

A Direct Approach for Detection of Bottom Topography in Shallow Water

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Abstract

We propose a fast, stable, and direct analytic method to detect underwater channel topography from surface wave measurements, based on one-dimensional shallow water equations. The technique requires knowledge of the free surface and its first two time derivatives at a single instant t^* above the fixed, bounded open segment of the domain. We first restructure the forward shallow water equations to obtain an inverse model in which the bottom profile is the only unknown, and then discretize this model using a second-order finite-difference scheme to infer the floor topography. We demonstrate that the approach satisfies a Lipschitz stability and is independent of the initial conditions of the forward problem. The well-posedness of this inverse model requires that, at the chosen measurement time t^* , the discharge be strictly positive across the fixed portion of the open channel, which is automatically satisfied for steady and supercritical flows. For unsteady subcritical and transcritical flows, we derive two empirically validated sufficient conditions ensuring strict positivity after a sufficiently large time. The proposed methodology is tested on a range of scenarios, including classical benchmarks and different types of inlet discharges and bathymetries. We find that this analytic approach yields high approximation accuracy and that the bed profile reconstruction is stable under noise. In addition, the sufficient conditions are met across all tests.

Keywords—Bathymetry, Inverse problem, Shallow water equations, Free surface measurements, Direct inverse approach, Analytic reconstruction.

1 Introduction

Accurate knowledge of underwater geometry in rivers and coastal zones is critical for coastal engineering applications, including the study of open channel flow hydrodynamics [33, 36] and the prediction of tsunami run-up in diverse settings [35, 34, 32]. This prediction direction is referred to as the direct problem and, importantly, utilizes bathymetry information to explore its impact on surface wave deformation, as well as changes in velocity and discharge across the domain. The selection of an appropriate forward model depends on the physical context and flow characteristics. For inviscid flows, we can employ the full incompressible Euler system or adopt its asymptotic simplification, including the Saint-Venant (shallow water) equations [36, 37].

Detecting the bottom profile with good accuracy is a challenging problem that often requires direct measurement techniques, which are time-consuming and costly. One of the primary approaches uses devices based on acoustic waves [38]. However, this technique is slow, incurs high operational costs, and has a risk of stranding [43, 39]. As a second technique, in [41, 42] the authors propose collecting bed topography measurements using airborne LiDAR. Although faster than sonar surveys, this approach remains costly and is limited by water clarity and flight restrictions [42, 43]. A comprehensive overview of additional direct bathymetric survey methods and their limitations can be found in the review paper [40] and the references therein.

As an alternative to these direct measurement techniques, we can infer the bottom profile using free surface measurements, which are significantly more accessible than direct underwater data [42, 44]. This strategy is formulated as an inverse problem based on a suitably chosen governing equations. For different models used to detect the bed profile from free surface data, see the review paper [45]. In this work, we formulate the recovery of bathymetry as an inverse problem for the one-dimensional shallow water equations. Importantly, our approach does not rely on the zero-inertia approximation [47] nor the small wave amplitude assumption [48]. In this direction, there are mainly two approaches used to solve this inverse problem: direct numerical methods (one-shot) [10, 13] and iterative schemes based on PDE-constrained optimization [6, 17, 46]. In the one-shot reconstruction paradigm, one first discretizes the forward shallow water equations and then manipulates the discrete system by swapping the roles of the free surface elevation and the bottom profile. This scheme requires access to the free surface elevation, the inlet discharge, and the bed elevation at the upstream boundary of the open channel. However, this method assumes steady-state flow. Moreover, there is no rigorous analysis of its convergence, stability, or independence from the initial conditions of the forward model. The only exception is [13], where a pseudo-analytic solution of the bed profile is derived in the steady one-dimensional case. On the other hand, the optimization approaches are free from the steady-state assumption; however, they require the same measurements as the direct methods, collected over a sufficiently long time interval. In particular, in [17, 46], a limited number of observation locations of the free surface is considered, yet the numerical experiments show that increasing the number of observation points improves convergence. In addition to the measurements used by the direct approach, the optimization-based approaches implicitly necessitate specified initial velocity and outlet depth to numerically solve the shallow water system and its adjoint. Moreover, a smallness condition on the amplitudes of the free surface and the bathymetry is required for the convergence in [17, 46].

In this work, we follow the one-shot approach introduced in [10]. Importantly, we refrain from assuming steady-state flow. Moreover, instead of manipulating a discrete shallow-water scheme, we first derive an inverse system from the continuous forward model, where the bottom profile is the only unknown. We then discretize this inverse system using a second-order finite-difference method. Crucially, our formulation is inherently independent of the initial conditions of the forward model and relies on minimal measurements that naturally extend those of the steady-state method [10]. Furthermore, we prove a Lipschitz stability estimate for our approach, which is an important guarantee for this class of ill-posed inverse problems. In the end, we confirm the accuracy and robustness of the approach through various numerical tests. The Python code is available in [24], allowing the reproduction of all numerical results of this paper and the development of further test cases.

1.1 Main results

We denote the elevation of the free surface by ζ . Assume that, at a single time t^* , we have access to the following three free surface measurements:

$$S_m(t^*) = (\zeta, \partial_t \zeta, \partial_t^2 \zeta)|_{\{t^*\} \times I}, \quad (1)$$

throughout a fixed channel interval $I = (a_1, a_2)$. Equivalently, we may replace the single instant measurements (1) by knowing ζ on a small time interval $(t^* - \epsilon, t^* + \epsilon) \times I$. Note that these free surface measurements are less restrictive than assuming access to a particular free surface profile over an interval of time across the domain, such as standing periodic waves in [28]. By supplementing (1) with the inlet discharge information $(q(a_1, t^*), \partial_t q(a_1, t^*))$ and the bed elevation at the upstream boundary $b(a_1)$, see Figure 1, our main results can then be summarized as follows:

1. Under the nondegeneracy condition of the discharge (5), we derive from the forward shallow water system (3) a direct inverse model (6) for the bed profile detection. Note that this condition on the discharge holds automatically in supercritical and steady flows. In particular, for the steady-state case, we obtain a fully analytic expression for the bottom profile (9) by integrating the inverse model (6). Crucially, the required

inlet discharge $q(a_1) = \text{constant}$ can be replaced by access to the bed elevation at a specific location x_* (10). Moreover, the measurements (1) reduce to the free surface elevation alone.

2. We establish two sufficient conditions (12) and (24) that guarantee the nondegeneracy assumption of the discharge for any sufficiently large time in subcritical and transcritical cases. These criteria relate inlet/outlet velocities and depths, plus a term involving the bed profile variation, which is controlled by the right- and left-going characteristic speeds. We then prove a Lipschitz stability estimate (29) for the inverse model (6).
3. Implementing the inverse model (6) via a second-order finite-difference method, we perform several numerical tests that include classical benchmarks to demonstrate both the high accuracy and robustness of the method. Moreover, in all these numerical tests, and depending on the flow, one or both sufficient conditions (12) and (24) hold. These experiments also illustrate the influence of initial conditions, boundary data, and bed geometry on flow characteristics.

1.2 Outline of the paper

In Sect. 2, we briefly introduce the one-dimensional shallow water equations and develop the corresponding inverse model for recovering the channel bed using the three free surface measurements taken at a single time (1). We then impose a strict positivity condition on the discharge across the open channel to guarantee the feasibility of a one-shot method. The section concludes with the steady-flow scenario, where we solve the inverse model to derive an explicit analytic reconstruction formula for the bed topography. In Sect. 3, we establish two (practically feasible) sufficient conditions under which the flow rate remains strictly positive in subcritical and transcritical flows. These two conditions provide a guarantee of nondegeneracy for the inverse model after a sufficiently large time. Both conditions depend on the ability of the characteristic speeds to dominate bottom variation. We end this section by proving a Lipschitz stability estimate for our reconstruction approach. In Sect. 4 we conduct several numerical tests to qualify the bathymetry reconstruction accuracy and the method stability. Each test verifies the two sufficient conditions to confirm that the discharge remains strictly positive after a sufficiently large time. Finally, Sect. 5 summarizes our findings, provides some further comments, and outlines possible directions for future work.

2 The Forward and the inverse problems

This section begins with a brief overview of the one-dimensional shallow water equations and their principal features. Moreover, we present two mathematically equivalent formulations of this system and highlight an important physical distinction that we will exploit in our bed profile detection. In Subsection 2.2, we algebraically manipulate the forward system to derive an inverse model that only needs (1) to infer the bathymetry. We conclude Subsection 2.2 by deriving an explicit analytic expression for the bed profile in the steady-flow case, where the usual inlet discharge requirement can be substituted by knowing the bed elevation at a second specific spatial point.

2.1 Governing equations

We consider the motion of a layer of an ideal, inviscid, incompressible, and irrotational fluid in an open channel. The fluid domain is bounded below by a solid bottom and above by a free surface, with both boundaries assumed to be parametrized, see Figure 1. Assuming the shallow water regime, $\sigma = \frac{h}{\lambda} \ll 1$, where h is the depth of the flow and λ is the wavelength. Averaging the two-dimensional Euler equations over the vertical axis yields the non-dispersive $\mathcal{O}(\sigma^2)$ one-dimensional shallow water equations [1, 2]:

$$\begin{cases} \partial_t \zeta + \partial_x ((\zeta - b) u) = 0, \\ \partial_t u + u \partial_x u + g \partial_x \zeta = 0, \end{cases} \quad (2)$$

where u is the depth-averaged horizontal velocity, $\zeta = h + b$ is the free surface elevation, b is the bed topography (bathymetry) to be reconstructed, and g is the gravitational acceleration set to $g = 9.812 \text{ m/s}^2$.

In system (2), the friction resistance of the bathymetry b is neglected under the inviscid flow assumption. Nonetheless, numerical investigations have shown that the friction term exerts a negligible influence on both the forward and inverse problems, see [3, 4, 5, 6]. Therefore, the Euler-based model (2) is practically valid for the study of the impact of bathymetry on surface wave deformation and for reconstructing bed topography from a given admissible measurements, under the assumption of the shallow-water regime. With straightforward algebraic manipulations, one can transform the above system (2) posed in terms of velocity and free surface elevation, to the following equivalent (discharge, depth) format:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h} \right) + g h \partial_x (h + b) = 0, \end{cases} \quad (3)$$

where $q = h u$ is the discharge (flow rate). Although the systems (2) and (3) are mathematically equivalent, they are physically different. For the moment, in practical applications, the discharge q in the domain can be measured by direct measurement [7]. As an alternative approach, we can use the first equation of the above system (3) ($\partial_x q = -\partial_t \zeta$) to infer the discharge in the domain only by knowing the first time derivative of free surface elevation ($\partial_t \zeta$), along with the inlet discharge at the upstream of the channel. This provides an easier method for approximating discharge using free surface measurements. In contrast, determining the depth-averaged horizontal velocity u is more involved and may require a known relation that connects u to the fluid's surface velocity (see, for example, [8]), which, in turn, should be approximated using direct measurement techniques [9]. In this study, we leverage the ability to reconstruct the discharge from free surface measurements in order to infer the bottom topography.

In [10], a direct algorithm for channel bathymetry was introduced, wherein the bed profile b is reconstructed by a finite-difference discretization of (3) in which the roles of the free surface elevation ζ and the bed topography b are interchanged within the momentum equation. Subsequently, this methodology was extended to two-dimensional shallow water in [11] and using finite-element discretization in [12]. All these algorithms presuppose steady-state flow and require knowledge of the free surface elevation, the inlet discharge, and the value of the bed topography at the upstream. To date, no rigorous proof of convergence has been provided. Furthermore, the dependency on the chosen initial conditions for these algorithms is not studied. As an alternative approach, the authors of [10] proposed a pseudo-analytic solution obtained by algebraic manipulation of (3) to yield an equation for b [13], which is then approximated using a first-order finite-difference scheme. Although this pseudo-analytic approach maintains the same prerequisite measurements as well as the steady-state assumption, it offers provable convergence, stability, and independence from the initial data.

One of the primary objectives of this paper is to develop a direct, fast, and stable analytical approach for identifying channel bathymetry in unsteady-state flow, utilizing only inlet discharge and free surface elevation measurements collected over a short time interval.

2.2 The inverse model

In this part, we derive the inverse equation. The main idea relies on an algebraic reordering of system (3) and observing that the continuity equation of (3) allows for the reconstruction of the discharge at any instant t over (a_1, a_2) only by knowing $q(t, a_1)$ and $\partial_t \zeta(t, \cdot)$, see Figure 1.

Consider an arbitrary open interval $I = (a_1, a_2) \subset \mathbb{R}$ within the open channel over which we aim to detect the bed topography. Assume the bottom profile b is sufficiently smooth such that the shallow water system (2) admits a unique solution (ζ, u) on $(0, t_f) \times I$, for some maximal time t_f during which ζ and q remain sufficiently smooth. Local well-posedness of (2) on the unbounded domain $\mathbb{R}$ is proved in [2] and on the half-line $\mathbb{R}_+$ in [20]. The local well-posedness on a bounded interval I with non-vanishing velocity at ∂I is established in [14, 15]. Throughout this paper, we assume that the system (2) admits a solution and that (ζ, q) is sufficiently smooth on $(0, t_f) \times I$.

Instead of directly dealing with system (3) as in [10], we transform it into the following system:

$$\begin{cases} q(t, x) = q(t, a_1) - \int_{a_1}^x \partial_t \zeta(t, y) dy, \quad \forall (t, x) \in (0, t_f) \times I, \\ \partial_x \left(\frac{q^2}{h} \right) = -g h \partial_x \zeta - \partial_t q, \quad \forall (t, x) \in (0, t_f) \times I. \end{cases} \quad (4)$$

As mentioned earlier, assuming access to the inlet discharge $q(t, a_1)$ at an instant $t \in (0, t_f)$, we can observe that from the first equation of (4), the discharge $q(t, x)$ at any point $x \in I$ can be directly obtained by knowing $\partial_t \zeta(t, \cdot)$ over (a_1, x) . The hypothesis of a known inlet discharge is a standard assumption in the inverse problem of bottom topography detection [10, 6, 17], as well as in the control of the wave profile using inlet and outlet data [18].

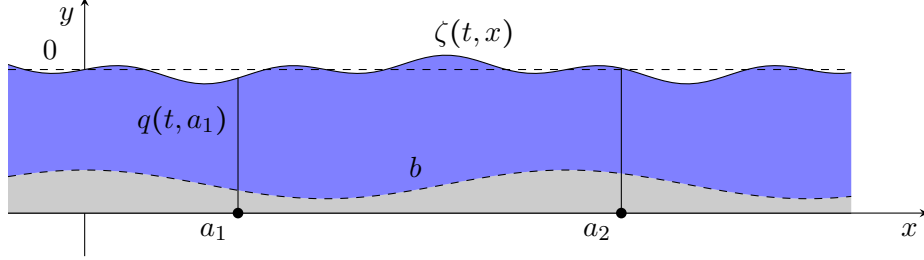


Figure 1: Scheme for the inverse problem

To develop a one-shot approach for bathymetry detection in unsteady flows, we impose the following nondegeneracy assumption:

$$\exists \beta > 0, \exists t^* \in (0, t_f), \forall x \in I, \quad q(t^*, x) \geq \beta. \quad (5)$$

Then the inverse model can be written as follows:

$$\begin{cases} q(t^*, x) = q(t^*, a_1) - \int_{a_1}^x \partial_t \zeta(t^*, y) dy, & \forall x \in I, \\ \partial_t q(t^*, x) = \partial_t q(t^*, a_1) - \int_{a_1}^x \partial_t^2 \zeta(t^*, y) dy, & \forall x \in I, \\ \partial_x \phi(t^*, x) = -g (\partial_x \zeta q^2)(t^*, x) \frac{1}{\phi(t^*, x)} - \partial_t q(t^*, x), & \forall x \in I, \end{cases} \quad (6)$$

where $\phi = \frac{q^2}{h}$ and t^* is given by (5). Under the assumption (5), the only free surface measurements required to solve the third equation in (6) are $(\zeta, \partial_t \zeta, \partial_t^2 \zeta)|_{\{t^*\} \times I}$. It should be mentioned that these three measurements correspond precisely to those considered in [19] to detect the bottom profile b using the Euler model when assuming a periodic velocity potential. After solving the equation for ϕ in (6), the bottom topography can be computed as follows:

$$b(x) = \zeta(t^*, x) - \frac{q^2(t^*, x)}{\phi(t^*, x)}, \quad \forall x \in I. \quad (7)$$

Remark 2.1. Although the work adopts the context of measurements in a single instant, it can be applied to any time interval $(0, t_f)$. If we have ζ and $q(\cdot, a_1)$ over $(0, t_f)$, then we can reconstruct $q(t, x)$ over $(0, t_f) \times I$ via the continuity equation (first equation of (4)) and numerically computing its time derivative, or directly evaluating $\partial_t q(t, x)$ from the second equation of (6). Finally, by choosing any t^* such that the discharge is strictly positive (5), we approximate the inverse equation on ϕ in the above system (6) to infer b (7).

Compared to optimization-based approaches and direct steady-state schemes, the major difference is the nondegeneracy assumption (5). However, this requirement is naturally met in open channels, where the fluid flows from upstream (a_1) to downstream (a_2). For the moment, assuming a globally supercritical flow: for any $(t, x) \in (0, t_f) \times I$, we have

$$u(t, x) > \sqrt{g h(t, x)}. \quad (8)$$

Then, the inequality (5) holds globally, since $q = h u$ and $h > \varepsilon$ for some positive constant ε . The same reasoning applies for flow with strong supercritical-type properties [15] and for the strengthened supercritical-type conditions considered in [16].

The system (6), along with knowledge of $b(a_1)$, can be seen as the natural extension of the steady-state case considered in [10, 13]. Moreover, based on (6), where the friction term is neglected, we can derive an exact

analytic solution for b as follows:

Assume the steady state ($\partial_t = 0$). Then (6) reduces to

$$\begin{cases} q(t, x) = q(0, a_1) = q, & \forall x \in I, \\ \phi(x) \partial_x \phi(x) = -g (\partial_x \zeta q^2)(x), & \forall x \in I, \end{cases}$$

straightforward integration over (a_1, x) for any $x \in I$, yields

$$\begin{cases} \frac{1}{h^2(x)} = \frac{1}{h^2(a_1)} - \frac{2g}{q^2}(\zeta(x) - \zeta(a_1)), \\ b = \zeta - h. \end{cases} \quad (9)$$

In addition, if we assume that the bottom topography is known at a second point x_* in $\{x \in I, \zeta(x) \neq \zeta(a_1)\}$ (for example, if possible, a_2). Then the inlet discharge q in (9) can be replaced by:

$$q^2 = -2g \frac{(\zeta(x_*) - \zeta(a_1)) h^2(a_1) h^2(x_*)}{h^2(a_1) - h^2(x_*)}, \quad (10)$$

which consequently yields an analytic bathymetry reconstruction independent of the inlet discharge.

We have introduced our analytic approach for the inverse problem of bathymetry detection in shallow water for both steady and unsteady cases. Although strictly positive discharge over the domain holds naturally in the steady and supercritical cases, in the next section, we demonstrate that this condition remains valid for subcritical and transcritical flows.

3 Sufficient Conditions and Stability

In this section, we begin by presenting the main Theorem 3.1, which delivers a sufficient condition for the nondegeneracy requirement (5) for subcritical flow cases. This criterion is based on an inequality linking the inlet data, outlet data, and bed profile variation with the characteristic propagation speeds. As a direct consequence of this theorem, we derive Corollary 3.1, which introduces a second sufficient condition for (5) that allows transcritical flows. Importantly, this alternative condition (24) depends only on inlet data, which matches the focus of this paper. Crucially, it reveals that bed profile variation is controlled by the right-going characteristic speed. Finally, in Subsection 3.2, we establish a Lipschitz stability estimate for our reconstruction method and illustrate the role of free surface smoothness in error amplification.

3.1 Nondegeneracy Sufficient Conditions

In the previous Section 2.2, the inverse approach was introduced under the nondegeneracy of discharge (5). Although this assumption is automatically fulfilled for supercritical flows, Theorem 3.1 below establishes that it continues to hold in the strong subcritical cases considered in [20, 16]. Specifically, we assume that there exist two positive constants $c_1, c_2 > 0$ such that for all $(t, x) \in (0, t_f) \times I$, we have

$$\lambda_1 = u(t, x) + \sqrt{g h(t, x)} \geq c_1 \quad \text{and} \quad \lambda_2 = u(t, x) - \sqrt{g h(t, x)} \leq -c_2. \quad (11)$$

Theorem 3.1. *Let $\varepsilon > 0$ such that $h(t, x) \geq \varepsilon$, $\forall (t, x) \in (0, t_f) \times I$. Assume that the bed profile $b \in C^1(I)$ and that the fluid flow satisfies the hypothesis (11). If there exists $\rho > 0$, such that for any $t \in \left(\frac{a_2 - a_1}{c_0}, t_f\right)$, where $c_0 = \min(c_1, c_2)$, the following condition holds:*

$$u(s, a_1) + u(\tau, a_2) \geq \rho - 2\sqrt{g} \left(\sqrt{h(s, a_1)} - \sqrt{h(\tau, a_2)} \right) + g \left(\frac{1}{c_1} + \frac{1}{c_2} \right) \sum_{i \geq 1} b(\gamma_{i+1}) - b(\gamma_i), \quad \forall (s, \tau) \in (0, t), \quad (12)$$

where $(\gamma_i)_i$ are such that $b'(\gamma_i) = 0$ and $\bigcup_{i \geq 1} (\gamma_i, \gamma_{i+1}) = \{x \in I, b'(x) > 0\}$. Then, for any $t \in \left(\frac{a_2 - a_1}{c_0}, t_f\right)$, we have

$$q(t, x) \geq \varepsilon \frac{\rho}{2}, \quad \forall x \in I. \quad (13)$$

Proof. The proof exploits the fact that information within the domain can be obtained from the upstream and downstream boundaries (see Figure 1) via the characteristic speeds λ_1 and λ_2 defined in (11).

We first recast system (3) in its equivalent balance-law formulation:

$$\begin{cases} \partial_t R_1 + \lambda_1 \partial_x R_1 = -g b', & \text{in } (0, t_f) \times I, \\ \partial_t R_2 + \lambda_2 \partial_x R_2 = -g b', & \text{in } (0, t_f) \times I, \end{cases} \quad (14)$$

where R_1 and R_2 are the Riemann invariants, defined as follows:

$$R_1 = u + 2\sqrt{gh} \quad \text{and} \quad R_2 = u - 2\sqrt{gh}.$$

Assuming that the flow hypothesis (11) and the condition (12) are satisfied, fix an arbitrary point $(t, x) \in \left(\frac{a_2 - a_1}{c_0}, t_f\right) \times I$. We now proceed to establish (13). By applying the method of characteristics, the system (14) can be expressed in its characteristic form as follows

$$\begin{cases} \partial_s R_1(s, X_1(s)) = -g b'(X_1(s)), & \forall s \in (0, t), \\ X_1'(s) = \lambda_1(s, X_1(s)), & \forall s \in (0, t); \quad X_1(t) = x, \end{cases} \quad (15)$$

$$\begin{cases} \partial_s R_2(s, X_2(s)) = -g b'(X_2(s)), & \forall s \in (0, t), \\ X_2'(s) = \lambda_2(s, X_2(s)), & \forall s \in (0, t); \quad X_2(t) = x. \end{cases} \quad (16)$$

Integrating the second equations of systems (15) and (16), we obtain

$$\begin{cases} x - X_1(\tau) = \int_{\tau}^t \lambda_1(s, X_1(s)) \, ds, & \forall \tau \in (0, t), \\ x - X_2(\tau) = \int_{\tau}^t \lambda_2(s, X_2(s)) \, ds, & \forall \tau \in (0, t). \end{cases}$$

Applying hypothesis (11) yields the following inequalities

$$\begin{cases} X_1(\tau) \leq x - c_0(t - \tau), & \forall \tau \in (0, t), \\ X_2(\tau) \geq x + c_0(t - \tau), & \forall \tau \in (0, t). \end{cases}$$

Using $t > \frac{a_2 - a_1}{c_0}$, we get

$$\begin{cases} X_1(0) \leq x - c_0 t < a_1, \\ X_2(0) \geq x + c_0 t > a_2. \end{cases} \quad (17)$$

By the mean-value theorem, since X_1 is strictly increasing on $(0, t)$ and X_2 is strictly decreasing on $(0, t)$, along with (17), there exist unique τ_1 and τ_2 in $(0, t)$ that depend on x and t , satisfying

$$X_1(\tau_1) = a_1 \quad \text{and} \quad X_2(\tau_2) = a_2.$$

Integrating the first equations of systems (15) and (16) over (τ_1, t) and (τ_2, t) , respectively, we arrive at

$$\begin{cases} R_1(t, x) - R_1(\tau_1, a_1) = -g \int_{\tau_1}^t b'(X_1(s)) \, ds, \\ R_2(t, x) - R_2(\tau_2, a_2) = -g \int_{\tau_2}^t b'(X_2(s)) \, ds. \end{cases} \quad (18)$$

In the following steps of the proof, we focus only on the Riemann invariant R_1 ; analogous reasoning applies to R_2 . We have

$$R_1(t, x) = u(\tau_1, a_1) + 2\sqrt{gh(\tau_1, a_1)} - g \int_{\tau_1}^t b'(X_1(s)) \, ds. \quad (19)$$

On the other hand, using the change of variable $r = X_1(s)$, the right-hand integral that involves b in the above equation (19) can be rewritten as follows

$$\begin{aligned}
\int_{\tau_1}^t b'(X_1(s)) \, ds &= \int_{a_1}^x b'(r) \frac{1}{\lambda_1(X_1^{-1}(r), r)} \, dr \\
&= \int_{\{(a_1, x), b' > 0\}} \frac{b'}{\lambda_1} + \int_{\{(a_1, x), b' < 0\}} \frac{b'}{\lambda_1} \\
&\leq \int_{\{(a_1, x), b' > 0\}} \frac{b'}{\lambda_1} \\
&\leq \frac{1}{c_1} \int_{\{I, b' > 0\}} b'.
\end{aligned} \tag{20}$$

Substituting the above inequality (20) into (19), we obtain the following inequality

$$R_1(t, x) \geq u(\tau_1, a_1) + 2\sqrt{g h(\tau_1, a_1)} - \frac{g}{c_1} \int_{\{I, b' > 0\}} b'. \tag{21}$$

Following similar arguments, with attention to the fact that X_2 is a strictly decreasing function over $(0, t)$, we derive the following lower bound for R_2

$$R_2(t, x) \geq u(\tau_2, a_2) - 2\sqrt{g h(\tau_2, a_2)} - \frac{g}{c_2} \int_{\{I, b' > 0\}} b'. \tag{22}$$

Summing the above inequalities (21) and (22) side by side, we obtain

$$2u(t, x) \geq u(\tau_1, a_1) + u(\tau_2, a_2) + 2\sqrt{g} \left(\sqrt{h(\tau_1, a_1)} - \sqrt{h(\tau_2, a_2)} \right) - g \left(\frac{1}{c_1} + \frac{1}{c_2} \right) \int_{\{I, b' > 0\}} b'.$$

Since b' is continuous on I , then $\{I, b' > 0\}$ is an open set of $\mathbb{R}$. Therefore, the above integral of b' can be written as the Riemann-sum in (12). Hence, the condition (12) implies

$$2u(t, x) \geq \rho.$$

Finally,

$$q(t, x) = h(t, x)u(t, x) \geq \varepsilon \frac{\rho}{2},$$

which completes the proof. $\square$

Before presenting a sufficient condition that utilizes only the inlet data to guarantee the nondegeneracy assumption (5), the following remarks are crucial.

Remark 3.1. Under the hypothesis (11) and regularity conditions on the bottom profile, initial data, and boundary data of the system (3), a result of local existence in $\mathbb{W}^m(t_f) = \cap_{j=0}^m C^j([0, t_f], H^{m-j}(\mathbb{R}_+)^2)$ for $m \geq 2$ is established in [20, Theorem 2.25]. Moreover, a direct continuity argument implies that, if there exists $\bar{t} \in (0, t_f)$ and $\beta > 0$ such that $q(\bar{t}, x) \geq \beta, \forall x \in I$, then there exists a neighborhood $V_{\bar{t}}$ of $\bar{t}$ in $(0, t_f)$ such that $q(t, x) \geq \beta/2, \forall (t, x) \in V_{\bar{t}} \times I$.

Remark 3.2. Theorem 3.1 remains valid if we only require the flow hypothesis (11) and the sufficient condition (12) on a shifted intervals (t_s, t_f) and (t_s, t) , respectively, where $t_f > t_s + \frac{a_2 - a_1}{c_0}$. Following the same steps in the proof, we then obtain (13) for any $t \in \left(t_s + \frac{a_2 - a_1}{c_0}, t_f\right)$. This extension is particularly important when hypothesis (11) does not hold in the neighborhood of the initial data $t = 0$. See **Test 2**.

Remark 3.3. The sufficient condition (12) that guarantees a positive discharge (5) requires considering bed profiles with a few variations (see Figure 1.10 in [2]) and a maximum amplitude $\max b$ remains small relative to the wave speeds c_1, c_2 . In particular, this scenario meets the cases of detecting low-amplitude bed obstacles such as those considered in [5, 6, 10, 17].

Remark 3.4. Theorem 3.1 provides a general condition to guarantee (13). However, in open channels where the fluid flows from upstream a_1 to downstream a_2 , the left-going wave speed c_2 can be small. Hence, the term involving the bottom variation in (12) may result in a large value if $\max b$ is not sufficiently small. Furthermore, the well-posedness and observation time t_f should be large enough.

The following result illustrates a sufficient condition requiring only inlet data. This hypothesis is motivated by Remark 3.4 and by the aim of considering transcritical flow as well. We assume that there exist positive constant $c_1 > 0$ such that for all $(t, x) \in (0, t_f) \times I$, we have

$$\lambda_1(t, x) \geq c_1, \quad (23)$$

where λ_1 is given by (11).

Corollary 3.1. Let $\varepsilon > 0$ such that $h(t, x) \geq \varepsilon$, $\forall (t, x) \in (0, t_f) \times I$. Assume that the bed profile $b \in C^1(I)$ and that the fluid flow satisfies the hypothesis (23). If there exists $\rho > 0$, such that for any $t \in \left(\frac{a_2 - a_1}{c_1}, t_f\right)$, the following condition holds:

$$u(s, a_1) \geq \rho - 2\sqrt{g} \left(\sqrt{h(s, a_1)} - \sqrt{\max_{(0, t_f) \times I} h} \right) + \frac{g}{c_1} \sum_{i \geq 1} b(\gamma_{i+1}) - b(\gamma_i), \quad \forall s \in (0, t), \quad (24)$$

where $(\gamma_i)_i$ are such that $b'(\gamma_i) = 0$ and $\bigcup_{i \geq 1} (\gamma_i, \gamma_{i+1}) = \{x \in I, b'(x) > 0\}$. Then, for any $t \in \left(\frac{a_2 - a_1}{c_1}, t_f\right)$, we have

$$q(t, x) \geq \varepsilon \rho, \quad \forall x \in I. \quad (25)$$

Proof. The result follows directly from the proof of Theorem 3.1, in particular from inequality (21). $\square$

Remark 3.5. Compared to condition (12), condition (24) is stated in terms of the difference between the inlet depth and the maximum depth, rather than the change of flow depth between the inlet and the outlet (12). However, both criteria quantify the relative change of the flow depth from the inlet depth. Furthermore, numerical experiments in Section 4 confirm that the sufficient condition (24) holds, including the hydraulic falls scenarios [5].

Remark 3.6. Although the sufficient condition (24) imposes constraints on the smoothness and amplitude of the bathymetry, it should be mentioned that this term is controlled by the right-going wave speed c_1 .

We note that the conclusions of Remarks 3.2-3.4 extend directly to the above Corollary 23.

Two sufficient conditions (12), (24) that ensure the validity of the direct-approach assumption (5) are introduced. These conditions rely on an inequality that balances the inlet or outlet data with the smoothness of the bottom and the speed of wave propagation. Alternatively, assumption (5) can be verified directly by discretizing the first equation in (6) and employing time-series data of inlet discharge and free surface elevation over the interval $(0, t_f)$, see Remark 2.1. We now proceed to analyze the sensitivity of our analytic reconstruction method.

3.2 Sensitivity analysis

We demonstrate that the mapping from the free surface measurements (1) to the reconstructed bed profile b is Lipschitz continuous in the L^1 -norm.

Assuming that one of the hypotheses (11)–(12) or (23)–(24) is satisfied. Then, we have (5): there exists $\beta > 0$ and $t^* \in (0, t_f)$ such that

$$q(t^*, x) \geq \beta, \quad \forall x \in I.$$

Since $(q, \partial_t q)|_{\{t^*\} \times I}$ are linked to $(\partial_t \zeta, \partial_t^2 \zeta)|_{\{t^*\} \times I}$ by the inlet discharge through the first and second equations in (6), respectively, without loss of generality, we investigate the sensitivity using the measurements

$$M_s = (b(a_1), (\zeta, q, \partial_t q)|_{\{t^*\} \times I}). \quad (26)$$

Let $\widetilde{M}_s$ denotes a perturbed measurement of M_s , and let $\widetilde{\phi}$ be the corresponding error solution of (6). Then, we have the following two systems associated with M_s and $\widetilde{M}_s$, respectively:

$$\begin{cases} \partial_x \phi(t^*, \cdot) = -g \frac{\partial_x \zeta(t^*, \cdot) q^2(t^*, \cdot)}{\phi(t^*, \cdot)} - \partial_t q(t^*, \cdot), & \text{in } I, \\ \phi(t^*, a_1) = \frac{q^2(t^*, a_1)}{h(t^*, a_1)}, \end{cases} \quad (27)$$

$$\begin{cases} \partial_x \widetilde{\phi}(t^*, \cdot) = -g \frac{\partial_x \widetilde{\zeta}(t^*, \cdot) (\widetilde{q}(t^*, \cdot))^2}{\widetilde{\phi}(t^*, \cdot)} - \partial_t \widetilde{q}(t^*, \cdot), & \text{in } I, \\ \widetilde{\phi}(t^*, a_1) = \frac{(\widetilde{q})^2(t^*, a_1)}{\widetilde{h}(t^*, a_1)}. \end{cases} \quad (28)$$

Needless to say that throughout this section, the solution is assumed to be sufficiently regular to make sense of the terms of the above systems (27) and (28). Such regularity is assured as soon as we place ourselves in the conditions of Theorem 2.25 in [20] since then the solution belongs to $\mathbb{W}^m(t_f)$, $m \geq 2$.

For simplicity, we omit in the following the explicit dependence on t^* . However, it should be mentioned that the analysis refers to the fixed instant t^* . In addition, the perturbations $\widetilde{q}$ and $\partial_t \widetilde{q}$ arise directly from the errors $\partial_t \widetilde{\zeta}$ and $\partial_t^2 \widetilde{\zeta}$ via the first and the second equations in (6), respectively. In particular, $\partial_t \widetilde{q}$ is defined by

$$\partial_t \widetilde{q}(x) := \partial_t \widetilde{q}(t^*, x) = \partial_t \widetilde{q}(t^*, a_1) - \int_{a_1}^x \partial_t^2 \widetilde{\zeta}(t^*, y) dy, \quad \forall x \in I.$$

Using this simplified notation, we have the following stability estimate.

Proposition 3.1. *Let $b, \widetilde{b} \in W^{1,1}(I)$, $\zeta, \widetilde{\zeta} \in W^{1,\infty}(I)$; $q, \widetilde{q}, \partial_t q, \partial_t \widetilde{q} \in W^{1,1}(I)$, and $\varepsilon > 0$ such that $h, \widetilde{h} \geq \varepsilon$. Under the assumption (5), we have Lipschitz stability for the detection of the bed profile using M_s (26). Precisely, there exist positive constants $C_1, C_2, C_3, C_4 > 0$, such that*

$$\|b - \widetilde{b}\|_{L^1(I)} \leq C_{1,2} E + \frac{1}{\beta^2} \|\widetilde{h}(q + \widetilde{q})\|_{L^\infty(I)} \|q - \widetilde{q}\|_{L^1(I)} + \|\zeta - \widetilde{\zeta}\|_{L^1(I)}, \quad (29)$$

where

$$C_{1,2} = \frac{1}{C_1} \left(e^{C_2(a_2 - a_1)} - 1 \right), \quad (30)$$

$$\begin{aligned} E = & \frac{q^2(a_1)}{h(a_1)\widetilde{h}(a_1)} \left(|b(a_1) - \widetilde{b}(a_1)| + |\zeta(a_1) - \widetilde{\zeta}(a_1)| \right) + \frac{q(a_1) + \widetilde{q}(a_1)}{\widetilde{h}(a_1)} |q(a_1) - \widetilde{q}(a_1)| \\ & + C_3 \|q - \widetilde{q}\|_{L^1(I)} + C_4 \|\partial_x(\zeta - \widetilde{\zeta})\|_{L^1(I)} + \|\partial_t q - \partial_t \widetilde{q}\|_{L^1(I)}, \end{aligned} \quad (31)$$

and

$$\begin{cases} C_1 = g \|\partial_x \zeta\|_{L^\infty(I)}, \\ C_2 = \frac{g}{\beta^2} \|\partial_x \zeta\|_{L^\infty(I)} \|h \widetilde{h}\|_{L^\infty(I)}, \\ C_3 = \frac{g}{\beta^2} \|\widetilde{h}\|_{L^\infty(I)} \|(\partial_x \widetilde{\zeta})(q + \widetilde{q})\|_{L^\infty(I)}, \\ C_4 = \frac{g}{\beta^2} \|\widetilde{h}\|_{L^\infty(I)} \|q^2\|_{L^\infty(I)}. \end{cases} \quad (32)$$

Proof. For ease of reading, the detailed proof is moved to Appendix A. Its core argument applies Grönwall's inequality [21] to establish an L^1 stability bound under perturbations of the variable coefficients and initial Cauchy data of the ODE (27). $\square$

We end this section with the following remark.

Remark 3.7. *In the above stability result (29), the constant $C_{1,2}$ (30) is an exponential constant that depends on the Lipschitz constant $\|\partial_x \zeta\|_{L^\infty(I)}$. Specifically, as $\|\partial_x \zeta\|_{L^\infty(I)}$ grows due to roughness or noise in the free surface data ζ , this stability constant increases sharply. This reflects the impact of the smoothness of the free surface on the detection of bed topography. Nevertheless, in our numerical tests, this issue is resolved by applying a smoother to prevent the introduction of large gradient values of the free surface in the inverse model (6); see the details in **Test 5**.*

4 Numerical results

We divide this section into three subsections. In the first Subsection 4.1, we briefly recall the finite-volume discretization [22] that we employ to approximate the forward model (3) and thus produce the required free surface measurements (1). In Section 4.2, we provide the detailed algorithm for approximating the inverse model (6) using a second-order finite-difference scheme. Finally, Section 4.3 presents five numerical tests that confirm the validity of the assumption of nondegeneracy (5) and its sufficient conditions (12)–(24). Furthermore, these tests demonstrate the high accuracy of the method and stability under noisy measurements. The Python script accompanying this section is given in [24].

4.1 Discretization of the forward system

We adopt the finite-volume scheme, detailed in [22, 23], which simultaneously preserves steady states and non-negativity of fluid depth h by incorporating a correction for dry regions. For further details, see the primary reference for this subsection [22]. A corresponding Python script is provided in [24] (file name: needed functions for code).

Based on [22], we first express the system (3) in the following equivalent form:

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{\zeta - b} + \frac{g}{2} (\zeta - b)^2 \right) = -g (\zeta - b) b'. \end{cases} \quad (33)$$

Let $U = (\zeta, q)^T$ and define

$$F(U, b) = \left(q, \frac{q^2}{\zeta - b} + \frac{g}{2} (\zeta - b)^2 \right)^T \quad \text{and} \quad S = (0, -g (\zeta - b) b')^T.$$

Then, the above system (33) can be written in the following vector form

$$\partial_t U + \partial_x F = S.$$

For the sake of simplicity, we denote the grid points by $x_\alpha = \alpha \Delta x$, where Δx is a chosen spatial step, and we denote finite-volume cells by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. Then, the main steps of the second-order central-upwind scheme [22] can be summarized as follows.

The semi-discrete system of ODEs is given by

$$\frac{d}{dt} \bar{U}_j(t) = - \frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x} + \bar{S}_j(t), \quad \forall t \in (0, t_f), \quad (34)$$

where

$$\bar{U}_j(t) \approx \frac{1}{\Delta x} \int_{I_j} U(t, x) dx, \quad \forall t \in (0, t_f).$$

To allow for discontinuous bathymetry, we approximate the bottom profile using a locally piecewise-linear approximation over each cell:

$$\check{b}(x) = b_{j-\frac{1}{2}} + \left(b_{j+\frac{1}{2}} - b_{j-\frac{1}{2}} \right) \frac{x - x_{j-\frac{1}{2}}}{\Delta x}, \quad x \in I_j. \quad (35)$$

where

$$b_{j+\frac{1}{2}} := \frac{b(x_{j+\frac{1}{2}} + 0) + b(x_{j+\frac{1}{2}} - 0)}{2}. \quad (36)$$

Here, $x_{j+\frac{1}{2}} + 0$ and $x_{j+\frac{1}{2}} - 0$ are the right and left limits at $x_{j+\frac{1}{2}}$, respectively [22]. Then

$$b_j := \check{b}(x_j) = \frac{1}{\Delta x} \int_{I_j} \check{b}(x) dx = \frac{b_{j+\frac{1}{2}} + b_{j-\frac{1}{2}}}{2}. \quad (37)$$

For ease of reading, we omit the explicit time dependence (t) in the following steps. Using (36) and (37), the terms in (34) are given by

$$\bar{S}_j^{(2)} \approx -g(\bar{\zeta}_j - b_j) \frac{b_{j+\frac{1}{2}} - b_{j-\frac{1}{2}}}{\Delta x} \quad (38)$$

and

$$\begin{aligned} H_{j+\frac{1}{2}} = & \frac{A_{j+\frac{1}{2}}^+ F(U_{j+\frac{1}{2}}^-, b_{j+\frac{1}{2}}) - A_{j+\frac{1}{2}}^- F(U_{j+\frac{1}{2}}^+, b_{j+\frac{1}{2}})}{A_{j+\frac{1}{2}}^+ - A_{j+\frac{1}{2}}^-} \\ & + \frac{A_{j+\frac{1}{2}}^+ A_{j+\frac{1}{2}}^-}{A_{j+\frac{1}{2}}^+ - A_{j+\frac{1}{2}}^-} (U_{j+\frac{1}{2}}^+ - U_{j+\frac{1}{2}}^-). \end{aligned} \quad (39)$$

Here,

$$U_{j+\frac{1}{2}}^\pm = \bar{U}_{j+\frac{1}{2} \pm \frac{1}{2}} \mp \frac{\Delta x}{2} (\partial_x U)_{j+\frac{1}{2} \pm \frac{1}{2}}, \quad (40)$$

where $(\partial_x U)_j$ is computed using the generalized minmod limiter

$$(\partial_x U)_j = \text{minmod} \left(\theta \frac{\bar{U}_j - \bar{U}_{j-1}}{\Delta x}, \frac{\bar{U}_{j+1} - \bar{U}_{j-1}}{2\Delta x}, \theta \frac{\bar{U}_{j+1} - \bar{U}_j}{\Delta x} \right), \quad \theta \in [1, 2], \quad (41)$$

defined by

$$\text{minmod}(z_1, z_2, z_3) = \begin{cases} \min(z_1, z_2, z_3), & \text{if } z_i > 0, \forall i \in \{1, 2, 3\}, \\ \max(z_1, z_2, z_3), & \text{if } z_i < 0, \forall i \in \{1, 2, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

In (39), the local speeds of propagation at each interface $x_{j+\frac{1}{2}}$ are defined as

$$\begin{cases} A_{j+\frac{1}{2}}^+ = \max \left\{ u_{j+\frac{1}{2}}^+ + \sqrt{g h_{j+\frac{1}{2}}^+}, u_{j+\frac{1}{2}}^- + \sqrt{g h_{j+\frac{1}{2}}^-}, 0 \right\}, \\ A_{j+\frac{1}{2}}^- = \min \left\{ u_{j+\frac{1}{2}}^+ - \sqrt{g h_{j+\frac{1}{2}}^+}, u_{j+\frac{1}{2}}^- - \sqrt{g h_{j+\frac{1}{2}}^-}, 0 \right\}. \end{cases} \quad (42)$$

It should be mentioned that the central-upwind algorithm outlined in steps (34)-(42) does not inherently preserve positivity of the local depth values $h_{j+\frac{1}{2}}$. In [22], the authors introduced corrections to steps (40)-(41) to ensure $h_{j+\frac{1}{2}} > 0$. For brevity, the positivity-preserving details are omitted; we refer the reader to the process (2.15)-(2.21) in [22] or the Python script of this work [24].

For time discretization of (34), we adopt the third-order strong stability preserving Runge-Kutta scheme (SSP-RK) [25]. Numerical tests indicate that this solver outperforms the first-order forward Euler method and enables avoiding unnecessary errors in free surface data. We refer the reader to the time discretization using both solvers in [24], where the time step is chosen as in [22, Theorem 2.1]. Specifically,

$$\Delta t = c \frac{\Delta x}{2A}, \quad (43)$$

where

$$A = \max_j \left\{ \max \left\{ A_{j+\frac{1}{2}}^+, -A_{j+\frac{1}{2}}^- \right\} \right\}, \quad \text{and } c \in (0, 1).$$

4.2 Discretization of the inverse system

It should be mentioned that, in practice, acquiring the measurements (1) at a time t^* requires knowledge of the free surface elevation in a neighborhood of t^* , or, equivalently, a short time series of observations around t^* , to approximate the time derivatives accurately. Moreover, from the sufficient conditions (12) and (24), it can be observed that the time at which the nondegeneracy (5) holds lies in the second half of the well-posedness interval $(0, t_f)$. For this reason, the time t^* should be chosen close to t_f ; for example,

$$t^* = t_f - k \Delta t,$$

for a small $k \in \mathbb{N}$. In our numerical experiments (detailed in the next subsection), we set $t^* = t_f$.

Without loss of generality, and to allow for different instants of measurements, we extract the free surface data (1) from the computed free surface elevation ζ over $(0, t_f)$ based on the forward scheme (34)-(43). Then, we approximate the first equation in (6) using a second-order central finite-difference scheme in time and the explicit second-order Heun method in space (see [26] and the reference therein). This procedure yields numerical approximations for the discharge $q(t, x)$ and its temporal derivative $\partial_t q(t, x)$ from the measured free surface profile and the upstream discharge, see Remark 2.1. Next, for a given t^* (for example $t^* = t_f$) (5), we use the explicit second-order Heun scheme to approximate the inverse equation in (6).

1. Consider the following time mesh:

$$t_0 = 0 < t_1, t_2, \dots, t_{N-1} < t_N = t_f,$$

with $t^* = t_{n^*}$, for some index $n^* \in \{0, 1, \dots, N\}$. At any time t_n , the surface's time derivative $\partial_t \zeta(t_n, x)$ is approximated using the following nonuniform central finite-difference formula

$$\partial_t \zeta(t_n, x_j) \approx (\partial_t \zeta)_j^n = \frac{-\delta_p^2 \zeta_j^{n-1} + (\delta_p^2 - \delta_m^2) \zeta_j^n + \delta_m^2 \zeta_j^{n+1}}{\delta_m \delta_p (\delta_m + \delta_p)}, \quad (44)$$

where

$$\begin{cases} \delta_p = t_{n+1} - t_n, \\ \delta_m = t_n - t_{n-1}. \end{cases}$$

Here, the time step is nonuniform by construction (43). Furthermore, the values ζ_j^{-1} and ζ_j^{N+1} are approximated by introducing a single ghost point just outside each boundary (0 and t_f) and reconstructing the necessary value (ζ_j^{-1} and ζ_j^{N+1}) via quadratic Lagrange extrapolation from the three adjacent interior grid points.

2. We now proceed to solve the first equation in (6): $\partial_x q = -\partial_t \zeta$.

$$q_{j+1}^n = q_j^n - \frac{\Delta x}{2} ((\partial_t \zeta)_j^n + (\partial_t \zeta)_{j+1}^n). \quad (45)$$

3. We have ζ and q on the time-space domain $(0, t_f) \times I$; we then compute the temporal derivative $\partial_t q$ and the spatial derivative $\partial_x \zeta$ by applying the second-order finite-difference scheme (44). To approximate $\partial_x \zeta$, we switch the roles of time and space in (44). Moreover, as noted in Remark 2.1, $\partial_t q$ can be equivalently reconstructed from the knowledge of $\partial_t^2 \zeta$ and the time derivative of the inlet discharge $q_t(t, a_1)$, via the second equation in (6).

4. The algorithm concludes by solving the inverse equation in (6) using a given measurement time $t^* = t_{n^*}$ (5):

- Predictor step

$$\begin{cases} \Phi_1 = -g (\partial_x \zeta)_j^{n^*} (q_j^{n^*})^2 \frac{1}{\phi_j^{n^*}} - (\partial_t q)_j^{n^*} \\ \Phi_1^* = \phi_j^{n^*} + \Delta x \Phi_1 \end{cases} \quad (46)$$

- Corrector step

$$\Phi_2 = -g (\partial_x \zeta)_{j+1}^{n^*} (q_{j+1}^{n^*})^2 \frac{1}{\Phi_1^*} - (\partial_t q)_{j+1}^{n^*} \quad (47)$$

- Update step

$$\phi_{j+1}^{n^*} = \phi_j^{n^*} + \frac{\Delta x}{2} (\Phi_1 + \Phi_2). \quad (48)$$

The step 4 starts with a given value of upstream bed profile $b(a_1)$. Finally, the bathymetry is given by (7).

From the above algorithm 1–4, it is clear that the inverse system is solved solely using the single instant free surface measurements in (1), as well as the inlet discharge and the bed topography at the upstream of the open channel. In the following, we present numerical tests that support this approach and ensure the validity of the nondegeneracy assumption (5).

4.3 Numerical tests

In the following numerical tests, we consider a computational domain of length 25 m in the open channel (Figure 1). For each test, the bottom profile, initial data, and boundary conditions are prescribed to perform the finite-volume discretization of the forward model (34)-(43), which produces the surface measurement vector S_m (1). We then inject these surface measurements, along with the inlet discharge and the upstream bed profile, into the inverse algorithm 1-4, resulting in an approximation of the bottom profile. Before performing the final step of this analytical approach (46)-(48), we first verify the general assumption (5) and its sufficient conditions (12)-(24). To measure the accuracy of the method, we evaluate the relative L^∞ and L^2 error norms, defined by

$$\begin{cases} L_r^\infty = \frac{\|b - b_{ex}\|_{L^\infty(I)}}{\|b_{ex}\|_{L^\infty(I)}}, \\ L_r^2 = \frac{\|b - b_{ex}\|_{L^2(I)}}{\|b_{ex}\|_{L^2(I)}}, \end{cases} \quad (49)$$

where b_{ex} stands for the exact bottom profile and b for the approximated bottom profile. For all the following numerical tests, we choose the time step Δt according to (43), where the parameter $c = 0.9$. Furthermore, θ in (41) is set to $\theta = 1$.

Test 1: Steady subcritical flow over a bump

We begin our numerical investigation with the classical steady subcritical flow over a bump (see, for example, [27, 10]). The bump profile is given by

$$b(x) = \begin{cases} 0.2 - 0.05(x - 10)^2, & x \in (8, 12), \\ 0, & \text{otherwise,} \end{cases} \quad (50)$$

while the upstream discharge $q(t, 0)$ and downstream depth $h(t, 25)$ are fixed to :

$$q(t, 0) = 4.42 \text{ m}^3/\text{s} \quad \text{and} \quad h(t, 25) = 2 \text{ m}. \quad (51)$$

The initial conditions are set to:

$$q(0, x) = 4.42 \text{ m}^3/\text{s} \quad \text{and} \quad \zeta(0, x) = 2 \text{ m}.$$

We observed numerically that this classical case falls into the strong subcritical regime (11) with $c_1 = 6$ and $c_2 = 1.4$. As noted in Remark 3.4, the left-going wave speed c_2 is smaller than the right-going wave speed c_1 .

We discretize the 25 m channel domain with a uniform grid $\Delta x = \frac{25}{N_x}$, where $N_x = 100$, which corresponds to four surface observations per meter. To guarantee that the flow arrives at steady state, we set the final time of simulation to $t_f = 400 \text{ s}$. Figure 2 shows the discharge profile produced by the direct solver (34)-(43) alongside the discharge reconstructed using the measured free surface elevation and the inlet discharge (45). This Figure 2 demonstrates that both the forward-computed and the reconstructed discharges satisfy the steady state. Moreover, the reconstructed has a significantly smaller deviation from the steady state value $q = 4.42$ than the simulated. However, increasing N_x further improves the simulated discharge convergence toward the steady-state discharge. Finally, it should be mentioned that this is a simple case for which the required nondegeneracy (5) holds automatically.

Since we are in the steady state flow case, from (9)-(10) we have an analytic solution for the reconstructed bed profile. In Figure 3, we plot the actual bathymetry, the analytic solution, and the free surface at time $t_f = 400$. The relative L^∞ and L^2 errors defined in (49) are very small. Precisely, the relative L^∞ error is 1.98 %, and the relative L^2 error is 1.28 %. Moreover, these errors are only introduced by the forward solver (34)-(43). Furthermore, when we increase N_x , these small errors vanish, yielding an exact reconstruction of the bed profile. This, in particular, confirms the accuracy of the forward discretization scheme.

For the sake of comparison with [10, 13] and although we have an analytical solution, we apply our general inverse approach (44)-(48) to this particular case. Figure 4 shows that the reconstructed bathymetry is in good agreement with the true channel bed profile. Furthermore, the relative errors (49) are small: the relative L^∞

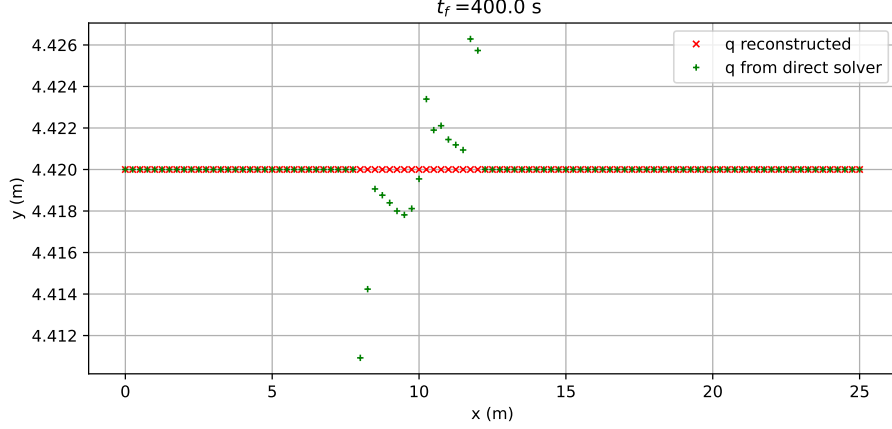


Figure 2: Comparison of the forward and reconstructed discharge profiles.

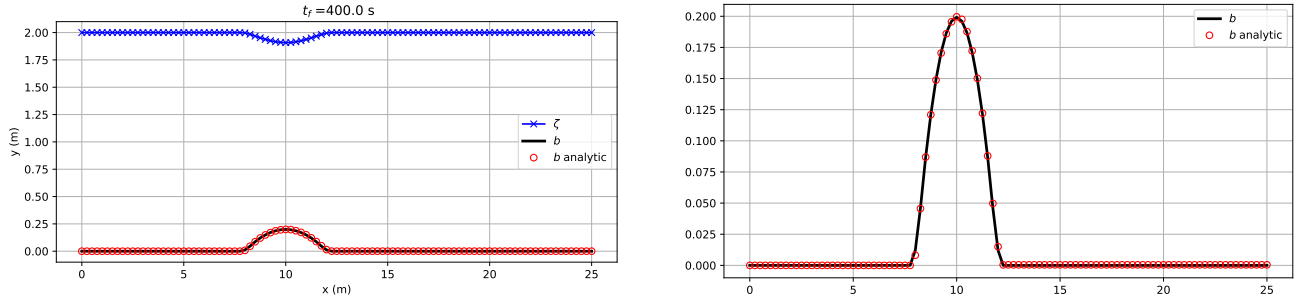


Figure 3: Comparison of the analytic bed topography and true bathymetry.

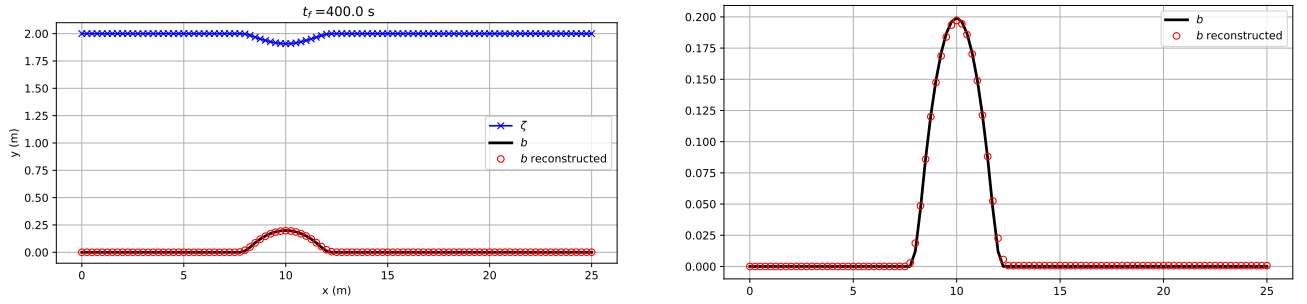


Figure 4: Comparison of the reconstructed bed topography and true bathymetry.

error is 5.26 % and the relative L^2 error is 2.26 %. As in the analytic case, these errors diminish sharply as N_x increases. For example, at $N_x = 200$ (which still represents a realistic number of observations), the relative L^∞ error drops to 2.58% and the relative L^2 error to 0.98 %.

Concluding the first test, we emphasize that our method does not depend on the flow having reached the steady state. Even for smaller t_f such as $t_f \geq 1$ s, we obtain similar results and observations compared to the steady state $t_f = 400$ s. Furthermore, it should be noted that, in all our numerical tests for this case, the sufficient conditions (12) and (24) are satisfied over the time intervals $(0, t_f)$. For brevity, the numerical details are omitted; for any further numerical tests, we refer the reader to the Python script of this paper [24].

Test 2: Unsteady subcritical flow over a bump and impact of initial data

In this test, we evaluate the adaptability of our method (see Remark (3.2)) by examining a scenario in which

the hypothesis (11) fails locally in time. Moreover, we illustrate the impact of the chosen initial data on the flow features.

To enable a direct comparison with the previous test, we retain the bottom topography given by (50) and impose the same boundary conditions (51). Instead, we initialize the flow with the stationary initial conditions:

$$q(0, x) = 0 \text{ m}^3/\text{s}, \quad \text{and} \quad \zeta(0, x) = 2 \text{ m} \quad (52)$$

This scenario reflects an unsteady inflow from upstream meets the stationary flow over $(0, 25)$. For the numerical implementation of this case, we use the same spatial step $\Delta x = \frac{25}{100}$; however, we only simulate up to $t_f = 50 \text{ s}$, well before the steady state is reached, see Figure 5.

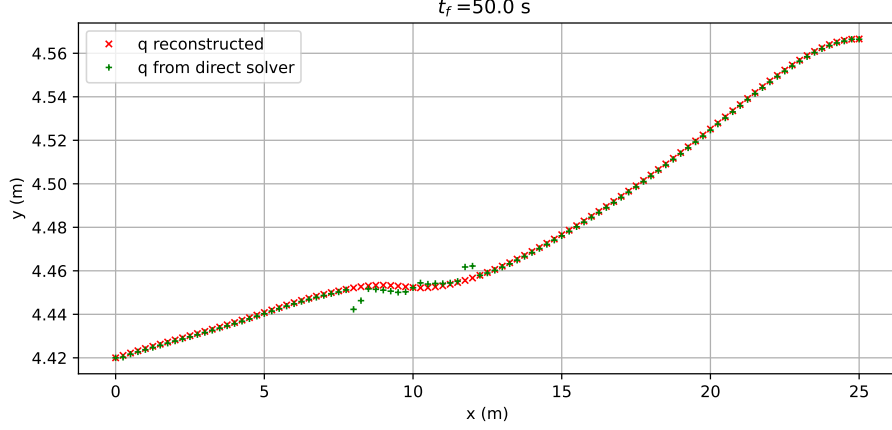


Figure 5: Comparison of the forward and reconstructed discharge profiles.

In the previous test, the strong subcriticality hypothesis (11) is verified on $(0, t_f)$ for any $t_f \geq 1 \text{ s}$. However, due to the modified initial conditions (52), this hypothesis is invalidated around $t = 20 \text{ s}$. As illustrated in Figure 6, the left-going characteristic speed λ_2 crosses zero twice, thereby breaking hypothesis (11) and thus invalidating the corresponding sufficient condition (12). Based on Remark 3.2, we only require (11) to hold from a shifted time t_s . Hence, if we choose $t_s = 27.39 \text{ s}$, hypothesis (11) is satisfied for any $t \in (27.39, t_f)$ with $c_1 = 6.3$ and $c_2 = 1.15$. Furthermore, $t_s + 25/c_0 = 49.12 < t_f$. Note that we can choose any $t_s > 27$ and take t_f large enough. Using this shift, both hypotheses (11) and (23), as well as the associated sufficient conditions (12) and (24), are satisfied, which guarantees the nondegeneracy (13) and (25) for any $t \in (49.12, 50)$.

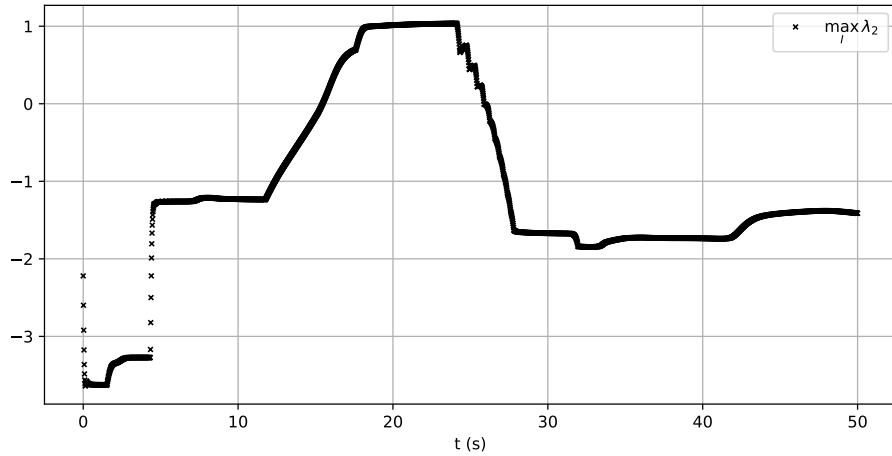


Figure 6: The maximum in space of the left-going characteristic speed.

We now compare the reconstructed and exact bathymetries. Figure 7 shows that the bed profile is accurately detected. The relative L^∞ error is 5.22% and the relative L^2 error is 3.35%. These errors decrease rapidly as N_x increases; for example, if we choose $N_x = 200$, the relative L^∞ error is 2.51% and the relative L^2 error is 1.13%.

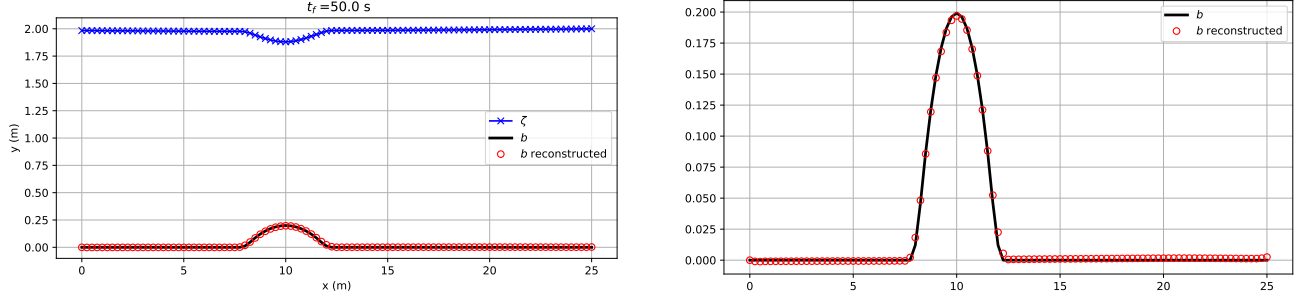


Figure 7: Comparison of the reconstructed bed topography and true bathymetry.

In both tests, we showed that the sufficient conditions (12) and (24) remain valid and adaptable when considering subcritical flows. Moreover, we demonstrated that the approach provides high reconstruction accuracy. Before extending our study to other types of flow, it should be mentioned that the results of the above two tests also apply to different bed shapes, such as the Gaussian bump examined in [28, 17]. For brevity, we do not present these supplementary simulations here; interested readers can produce the results using [24].

Test 3: Unsteady supercritical flow over a sandbar

Assuming a supercritical flow ($Fr > 1$) automatically implies the nondegeneracy assumption (5), eliminating the need for supplementary sufficient conditions (12) and (24). The Froude number (Fr) is defined by

$$Fr(t, x) = \frac{u(t, x)}{\sqrt{g h(t, x)}}, \quad \forall (t, x) \in (0, t_f) \times I. \quad (53)$$

In this test, we consider global supercritical flow. Precisely

$$Fr(t, x) > 1, \quad \forall (t, x) \in (0, t_f) \times I. \quad (54)$$

However, similar analysis can be applied to the strong supercritical flow [15] and the strengthened supercritical flow [16], since all these flow hypotheses lead to $u(t, x) > 0$ for every $t \in (0, t_f)$ and $x \in I$. In order to produce the surface data required for our analytical approach, we adopt the sandbar bed profile [28, 17]. For a given $A \in \mathbb{R}_+$, the sandbar bathymetry is defined by

$$b(x) = A \left(\tanh \left(2 \left(\frac{6x}{25} - 3 + B \right) \right) - \tanh \left(2 \left(\frac{6x}{25} - 3 - B \right) \right) \right), \quad \forall x \in I, \quad (55)$$

where $B = \frac{3\pi}{5}$. At the upstream boundary, we prescribe a constant inlet depth and a time-periodic inlet discharge:

$$q(t, 0) = 2.5 + 0.3 \sin \left(\frac{2\pi t}{10} \right) \text{ m}^3/\text{s} \quad \text{and} \quad h(t, 0) = 0.6 \text{ m}.$$

The last step before executing the forward solver is to define the initial state. Here, we consider the following non-stationary initial conditions

$$q(0, x) = 2.5 \text{ m}^3/\text{s}, \quad \text{and} \quad \zeta(0, x) = 0.6 + 0.15 e^{-\left(\frac{x-5}{1.2}\right)^2} \text{ m}.$$

In the following figures of this test, we fix the sandbar parameter (55) to $a = 0.1$. Numerical tests indicate that the flow remains supercritical and meets condition (8) until $t_f = 12$ s; beyond $t_f = 13$ s, global supercriticality (8) is lost. However, the nondegeneracy assumption (5) holds for any t_f . Without loss of generality, we chose $t_f = 12$ s. For the spatial discretization, we keep the same grid step $\Delta x = \frac{25}{100}$.

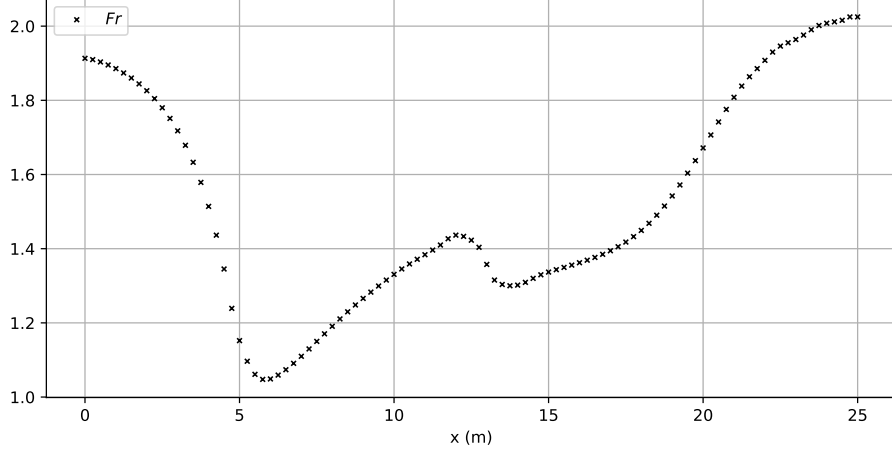


Figure 8: Spatial profile of the Froude number (53) at $t_f = 12$ s.

Figure 8 presents the Froude number distribution along the channel at t_f , clearly demonstrating supercritical flow at t_f . Although the flow is globally supercritical up to 12 s, we recall that our approach only requires a single time for which strict positivity (5) holds. Hence, we only focus on t_f .

Using the generated surface measurements (1) and the inlet discharge, we reconstruct the spatial discharge profile (45). Figure 9 presents the forward-computed discharge and the discharge reconstructed from surface measurements at $t_f = 12$ s.

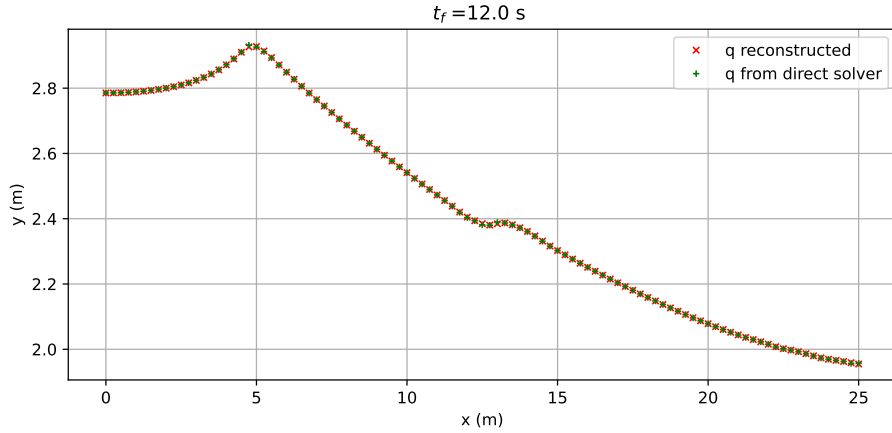


Figure 9: Comparison of the forward and reconstructed discharge profiles.

We now illustrate the accuracy of our approach to reconstruct the bottom profile. In Figure 10, it can be seen that the bed profile is very well recovered. Moreover, the relative L^∞ error is 1.28% and the relative L^2 error is 0.64%. Importantly, for a grid resolution of $N_x = 100$, the error of estimation achieved reflects excellent reconstruction performance.

At the end of this test, it should be mentioned that the reconstruction accuracy remains consistent across all time instants $t \in (0, 12)$ and even for $t \geq 13$ when hypothesis (54) is no longer valid. Furthermore, an equivalent accuracy is achieved for different values of the sandbar amplitude A (55), which underlines the flexibility of our method.

Test 4: Hydraulic fall of unsteady transcritical flow over a Gaussian topography

In this test, we examine an unsteady hydraulic fall scenario [5]. Consistent with our earlier tests, the bottom topography, inlet/outlet conditions, and initial state are specified to produce the single time measurements (1). We recall that our bathymetry-reconstruction approach does not rely on downstream observations or particular

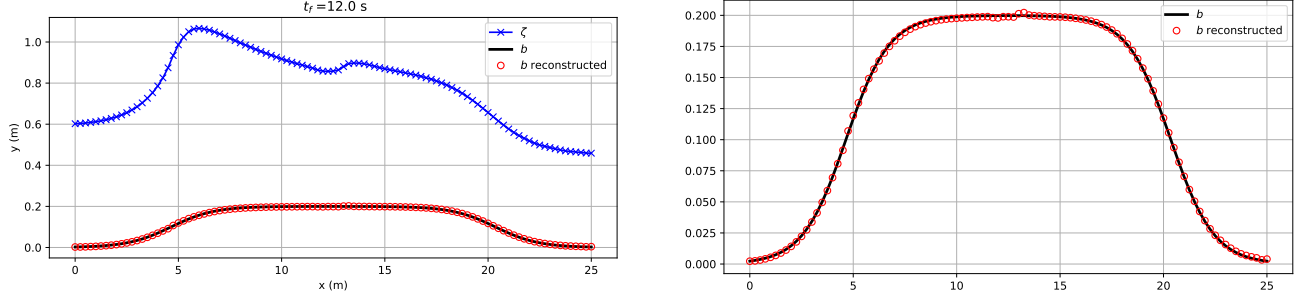


Figure 10: Comparison of the reconstructed bed topography and true bathymetry.

initial data; it only requires inlet discharge, upstream bottom value, and the free surface measurements given in (1). We simulate the hydraulic fall over the following Gaussian bottom profile [28]:

$$b(x) = 0.2 \operatorname{sech}\left(2\left(\frac{6x}{25} - 3\right)\right), \quad \forall x \in I.$$

We initialize the flow with

$$q(0, x) = 1.5 \text{ m}^3/\text{s}, \quad \text{and} \quad \zeta(0, x) = 0.7 \text{ m}.$$

In order to avoid any steady-state constraint, the inlet discharge is specified as follows

$$q(t, 0) = 1.5 + 0.2 \sin\left(\frac{2\pi t}{10}\right) \text{ m}^3/\text{s}.$$

With $\Delta x = \frac{25}{100}$, we run the forward solver (34)-(43) up to $t_f = 20 \text{ s}$. From the obtained surface measurements (1) and the given inlet flow rate, we reconstruct the discharge profile over the domain. As shown in Figure 11, the recovered discharge aligns with the direct-solver output discharge. In Figure 12, we plot the Froude number at t_f , revealing a single crossing of unity and thus confirming that the flow is transcritical.

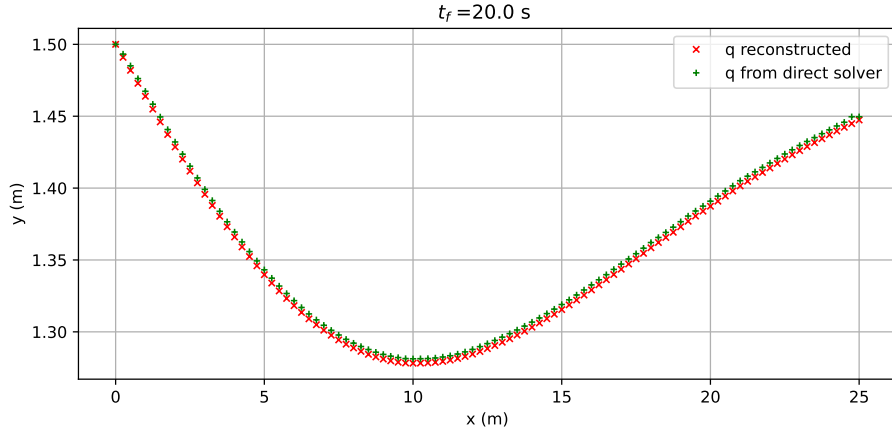


Figure 11: Comparison of the forward and reconstructed discharge profiles.

Applying the algorithm (44)–(48) produces an estimated bottom profile whose shape matches the true bathymetry, as shown in Figure 13. Quantitatively, we measure a relative L^∞ error of 1.88 % and a relative L^2 error of 2.46 %. Spatial refinement further reduces these small errors: with $N_x = 200$, the relative L^∞ error drops to 0.93 % and the relative L^2 error to 1.21 %, reflecting the robustness and precision of the method.

Regarding Corollary 3.1, although Figure 11 already indicates a strictly positive discharge, we carry out a numerical verification of the sufficient condition (24) in order to rigorously confirm nondegeneracy. Our computation shows that this sufficient condition is met for $\rho = 0.2$.

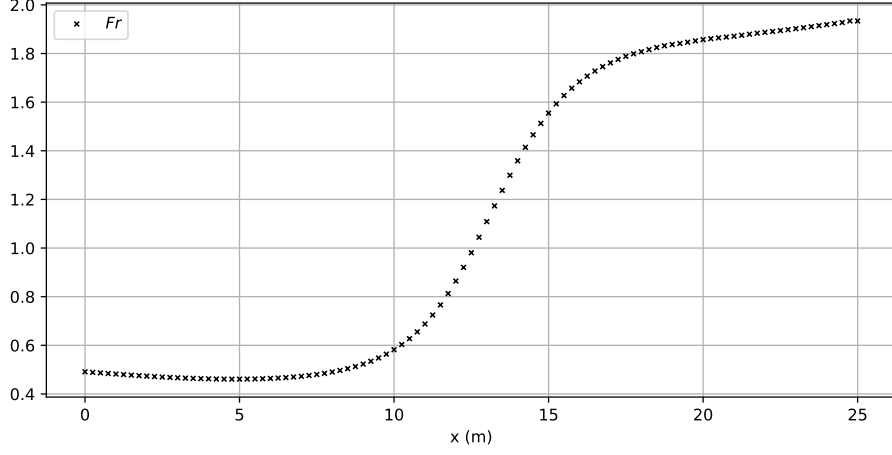


Figure 12: Spatial profile of the Froude number (53) at $t_f = 20$ s.

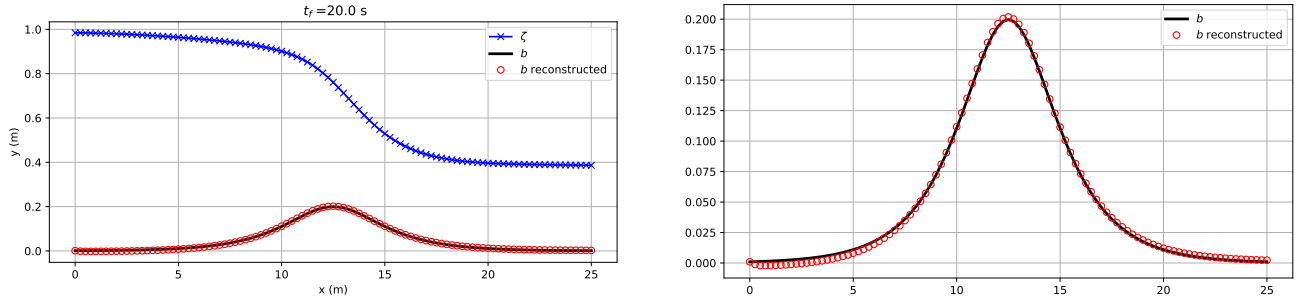


Figure 13: Comparison of the reconstructed bed topography and true bathymetry.

Test 5: Impact of noisy free surface on stability

In practice, free surface data are noisy; therefore, analyzing the response of our approach to such noise is important for the validation. Although Proposition 3.1 ensures stability in theory, the associated exponential factor grows with the Lipschitz constant $\|\partial_x \zeta\|_{L^\infty(I)}$, which tends to be large for noisy data, see Remark 3.7. Consequently, noisy free surface measurements yield larger Lipschitz constants and hence larger stability bounds. Here, we show that applying a cubic spline filter [29] to smooth the noisy free surface effectively controls this Lipschitz constant (30) and enables accurate bathymetry reconstruction.

Unlike the previous examples, this test uses a non-standard bed geometry consisting of two bumps (50) and a cosine function. To generate the surface data (1), we use the stationary initial conditions given in (52) and the downstream depth (51). For the inlet discharge, we introduce a time-dependent perturbation to the upstream discharge in (51). Precisely

$$q(t, 0) = 4.42 + 0.2 \sin\left(\frac{2\pi t}{10}\right) \text{ m}^3/\text{s}.$$

Figure 14 compares the reconstructed bathymetry to the actual profile for $t_f = 40$ s and $N_x = 100$. The bed profile is recovered with a relative L^∞ error of 5.78 % and a relative L^2 error of 3.06 %, demonstrating the good accuracy of the approach to recover a wide variety of bed geometries.

Prior to adding noise, it should be mentioned that this test closely mirrors **Test 2**. In particular, we find that the strong subcriticality hypothesis (11) becomes valid only after a time shift t_s . However, unlike in **Test 2**, beyond this instant, only the second sufficient condition (24) holds. This is because, in this test, the downstream characteristic speed c_2 is small even after shift (equal to 0.2) and cannot control the bottom profile variation in (12) (see Remark 3.4).

In the following, we introduce 2 % of noise based on the depth of the flow to the free surface. Figure 15 illustrates the noisy free surface alongside the exact profile. Injecting this noisy surface directly into (44)-(48)

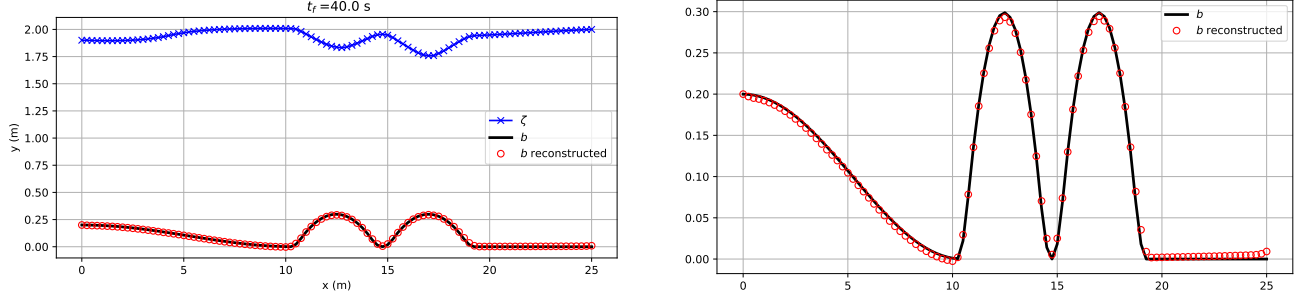


Figure 14: Comparison of the reconstructed bed topography and true bathymetry.

results in large approximation errors, as highlighted in Remark 3.7. Therefore, we first smooth the noisy free surface using a cubic-spline smoother [29], see Figure 15. This smoothed surface is then used in (44)-(48). In Figure 16, the exact bottom elevation and the reconstructed profile from the smoothed data are compared, demonstrating that our method recovers the bathymetry despite the initial noise. Moreover, even with the 2 % of noise, reconstruction accuracy remains high: the relative L^∞ error increases to 10 % and the relative L^2 error to 8 %, corresponding only to 5 % decay compared to the noise-free case. Although the error magnitudes depend implicitly on N_x , the smoothing parameter, and noise shape, the bed topography is generally well reconstructed.

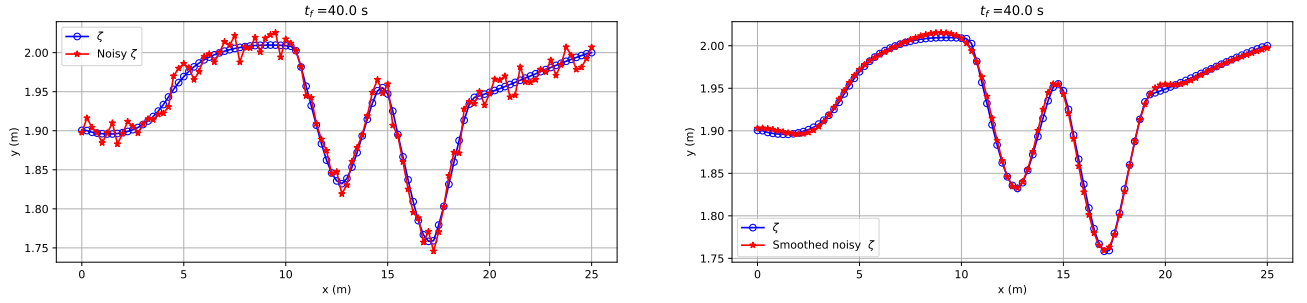


Figure 15: Exact, noisy, and cubic-Spline-smoothed free surface profiles

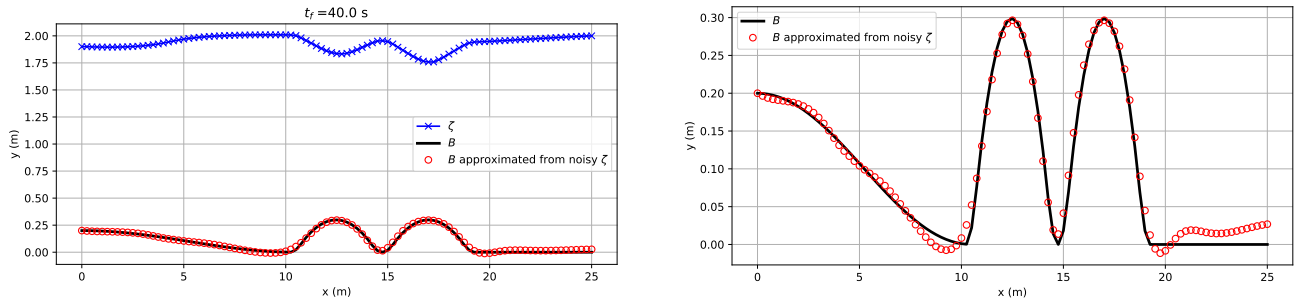


Figure 16: Comparison of the true and reconstructed bathymetry from noisy free surface

At the end of this subsection, we emphasize that our reconstruction method delivers comparable performance under different settings, such as the hypothetical hydrograph inlet discharge considered in [18], as well as for more complex bottom profiles. For any further numerical simulations, we encourage readers to utilize [24].

5 Conclusions, further comments, and future work

We have provided a direct, stable, and analytic approach to infer the bed topography using the one-dimensional shallow water equations in both steady and unsteady flows. The technique requires an open domain setting and knowledge of the free surface elevation and its first two time derivatives over the spatial domain at a single time t^* (1), as well as the flow rate and the bed profile at the upstream boundary, see Figure 1. This time t^* should satisfy the nondegeneracy assumption (5). We illustrated that this condition automatically holds for supercritical flow. To extend this guarantee to subcritical and transcritical flows, we established two sufficient conditions (12) and (24) that connect the bottom profile variation with the speed of propagation and inlet/outlet data, via the Riemann characteristics. We further proved a Lipschitz stability result, demonstrating that the bottom profile reconstruction depends continuously on the given measurement data, thus quantifying the method's robustness.

In Subsection 4.3, five numerical tests have been detailed for several bottom profiles and inlet discharges, confirming both the nondegeneracy assumption (5) and its sufficient conditions (12)-(24). In all these numerical experiments, the relative L^∞ and L^2 errors (49) were computed to rigorously quantify the accuracy of the method. We found that, in all tests, the bed profile is reconstructed with low relative errors. Even when subject to a free surface with noise or a free surface with a high Lipschitz constant $\|\partial_x \zeta\|_{L^\infty(I)}$, the method remains robust and the bottom profile is well recovered.

The work methodology exploits the exact continuity equation (6) to recover the spatial discharge profile from the first time derivative of the free surface and the inlet discharge (45). Since this equation follows directly from the full Euler equations without relying on any asymptotic regime, it applies beyond the classical shallow-water approximation and can be adapted to deeper-water models that enforce exact mass conservation [30]. Therefore, the methodology has the potential to extend to deeper-water regimes. In particular, the analytic solution in the steady-state scenario (9)-(10) also holds for the Peregrine model [31], because under the assumption of time independence ($\partial_t = 0$), this system is equivalent to the shallow water system. This demonstrates that our analytic reconstruction (9)-(10) remains valid in intermediate-depth.

A promising direction for future work is to extend the work approach to open-channel flows that admit no stagnation points. Specifically, by relaxing the nondegeneracy assumption (5) to a no-stagnation requirement, that is, for every spatial location $x \in I$ there exists a time $t(x) \in (0, t_f)$ such that $|q(t(x), x)| > 0$. Developing a reconstruction algorithm based on the inverse system (6) and proving its convergence and stability is of interest. However, it should be mentioned that this direction may require different time intervals of measurement of the free surface data S_m (1).

Acknowledgements

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Appendix

A Proof of stability

Here, we provide the details for the proof of Proposition 3.1. We recall that, for simplicity, we omit in the following the explicit dependence on t^* .

Let $t^* \in (0, t_f)$, such that (5) holds. From (27) and (28), we have

$$\begin{cases} \partial_x \phi = -g \frac{\partial_x \zeta q^2}{\phi} - \partial_t q, & \text{in } I, \\ \partial_x \tilde{\phi} = -g \frac{\partial_x \tilde{\zeta} (\tilde{q})^2}{\tilde{\phi}} - \partial_t \tilde{q}, & \text{in } I. \end{cases} \quad (56)$$

Subtracting the equations of the above system (56), we derive

$$\begin{aligned}
\partial_x(\phi - \tilde{\phi}) &= -g \partial_x \zeta q^2 \left(\frac{1}{\phi} - \frac{1}{\tilde{\phi}} \right) + g \frac{1}{\tilde{\phi}} (\partial_x \tilde{\zeta} (\tilde{q})^2 - \partial_x \zeta q^2) - (\partial_t q - \partial_t \tilde{q}) \\
&= -g \partial_x \zeta \frac{h \tilde{h}}{\tilde{q}^2} (\tilde{\phi} - \phi) + g \frac{\tilde{h}}{\tilde{q}^2} \left(\partial_x \tilde{\zeta} (\tilde{q}^2 - q^2) - q^2 (\partial_x \zeta - \partial_x \tilde{\zeta}) \right) - (\partial_t q - \partial_t \tilde{q}).
\end{aligned}$$

Hence,

$$|\partial_x(\phi - \tilde{\phi})| \leq g |\partial_x \zeta| \frac{h \tilde{h}}{\beta^2} |\phi - \tilde{\phi}| + g \frac{\tilde{h}}{\beta^2} \left(|\partial_x \tilde{\zeta}| |q + \tilde{q}| |q - \tilde{q}| + q^2 |\partial_x \zeta - \partial_x \tilde{\zeta}| \right) + |\partial_t q - \partial_t \tilde{q}|. \quad (57)$$

For ease of reading, we adopt the following notations

$$\|\cdot\|_{L^1(I)} = \|\cdot\|_1 \quad \text{and} \quad \|\cdot\|_{L^\infty(I)} = \|\cdot\|_\infty.$$

Employing this notation and integrating the above inequality (57) over (a_1, x) for any $x \in I$, we get

$$\begin{aligned}
\int_{a_1}^x |\partial_x(\phi - \tilde{\phi})| dy &\leq \frac{g}{\beta^2} \|\partial_x \zeta\|_\infty \|h \tilde{h}\|_\infty \int_{a_1}^x |\phi - \tilde{\phi}| dy \\
&\quad + \frac{g}{\beta^2} \|\tilde{h}\|_\infty \left(\|\partial_x \tilde{\zeta} (q + \tilde{q})\|_\infty \|q - \tilde{q}\|_1 + \|q^2\|_\infty \|\partial_x (\zeta - \tilde{\zeta})\|_1 \right) \\
&\quad + \|\partial_t q - \partial_t \tilde{q}\|_1.
\end{aligned} \quad (58)$$

By substituting the constants in (32), the preceding inequality (58) is equivalent to

$$\int_{a_1}^x |\partial_x(\phi - \tilde{\phi})| dy \leq C_2 \int_{a_1}^x |\phi - \tilde{\phi}| dy + C_3 \|q - \tilde{q}\|_1 + C_4 \|\partial_x (\zeta - \tilde{\zeta})\|_1 + \|\partial_t q - \partial_t \tilde{q}\|_1.$$

On the other hand, since ϕ and $\tilde{\phi}$ are in $W^{1,1}(I)$, we have

$$|\phi(x) - \tilde{\phi}(x)| \leq |\phi(a_1) - \tilde{\phi}(a_1)| + \int_{a_1}^x |\partial_x(\phi - \tilde{\phi})| dy, \quad \forall x \in I.$$

Then, for any $x \in I$, we get

$$|\phi(x) - \tilde{\phi}(x)| \leq |\phi(a_1) - \tilde{\phi}(a_1)| + C_3 \|q - \tilde{q}\|_1 + C_4 \|\partial_x (\zeta - \tilde{\zeta})\|_1 + \|\partial_t q - \partial_t \tilde{q}\|_1 + C_2 \int_{a_1}^x |\phi - \tilde{\phi}| dy.$$

With algebraic manipulation, we can obtain the following

$$\begin{aligned}
|\phi(a_1) - \tilde{\phi}(a_1)| &= \left| \frac{q^2(a_1)}{h(a_1)} - \frac{\tilde{q}^2(a_1)}{\tilde{h}(a_1)} \right| \\
&= \frac{1}{h(a_1) \tilde{h}(a_1)} |\tilde{h}(a_1) q^2(a_1) - h(a_1) \tilde{q}^2(a_1)| \\
&= \frac{1}{h(a_1) \tilde{h}(a_1)} |q^2(a_1) (\tilde{h}(a_1) - h(a_1)) + h(a_1) (q^2(a_1) - \tilde{q}^2(a_1))| \\
&\leq \frac{q^2(a_1)}{h(a_1) \tilde{h}(a_1)} (|b(a_1) - \tilde{b}(a_1)| + |\zeta(a_1) - \tilde{\zeta}(a_1)|) + \frac{q(a_1) + \tilde{q}(a_1)}{\tilde{h}(a_1)} |q(a_1) - \tilde{q}(a_1)|.
\end{aligned}$$

Hence, the definition of E in (31) yields

$$|\phi(x) - \tilde{\phi}(x)| \leq E + C_2 \int_{a_1}^x |\phi - \tilde{\phi}| dy, \quad \forall x \in I.$$

Applying Grönwall's inequality [21], we obtain

$$\|\phi - \tilde{\phi}\|_1 \leq \frac{E}{C_2} (e^{C_2(a_2 - a_1)} - 1). \quad (59)$$

We now use the definitions of ϕ and $\tilde{\phi}$ to transform the above stability (59) into a stability estimate in terms of the bathymetry profile b . We first link the bottom profile error to the error on ϕ as follows:

$$\begin{aligned}
|b - \tilde{b}| &= |(h - \tilde{h}) - (\zeta - \tilde{\zeta})| \\
&\leq |h - \tilde{h}| + |\zeta - \tilde{\zeta}| \\
&= \frac{1}{\phi \tilde{\phi}} |q^2 \tilde{\phi} - \tilde{q}^2 \phi| + |\zeta - \tilde{\zeta}| \\
&\leq \frac{h \tilde{h}}{\tilde{q}^2} |\tilde{\phi} - \phi| + \frac{\tilde{h}}{\tilde{q}^2} |q^2 - \tilde{q}^2| + |\zeta - \tilde{\zeta}|.
\end{aligned}$$

By integration over I , we get

$$\|b - \tilde{b}\|_1 \leq \frac{\|h \tilde{h}\|_\infty}{\beta^2} \|\phi - \tilde{\phi}\|_1 + \frac{1}{\beta^2} \|\tilde{h}(q + \tilde{q})\|_\infty \|q - \tilde{q}\|_1 + \|\zeta - \tilde{\zeta}\|_1. \quad (60)$$

Injecting (59) into the above inequality (60) and using the definition of C_1 in (32), results in (29), thus completing the proof.

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