Note on the Number of Almost Ordinary Triangles

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Abstract

Let X be a set of n points in the plane, not all on a line. According to the Gallai-Sylvester theorem, X always spans an *ordinary line*, i.e., one that passes through precisely 2 elements of X. Given an integer $c \geq 2$, a *line* spanned by X is called *c-ordinary* if it passes through at most c points of X. A *triangle* spanned by 3 noncollinear points of X is called *c-ordinary* if all 3 lines determined by its sides are *c-ordinary.* Motivated by a question of Erdős, Fulek *et al.* [15] proved that there exists an absolute constant $c > 2$ such that if X cannot be covered by 2 lines, then it determines at least one c-ordinary triangle. Moreover, the number of such triangles grows at least linearly in n. They raised the question whether the true growth rate of this function is superlinear.

We prove that if X cannot be covered by 2 lines, and no line passes through more than $n - t(n)$ points of X, for some function $t(n) \to \infty$, then the number of 17-ordinary triangles spanned by X is at least constant times $n \cdot t(n)$, i.e., superlinear in n. We also show that the assumption $t(n) \to \infty$ is necessary. If we further assume that no line passes through more than $n/2 - t(n)$ points of X, then the number of 17-ordinary triangles grows superquadratically in n. This statement does not hold if $t(n)$ is bounded. We close this paper with some algorithmic results. In particular, we provide a $O(n^{2.372})$ time algorithm for counting all c-ordinary triangles in an n-element point set, for any $c < n$.

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1 Introduction

Given a finite point set X, a connecting line is called *ordinary* if it is incident to precisely two points of X. The question of whether every noncollinear point set determines an ordinary line was first considered by Sylvester [28] at the end of the 19th century. The first proofs of existence of such a line were found independently by Gallai and by Melchior [23] in the 1940s. This result is now commonly referred to as the *Sylvester–Gallai Theorem*: Every set of n noncollinear points in the plane admits an ordinary line. Many related problems and statements can be found in [4, 6, 7, 8, 14, 16, 20, 25].

Motzkin [24] was the first to show that the number of ordinary lines tends to infinity with n. Kelly and Moser [19] proved that the dependence is in fact linear: They showed that n noncollinear points in the plane determine at least $3n/7$ ordinary lines, which is tight for $n = 7$. Csima and Sawyer [9] raised this bound to $6n/13$ for $n \geq 8$. Finally, Green and Tao [16] proved that if n is sufficiently large, then there are at least $n/2$ ordinary lines. On the other hand, there are arbitrarily large points sets with no more than $n/2$ ordinary lines; see, e.g., [6, Ch. 7.2].

Given a finite point set X in the plane and a constant $c \geq 2$, a connecting line is called a *c-ordinary line* if it is incident to at most c points of X; see [5, 15]. Three noncollinear elements of X make a *c-ordinary triangle* if each of the three lines induced by them is c-ordinary.

Motivated by a question of Erdős (see in [4]), de Zeeuw [5, 31] asked whether there exists a constant $c \geq 2$ such that every n-element point set X in the plane which cannot be covered by two lines has a c-ordinary triangle? Clearly, the latter assumption is necessary, because if all points of X lie on the union of two lines and both of these lines contain more than c points, then X spans no c ordinary triangle. Fulek, Mojarrad, Naszódi, Solymosi, Stich, and Szedlák et al. [15] gave a positive answer to de Zeeuw's question, with $c = 12000$. The value of c was reduced to 11 by Dubroff [12]. On the other hand, it is known that the statement is not true with $c = 2$ (see, e.g., [16, Fig. 4])

It follows from the proof of Fulek et al. [15] that the number of c-ordinary triangles grows linearly in n. They raised the question whether, for a suitable constant $c > 2$, the true growth rate is superlinear. The next statement shows that the problem needs to be carefully formulated, otherwise the answer is negative.

▶ **Proposition 1.** *Let $c, k \geq 2$, and n be any integers satisfying $n \geq 2k + c$.*

Then there exists an n-element point set X that cannot be covered by k lines. and the number of c-ordinary triangles spanned by X is at most $2k^2n$.

In particular, for a fixed k, the number of c-ordinary triangles spanned by X is at most linear in n. However, if we assume that there is no line that contains all but a constant number of points of X, then the number of c-ordinary triangles indeed grows superlinearly in n. More precisely, we have the following.

▶ **Theorem 2.** *Let $t(n) \leq 0.1n$ be a function of n, with $\lim_{n\to\infty} t(n) = \infty$. Let X be an n-element point set in the plane that cannot be covered by two lines, and suppose that X has at most $n - t(n)$ collinear points.*

Then, for $c = 17$, the number of c-ordinary triangles spanned by X is at least $\Omega(n \cdot t(n))$. This bound is tight, apart from the multiplicative constant hidden in the Ω-notation.

We can substantially improve on the above bound under the assumption that no line contains more than half of the points. In fact, it follows from our next theorem that if the maximum number of collinear points is smaller than $(1/2 - \varepsilon)n$, for a small $\varepsilon > 0$, then a positive fraction of all triangles are c-ordinary.

▶ **Theorem 3.** *Let $t(n) \leq 0.1n$ be a function of n, with $\lim_{n\to\infty} t(n) = \infty$. Let X be an n-element point set in the plane with no more than $n/2 - t(n)$ collinear points.*

Then, for $c = 17$, the number of c-ordinary triangles spanned by X is at least $\Omega(n^2 \cdot t(n))$. This bound is tight, apart from the multiplicative constant hidden in the Ω-notation.

Of course, if every line contains fewer than $n/2$ points, X cannot be covered by two lines. It is very likely that the above results remain true with a much smaller value of c, perhaps already with $c = 3$.

There are several interesting algorithmic questions related to the detection and enumeration of almost ordinary triangles in a finite point set in the plane. We present two such results.

▶ **Proposition 4.** *Given a set X of n points in the plane and a positive integer τ, all τ-ordinary triangles spanned by X can be reported in $O(n^3)$ time. The order of magnitude of this bound cannot be improved in the worst case.*

The problem gets easier if we need to find just one τ-ordinary triangle or count the number of such triangles.

▶ **Proposition 5.** *Given set X of n points in the plane and a positive integer τ, deciding whether X spans a τ-ordinary triangle and finding one, if it does, or counting all τ-ordinary triangles, can be done in $O(n^\omega)$ time, where $\omega < 2.372$ is the exponent of the best algorithm for matrix multiplication.*

Our note is organized as follows. In Section 2, we collect some classical results and technical lemmas needed for the proofs. The proofs of all combinatorial results will be presented in Section 3, while in the last section, we establish Propositions 4 and 5.

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2 Preliminaries

In 1951, Dirac [10] conjectured that every set X of n noncollinear points in the plane has a point incident to at least $\lfloor n/2 \rfloor$ distinct lines connecting it to other elements of X. The same conjecture was made, independently, by Motzkin. Although the conjecture turned out to be false, it is quite possible that it is true up to an additive constant. Ten years later, Erdős [13] formulated a weaker statement, usually referred to, as the "weak Dirac conjecture:" There is a constant $c' > 0$ such that every n-element noncollinear set X has a point incident to at least $c'n$ distinct lines connecting it to other elements of X. In 1983, this weaker conjecture was proved in two seminal papers by Beck [2] and, independently, by Szemerédi and Trotter [29].

One of the initial observations used in [2] was the following.

▶ **Lemma 6.** (Beck [26, Obs. 15]). *Let P be a set of n noncollinear points in the plane. Then at least half the lines spanned by P contain at most 3 points.*

For the proofs of Theorems 2 and 3, we need four other statements concerning incidences between points and lines, due to Langer, Payne and Wood, and de Zeeuw. All of them are refinements of some technical lemmas from Beck's paper [2]; see also [17, 27].

▶ **Lemma 7.** (Langer [21, Prop. 11.3.1]). *Given n points in the plane, with at most $2n/3$ collinear, the number of incidences between these points and their connecting lines is at least $n(n+3)/3$.*

▶ **Lemma 8.** (Payne and Wood [26, Thm. 12]). *Let X be a set of n points in the plane. If at most m points of X are collinear, then X determines at least $\frac{1}{98}n(n-m)$ distinct connecting lines.*

▶ **Lemma 9.** (de Zeeuw [31, Thm. 2.1]). *Given a set X of n points in the plane, at least one of the following holds:*

1. *Some connecting line is incident to at least γn points in X, where $\gamma = (6+\sqrt{3})/9 = 0.859\ldots$.*
2. *There are at least $n^2/9$ lines spanned by X.*

▶ **Lemma 10.** (de Zeeuw [31, Cor. 3.2]). *Let X be a set of n points in the plane, with at most $2n/3$ collinear and let $\mathcal{L}$ be the set of lines spanned by X.*

Then for any $k \geq 5$, the number of lines in $\mathcal{L}$ incident to at least k points of X is at most $4|\mathcal{L}|/(k-2)^2$.

For the algorithmic part, we use the following classical result on triangle detection and triangle counting.

▶ **Lemma 11.** (Itai and Rodeh [18].) *Deciding whether a graph contains a triangle and finding one if it does, or counting all triangles in a graph, can be done in $O(n^{\omega})$ time, where ω is the infimum of all values τ such that two $n \times n$ real matrices can be multiplied in $O(n^{\tau})$ time.*

The best known upper bound for ω is roughly 2.372; see [11, 30]. The significance of the asymptotically fastest known algorithms for matrix multiplication is mainly theoretical: most of them are impractical [22, Ch. 10.2.4]. On the other hand, currently there is no combinatorial algorithm for triangle detection that runs in truly subcubic time [1].

3 Proofs of Theorems 2 and 3

We start by giving two very similar constructions of point sets with many collinear points that cannot be covered by k lines, for a fixed $k \geq 2$, but determine only a small number of c-ordinary triangles. The first construction proves Proposition 1.

Proof of Proposition 1. Let $n \geq 2k + c$, and place $n - 2k + 1$ arbitrary points on the x-axis. Pick the remaining $2k - 1$ points in such a way that no 3 of them are collinear, and none of the $\binom{2k-1}{2}$ lines spanned by them passes through any of the points selected on the x-axis; see Fig. 1. Let X denote the resulting n-element set and Y the set of $2k - 1$ points not on the x-axis ($Y \subset X$).

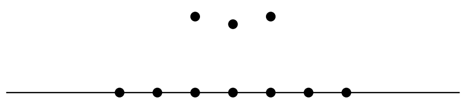

Figure 1 A counterexample to superlinearity (here $n = 10$ and $k = 2$).

Every c-ordinary triangle must have at least two vertices in Y, and the number of such triangles is at most

$$\binom{2k-1}{2} n < 2k^2 n.$$

Indeed, a c-ordinary triangle cannot have two vertices on the x-axis, because then that side is not c-ordinary, as $n - (2k - 1) \geq c + 1$. On the other hand, it is easy to see that the minimum number of lines that cover X is $k + 1$. ◀

The following slight variation of the above construction shows that the order of magnitude of the lower bound in Theorem 2 cannot be improved, even if we only require that our point set cannot be covered by k lines, for any fixed $k \geq 2$.

▶ **Proposition 12.** *Let $t(n) \leq 0.1n$ be a function of n with $\lim_{n\to\infty} t(n) = \infty$, and let $k, c \geq 2$ be fixed integers.*

Then there exists an n-element point set X in the plane with at most $n - t(n)$ points on a line such that X cannot be covered by k lines, but the number of c-ordinary triangles spanned by X is only $O(n \cdot t(n))$.

Proof of Proposition 12. Since $\lim_{n\to\infty} t(n) = \infty$, we may assume that $t(n) \geq 2k + c - 2$, provided that n is sufficiently large. Select $n - t(n)$ points arbitrarily on the line $\ell_0 : y = 0$. Pick another $t(n) - 2k + 3$ points on the line $\ell_1 : y = 1$, and the remaining $2k - 3$ points not on these lines and in such a way that the only collinear triples are contained in one of the lines $\ell_i, i = 0, 1$; see Fig. 2. Let X denote the resulting n-element set and Y the set of $2k - 3$ points not on ℓ_0 or ℓ_1 ($Y \subset X$). Clearly, X cannot be covered by k lines.

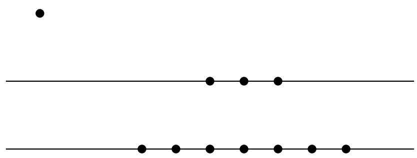

Figure 2 A set of 11 points spanning 21 ordinary triangles ($n = 11$ and $k = 2$).

Every c-ordinary triangle must have at least one vertex in Y, and at most one vertex on each of the lines ℓ_0 and ℓ_1. Indeed, an ordinary triangle cannot have two vertices on ℓ_0 or on ℓ_1, since these lines are not c-ordinary, because $n - t(n) \geq c + 1$ and $t(n) - 2k + 3 \geq c + 1$. Therefore, X spans at most $(2k - 3) \cdot n \cdot t(n) = O(n \cdot t(n))$ c-ordinary triangles. ◀

Denote by $\ell(x, y)$ the line incident to x and y, with $x, y \in X$, $x \neq y$. The following auxiliary lemma is used in the proofs of Theorems 2 and 3.

▶ **Lemma 13.** *Let X be a set of n points in the plane. Set $c = 17$ and $\alpha = 0.2$. Let ℓ be a line incident to at least αn points in X, $L = X \cap \ell$ and $Y = X \setminus L$. Let $p, q \in Y$ such that $\ell(p, q)$ is c-ordinary with respect to X.*

Then there exist at least $\alpha n/3$ points $r \in L$ such that p, q, r form a c-ordinary triangle in X.

Proof. Define

$$X_p = \{x \in L : |\ell(x, p) \cap X| \geq c + 1\}, \text{ and } X_q = \{x \in L : |\ell(x, q) \cap X| \geq c + 1\}.$$

By definition, any point of X_p determines a line in $\mathcal{L}$ that is incident to at least $c - 1$ points in $X \setminus (L \cup \{p\})$, and furthermore, these point sets are distinct for each $x \in X_p$. Hence, we have $(c - 1)|X_p| \leq n - |L|$, which gives

$$|X_p| \leq \frac{n - |L|}{c - 1} \leq \frac{1 - \alpha}{c - 1} n,$$

by the assumption of this first case. Similarly, we have

$$|X_q| \leq \frac{1 - \alpha}{c - 1} n.$$

Recall that $\alpha = 0.2$ and $c = 17$. A straightforward calculation implies that

$$\alpha - \frac{2(1 - \alpha)}{c - 1} = \frac{\alpha}{2},$$

whence there exist at least

$$|L| - |X_p| - |X_q| - 1 \geq \alpha n - \frac{2(1 - \alpha)}{c - 1} n - 1 \geq \frac{\alpha}{3} n$$

points $r \in L$ such that $r \notin (X_p \cup X_q)$ and p, q, r are noncollinear. Furthermore, p, q, r form a c-ordinary triangle, as required. ◀

Proof of Theorem 2. Assume that n is sufficiently large: $n \geq n_0 = 10^5$. Set

$$c = 17 \text{ and } \alpha = 0.2. \tag{1}$$

Let $\mathcal{L}$ denote the set of lines determined by X. As in the earlier existence proofs [12, 15], we distinguish two cases.

(i) There is a line $\ell \in \mathcal{L}$ such that $|\ell \cap X| \geq \alpha n$.
(ii) For every line $\ell \in \mathcal{L}$, we have $|\ell \cap X| \leq \alpha n$.

Case **(i)**. Let $L := \{x \in \ell \cap X\}$ and $Y := X \setminus L$. Since $Y := X \setminus L$ is noncollinear by assumption, by the result of Kelly and Moser [19], Y determines $z \geq 3|Y|/7$ ordinary lines (in Y), say,

$$\{p_j, q_j\}, j = 1, \dots, z.$$

Note that $|Y| \geq t(n)$, thus $z = \Omega(t(n))$. Note also that every ordinary line in Y is 3-ordinary in X. By Lemma 13, each pair $\{p_j, q_j\}$ occurs in at least $|L|/3 = \Omega(n)$ c-ordinary triangles with the third vertex on ℓ. Since triangles from different pairs are distinct, this completes the proof in Case (i). Indeed, we have $z|L|/3 = \Omega(n \cdot t(n))$, as required.

Case **(ii)**. We sharpen the proof of Theorem 1.1 in [12], to conclude that there are $\Omega(n^3) = \Omega(n \cdot t(n))$ c-ordinary triangles with $c = 17$. The main tools in the proof are incidence bounds for points-and-lines and averaging.

Let X' be the set of points in X that are incident to at least $33n/100$ lines determined by X.

▶ **Lemma 14.** *We have* $|X'| > n/1000$.

Proof. Recall that $\alpha = 0.2$, hence X satisfies the assumptions of Lemma 7. Let $\mathcal{L}$ denote the set of lines determined by X. Assume for contradiction that $|X'| \leq n/1000$. Let $I = I(X, \mathcal{L})$ denote the set of incidences between X and $\mathcal{L}$. We have

$$|I| \leq |X'| \cdot (n-1) + |X \setminus X'| \cdot \frac{33n}{100} \leq \frac{n}{1000} \cdot n + \frac{999n}{1000} \cdot \frac{33n}{100} = \frac{33067n}{100000} n^2 < \frac{1}{3} n^2,$$

contradicting Lemma 7. ◀

▶ **Lemma 15.** *Every point* $p \in X'$ *is incident to at least* $0.27n$ *c-ordinary lines spanned by* X.

Proof. We claim that p is incident to at most n/c lines that are not c-ordinary. Assume that this is not the case, i.e., there are more than n/c lines, each incident to at least c points in $X \setminus \{p\}$. Then there are at least

$$\frac{n}{c} \cdot c + 1 = n + 1$$

points in X, a contradiction. Recall that $c = 17$. By the previous claim and Lemma 14, it follows that p is incident to at least

$$\frac{33n}{100} - \frac{n}{17} \geq 0.27\, n$$

c-ordinary lines, as required. ◀

We can now finalize the proof of Theorem 2. The assumption of Lemma 10 is satisfied and so the number of lines in $\mathcal{L}$ that are not c-ordinary is at most

$$\frac{4}{(c-1)^2}\,|\mathcal{L}| = \frac{4}{256}\,|\mathcal{L}| \le \frac{1}{64}\binom{n}{2} \le \frac{1}{128}\,n^2. \tag{2}$$

Let $p \in X'$ be an arbitrary point in X', and let $\mathcal{L}(p)$ denote the set of c-ordinary lines incident to p. By Lemma 15, we have $|\mathcal{L}(p)| \ge 0.27n$. Let $X(p) \subseteq X$ denote the set of points in X that are incident to at least one line in $\mathcal{L}(p)$. Obviously, we have $|X(p)| \ge 0.27n$. Let $\mathcal{L}(X(p))$ denote the set of lines spanned by $X(p)$.

Note that $\alpha = 0.2 < \gamma \cdot 0.27$, where γ is the constant in Lemma 9. Thus, $X(p)$ does not satisfy the first condition of Lemma 9. Therefore, it must satisfy the second condition:

$$|\mathcal{L}(X(p))| \ge \frac{|X(p)|^2}{9} \ge \frac{n^2}{124}.$$

At most $n-1$ lines in $\mathcal{L}(X(p))$ are incident to p. Thus, at least

$$\frac{n^2}{124} - n \ge \frac{n^2}{125}$$

lines in $\mathcal{L}(X(p))$ are not incident to p. From (2), it follows that $X(p)$ determines

$$y(p) \ge \left(\frac{1}{125} - \frac{1}{128}\right)n^2 \ge \frac{1}{6000}\,n^2$$

c-ordinary lines that are not incident to p. By construction, any pair of points in $X(p)$ that belong to one of these lines, together with p, forms a c-ordinary triangle, so p participates in $\Omega(n^2)$ c-ordinary triangles. In view of Lemma 14, the number of c-ordinary triangles with at least one vertex in X' is at least

$$\Omega\left(\sum_{p \in X'} y(p)\right) = \Omega(n^3) = \Omega(n \cdot t(n)).$$

The tightness of this bound follows from Proposition 12. This concludes the proof of Theorem 2. ◀

Proof of Theorem 3. Set $c = 17$ and $\alpha = 0.2$. Let $\mathcal{L}$ denote the set of lines determined by X. As before, we distinguish two cases in the proof:

(i) There is a line $\ell \in \mathcal{L}$ such that $|\ell \cap X| \ge \alpha n$.
(ii) For every line $\ell \in \mathcal{L}$, we have $|\ell \cap X| \le \alpha n$.

Case **(i)**. Let ℓ be one of the lines passing through the largest number of elements of X. Let $L := \{x \in \ell \cap X\}$ and $Y := X \setminus L$. Note that $|Y| \ge n/2 + t(n)$. Let $\mathcal{L}'$ denote the set of lines spanned by Y. Since at most $n/2 - t(n)$ points of Y are collinear, by Lemma 8, Y spans at least

$$\frac{|Y| \cdot (|Y| - |L|)}{98} \ge \frac{(n/2 + t(n)) \cdot 2t(n)}{98} = \Omega(nt(n))$$

lines.

By assumption, Y is noncollinear, so we can apply Beck's Lemma 6. We obtain that at least half of the lines spanned by Y contain at most 3 points of Y. Thus, among all lines spanned by Y, at least $\Omega(nt(n))$ are 4-ordinary in X. Letting $\mathcal{L}'_c$ denote the set of c-ordinary lines with respect to X that are spanned by Y, we have $|\mathcal{L}'_c| = \Omega(nt(n))$. For each line $\ell' \in \mathcal{L}'_c$, fix two arbitrary points $p, q \in \ell'$.

By Lemma 13, each pair p, q occurs in at least $|L|/3 = \Omega(n)$ c-ordinary triangles with the third vertex on ℓ. Since any two triangles that belong to different pairs are distinct, this completes the proof in Case (i). Indeed, we have $|\mathcal{L}'_c| \cdot |L|/3 = \Omega(n^2 \cdot t(n))$, as required.

Case **(ii)**. Same proof as for case (ii) in Theorem 2 applies. The number of c-ordinary triangles found is $\Omega(n^3) = \Omega(n^2 \cdot t(n))$.

For the tightness, distribute the n points on three parallel lines incident to $n/2 - t(n)$, $n/2 - t(n)$, and $2t(n)$ points, respectively, and observe that any c-ordinary triangle must have a vertex on each of these lines. Therefore, the number of c-ordinary triangles is at most

$$\frac{n}{2} \cdot \frac{n}{2} \cdot 2t(n) = \frac{1}{2} \cdot n^2 \cdot t(n),$$

as claimed. ◀

4 Algorithmic aspects

In this section, we establish Propositions 4 and 5.

Proof of Proposition 4. Let $\mathcal{L}$ be the set of lines spanned by X. For every line $\ell \in \mathcal{L}$, let $I(\ell)$ denote the number of incidences of ℓ with points in X.

In the first phase, the algorithm determines the counts $I(\ell)$ over all $\ell \in \mathcal{L}$ by using the standard point-line duality [3, Chap. 8], after computing the dual line arrangement. In this setting, the count $I(\ell)$, where $\ell = \ell(a, b)$ is the line spanned by $a, b \in X$, is equal to the degree of the vertex that is the intersection point of the dual lines $D(a)$ and $D(b)$. Since the dual line arrangement can be computed in $O(n^2)$ time, the first phase takes $O(n^2)$ time. Alternatively, computing $I(\ell)$ for all $\ell \in \mathcal{L}$ can be done by brute force in cubic time, working in the primal setting; this would also be fine and within the overall time complexity.

In the second phase, the algorithm scans all triples a, b, c of points in X; for each such triple it determines whether the lines $\ell(a, b)$, $\ell(b, c)$, and $\ell(c, a)$ are τ-ordinary. If so, the triangle Δabc is τ-ordinary and, hence, is included in the output. Duplicates can be easily filtered out. Since the number of triples is $O(n^3)$, the second phase, and thus the entire algorithm, takes $O(n^3)$ time.

Note that for a set of n points in general position, all $\binom{n}{3}$ triangles are ordinary. Similarly, under broad conditions as in Theorem 3, the number of τ-ordinary triangles is cubic when $\tau = 17$. It follows that the algorithm is asymptotically optimal in the worst case. ◀

Proof of Proposition 5. As in the proof of Theorem 4, in the first phase, the algorithm determines the counts $I(\ell)$ over all $\ell \in \mathcal{L}$ in $O(n^2)$ time. In the second phase, the algorithm constructs an undirected graph G on vertex set X, where two vertices are connected by an edge if and only if the connecting line is τ-ordinary. Since $|X| = n$, constructing G takes $O(n^2)$ time based on the previous information. Note that triangles in G are in one-to-one correspondence to τ-ordinary triangles spanned by X. Then it runs the algorithms of Itai and Rodeh (cf. Lemma 11) for triangle detection in G (respectively, for triangle counting), and returns with the same answer. This last step takes $O(n^\omega)$ time, thus the overall running time is $O(n^\omega)$. ◀

Given X, finding the smallest τ for which there is a τ-ordinary triangle spanned by X is easily accomplished in $O(n^{\omega} \log \tau)$ time: use the detection algorithm from Theorem 5 for $\tau = 2, 4, 8, \ldots$ until the answer is positive, followed by binary search in the interval between the last negative answer and the first positive answer.

It would be interesting to know whether τ-ordinary triangle detection can be done in quadratic time or by a truly subcubic combinatorial algorithm.

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