graphs embedded in surfaces such that every face is bounded by four edges. More precisely, we prove the following.

**Theorem 1.1.** *If $G \ncong K_{2,3}$ is a connected, non-bipartite quadrangulation of a surface $S$ in which every 4-cycle is facial, then $\mathsf{L}(G)$ is a double cover of $S$. Conversely, if $\mathsf{L}(G)$ is a closed surface, no neighborhood of $G$ dominates another, and $G$ is $K_{2,3}$-free, then $G$ is a quadrangulation of $\mathsf{L}(G)/\mathbb{Z}_2$ in which all 4-cycles bound a face.*

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As a corollary of our main result, we classify all Lovász complexes of quadrangulations of surfaces (subject to mild assumptions, see Corollary 3.4) and, conversely, all graphs whose Lovász complex is a surface (also subject to mild assumptions, see Corollary 4.2).

The chromatic number, denoted $\chi$, of surface quadrangulations has been studied extensively. Hutchinson [9] showed that all quadrangulations of orientable surfaces with sufficiently large edge-width are 3-colorable. The non-orientable case exhibits different behavior: Youngs [19] proved that every quadrangulation of the projective plane is either bipartite or 4-chromatic. This result was later extended to other non-orientable surfaces by Archdeacon *et al.* [1], and independently by Mohar and Seymour [14]. A quadrangulation $G$ of a non-orientable surface $S$ is *odd* if it contains an odd cycle $C$ such that cutting $S$ along $C$ results in an orientable surface.

**Theorem 1.2** ([1, 14])**.** *If $G$ is an odd quadrangulation of a non-orientable surface, then $\chi(G) \geq 4$.*

The proof in [1, 14] uses the *winding number*, so it is clearly topological, yet it is not obvious how it relates to the theory of *topological methods* originating in Lovász's proof of Kneser's conjecture [11]; see [6, 10, 12, 13]. A modern interpretation of Lovász's bound uses the $\mathbb{Z}_2$-*index* (definitions are given in Section 2).

**Theorem 1.3** ([6, 10, 12, 13])**.** *If $G$ is a (non-empty) graph, then $\chi(G) \geq \operatorname{ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) + 2$.*

In the final part of our paper, we prove the following strengthening of Theorem 1.2.

**Theorem 1.4.** *If $G$ is an odd quadrangulation of a non-orientable surface, then $\operatorname{ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) \geq 2$.*

A weaker topological bound on the chromatic number can be obtained by replacing the $\mathbb{Z}_2$-index in Theorem 1.3 by the $\mathbb{Z}_2$-*coindex* (see Section 2 for a definition). Spaces where these two invariants do not coincide were called *non-tidy* spaces by Matoušek [12, pp. 100–101]. While examples of non-tidy spaces are known (see, e.g. [4, 12]), we are unaware of explicit examples of graphs whose Lovász complex is non-tidy.

Using the universal cover, one can show that $\operatorname{coind}_{\mathbb{Z}_2}(S) = 1$ for any closed surface $S$ other than the sphere; see, e.g., [12, p. 100]. Therefore, by Theorem 1.1, if $G$ is an odd quadrangulation of a non-orientable surface other than the projective plane, with all 4-cycles facial, then $\operatorname{coind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 1$. On the other hand, Theorem 1.4 implies that $\operatorname{ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 2$. We conclude that $\mathsf{L}(G)$ is non-tidy for such graphs.

## 2. Preliminaries

A *graph* $G$ is a pair of sets $V(G)$ and $E(G)$, the set of *vertices* and *edges* of $G$, respectively. If $G$ and $H$ are isomorphic graphs, we write $G \cong H$.

We shall assume throughout that $V(G)$ is finite. We will say that a graph is $K_{2,3}$-*free* if it does not contain the complete bipartite graph $K_{2,3}$ as a subgraph. A $k$-*coloring* of a graph $G$ is a function $c : V(G) \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ for every edge $\{u, v\} \in E(G)$. The *chromatic number* of $G$ is the smallest $k$ for which a $k$-coloring of $G$ exists. Let $G$ be a graph, $v \in V(G)$ a vertex, and $A \subseteq V(G)$ a set of vertices. The *neighborhood* of $v$ is denoted by $N(v)$. We will say that the neighborhood of $v$ *dominates* the neighborhood of $u$ if $N(u) \subseteq N(v)$. The set of *common neighbors* of $A$ is

$$\nu(A) := \{v \in V(G) \mid \{u, v\} \in E(G) \text{ for all } u \in A\}.$$

For more background on graph-theoretic notions, see Bondy and Murty [3].

If $X$ and $Y$ are homeomorphic topological spaces, we write $X \cong Y$. A *double cover* of a topological space $X$ is a $\mathbb{Z}_2$-space $\widetilde{X}$ such that the orbit space $\widetilde{X}/\mathbb{Z}_2$ is homeomorphic to $X$. For $k \geq 1$, a topological space $X$ is $k$-*connected* if the homotopy groups $\pi_1(X), \dots, \pi_k(X)$ are all trivial. A *surface* is a connected compact Hausdorff topological space $S$ locally homeomorphic to an open disc. According to the classification of surfaces, a closed surface is uniquely determined, up to homeomorphism, by its Euler characteristic and orientability. We denote the orientable surface of genus $k$ by $\mathbb{S}_k$, and the non-orientable surface of genus $k$ by $\mathbb{N}_k$. Note that $\mathbb{S}_k$ and $\mathbb{N}_k$ have Euler characteristic $2 - 2k$ and $2 - k$, respectively. For further topological background, see Hatcher [8].

A *quadrangulation* of a surface $S$ is a graph embedded in $S$ such that the boundary of each face is a 4-cycle. Following [1], a quadrangulation of a non-orientable surface is called *odd* if there exists an odd cycle in the graph whose removal (i.e., cutting the surface along the cycle) yields an orientable surface. A cycle in an embedded graph is *one-sided* if a small strip around it forms a Möbius strip. For more background on graphs embedded in surfaces, see Mohar and Thomassen [15].

A $\mathbb{Z}_2$-*space* is a topological space equipped with a free $\mathbb{Z}_2$-action. A canonical example of a $\mathbb{Z}_2$-space is the $n$-dimensional sphere $\mathbb{S}^n$ equipped with the antipodal action. Given $\mathbb{Z}_2$-spaces $X$ and $Y$, we write $X \xrightarrow{\mathbb{Z}_2} Y$ if there exists a continuous $\mathbb{Z}_2$-equivariant map from $X$ to $Y$. We define the following invariants:

$$\operatorname{ind}_{\mathbb{Z}_2}(X) = \min\{n \in \mathbb{N} \mid X \xrightarrow{\mathbb{Z}_2} \mathbb{S}^n\},$$
$$\operatorname{coind}_{\mathbb{Z}_2}(X) = \max\{n \in \mathbb{N} \mid \mathbb{S}^n \xrightarrow{\mathbb{Z}_2} X\}.$$

Note that the Borsuk–Ulam theorem can be succinctly expressed by the inequality $\operatorname{ind}_{\mathbb{Z}_2}(\mathbb{S}^n) \geq n$. Generalizations of the Borsuk–Ulam theorem to manifolds other than the sphere—also expressible using the $\mathbb{Z}_2$-index—have been studied in [2, 16, 7]. We will also make use of the *cohomological index* (also known as the *Stiefel–Whitney height*), denoted $\operatorname{cohom-ind}_{\mathbb{Z}_2}(X)$, defined as the height of the first Stiefel–Whitney class of $X$. The following

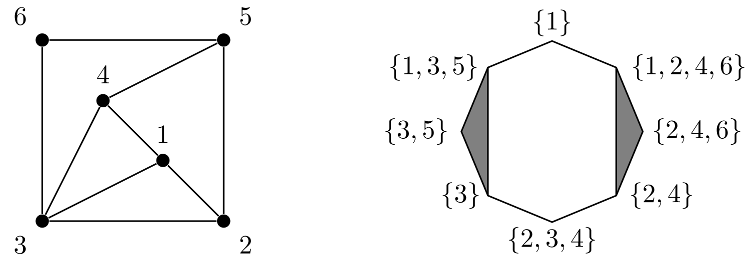


Figure 1. A graph (left) and its Lovász complex (right).

inequalities hold for any $k$-connected $\mathbb{Z}_2$-space $X$:

$$k + 1 \leq \mathrm{coind}_{\mathbb{Z}_2}(X) \leq \text{cohom-ind}_{\mathbb{Z}_2}(X) \leq \mathrm{ind}_{\mathbb{Z}_2}(X). \tag{1}$$

See Kozlov [10] or Matoušek [12] for more details.

An (abstract) *simplicial complex* $\mathsf{K}$ is a finite hereditary set system. Its vertex set is denoted by $V(\mathsf{K})$, and its barycentric subdivision by $\mathsf{sd}(\mathsf{K})$. Any abstract simplicial complex $\mathsf{K}$ can be realized as a topological space $|\mathsf{K}|$ in $\mathbb{R}^d$ for some $d$. The *neighborhood complex* of $G$ is the simplicial complex

$$\mathsf{N}(G) := \{A \subseteq V(G) \mid \nu(A) \neq \emptyset\}.$$

The *Lovász complex* of $G$, denoted $\mathsf{L}(G)$ (see Figure 1), is the subcomplex of the barycentric subdivision $\mathsf{sd}(\mathsf{N}(G))$ induced by the vertex set

$$V(\mathsf{L}(G)) = \{A \subseteq V(G) \mid \nu^2(A) = A\}.$$

For more on topological methods in combinatorics, see Matoušek [12], de Longueville [6], and Kozlov [10].

## 3. Lovász complex of quadrangulations

The goal of this section is to prove the "forward" direction of Theorem 1.1. We begin with two simple lemmas.

**Lemma 3.1.** *Let $G$ be a connected non-bipartite quadrangulation of a surface in which every 4-cycle is facial. Then either $G \cong K_{2,3}$, or $G$ is $K_{2,3}$-free and no neighborhood dominates another.*

*Proof.* Suppose $G$ contains $K_{2,3}$ with shores $\{a_1, a_2\}$ and $\{b_1, b_2, b_3\}$. Then the cycles $\{a_1, b_1, a_2, b_2\}$,$\{a_1, b_2, a_2, b_3\}$ and $\{a_1, b_3, a_2, b_1\}$ must all bound faces. Each edge $\{a_i, b_j\}$ is already in two faces, so cannot be in any other faces. Therefore, $G \cong K_{2,3}$.

Now suppose $G$ is $K_{2,3}$-free. Let $a, b$ be two distinct vertices of $G$ such that $N(b) \subseteq N(a)$. As $G$ is a quadrangulation, $|N(b)| \geq 2$. If $|N(b)| \geq 3$, then $|\,\nu(\{a, b\})| = |N(b)| \geq 3$, so $G$ contains a 4-cycle which does not bound a face.

It remains to consider the case $|N(b)| = 2$, so suppose $N(b) = \{c, d\}$. As $G$ is non-bipartite, $G$ is not a 4-cycle. The vertex $b$ has only $c, d$ as

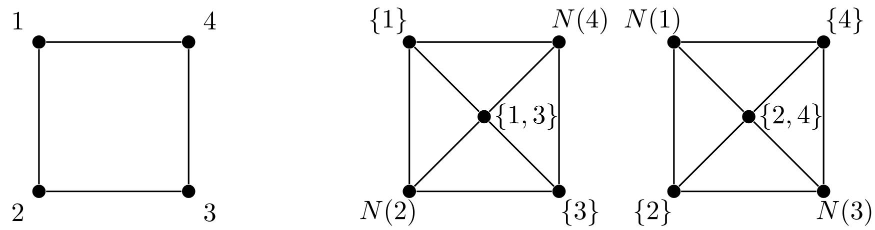


FIGURE 2. A face of a quadrangulation (left) and the corresponding faces of the Lovász complex (right).

neighbors, but the edge $\{c, b\}$ is in two faces by definition. The second face must also include $\{b, d\}$ because $b$ is not incident to other edges, and each face must have two edges incident to each vertex. Such a face has one more vertex $e \neq a$ such that $\{c, e\}$ and $\{d, e\}$ are edges of $G$. But that implies $|\,\nu(\{c, d\})| \geq 3$, so we again conclude that $G$ contains a 4-cycle which does not bound a face. $\square$

**Lemma 3.2.** *Let $G \ncong K_{2,3}$ be a connected non-bipartite quadrangulation of a surface in which every 4-cycle is facial. Then $A \in V(\mathsf{L}(G))$ if and only if $A$ satisfies one of the following conditions:*

1. *$A$ is a pair of opposite vertices in a face of $G$;*
2. *$A = N(v)$, for some vertex $v \in V(G)$;*
3. *$A = \{v\}$, for some vertex $v \in V(G)$.*

*Proof.* For the sake of readability, we call $A \in V(\mathsf{L}(G))$ a *diagonal*, *singleton*, or *neighborhood* if $A$ satisfies condition (1), (2) or (3), respectively.

We will first show that if $A \subseteq V(G)$ satisfies one of the conditions (1)–(3), then $A \in V(\mathsf{L}(G))$. Since no neighborhood dominates another, $\nu(N(a)) = \{a\}$. Hence, $\{a\} \in V(\mathsf{L}(G))$ and $N(a) \in V(\mathsf{L}(G))$ for any $a \in V(G)$.

Now consider a 4-cycle in $G$ with vertices $a, b, c, d$ (in cyclic order), and set $A = \{a, c\}$. Then $\nu(A) = \{b, d\}$ and $\nu(\{b, d\}) = A$ by the hypothesis. Therefore,

$$\nu^2(A) = \nu(\{b, d\}) = A,$$

which proves that $A \in V(\mathsf{L}(G))$.

It remains to show that if $A \in V(\mathsf{L}(G))$, then $A$ satisfies one of the conditions (1)–(3). Suppose $A \in V(\mathsf{L}(G))$ and $A$ does not satisfy the conditions (1) and (3). Then $|A| \geq 2$, and $A$ is not a diagonal, so $A$ has only one common neighbor $w$. So $N(w) = \nu(\{w\}) = \nu^2(A) = A$. We conclude that $A$ satisfies condition (2). $\square$

We are now ready to prove the "forward" direction of Theorem 1.1.

**Theorem 3.3.** *Let $G \ncong K_{2,3}$ be a connected non-bipartite quadrangulation of a surface $S$ in which every 4-cycle is facial. Then $\mathsf{L}(G)$ is homeomorphic to a double cover of $S$.*

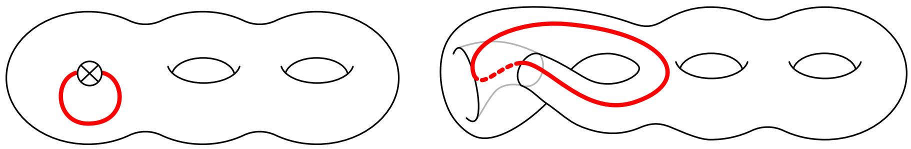

FIGURE 3. Non-orientable surfaces of odd (left) and even (right) genus, with one-sided closed curves highlighted in thick red.

*Proof.* Lemma 3.2 implies that the maximal simplices of $\mathsf{L}(G)$ are triangles. Each face of $G$ therefore induces eight faces in $\mathsf{L}(G)$ (see Figure 2). It follows that $\mathsf{L}(G)$ is a triangulation of a double cover of $S$. Indeed, the link of every vertex of $\mathsf{L}(G)$ is a cycle: the link of a diagonal is a 4-cycle, while the links of a singleton $v$ and of a neighborhood $N(v)$ are isomorphic to subdivisions of the link of $v$ in $G$, which is a cycle. □

Theorem 3.3 implies that the Euler characteristic of $\mathsf{L}(G)$ is twice that of $S$. Moreover, the double cover of an orientable surface must also be orientable, which can be seen by considering the Jacobians of transition functions of an atlas.

Nakamoto *et al.* [17] defined an algebraic invariant of quadrangulations of surfaces called *cycle parity*, denoted $\rho$, and used it to classify non-bipartite quadrangulations of closed non-orientable surfaces. They showed that $\rho$ falls into one of six types: $A$, $B$, $C$, $D$, $E$, or $F$ [17, Theorems 6–8]. In particular, if the embedding of $G$ contains a Möbius strip, it contains a one-sided cycle (see Figure 3). Consequently, $\mathsf{L}(G)$ is orientable if and only if all one-sided cycles are odd. Indeed, in the double cover of $G$, each one-sided cycle lifts either to two one-sided cycles (if even) or to a single two-sided cycle (if odd), which occurs precisely when $\rho_G$ is of type $A$, $B$, or $F$.

**Corollary 3.4.** *Let $G \ncong K_{2,3}$ be a connected non-bipartite quadrangulation of a surface $S$ in which every 4-cycle is facial.*

(1) *If $S \cong \mathbb{S}_k$, then $\mathsf{L}(G) \cong \mathbb{S}_{2k-1}$.*
(2) *If $S \cong \mathbb{N}_k$ and $G$ contains no even one-sided cycle, then $\mathsf{L}(G) \cong \mathbb{S}_{k-1}$.*
(3) *If $S \cong \mathbb{N}_k$ and $G$ contains an even one-sided cycle, then $\mathsf{L}(G) \cong \mathbb{N}_{2k-2}$.*

We conclude the section by noting that the requirement that all 4-cycles be facial is necessary for the Lovász complex to be a surface.

**Observation 3.5.** *Let $G$ be a non-bipartite quadrangulation of a surface such that some 4-cycle is not facial. Then $\mathsf{L}(G)$ is not a surface.*

*Proof.* Let $a, b, c, d$ be the vertices of a cycle that is not a face in $G$. Then the edge $\{a, b\}$ is in at least three cycles in $G$, since it is in two faces. So the edge $\{\{a\}, N(b)\}$ is in three triangles of $\mathsf{L}(G)$, with the third vertex of the triangle being the diagonal containing $a$ of each of the three 4-cycles. □

## 4. Graphs whose Lovász complex is a surface

We now prove the "backward" direction of Theorem 1.1.

**Theorem 4.1.** *Let $G$ be a $K_{2,3}$-free graph such that no neighborhood dominates another. If $\mathsf{L}(G)$ is a surface, then $G$ is a quadrangulation of $\mathsf{L}(G)/\mathbb{Z}_2$ in which every 4-cycle is facial.*

*Proof.* Singletons and neighborhoods are in $V(\mathsf{L}(G))$ by Lemma 3.2. Therefore, any triangle in $\mathsf{L}(G)$ is of the form $\{\{u\}, A, N(v)\}$, where $u, v$ are vertices of $G$ and $u \in A \subset N(v)$. Without using the hypothesis that $G$ is $K_{2,3}$-free, we can already say that $A$ is contained in an even number of triangles. Indeed, when one considers the union of the triangles containing $A$, the edges $\{A, \{u\}\}$ and $\{A, N(v)\}$ incident to $A$ must alternate, as $N(v)$ cannot be a singleton.

For $1 \leq i \leq j$, note $N(v_i)$ the neighborhoods containing $A$, and $\{u_i\}$ the singletons contained in $A$. Equivalently, $u_i$ are the elements of $A$ and $v_i$ the common neighbors of $A$. Since $\{u\} \subsetneq A$, $j \geq 2$. If $j \geq 3$, three elements of $A$ and two common neighbors would produce a $K_{2,3}$. So $A$ is contained in four triangles.

Consider the subgraph $\widetilde{G}$ of $\mathsf{L}(G)$ induced by singletons and neighborhoods. This is a symmetric quadrangulation of $\mathsf{L}(G)$. No neighborhood dominates another, so the free $\mathbb{Z}_2$-action associating $\{v\}$ to $N(v)$ is well-defined, and the orbit space $\widetilde{G}/\mathbb{Z}_2$ is isomorphic to $G$. Indeed, $\{\{v\}, N(u)\} \in \mathsf{L}(G)$ if and only if $\{\{u\}, N(v)\} \in \mathsf{L}(G)$. Since $\mathbb{Z}_2$ is a finite group acting on the surface $\mathsf{L}(G)$, the orbit space $\mathsf{L}(G)/\mathbb{Z}_2$ is also a surface. □

Note that if we drop the hypothesis of $G$ being $K_{2,3}$-free from Theorem 4.1, then $G$ is embedded in $\mathsf{L}(G)$ with all faces bounded by an even cycle (not necessarily a 4-cycle).

The following is a straightforward corollary of Theorem 4.1 and the classification of surfaces.

**Corollary 4.2.** *Let $G$ be a $K_{2,3}$-free graph such that no neighborhood dominates another.*

(1) *If $\mathsf{L}(G) \cong \mathbb{S}_{2k}$, then $G$ is a quadrangulation of $\mathbb{N}_{2k+1}$ such that every 4-cycle of $G$ bounds a face, and $G$ contains no even one-sided cycle.*
(2) *If $\mathsf{L}(G) \cong \mathbb{N}_{2k}$, then $G$ is a quadrangulation of $\mathbb{N}_{k+1}$ such that every 4-cycle of $G$ bounds a face, and $G$ contains an even one-sided cycle if and only if $k$ is odd.*
(3) *If $\mathsf{L}(G) \cong \mathbb{S}_{2k-1}$, then $G$ is a quadrangulation of either $\mathbb{S}_k$ or $\mathbb{N}_{2k}$ such that every 4-cycle of $G$ bounds a face, and $G$ contains an even one-sided cycle.*

## 5. Cohom-index of the Lovász complex of quadrangulations

The goal of the last section is to prove Theorem 1.4. Rather than working with the $\mathbb{Z}_2$-index directly, we will prove a result on the cohom-index, and apply the inequality (1).

We adapt the proof in Archdeacon *et al.* [1].

**Theorem 5.1.** *Let $G$ be a non-bipartite quadrangulation of a non-orientable surface $S$ in which every 4-cycle is facial. Then* $\operatorname{cohom-ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 2$ *if $G$ is odd, otherwise* $\operatorname{cohom-ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 1$.

*Proof.* Suppose $G$ has vertices $v_1, \dots, v_n$. Let $\widetilde{G}$ be the subgraph of the 1-skeleton of $\mathsf{L}(G)$ induced by the singletons and the neighborhoods. Note that $\widetilde{G}$ is a quadrangulation of $\mathsf{L}(G)$, and denote the set of its faces by $F$. Define a labelling $\lambda : V(\widetilde{G}) \to \{\pm 1, \dots, \pm n\}$ by $\lambda(\{v_i\}) = i$ and $\lambda(N(v_i)) = -i$.

Let $C$ be an orientizing $\ell$-cycle in $G$. Using the same idea as [1], define two orientations on couples of faces and edges (in that face); one from a fixed orientation of the orientable surface $S \setminus C$, the other from the labeling $\lambda$ by orienting $\{u, v\}$ from $u$ to $v$ if $|\lambda(u)| < |\lambda(v)|$. For each face $f \in F$, let $\varphi(f)$ be the difference between the number of edges in $f$ where the two orientations agree and the number where they disagree, as in [1]. We obtain the congruence

$$\sum_{f \in F} \varphi(f) \equiv 2\ell \pmod 4. \tag{2}$$

If the vertices in a face $f$ are not in cyclic order with respect to $|\lambda(\cdot)|$, then $\varphi(f) = 0$, and if they are, then $\varphi(f) \in \{-2, 2\}$. Therefore, if $r$ is the number of faces in $\widetilde{G}$ in cyclic order,

$$\sum_{f \in F} \varphi(G) \equiv 2r \pmod 4. \tag{3}$$

It follows from (2) and (3) that $r$ and $\ell$ have the same parity.

We will now triangulate $\widetilde{G}$ by dividing each face into two triangles (in a symmetric way). We wish to count the parity of the number of triangles whose middle vertex (by absolute value) has a different sign from the other two. Such triangles are called *gray* in Matoušek [12, pp. 103–106]. It is not hard to verify that the only faces of $\widetilde{G}$ with exactly one gray triangle have vertices in cyclic order. Therefore, the number of gray triangles has the same parity as $r$, which has the same parity as $\ell$.

We conclude that if $G$ is odd, then $\operatorname{cohom-ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 2$, otherwise $\operatorname{cohom-ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 1$. □

Theorem 1.4 is an easy consequence of Theorem 5.1.

*Proof of Theorem 1.4.* Let $G$ be an odd quadrangulation of a non-orientable surface $S$. If $G$ contains non-facial 4-cycles, then $\mathsf{L}(G)$ is not a surface, as established in Observation 3.5. To address this, consider the subcomplex

$\mathsf{K} \subseteq \mathsf{L}(G)$ induced by the diagonals, singletons, and neighborhoods. This subcomplex $\mathsf{K}$ forms a double cover of $S$, and by an argument analogous to that in the proof of Theorem 5.1, we find that $\text{cohom-ind}_{\mathbb{Z}_2}(\mathsf{K}) = 2$. Applying the monotonicity of the index along with the inequality (1), we conclude that $\text{ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) \geq \text{ind}_{\mathbb{Z}_2}(\mathsf{K}) \geq 2$. □

It is worth noting that while the level of precision provided by Nakamoto *et al.* [17] is not required to determine the homeomorphism class of the Lovász complex, it *is* essential for determining the $\mathbb{Z}_2$-index. In particular, two quadrangulations with homeomorphic Lovász complexes may have different $\mathbb{Z}_2$-index or chromatic number.

It is shown in Gonçalves *et al.* [7] that the cohom-index of a surface is equal to its $\mathbb{Z}_2$-index. The following corollary follows.

**Corollary 5.2.** *Let $G$ be a non-bipartite quadrangulation of a non-orientable surface $S$ in which each 4-cycle is facial.*

(1) *If $S \cong \mathbb{N}_{2k+1}$ and $\rho_G$ is of type $A$ or $B$, then* $\text{ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 2$.
(2) *If $S \cong \mathbb{N}_{2k+1}$ and $\rho_G$ is of type $C$, then* $\text{ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 1$.
(3) *If $S \cong \mathbb{N}_{2k}$ and $\rho_G$ is of type $D$ or $E$, then* $\text{ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 1$.
(4) *If $S \cong \mathbb{N}_{2k}$ and $\rho_G$ is of type $F$, then* $\text{ind}_{\mathbb{Z}_2}(\mathsf{L}(G)) = 2$.

Note that Musin [16] extended the Borsuk–Ulam theorem to manifolds other than the sphere; he called them *BUT-manifolds*. The above corollary implies that $\mathsf{L}(G)$ with its canonical $\mathbb{Z}_2$-action is a 2-dimensional BUT-manifold if and only if $\rho_G$ is of type $A$, $B$, or $F$.

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Université Paris Cité, CNRS, IRIF, F-75006, Paris, France.
*Email address*: {arana,matej}@irif.fr