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Holographic description of 4d Maxwell theories and their code-based ensembles

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Abstract: We formulate a precise holographic duality between an ensemble of 4d $\mathrm{U}(1)^g$ Maxwell theories living on a spin four-manifold M_4 and an Abelian BF-type 2-form gauge theory of level N, summed over all five-manifolds with boundary M_4. The elements of the boundary ensemble are Abelian gauge theories specified by self-dual symplectic codes over $\mathbb{Z}_N$, that parameterize topological boundary conditions in the 5d TQFT. Similarly, the equivalence classes of topologies distinguished by the 5d theory are parameterized by orthogonal self-dual codes. Hence the holographic duality can be reformulated in the language of quantum stabilizer codes. This duality is closely related to the holographic relationship between ensembles of Narain conformal field theories in 2d and level-N Abelian Chern-Simons theories in 3d. In both contexts, the duality extends to correlation functions. In the large-N limit, we find that the boundary ensemble average converges to an integral over the moduli space of the gauge couplings and, when finite, is equal to an Eisenstein series of the orthogonal group, a version of the Siegel-Weil formula that appears in the 2d/3d context. As a spinoff, we clarify the holographic relationship between the gauge group of the 4d $\mathcal{N}=4$ super Yang-Mills theory and the boundary conditions of the singleton sector in the bulk.

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1 Introduction

Nearly three decades after the emergence of holography as a fundamental framework for string theory and quantum gravity, there are still many aspects of the holographic correspondence that are incompletely understood. In particular, over the past few years it has been realized that the paradigm of holography as a relation between a single quantum field theory and a gravitational theory requires modification. In a variety of low-dimensional examples [1–3], it has been found that gravitational theories are dual not to a single quantum field theory, but to a whole ensemble of QFTs.

In this paper we will establish a new class of holographic dualities between ensembles of four-dimensional Abelian gauge theories and five-dimensional topological gravity theories. We will relate ensemble averages of $\mathrm{U}(1)^g$ Maxwell theories living on a closed 4-manifold $\mathcal{M}_4$ to five-dimensional topological quantum field theories (TQFTs), summed over 5-manifolds that end on $\mathcal{M}_4$. The TQFTs in question are 5d BF-type Chern-Simons theories involving 2-form gauge fields B_2 and C_2.

The composition of the ensemble and the set of inequivalent topologies appearing in the sum are controlled by the level N of the Chern-Simons theory. At level $N = 1$, the ensemble consists of a single boundary theory – as in conventional holography – and the sum over bulk 5-manifold topologies is trivial, because the bulk theory then does not distinguish between topologies. For $N > 1$, the ensembles that arise can be described in terms of codes. These are self-dual symplectic codes of length $2g$ over $\mathbb{Z}_N$ that parametrize topological boundary conditions of the 5d theory. Each 4d partition function in the ensemble corresponds to a stabilizer state, i.e. a quantum code defined in terms of the symplectic one. Likewise, the set of equivalence classes of topologies that are distinguished by the bulk CS theory are associated with self-dual orthogonal codes and corresponding stabilizer states.

We can schematically represent our holographic relation as follows:

$$\langle Z_{\mathrm{Maxwell}} \rangle = \sum_{\mathrm{topologies}} Z_{5\mathrm{d\,CS}} \tag{1.1}$$

where the left side is the ensemble averaged partition function and the $Z_{5\mathrm{d\,CS}}$ on the right is a wavefunction of the bulk CS theory on a topology with a fixed boundary. This holographic duality extends to include correlation functions of $\mathrm{U}(1)$ primaries. Each partition function appearing in the boundary ensemble is invariant under the 4d modular group $\mathrm{O}(n, n, \mathbb{Z})$. There is an additional group $\mathrm{Sp}(2g, \mathbb{Z}_N)$ inherited from the bulk, where it is a 0-form symmetry, that maps theories in the ensemble to each other. This is not the S-duality group, which is a symmetry of each 4d theory. On the right-hand side, each bulk wavefunction is invariant under $\mathrm{Sp}(2g, \mathbb{Z}_N)$, while the set of 5d topologies decomposes into orbits of $\mathrm{O}(n, n, \mathbb{Z})$. The combined invariance under both $\mathrm{Sp}(2g, \mathbb{Z}_N)$ and $\mathrm{O}(n, n, \mathbb{Z})$ when the level N is square-free constrains both sides to be equal up to normalization, which can be fixed on physical grounds [4].

Our setup and results have closely related analogues in 2d/3d, where ensembles of Narain conformal field theories (CFTs) are known to be dual to Abelian Chern-Simons theories. In that context, it has been established that the CFT partition function, averaged over a discrete set of Narain moduli specified by codes, is reproduced by the path integral of a Chern-Simons theory, summed over all possible topologies with a fixed boundary [4, 5]. It has been known for a long time that the partition function of the 2d Narain CFT for worldsheet modulus Ω_{ij} and Narain modulus E_{IJ} is closely related to the partition function of 4d $\mathrm{U}(1)^g$ Maxwell with coupling matrix Ω_{ij} on a 4-manifold with modulus E_{IJ}

[6, 7]. In the 4d Maxwell case, the spacetime and target space modular groups, respectively $\mathrm{O}(n,n,\mathbb{Z})$ and $\mathrm{Sp}(2g,\mathbb{Z})$, play roles that are reversed relative to their roles in Narain CFTs. Thus the elements of the Maxwell ensemble, parameterized by symplectic codes, correspond to equivalence classes of topologies in the 3d CS theory. Similarly, distinct Narain theories in the ensemble, specified by orthogonal codes, correspond to equivalence classes of the 5d topologies appearing in the sum on the right side of (1.1). The origin of this Narain-Maxwell duality lies in 6d/7d, the duality between the 6d Abelian self-dual two-form gauge theory on $\Sigma \times \mathcal{M}_4$ and the 7d three-form CS theory [7, 8].

It is especially interesting to consider the large-level limit of the Maxwell ensemble. As $N \to \infty$, the ensemble-averaged partition function becomes an integral over the entire moduli space of $\mathrm{U}(1)^g$ gauge couplings (which is identical to the fundamental domain of genus-g Riemann surfaces), and the sum over 5d topologies approaches a sum over 5d handlebodies, yielding an Eisenstein series of the orthogonal group. This is similar to 2d/3d result, where the large-N limit yields the Siegel-Weil formula, and reproduces the results of [2, 3].

Thanks to the exact solvability of both sides of our duality, we will be able to address some central questions about ensemble holography: When does an ensemble of boundary theories have a holographic dual? How are the ensemble weights determined? Which topologies should be summed over in the bulk theory and with what weights?

Addressing these questions helped us clarify the conventional holographic dictionary of $\mathcal{N} = 4$ SYM dual to Type IIB supergravity. It is well known that the bulk includes a topological sector of the kind we discuss in this paper [9]. We will see that different boundary conditions for the TQFT fields correspond to different theories at the boundary: $\mathrm{U}(N)$ SYM theory or one of the $\mathrm{SU}(N)$ theories.

The paper is organized as follows. In Section 2, we carry out the holomorphic quantization of 5d Abelian Chern-Simons theory on $M_4 \times \mathbb{R}$, where M_4 is a closed spin four-manifold. In section 3, we discuss the relation to SymTFT and the connection to codes. In section 4, we discuss the holographic duality between ensembles of Maxwell theories and the 5d Chern-Simons theory summed over topologies. Section 5 reconsiders the conventional duality between $\mathcal{N} = 4$ SYM and the type IIB string on $AdS_5 \times S^5$, and describe the relation between the boundary conditions of the TQFT and the gauge group. We conclude in Section 6 with a discussion of how ensemble duality can be extended to more complex cases involving a dynamical graviton. Some derivations and detailed calculations have been relegated to a series of appendices.

2 Quantization of 5d BC theory

In this section we construct the Hilbert space $\mathcal{H}_{M_4}$ of topological five-dimensional theory

$$\frac{N}{2\pi} \sum_{I=1}^{g} \int B_2^I \wedge dC_2^I \tag{2.1}$$

quantized on $M_4 \times \mathbb{R}$, where M_4 is a Euclidean orientable spin four-manifold. We first notice that $\mathcal{H}_{M_4}$ depends only on the 2-cohomology structure of M_4; hence we can morph

M_4 into M_4' with trivial odd cohomology while preserving the even cohomology structure, such that $\mathcal{H}_{M_4}$ and $\mathcal{H}_{M_4'}$ are isomorphic. Thus, in what follows we assume M_4 has trivial $H^1(M_4, \mathbb{Z})$ and $H^3(M_4, \mathbb{Z})$. With this choice of M_4, the theory (2.1) is a dimensional reduction of the seven-dimensional Chern-Simons-type theory

$$\frac{N}{4\pi} \int H_3 \wedge dH_3 \tag{2.2}$$

compactified on a Riemann surface Σ of genus g. Alternatively, (2.2) compactified on M_4 is a conventional three-dimensional Chern-Simons theory

$$\frac{K_{ij}}{4\pi} \int A^i \wedge dA^j, \tag{2.3}$$

with $K_{ij} = N\,\eta_{ij}$, where η is the integer-valued intersection form on $H^2(M_4, \mathbb{Z})$. Since by assumption M_4 is spin, η is even and the 3d theory (2.3) is bosonic. As was pointed out in [7], the Hilbert space $\mathcal{H}_{M_4}$ of (2.1) on $M_4 \times \mathbb{R}$ will be isomorphic to the Hilbert space $\mathcal{H}_\Sigma$ of (2.3) on $\Sigma \times \mathbb{R}$, both being subspaces of the Hilbert space of (2.2) on $M_6 \times \mathbb{R}$, where $M_6 = \Sigma \times M_4$. In what follows we will simply refer to the Hilbert space $\mathcal{H}_{M_4} \simeq \mathcal{H}_\Sigma$ as $\mathcal{H}$.

Our main focus will be on holography, in the context of which M_4 will be the boundary of a five-dimensional bulk manifold X_5. Such an M_4 must have signature $\sigma = b_2^+ - b_2^- = 0$,[1] in which case η can always be brought to the form

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}, \qquad n = b_2^+ = b_2^-\,. \tag{2.4}$$

We will refer to n as the "genus" of M_4.

For a closed, oriented, simply-connected Euclidean spin 4-manifold M_4, Freedman's theorem [10] then implies that M_4 is homeomorphic to $\#^n\, \mathbb{S}^2 \times \mathbb{S}^2$, the connected sum of n copies of $\mathbb{S}^2 \times \mathbb{S}^2$.

To describe the Hilbert space of the theory (2.1) on $M_4 \times \mathbb{R}$, it will be convenient to make use of the equivalence of $\mathcal{H}_{M_4}$ and $\mathcal{H}_\Sigma$. From (2.4), the Chern-Simons theory (2.3) resulting from the compactification of the 7d theory (2.2) on M_4 takes the form

$$\frac{N}{4\pi} \sum_{i=1}^{n} \int A^i \wedge dB^i + B^i \wedge dA^i. \tag{2.5}$$

This theory was recently studied in detail in [4, 5]. A basis for $\mathcal{H}_\Sigma$, and hence for $\mathcal{H}_{M_4}$, can be defined by choosing a handlebody X_3 ending on $\partial X_3 = \Sigma$. The basis states

$$|(\alpha, \beta)\rangle_{3d}, \qquad \alpha_I^i, \beta_I^i \in \mathbb{Z}_N, \tag{2.6}$$

are given by the 3d path integral on X_3 with Wilson lines of A^i and B^i wrapping the non-shrinkable cycles of X_3 α_I^i and β_I^i times correspondingly. The subscript 3d in (2.6) serves as a reminder that this basis for $\mathcal{H}$ has a natural origin in 3 dimensions.

[1]This is because such an X_5 would be a cobordism between M_4 and the empty set (which has signature 0), and the signature is a complete cobordism invariant of spin 4-manifolds.

The 3d Chern-Simons theory (2.5) has an explicit $\mathrm{O}(n,n,\mathbb{Z}_N)$ symmetry that acts on the gauge fields A^i and B^i while preserving the quadratic form $K_{ij} = N\eta_{ij}$. The action of $\mathrm{O}(n,n,\mathbb{Z}_N)$ on $\mathcal{H}$ is implemented by surface operators, and is straightforward in the basis (2.6): an element $h \in \mathrm{O}(n,n,\mathbb{Z}_N)$ acts via a unitary operator

$$U_h|(\alpha,\beta)\rangle_{3d} = |h(\alpha,\beta)\rangle_{3d}, \tag{2.7}$$

where on the right side h acts on (α^i,β^i) as a fundamental vector mod N. The action of $\mathrm{Sp}(2g,\mathbb{Z}_N)$ on (2.6), which is derived from the representation of the mapping class group $\mathrm{Sp}(2g,\mathbb{Z})$ on Σ, is more involved and can be found in Appendix A.

From the point of view of the 5d theory, the basis (2.6) simplifies the action of the mapping class group of M_4 while the action of the symmetry group of the bulk theory $\mathrm{Sp}(2g,\mathbb{Z}_N)$ is convoluted. (If the original M_4 is not connected or has nontrivial odd cohomology, its mapping class group is a subgroup of $\mathrm{O}(n,n,\mathbb{Z})$. Nevertheless since $\mathcal{H}$ only depends on the 2-cohomology of M_4, the action of the whole $\mathrm{O}(n,n,\mathbb{Z})$ is well-defined.) To make the $\mathrm{Sp}(2g,\mathbb{Z}_N)$ symmetry of the 5d theory (2.1) manifest we introduce a different basis for $\mathcal{H}$,

$$|(a,b)\rangle_{5d} = \frac{1}{N^{gn/2}} \sum_{\alpha,\beta\in\mathbb{Z}_N^{gn}} e^{\frac{2\pi i}{N}\mathrm{Tr}(a\,\alpha)}\,\delta_{b,\beta^T}\,|(\alpha,\beta)\rangle_{3d}, \qquad a_i^I, b_i^I \in \mathbb{Z}_N. \tag{2.8}$$

One can check straightforwardly that the action of $\gamma \in \mathrm{Sp}(2g,\mathbb{Z}_N)$ on (2.8) is analogous to (2.7),

$$U_\gamma|(a,b)\rangle_{5d} = |\gamma(a,b)\rangle_{5d}, \tag{2.9}$$

where γ acts on (a^I,b^I) as a fundamental vector. This action is realized by 4d surface operators of the 5d theory on $M_4\times\mathbb{R}$. The price to pay for the simplicity of (2.9) is the convoluted form of the transformation of (2.8) under $\mathrm{O}(n,n,\mathbb{Z}_N)$, which can be found in Appendix A.

The subscript $5d$ in (2.8) indicates that the corresponding states have a natural interpretation in 5d as path integrals of (2.1) over a five-dimensional "handlebody" X_5 with the surface operators of B_2^I, C_2^I wrapping n non-shrinkable 2-cycles a_i^I and b_i^I times correspondingly. By the "handlebody" X_5 here we mean the geometry homeomorphic to a connected sum of n copies of $B_3\times S^2$ (where B_3 is a 3-ball), which is characterized by the contractibility of a maximal set of n nonintersecting 2-cycles of M_4.

The basis (2.6) is well-defined for spin M_4 of any signature σ, in which case $\alpha\in\mathbb{Z}_N^n$ and $\beta\in\mathbb{Z}_N^{\bar{n}}$, where $n\equiv b_2^+$, $\bar{n}\equiv b_2^-$, such that (α,β) labels an element of $H_2(M_4,\mathbb{Z}_N)$. This basis was used to quantize the 5d theory (2.1) in [11].

In fact the basis (2.8) has a natural interpretation in 3d as well. It is the basis of states with a fixed holonomy of the gauge filed A^i over $2g$ cycles of Σ, which means these states diagonalize the action of the Wilson loops of A^i. In the SymTFT context, this is the basis of distinct twisted partition functions of a given boundary theory [12]. Similarly, the basis (2.3) in 5d is the basis of states with fixed B_2 flux over all 2-cycles of M_4. Its SymTFT interpretation is also as a set of twisted partition functions of the boundary 4d theory.

2.1 Holomorphic quantization

In the previous section we discussed the structure of the Hilbert space $\mathcal{H}$ without specifying the explicit form of the wavefunctions. The latter depends on the boundary conditions at Σ (in the 3d theory) or M_4 (in the 5d theory). Below we follow the standard holomorphic quantization of the 3d Chern-Simons theory [13, 14] and generalize it to higher d. The idea is to start with the 7d theory (2.2), which is very similar to chiral CS theory in 3d, and quantize it first. Then the wavefunctions of the 3d theory (2.5) and the 5d theory (2.1) can be obtained by dimensional reduction. More details can be found in Appendix B.

We begin with the 7d theory (2.2) on $M_6 \times \mathbb{R}$ in a gauge where all components of the 3-form field H_3 along $\mathbb{R}$ vanish. In this gauge one can expand H_3 into harmonic modes and cohomologically trivial fluctuations, generalizing the 3d case [13],

$$H_3 = \frac{i\pi}{N} \sum_{A,B} \zeta_A (\mathbf{\Omega}_2^{-1})^{AB} \omega_B^{(3)} + \text{c.c.} + \partial\chi. \tag{2.10}$$

Here $\omega_A^{(3)}$ is a basis of self-dual 3-forms on M_6, $\star\omega_A^{(3)} = i\,\omega_A^{(3)}$ and $\mathbf{\Omega}$ is the corresponding "modular" parameter of the 6d manifold. Following [5] we add the boundary term to the 7d action (2.2)

$$\frac{N}{4\pi} \int_{M_6} H_3 \wedge \star H_3. \tag{2.11}$$

With this boundary term it is consistent to fix the value of ζ_A at the boundary while allowing its complex conjugate $(\zeta_A)^*$ to fluctuate freely. The resulting wavefunction, which we construct explicitly in Appendix B, is a holomorphic function of ζ_A.

Next we consider $M_6 = \Sigma \times M_4$, with Σ and M_4 as above. There are a total of $2ng$ independent self-dual three-forms on M_6, see Appendix B,

$$\omega_I \wedge \omega_i^+, \qquad \omega_I^* \wedge \omega_i^-, \qquad I = 1\dots g,\ i = 1\dots n. \tag{2.12}$$

In this case the vector ζ_A is a combination of two holomorphic variables ξ_I^i and $\bar{\xi}_I^i$ such that

$$H_3 = \frac{i\pi}{\sqrt{N}} \sum_{I,i} \xi_I^i (\Omega_2^{-1})^{IJ} \omega_J^* \wedge \omega_i^+ - \frac{i\pi}{\sqrt{N}} \sum_{I,i} \bar{\xi}_I^i (\Omega_2^{-1})^{IJ} \omega_J \wedge \omega_i^- + \text{c.c.} + \partial\chi. \tag{2.13}$$

At the quantum level the complex conjugate variables ξ^* and $\bar{\xi}^*$ become operators canonically conjugate to ξ and $\bar{\xi}$ with respect to the measure $e^{-\pi\,\Omega_2^{-1}(|\xi|^2+|\bar{\xi}|^2)}$ inherited from (2.11) such that,

$$\xi^* \to \frac{\Omega_2}{\pi}\frac{\partial}{\partial\xi}, \qquad \bar{\xi}^* \to \frac{\Omega_2}{\pi}\frac{\partial}{\partial\bar{\xi}}. \tag{2.14}$$

To obtain wavefunctions of the 3d and 5d theories, one can start with the wavefunction in 7d and then dimensionally reduce it. Alternatively one can start directly in 3d or 5d and add corresponding boundary terms to (2.5) or (2.1). The holomorphic variables $\xi, \bar{\xi}$ will then emerge parameterizing the harmonic expansions of the 3d and 5d fields. Choosing a gauge in which all components along the non-compact direction vanish, we have in 3d

$$\begin{aligned} A^i &= \frac{i\pi(G^{-1/2})_{ij}}{\sqrt{2N}} \left(\xi_I^j(\Omega_2^{-1})^{IJ}\omega_J^* - \bar{\xi}_I^j(\Omega_2)^{-1}_{IJ}\omega_J\right) + \text{c.c.} + \partial\chi_A, \\ B^i &= \frac{i\pi(G^{-1/2})_{ij}}{\sqrt{2N}} \left((G-B)_{jk}\xi_I^k(\Omega_2)^{-1}_{IJ}\omega_J^* + (G+B)_{jk}\bar{\xi}_I^j(\Omega_2)^{-1}_{IJ}\omega_J\right) + \text{c.c.} + \partial\chi_B, \end{aligned} \tag{2.15}$$

and the fluctuating part $\partial\chi_{A,B}$ vanishes at the boundary. Similarly, in 5d

$$\begin{aligned} B_2^I &= \frac{i\pi(\Omega_2^{-1})^{IJ}}{\sqrt{N}} \left(\xi_J^i\omega_i^+ - \bar{\xi}_J^i\omega_i^-\right) + \text{c.c.} + \partial\chi_B, \\ C_2^I &= \frac{i\pi(\Omega_2^{-1})^{JK}}{\sqrt{N}} \left(\Omega_{IJ}^*\xi_K^i\omega_i^+ - \Omega_{IJ}\bar{\xi}_K^i\omega_i^-\right) + \text{c.c.} + \partial\chi_C. \end{aligned} \tag{2.16}$$

There is another, equivalent way to construct the wavefunctions. The Hilbert space $\mathcal{H}$ can be defined as a representation of the group of surface operators wrapping the three-cycles Γ in M_6,

$$W_{\Gamma(\mathrm{n})} = \exp\{2\pi i\int_\Gamma H_3\}, \qquad \Gamma^\vee = \sum_{I,i} \mathrm{n}_I^i\,\omega_I^{(1)}\wedge\omega_i^{(2)}, \qquad \mathrm{n}\in\mathbb{Z}_N^{4gn}. \tag{2.17}$$

Upon compactification on M_4, in 3d these operators become Wilson lines of the A and B gauge fields wrapping one-cycles of Σ, while after compactifying on Σ, in 5d they became the surface operators of B_2 and C_2 wrapping two-cycles of M_4. Mathematically, W_Γ are holomorphic differential operators of $\xi,\bar{\xi}$. Multiplication by $\xi,\bar{\xi}$ together with (2.14) form the Heisenberg algebra, while the surface operators (2.17) form a Heisenberg group

$$W_\Gamma W_{\Gamma'} = W_{\Gamma+\Gamma'}e^{\frac{i\pi}{N}(\Gamma\cap\Gamma')}, \qquad (\Gamma\cap\Gamma') = -\mathrm{Tr}(\mathrm{n}^T\eta\,\mathrm{n}'J). \tag{2.18}$$

Here n,n' are $2n$ by $2g$ matrices. The Stone-von Neumann theorem implies that such a representation is unique [15], which is another way to see that $\mathcal{H}_\Sigma$ and $\mathcal{H}_{M_4}$ are isomorphic. The wavefunctions forming the representation of (2.18) can be written explicitly in terms of the theta functions [15, 16], as discussed in Appendix B.

The explicit form of the wavefunctions was recently reviewed in [4, 5],

$$\Theta_{c_1c_2\ldots c_g}(\Omega,E) \;=\; \det(\Omega_2)^{n/2}\sum_{u_1,\ldots u_g} e^{i\pi u_I^T\Pi^{IJ}u_J+2\pi i\,(p_L^I\cdot\xi_I-p_R^I\cdot\bar{\xi}_I)+\pi(\xi\Omega_2^{-1}\xi^T+\bar{\xi}\Omega_2^{-1}\bar{\xi}^T)/2}. \tag{2.19}$$

Here

$$u_I = \frac{1}{\sqrt{2}}\begin{pmatrix} p_L^I+p_R^I \\ p_L^I-p_R^I\end{pmatrix} = \mathcal{O}(n_I\sqrt{N}+c_I/\sqrt{N}), \qquad c_I=(\alpha_I,\beta_I)\in\mathbb{Z}_N^{2n}, \tag{2.20}$$

$$\Pi_{ij}^{IJ} = (\Omega_1)_{IJ}\eta_{ij} + i(\Omega_2)_{IJ}\delta_{ij}, \tag{2.21}$$

and the sum goes over all $n_I \in \mathbb{Z}^{2n}$,. The orthogonal matrix $\mathcal{O} \in \mathrm{O}(n, n, \mathbb{R})$ is defined in terms of the "Narain data" $E = G + B$, metric G and B-field, specified by the metric on M_4,

$$\mathcal{O}(E) = G^{-1/2} \begin{pmatrix} \mathbb{1} & B \\ 0 & G \end{pmatrix}. \tag{2.22}$$

We stress that the dependence on the metric on Σ, encoded in Ω, as well as on the metric on M_4, encoded in $E = G + B$, comes from the boundary term (2.11) (or its dimensional reduction). The corresponding boundary conditions with $\xi, \bar{\xi}$ fixed and $\xi^*, \bar{\xi}^*$ fluctuating define a particular state $\langle \Omega, E, \xi, \bar{\xi}| \in \mathcal{H}^*$. The wavefunction is a matrix element between this state and a state in $\mathcal{H}$. An interpretation of this construction in terms of SymTFT will be discussed in the next section.

The modular $\mathrm{Sp}(2g, \mathbb{Z})$ and orthogonal $\mathrm{O}(n, n, \mathbb{Z})$ groups – mapping class groups of Σ and M_4, that act on $\mathcal{H}$ by $\mathrm{Sp}(2g, \mathbb{Z}_N)$ and $\mathrm{O}(n, n, \mathbb{Z}_N)$ discussed above – act on the boundary state as follows

$$U_\gamma |\Omega, E, \xi, \bar{\xi}\rangle = |\Omega_\gamma, E, \xi_\gamma, \bar{\xi}_\gamma\rangle, \tag{2.23}$$
$$\Omega_\gamma = (A\Omega + B)(C\Omega + D)^{-1}, \quad \xi_\gamma = \xi(C\Omega + D)^{-1}, \quad \bar{\xi}_\gamma = \bar{\xi}(C\Omega^* + D)^{-1}, \tag{2.24}$$

and

$$U_h |\Omega, E, \xi, \bar{\xi}\rangle = |\Omega, E_h, \xi_h, \bar{\xi}_h\rangle, \tag{2.25}$$
$$(E_h)^T = (AE_h^T + B)(CE_h^T + D)^{-1}, \quad \xi_h = O_L\, \xi, \quad \bar{\xi}_h = O_R\, \bar{\xi}, \tag{2.26}$$

where orthogonal matrices $(O_L, O_R) \in O(n, \mathbb{R}) \times O(n, \mathbb{R}) = O(2n, \mathbb{R}) \cap O(n, n, \mathbb{R})$ are defined by $(O_L \oplus O_R)(h, E) = H\mathcal{O}(E_h)\, h\, \mathcal{O}^{-1}(E) H^{-1}$ and H is the 2×2 Hadamard matrix tensor $\mathbb{1}_n$.

Actual wavefunctions in 3d and 5d will also include contributions from the fluctuating modes,

$$\langle \Omega, E, \xi, \bar{\xi}|(\alpha, \beta)\rangle_{3d} = \Psi_{c_1 \ldots c_g}(\Omega, E, \xi, \bar{\xi}) = \frac{\Theta_{c_1 c_2 \ldots c_g}}{\Phi}. \tag{2.27}$$

The ket states $\langle \Omega, E, \xi, \bar{\xi}|$ above are different in 3d and 5d theories by a scalar factor. In 3d theory Φ is determined by a scalar Laplacian on Σ [3, 17, 18],

$$\Phi_{3d} = (\det{}' \Delta_0)^{n/2}, \tag{2.28}$$

while in 5d theory Φ is determined by the scalar and vector Laplacians on the four-manifold [11]

$$\Phi_{5d} = \left(\frac{\det{}' \Delta_1}{\det{}' \Delta_0} \right)^{g/2}. \tag{2.29}$$

In both cases Φ does not carry any quantum charges and is invariant under both orthogonal and symplectic groups (with one group not acting on Σ or M_4 at all, while the other merely relabeling the cohomologies). For $g=1$, expression in (2.28) simplifies to $\Phi_{3d} = \tau_2^{n/2}|\eta(\tau)|^{2n}$.

A straightforward computation yields for the wavefunctions of the basis (2.8),

$$\langle \Omega, E, \xi, \bar{\xi}|(a,b)\rangle_{5d} = \Psi_{\mathrm{c}_1\dots\mathrm{c}_n}(\Omega, E, \xi, \bar{\xi}) = \frac{\Theta_{\mathrm{c}_1\dots\mathrm{c}_n}}{\Phi}. \tag{2.30}$$

Here $\mathrm{c}_i = (a_i, b_i) \in \mathbb{Z}^{2g}$ and

$$\begin{aligned}\Theta_{\mathrm{c}_1\dots\mathrm{c}_g} &= \det(G)^{g/2} \sum_{\mathrm{v}_1\dots\mathrm{v}_n} e^{-\pi|\mathrm{v}|^2 - 2\pi i \tilde{n}^T B \tilde{m} + \sqrt{2}\pi(\bar{\xi}\Omega_2^{-1/2}\mathrm{v} - \xi\Omega_2^{-1/2}\mathrm{v}^*) + \pi\bar{\xi}\Omega_2^{-1}\xi^T}, \\ \mathrm{v} &= G^{1/2}\Omega_2^{-1/2}(\tilde{n} + \Omega\tilde{m}), \quad \tilde{n}_i = (n_i N + a_i)/\sqrt{N}, \quad \tilde{m}_i = (m_i N + b_i)/\sqrt{N}, \quad n_i, m_i, a_i, b_i \in \mathbb{Z}_N^g.\end{aligned} \tag{2.31}$$

To conclude this section, we discuss the relation between the holomorphic quantization discussed above and the quantization scheme discussed in the appendix A of [19]. The D-dimensional spacetime there is assumed to be a cylinder $X_D = M_{D-1} \times \mathbb{R}$ with Minkowski signature. Starting from the BF-type Abelian theory

$$\frac{N}{2\pi}\int A_{p+1} \wedge dA_{D-p-2}, \tag{2.32}$$

and following [19] we add the boundary term on $\partial M_{D-1} \times \mathbb{R}$,

$$\frac{1}{4r^2}\int A_{D-p-2} \wedge \star A_{D-p-2}, \tag{2.33}$$

which yields the self-dual boundary condition

$$\frac{Nr^2}{\pi} A_{p+1} = \star A_{D-p-2}. \tag{2.34}$$

As we will show momentarily this is the same as the holomorphic quantization described above, in the particular case of vanishing $\xi = \bar{\xi}$. To see that, we first note that the near-boundary region $\partial M_{D-1} \times \mathbb{R}$ is related to $\Sigma \times \mathbb{R}$ for the 3d theory (or $M_4 \times \mathbb{R}$ for the 5d theory) after the Wick rotation. Next, the holomorphic quantization assumes that $\xi, \bar{\xi}$ are fixed at the boundary, while their complex conjugates $\xi^*, \bar{\xi}^*$ fluctuate freely. Focusing on the 3d case and ignoring for now that this condition renders A, B complex, by taking $\xi = \bar{\xi} = 0$ we find from (2.15)

$$B^i = iG_{ij} \star A^j - (G^{-1/2}BG^{1/2})_{ij}A^j. \tag{2.35}$$

After taking $n=1$, such that $G = r^2$ and $B = 0$ and rotating to Minkowski signature we recover (2.34) after identifying

$$A_{p+1} = \sqrt{\frac{\pi}{N}}A, \qquad A_{D-p-2} = \sqrt{\frac{N}{\pi}}B, \qquad D=3, \quad p=0. \tag{2.36}$$

At this point we can match the boundary term (B.9) to be equal on-shell to the difference between (2.32) and (2.5), while the boundary term (B.35) vanishes on-shell.

In a similar way, the holomorphic quantization of the 5d theory (2.1) with $\xi = \bar{\xi} = 0$, after rotating to Minkowski signature, leads to

$$\star(C_2 - \Omega\, B_2) = i(C_2 - \Omega\, B_2). \tag{2.37}$$

Focusing on the case with $g = 1$, for which we can replace $\Omega \to \tau$, and taking for simplicity $\tau_1 = 0$, this reduces to (2.34) after the identifications

$$A_{p+1} = \frac{\sqrt{\pi\tau_2}}{\sqrt{N}r} B_2, \qquad A_{D-p-2} = \frac{\sqrt{N}r}{\sqrt{\pi\tau_2}} C_2, \qquad D = 5, \quad p = 1. \tag{2.38}$$

To summarize, the self-duality boundary condition (2.34) is the counterpart of holomorphic quantization when the system is quantized in Minkowski signature on a cylinder, as is done in [14]. A generalization of (2.34) to allow non-zero $\xi, \bar{\xi}$ was recently discussed in [20].

3 Abelian TQFTs and codes

3.1 SymTFT, topological boundary conditions, and codes

As in the case of the 3d theory (2.5), which is a SymTFT of global $\mathbb{Z}_N^n$ symmetry in two dimensions, the 5d topological theory (2.1) is a SymTFT of global $\mathbb{Z}_N^g$ symmetry in four dimensions. Any four-dimensional theory with global $\mathbb{Z}_N^g$ symmetry can be coupled to B_2 and C_2 fields so that the wavefunctions

$$\langle \mathcal{B}_{4d} | (a, b) \rangle_{5d}, \tag{3.1}$$

are the conformal blocks of the 4d theory on M_4 (which is assumed to be of signature zero). Here $\langle \mathcal{B}_{4d}|$ is the state in the dual to the Hilbert space $\mathcal{H}^*_{M_4}$ of the 5d topological theory, created by the boundary conditions on the B_2, C_2 fields coupled to 4d theory. This is completely analogous to the 2d case, where

$$\langle \mathcal{B}_{2d} | (\alpha, \beta) \rangle_{3d} \tag{3.2}$$

are the conformal blocks of the 2d CFT [12]. The wavefunctions of 3d and 5d theories obtained in previous section using holomorphic quantization (2.27) and (2.30) can be recognized in this language to be the conformal blocks of 2d Narain and 4d Maxwell theories correspondingly,[2]

$$\langle \mathcal{B}_{2d/4d} | = \langle \Omega, E, \xi, \bar{\xi} |. \tag{3.3}$$

Another convenient 4d example is provided by $su(N)$ gauge theory, with center $\mathbb{Z}_N$ coupled to 5d B_2, C_2 fields [9, 24, 25].

We are primarily interested in the description of topological boundary conditions. It has been recently shown in [26] that the topological boundary conditions of a 3d Abelian Chern-Simons theory are naturally parameterized by classical even self-dual codes. Below

[2]The conformal blocks are themselves the partition functions of generalized Narain or Maxwell theories defined by non-self-dual lattices, e.g. [21, 22]. These are relative theories in the sense of [23].

we extend this result to 5d and show that topological boundary conditions of the 5d Abelian theory (2.1) are naturally parameterized by classical symplectic self-dual codes over $\mathbb{Z}_N$ of length $2g$. We first recall the 3d story, where topological boundary conditions are specified by a maximal (Lagrangian) non-anomalous subgroup $\mathcal{C}$ of the one-form symmetry group. For the theory (2.5) these are the subgroups of "codewords" $c \in \mathcal{C} \subset \mathbb{Z}_N^{2n}$ closed under addition mod N and satisfying the condition of "evenness"

$$\alpha \cdot \beta = 0 \text{ mod } N, \qquad c = (\alpha, \beta) \in \mathbb{Z}_N^n \times \mathbb{Z}_N^n, \tag{3.4}$$

and self-duality with respect to inner product (2.4) [4, 26]. In the language of anyons these conditions ensure that the anyons are all bosons and that the group of anyons is maximal. Trivial mutual braiding of all anyons is guaranteed by these conditions, as follows from the linearity of $\mathcal{C}$ and (2.18).

Similarly, in 5d, topological boundary conditions are specified by additive subgroups $\mathcal{L} \subset \mathbb{Z}_n^{2g}$ satisfying self-orthogonality

$$a_1 \cdot b_2 - a_2 \cdot b_1 = 0 \text{ mod } N, \qquad \mathrm{c}_k = (a_k, b_k) \in \mathcal{L} \subset \mathbb{Z}_N^n \times \mathbb{Z}_N^n, \tag{3.5}$$

and self-duality $\mathcal{L} = \mathcal{L}^\top$ with respect to

$$J = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}. \tag{3.6}$$

In the language of anyons the group $\mathcal{L}$ is a collection of anyons closed under fusion and with trivial mutual braiding, as follows from self-orthogonality and (2.18). The self-duality condition ensures that the group $\mathcal{L}$ is maximal, i.e. Lagrangian.

The conditions on $\mathcal{C}$ and $\mathcal{L}$ as outlined above are well-known in the literature [27–30]. What is new here is the interpretation of these groups as classical additive codes, which has several advantages. First, this approach makes an explicit connection with the extensive literature on "code CFTs" [4, 5, 26, 31–48]. In addition, classical self-dual codes are well studied and certain results from coding theory have immediate applications to SymTFT. For example, the central and very general result of [49] is that the code-based states

$$|\mathcal{C}\rangle \equiv \sum_{c \in \mathcal{C}^g} |c\rangle_{3d} \tag{3.7}$$

and

$$|\mathcal{L}\rangle \equiv \sum_{\mathrm{c} \in \mathcal{L}^n} |\mathrm{c}\rangle_{5d} \tag{3.8}$$

span an (over-)complete basis of $\mathrm{O}(n, n, \mathbb{Z})$- and $\mathrm{Sp}(2g, \mathbb{Z})$-invariant states in $\mathcal{H}$. A further advantage is an immediate connection to quantum information theory arising from the fact that (3.7) and (3.8) are quantum stabilizer states. We elaborate on this point below.

To illustrate the connection with codes, we recall that classical self-dual symplectic codes $\mathcal{L}$ over $\mathbb{Z}_N$ of length $2g$ define maximal (i.e. self-dual) quantum stabilizer codes on

g qudits of dimension N [50, 51].[3] Consider for example 4d gauge theory with the gauge group $su(2)$. After gauging the center $\mathbb{Z}_2$, one obtains an $su(2)/\mathbb{Z}_2$ gauge theory known as $\mathrm{SO}(3)_+$. Subsequently shifting θ by π yields the so-called $\mathrm{SO}(3)_-$ theory. These three theories correspond to three topological boundary conditions of the 5d theory (2.1) with level $N = 2$. The corresponding Lagrangian subgroups

$$\mathcal{L}_0 = (a, 0), \quad \mathcal{L}_1 = (0, b), \quad \mathcal{L}_2 = (a, a), \quad a, b \in \mathbb{Z}_2, \tag{3.9}$$

exhaust the list of all self-dual symplectic codes of length 2 over $\mathbb{Z}_2$. This description in terms of the subgroups $\mathcal{L}_k$ is of course exactly the same as in [28]. Using the interpretation of $\mathcal{L}_k$ as additive self-dual symplectic codes we readily recognize these as three self-dual stabilizer groups of one qubit, generated by σ_z, σ_x, and $\sigma_y = -i\sigma_z\sigma_x$ correspondingly.

The equivalence between the symplectic self-dual codes $\mathcal{L}$ and maximal (self-dual) stabilizer codes suggests that topological boundary conditions defined by $\mathcal{L}$ might admit an interpretation in terms of quantum information. This approach was taken in [32, 33, 38] for codes $\mathcal{C}$ when $N = 2$ (in this case symmetric (2.4) and symplectic (3.6) scalar products are equivalent). While the relation between $\mathcal{L}$ (or $\mathcal{C}$ for $N = 2$) and quantum stabilizer codes is mathematically correct and useful, e.g. it was used in [32] to classify all inequivalent codes in terms of graphs, it does not immediately provide a physically motivated interpretation.

To interpret topological boundary conditions in terms of quantum error correcting codes, we note that the states in $\mathcal{H}$ produced by these boundary conditions are given by (3.7) and (3.8). It is well-known that the Hilbert space of the Abelian TQFT is equivalent to the Hilbert space of generalized qudits. Thus, braiding of anyons in Abelian theories results only in Clifford gates [52]. That is to say, the line operators of the 3d theory wrapping all possible 1-cycles of Σ or surface operators of the 5d theory wrapping all possible 2-cycles of M_4 generate the generalized Pauli group of $2gn$ qudits of dimension N. Mathematically, this is equivalent to the statement that the generators W_Γ from (2.17) form the Heisenberg group (2.18). More specifically, taking X_3 to be the handlebody used in 3d theory (2.5) to define the basis (2.3), the line operators wrapping around contractable cycles of X_5 are the generalized Z gates while those wrapping non-shrinkable cycles are generalized X gates [4]. Choosing shrinkable one-cycles of X_3 to be the first n, denoted as $\omega_I^{(1)}$ in Appendix (B), we can write explicitly

$$W[c_{\mathrm{a}}, c_{\mathrm{b}}] := W_\Gamma, \qquad c_{\mathrm{a}} = (\mathrm{n}, \mathrm{m}), \quad c_{\mathrm{b}} = (\mathrm{p}, \mathrm{q}), \quad \mathrm{n}, \mathrm{m}, \mathrm{p}, \mathrm{q} \in \mathbb{Z}_N^{ng} \tag{3.10}$$

$$\Gamma^\vee = \sum \mathrm{n}_I^i\, \omega_I^{(1)} \wedge \omega_i^{(2)} + \mathrm{m}_I^i\, \omega_I^{(1)} \wedge \omega_{n+i}^{(2)} + \mathrm{p}_I^i\, \omega_{g+I}^{(1)} \wedge \omega_i^{(2)} + \mathrm{q}_I^i\, \omega_{g+I}^{(1)} \wedge \omega_{n+i}^{(2)},$$

$$W[c_{\mathrm{a}}, 0]|c\rangle_{3d} = e^{\frac{2\pi i}{N}\mathrm{Tr}(c_{\mathrm{a}}^T \eta c)}|c\rangle_{3d}, \tag{3.11}$$

$$W[0, c_{\mathrm{b}}]|c\rangle_{3d} = |c + c_{\mathrm{b}}\rangle_{3d}. \tag{3.12}$$

Similarly, in 5d surface operators wrapping shrinkable 2-cycle of X_5 used to define the basis (2.8) are generalized Z gates while those wrapping non-shrinkable 2-cycles are generalized

[3] Starting with the generalized Pauli matrices X and Z on the dimension N qudit, up to an overall phase the stabilzier generators are $Z^a X^b$ for each $(a, b) \in \mathcal{L} \subset \mathbb{Z}_N^g \times \mathbb{Z}_n^g$.

X gates,

$$W[\mathrm{c}_\alpha;\mathrm{c}_\beta] := W[c_\mathrm{a}, c_\mathrm{b}], \quad \mathrm{c}_\alpha = (-\mathrm{p}^T, \mathrm{n}^T), \quad \mathrm{c}_\beta = (\mathrm{m}^T, \mathrm{q}^T), \tag{3.13}$$

$$W[\mathrm{c}_\alpha; 0]|\mathrm{c}\rangle_{5d} = e^{\frac{2\pi i}{N}\mathrm{Tr}(\mathrm{c}_\alpha^T J \mathrm{c})}|\mathrm{c}\rangle_{5d}, \tag{3.14}$$

$$W[0;\mathrm{c}_\beta]|\mathrm{c}\rangle_{5d} = |\mathrm{c}+\mathrm{c}_\beta\rangle_{5d}. \tag{3.15}$$

Now we readily recognize that (3.7) and (3.8) are quantum stabilizer states defined by self-dual quantum stabilizer codes. Furthermore the latter are of the Calderbank–Shor–Steane (CSS) type [53, 54], defined in terms of classical codes $\mathcal{C}$ and $\mathcal{L}$ correspondingly. Thus the state (3.8) created by the topological boundary condition (maximal anyon condensation) defined by $\mathcal{L}$ is stabilized by all generalized Pauli group elements of the form

$$W[\mathrm{c}_\alpha;\mathrm{c}_\beta]|\mathcal{L}\rangle = |\mathcal{L}\rangle, \qquad \mathrm{c}_\alpha, \mathrm{c}_\beta \in \mathcal{L}. \tag{3.16}$$

This defines the state uniquely (up to normalization). Interpretation of the states produced by topological boundary conditions in 3d Abelian theory as CSS quantum stabilizer states was already given in [4, 55]. Here we extend it to 5d.[4]

The connection between anyon condensation and quantum stabilizer codes can be formulated more broadly, by relaxing the condition of self-duality of the underlying classical codes. Namely, the Hilbert space of a theory obtained via (partial) anyon condensation in an Abelian TQFT is always a quantum stabilizer code of CSS type, parameterized by a self-orthogonal classical code. The meaning of self-orthogonality, i.e. the choice of the inner product, depends on the theory. In 3d self-orthogonal means even; in 5d it means symplectic. Furthermore, if the anyons are condensed on a codimension one defect, also known as higher gauging [56], it becomes a projector on the code subspace. For example, the surface operator in the 3d theory obtained by gauging the 1-form symmetry $\mathcal{C}$ on a 2d surface is a projector of the form

$$\frac{1}{|\mathcal{C}|}\sum_{c_a, c_b \in \mathcal{C}} W[c_a, c_b]. \tag{3.17}$$

Here $\mathcal{C}$ is even but not necessarily self-dual.

We conclude by mentioning that topological boundary conditions specified by $\mathcal{L}$ can be understood in the literal sense in terms of the behavior of B_2, C_2 near the boundary [9]. In particular the simplest symplectic code $\mathcal{L}_0 = (*, 0) \subset \mathbb{Z}_N \times \mathbb{Z}_N$, where $*$ stands for any element of $\mathbb{Z}_N$, corresponds to the boundary condition

$$C_2 = 0, \tag{3.18}$$

while B_2 fluctuates freely (Neumann b.c.). Similarly $\mathcal{L}_1 = (0, *) \subset \mathbb{Z}_N \times \mathbb{Z}_N$ would correspond to $B_2 = 0$ and C_2 fluctuating freely at the boundary (Dirichlet b.c.). We note that these, and other boundary conditions are consistent with action (2.1) without any

[4]A construction of the state $|\mathcal{L}\rangle$ defined by a topological boundary condition in 5d theory, that is equivalent to (3.16) has previously appeared in [25], but without noting the connection to quantum stabilizer codes.

additional boundary terms. They assume real polarization, which is different from the holomorphic quantization discussed above. Imposing e.g. $C_2 = 0$ within the holomorphic quantization by fixing the values of $\xi, \bar{\xi}$ will not impose topological boundary conditions. To impose topological boundary condition in terms of $\xi, \bar{\xi}$ one can use the complexness of ket states $\langle \Omega, E, \xi, \bar{\xi}|$ and integrate the boundary wavefunction over $d\xi, d\bar{\xi}$ with the kernel $\langle \mathcal{L}|\Omega, E, \xi, \bar{\xi}\rangle$.

3.2 Topology and codes

States $|\mathcal{L}\rangle$ and $|\mathcal{C}\rangle$ were discussed above as those defined by topological boundary conditions in 5d and 3d theories correspondingly. As we show below these state have yet another interpretation. As was discussed in [55], up to an overall normalization $|\mathcal{L}\rangle$ are the states produced by the path integrals of 3d theory (2.5) on a 3d manifolds with a boundary. Here we revisit and generalize this result to 5d by showing that $|\mathcal{C}\rangle$ are the states produced by the 5d path integral on 5d topologies.

We start with the 3d case. In what follows we make reference to genus reduction as a way to construct 3d topologies with a boundary. We refer the reader to [55] for details. We also discuss "genus reduction" in 5d later in section 4.3.

Without loss of generality we can take $n = 1$; an arbitrary n can be restored by considering n-th tensor power of the resulting state. Up to an overall normalization, the vacuum state of the 3d theory $|0\rangle^g_{3d}$ – the path integral on a handlebody X_3 of genus g – is the code state $|\mathcal{L}_0\rangle$ where $\mathcal{L}_0$ is a self-dual symplectic code,

$$|0\rangle^g_{3d} = \frac{1}{N^{g/2}}|\mathcal{L}_0\rangle_{5d}, \qquad (a_1, \dots, a_g, 0, \dots, 0) \in \mathcal{L}_0, \qquad a_i \in \mathbb{Z}_N. \tag{3.19}$$

In what follows we skip the subscripts $3d, 5d$ for simplicity. A modular transformation $\gamma \in Sp(2g, \mathbb{Z})$ maps this state into another state – the path integral on another handlebody,

$$U_\gamma|0\rangle^g = \frac{1}{N^{g/2}}|\mathcal{L}_\gamma\rangle, \tag{3.20}$$

which is also a code state associated with another self-dual symplectic code $\mathcal{L}_\gamma$, where

$$\mathrm{c} = (a_1, \dots, a_g, b_1, \dots, b_g) \in \mathcal{L}_\gamma, \quad \text{iff} \quad \mathrm{c} = \gamma(a'_1, \dots, a'_g, 0, \dots, 0) \bmod N, \tag{3.21}$$

for arbitrary $a'_i \in \mathbb{Z}_N$. After the genus reduction the resulting state

$$^{\tilde{g}}\langle 0|U_\gamma|0\rangle^g = N^{-(g+\tilde{g})/2} \sum_{\mathrm{x} \in L} m(\mathrm{x})|\mathrm{x}\rangle \in \mathcal{H}^{g-\tilde{g}} \tag{3.22}$$

is defined in terms of the set L, that includes all codewords

$$\mathrm{x} = (a_1, \dots, a_{g-\tilde{g}}, b_1, \dots, b_{g-\tilde{g}}) \in L \subset (\mathbb{Z}_N \times \mathbb{Z}_N)^{g-\tilde{g}} \tag{3.23}$$

such that there exist $a_{g-\tilde{g}+1}, \dots a_g$ satisfying

$$(a_1, \dots a_g, b_1, \dots b_{g-\tilde{g}}, 0 \dots, 0) \in \mathcal{L}_\gamma. \tag{3.24}$$

This definition makes it clear L is closed under addition, i.e. it is an additive code. The multiplicity $m(\mathrm{c})$ is the number of different sets $a_{g-\tilde{g}+1}, \dots a_g$ for any given $\mathrm{c} \in L$. We show in Appendix C that $m(\mathrm{c}) = |M_0|$ is the same for all c and L is a self-dual symplectic code of length $g' = g - \tilde{g}$. With that we arrive at the desired result: path integral of the 3d theory on any topology ending on Σ of genus g' is a CSS quantum stabilizer state defined by a classical slef-dual symplectic code L,

$$^{\tilde{g}}\langle 0|U_\gamma|0\rangle^g = N^{-(g+\tilde{g})/2}|M_0||L\rangle. \tag{3.25}$$

The relation between $|\mathcal{L}\rangle$ and associated 3d topologies X_3, which could be many, can be understood geometrically using the representation of the wavefunction (2.31). Focusing on $g = 1$ for simplicity we readily see that $\mathcal{L}$ parameterizes the cohomologies of $\Sigma = \partial X_3$ mod N. For example for $\mathcal{L}_0 = (*, 0)$ the a-cycle of the torus is cohomologically trivial as which follows from the summation over all a_i in (2.31). Similarly $\mathcal{L}_1 = (0, *)$ corresponds to a solid torus with the trivial b-cycle. When $N = p^2$ the code $\mathcal{L}' = (p*, p*)$ corresponds to a non-handlebody topology with torsion on both a and b-cycles, see [55] where this example is discussed in detail. Saying the same differently, a symplectic self-dual code $\mathcal{L}$, through the Construction A [57], defines a symplectic self-dual lattice of one-cohomologies of $H_1(\Sigma, \mathbb{Z})$, generalizing the construction of handlebodies explained in [3].

The statement and its derivation for the 5d theory (2.1) is completely analogous. The vacuum state of $g = 1$ theory – the path integral on a 5d "handlebody" which is a direct sum of n $B_3 \times \mathbb{S}^2$ is

$$|0\rangle^n_{5d} = \frac{1}{N^{n/2}}|\mathcal{C}_0\rangle_{3d}, \qquad (\alpha_1, \dots, \alpha_n, 0, \dots, 0) \in \mathcal{C}_0, \qquad \alpha_i \in \mathbb{Z}_N. \tag{3.26}$$

It is clearly a code state for even self-dual code $\mathcal{C}_0$. A mapping class group transformation, or more generally any transformation from $h \in O(n, n, \mathbb{Z}_N)$, produces a state

$$U_h|0\rangle^n = \frac{1}{N^{n/2}}|\mathcal{C}_h\rangle, \tag{3.27}$$

which is a code state associated with another even self-dual code $\mathcal{C}_h$,

$$c = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \in \mathcal{C}_h, \quad \text{iff} \quad c = h(\alpha'_1, \dots, \alpha'_n, 0, \dots, 0) \bmod N. \tag{3.28}$$

The genus reduction of this state is given by

$$^{\tilde{n}}\langle 0|U_h|0\rangle^n = N^{-(n+\tilde{n})/2}|M_0||L\rangle, \tag{3.29}$$

where sets L_i, M_i are defined similarly to the 3d case discussed in the Appendix C. It is straightforward to see from the definition that L is an even code. To show that it is self-dual, one can follow the same logic as in the 3d case, evaluating its size $|L| = N^{n-\tilde{n}}$.

The geometric relation between $|\mathcal{C}\rangle$ and X_5 can be established using the wavefunction representation (2.19) in a way similar to the 3d case discussed above: the code $\mathcal{C}$ parameterizes two-cohomologies of $M_4 = \partial X_5$.

Using the isomorphism between 3d and 5d theories, we can formulate our findings above as follows. Any state produced by a path integral of the 3d theory on any given

3d topology is (up to an overall coefficient) the state produce by a topological boundary condition in 5d theory. Similarly, any state produced by the 5d path integral on any 5d topology is the state in 3d theory produced by topological boundary conditions, again up to an overall normalization. It would be very tempting to give this observation a geometric interpretation in terms of the 7d theory (2.2).

4 Holographic correspondence

The AdS/CFT correspondence, also known as holographic duality, is an equivalence between a (non-gravitational) field theory on living on a d-dimensional "boundary" manifold M, and a (gravitational) field theory in $d+1$ dimensions on a "bulk" manifold X_{bulk} that ends on $\partial X_{\text{bulk}} = M$. Given that the bulk theory is gravitational, the bulk path integral includes a sum over bulk manifolds of all possible topologies satisfying $\partial X_{\text{bulk}} = M$ [58]. The statement of holographic duality is often formulated as

$$Z_{\text{CFT}}[J] = Z_{\text{bulk}}[J], \tag{4.1}$$

where both bulk and boundary partition functions depend on J. The latter has different interpretations on the two sides of the duality. On the CFT side, J represents the classical external sources, while on the bulk side J specifies boundary conditions at $\partial X_{\text{bulk}} = M$ in the bulk path integral. We emphasize that on both sides J represents classical data "living" on M, which in turn specifies the coupling constants of the d-dimensional theory.

An older idea, originated in [59] and known as the TQFT/(R)CFT correspondence, establishes a similar yet distinct relation between bulk path integrals in a 3d TQFT and conformal blocks in a dual 2d CFT. Unlike the AdS/CFT correspondence, this relation, commonly known as the bulk-boundary correspondence, does not involve a sum over bulk geometries. To illustrate it, consider a $(2+1)$-dimensional topologically ordered system on a spatial disc D described by a TQFT $\mathcal{T}$. Depending on the boundary conditions at ∂D, such a system may exhibit massless edge modes. These modes can be described either by a $(1+1)$-dimensional theory on $\partial D \times \mathbb{R}$ or by a $(2+1)$-dimensional theory $\mathcal{T}$ on the space $X_{\text{bulk}} = D \times \mathbb{R}$ with boundary $\partial D \times \mathbb{R}$ [14, 59].

This scenario is of course very similar to the holographic duality described above, and it is frequently called holography in the condensed matter literature, but there is a crucial distinction that we would like to emphasize. The path integral in $2+1$ dimensions evaluates a particular conformal block of the $(1+1)$-d theory, not the entire modular-invariant partition function. In other words certain sectors of the $(1+1)$-d theory are missing. It is well known that to generate excitations in those sectors on the boundary, the $(2+1)$-d theory on the disc should be amended to include corresponding defects. Certain combinations of defects, such that the bulk fields have topological (gapped) boundary conditions at the defect worldline, will give rise to a full, modular-invariant theory at the boundary ∂D (without any light dynamical modes living on the defect itself). Since the bulk theory $\mathcal{T}$ is topological, the defect worldline can be fattened into $\mathbb{S}^1 \times \mathbb{R}$ as shown in Fig. 1, so that the disc D becomes an annulus. As a result, we obtain the description of the boundary CFT on $\partial D \times \mathbb{R}$ in terms of the so-called sandwich construction [12, 24, 60, 61]:

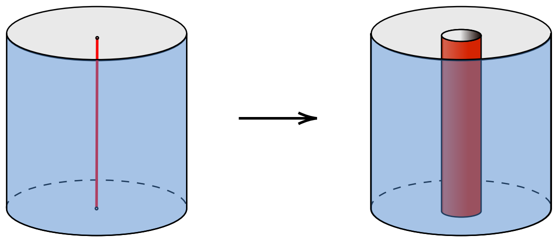

Figure 1. Left image: a $2+1$ topological system with a defect. A particular combination of defects gives rise to topological boundary conditions at the defect worldline and a CFT at the cylinder boundary. Right image: the same, after the defect worldline is fattened into a cylindrical shell (shown in red). The result is the sandwich construction: topological boundary condition at the boundary shown in red gives rise to a CFT at the *other* boundary.

the topological theory $\mathcal{T}$ on the annular cylinder $(\partial D\times\mathbb{R})\times[0,1]$ with topological boundary conditions at one of the boundaries gives rise to a CFT at the other. Often a topological theory admits several different possible choices of topological boundary conditions $\mathcal{C}$ (if it admits any). Choosing any one of them at one boundary (while the boundary conditions at the other boundary should be chosen to admit massless modes) and evaluating the path integral on the cylinder would yield the partition function of the CFT specified by $\mathcal{C}$,

$$Z_{\mathcal{C}} = Z_{\text{bulk}}. \tag{4.2}$$

This is of course similar to holography. But the crucial difference between holography and the sandwich construction is that, in the latter case, the details of the boundary theory are specified by the boundary conditions of the bulk fields at the *other* boundary, not the one where the boundary theory lives. This difference is both geometric and conceptual. Given the topological nature of $\mathcal{T}$, the "width" of the sandwich can be made arbitrarily small resulting in a d-dimensional theory [62]. In the case of holography the additional direction is dynamical and the bulk theory is $(d+1)$-dimensional. Another important differences is that in holography bulk theory is gravitational and hence includes a sum over all topologies of Z_{bulk}.

There is a particular scenario in which the distinction between the bulk-boundary correspondence and holography disappears — when the theory $\mathcal{T}$ is topologically trivial and non-anomalous. In this case its path integral on $D\times\mathbb{R}$ evaluates the modular-invariant 2d CFT partition function. This was first pointed out in [63], which formulated a holographic description of a 2d Narain (compact scalar) CFT in terms of a trivial 3d BF-type Abelian Chern-Simons theory of level $k=1$. In this paper we revisit this idea and provide a holographic description for 4d Maxwell theory in terms of a trivial 5d BF-type Abelian TQFT of level $N=1$.

Yet, the main approach we take in this paper is different. Inspired by [2, 3, 64] and following [26, 55] we promote $\mathcal{T}$ to a "gravitational" theory – TQFT gravity – by summing over all possible topologies of X_{bulk} with fixed ∂X_{bulk} when evaluating the path integral. Thus, in the case discussed above the path integral will be evaluated and summed over all 3d

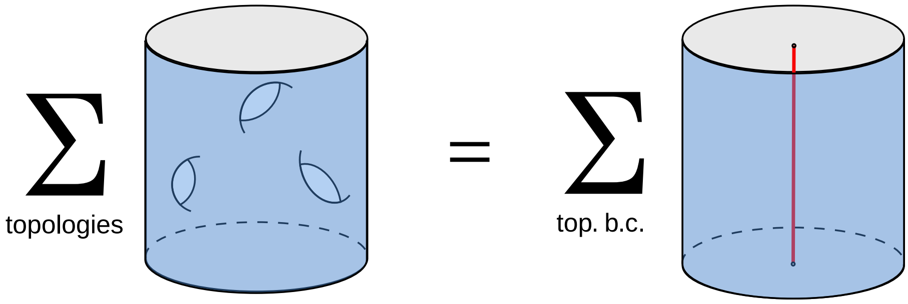

Figure 2. A schematic illustration of the sum over 3d topologies being equal to the sum over topological boundary conditions. The sum includes all 3d topologies, smooth topologies (handlebodies) as well as singular ones, obtained via genus reduction. Possible weights on both sides of the equality are omitted for visual simplicity.

topologies ending on $\partial D \times \mathbb{R}$. We emphasize that, as in the case of conventional holography, the bulk has only one boundary ∂D, where the CFT lives, and where boundary conditions for the bulk fields are specified. It has been shown in [55] that summing over all possible topologies of X_{bulk} is mathematically the same as inserting a linear combination with *positive* coefficients of all possible topological boundary conditions at the internal boundary of the sandwich construction, as we schematically illustrate in Fig. 2. After summing over topologies the bulk path integral will yield an ensemble-averaged CFT partition function

$$\langle Z_{\text{CFT}} \rangle \equiv \sum_{\mathcal{C}} \alpha_{\mathcal{C}} Z_{\mathcal{C}} = Z_{\text{TQFT gravity}}, \qquad \alpha_{\mathcal{C}} \geq 0. \tag{4.3}$$

This identity represents a holographic duality between TQFT gravity in the bulk and an ensemble of CFTs on the boundary.

4.1 Preliminaries of 4d Maxwell theory

We first recall the preliminaries of 4d Maxwell theory. On a spin four-manifold M_4 Maxwell is characterized just by the coupling constant $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$,[5]

$$S = \int_{M_4} \frac{1}{g^2} F \wedge \star F + \frac{i\theta}{8\pi^2} F \wedge F. \tag{4.4}$$

Theories with τ and τ' related by an $SL(2,\mathbb{Z})$ transformation are S-dual to each other, i.e. they are physically equivalent, although due to an anomaly the partition functions $Z_\tau[M_4]$ and $Z_{\tau'}[M_4]$ may differ by a phase [6, 71–73]. When the manifold has vanishing signature $\sigma = 0$ the anomaly cancels. The partition function was evaluated in [6, 7] to be[6]

$$Z_\tau[M_4] = \frac{\theta(\tau, E, \xi, \bar{\xi})}{\Phi_{5d}}, \quad \Phi_{5d} = \frac{(\det' \Delta_1)^{1/2}}{\det' \Delta_0}. \tag{4.5}$$

[5] Maxwell theory on non-spin manifolds is significantly more complicated as has been discussed in [65–70].

[6] There is an ambiguity in the power of τ_2 inside θ associated with the freedom to add a local term to the action [6]. We fix it by requiring Z_τ to be invariant under S-transformation $\tau \to -1/\tau$.

Here θ is a Siegel-Narain theta function defined in terms of the "Narain data" $E = G + B$, specified by the four-manifold M_4, see Appendix B.2 for details,

$$\theta(\tau, E, \xi, \bar{\xi}) = \tau_2^{n/2} \sum e^{i\pi\tau p_L^2 - i\pi\bar{\tau}p_R^2 + 2\pi i(p_L\xi - p_R\bar{\xi}) + \frac{\pi}{2\tau_2}(\xi^2+\bar{\xi}^2)}, \tag{4.6}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} p_L^I + p_R^I \\ p_L^I - p_R^I \end{pmatrix} = \mathcal{O}\,\vec{n}, \qquad \vec{n} \in \mathbb{Z}^{2n}, \tag{4.7}$$

and $\mathcal{O}$ is given by (2.22). The same theta-function after Poisson resummation can be written as follows,

$$\tilde{\theta}(\tau, E, z, \bar{z}) = \det(G)^{1/2} \sum e^{-\pi|\mathrm{v}|^2 - 2\pi i n^T B m + \sqrt{2}\pi(\bar{z}^T\mathrm{v} - z^T\mathrm{v}^*) + \pi\bar{z}^T z}, \tag{4.8}$$

$$\mathrm{v} = G^{1/2}(n + \tau m)/\sqrt{\tau_2}, \quad n, m \in \mathbb{Z}^n, \tag{4.9}$$

where

$$\theta(\tau, E, \xi, \bar{\xi}) = \tilde{\theta}(\tau, E, z, \bar{z}), \qquad \xi = z\sqrt{\tau_2},\ \bar{\xi} = \bar{z}\sqrt{\tau_2}. \tag{4.10}$$

The classical sources $G^{1/2}z$ and $G^{1/2}\bar{z}$ couple to $\mathrm{U}(1)\times\mathrm{U}(1)$-charges of Maxwell theory: fluxes of $(F^D - \bar{\tau}F)/\sqrt{\tau_2}$ and its conjugate evaluated through half of the two-cycles of M_4. Here $F^D = \tau_2 \star F + \tau_1 F$ and F form a doublet under $SL(2,\mathbb{Z})$. The lattice-independent factor $e^{\pi\bar{z}^T z}$ in (4.8) arises because we are evaluating field theory path integral, not the partition function; see [5, 74] for a detailed discussion of the difference between the two in the 2d case.

Appearance of the Siegel-Narain theta function in (4.5), which is up to the factor of Φ the same as the partition function of 2d Narain CFT, can be understood directly in field theory, by considering six-dimensional theory of the self-dual two-form on $\Sigma \times M_4$ [7, 8].

A generalization of the above discussion to Maxwell theory with gauge group $\mathrm{U}(1)^g$ is straightforward. In this case the coupling is a $g \times g$ "modular parameter" Ω, and the partition function on a spin four-manifold M_4 is given by

$$Z_\Omega[M_4] = \frac{\theta(\Omega, E, \xi, \bar{\xi})}{\Phi_{5d}} = \frac{\tilde{\theta}(\Omega, E, z, \bar{z})}{\Phi_{5d}}, \quad \xi = z\,\Omega_2^{1/2},\ \bar{\xi} = \bar{z}\,\Omega_2^{1/2}, \tag{4.11}$$

where $\theta, \tilde{\theta}$ are given by (2.19),(2.31) with $N = 1$ and vanishing c_i, c_i. We note that $\theta(\Omega, E, \xi, \bar{\xi}) = \tilde{\theta}(\Omega, E, z, \bar{z})$ is invariant under modular transformations (2.24), reflecting S-duality of 4d Maxwell theory, as well as under orthogonal transformations (2.26), which are the mapping class group transformations of the 4d manifold M_4. Under modular transformations variables $z, \bar{z}$ change by a unitary matrix $U \in \mathrm{U}(g) = O(2n,\mathbb{R}) \cap Sp(2n,\mathbb{R})$,

$$z \to z_\gamma = z\,U, \quad \bar{z} \to \bar{z}_\gamma = \bar{z}\,U^\dagger, \tag{4.12}$$

defined by polar decomposition of $\Omega_2^{1/2}(C\Omega + D)^{-1}$. Alternatively it can be defined by $S = H\mathcal{S}(\Omega_\gamma)\,\gamma\,\mathcal{S}^{-1}(\Omega)H^{-1}$, in full analogy with the matrices O_L, O_R enacting orthogonal transformations (2.26), $z \to z_h = O_L\,z$, $\bar{z} \to \bar{z}_h = O_R\,\bar{z}$.

The representation (4.8) and its generalization to higher g, given by (2.31) with $N = 1$, provides an alternative way to think about the Siegel-Narain theta function in $\mathbb{R}^{n,n}$.

Essentially it is a lattice theta function for a self-dual *symplectic* lattice $\mathbb{Z}^{2g} \in \mathbb{R}^{2g}$ equipped with the symplectic inner product (3.6) and conventional Euclidean metric. All such lattices are parameterized by modular parameters Ω subject to $\mathrm{Sp}(2g,\mathbb{Z})$ identifications [75], in terms of which the lattice-generating matrix can be chosen as follows:

$$\vec{\mathrm{v}} = \mathcal{S}\begin{pmatrix} \vec{n} \\ \vec{m} \end{pmatrix}, \quad \vec{n},\vec{m} \in \mathbb{Z}^g, \quad \mathcal{S} = \Omega_2^{-1/2}\begin{pmatrix} \mathbb{1} & \Omega_1 \\ 0 & \Omega_2 \end{pmatrix}. \tag{4.13}$$

In what follows we will refer to the space of modular parameters (symplectic lattices)

$$\mathcal{M}_g = Sp(2g,\mathbb{Z})\backslash Sp(2g,\mathbb{R})/\mathrm{U}(g,\mathbb{R}) \tag{4.14}$$

as the Σ-moduli space. This is the space of couplings of 4d $\mathrm{U}(1)^g$ gauge theories.

Similarly to Narain lattice, which is a lattice of $\mathrm{U}(1)^n \times \mathrm{U}(1)^n$ charges of states in 2d theory, symplectic lattice above is the lattice of $\mathrm{U}(1)^g \times \mathrm{U}(1)^g$ charges of 4d Maxwell theory [76].

4.2 $N = 1$: 5d holographic dual to Maxwell theory

We now proceed with the case of $N = g = 1$ 5d theory (2.1), which is trivial in the TQFT sense: it has no non-trivial surface operators (topological defects) and its Hilbert space on any M_4 is one-dimensional. Its path integral on any five-dimensional X_{bulk} with boundary $\partial X_{\text{bulk}} = M_4$ is (up to an overall normalization) the same, and is given by (2.27) with $c_1 = 0$ or (2.30) with $\mathrm{c}_i = 0$. It is straightforward to see that this is the same as the partition function of a 4d Maxwell theory on M_4,

$$Z_\tau[M_4] = \Psi_0(\tau,E,\xi,\bar{\xi}) = \Psi_{0\ldots0}(\tau,E,\xi,\bar{\xi}) = \frac{\tilde{\theta}(\tau,E,z,\bar{z})}{\Phi_{5d}}, \tag{4.15}$$

where Ψ_0 is as in (2.27) and $\Psi_{0\ldots0}$ is defined in (2.30). This relation, as well as a similar statement about 3d Chern-Simons theory and the 2d free boson, was already mentioned in [7], two years before the holographic correspondence was introduced. To our knowledge the statement about the TQFT in the bulk being holographically dual to the boundary CFT appears for the first time in [19], without emphasizing that the TQFT should be trivial. The complete statement, that the trivial AB theory (2.5) with $N = 1$ is dual to a compact scalar (an arbitrary Narain theory for $n > 1$), was put forward in [63], and more recently revisited in [5, 21, 77]. Holography for free field theories dual to trivial TQFTs in higher dimensions, including 4d Maxwell theory, was also discussed recently in [20].

We have evaluated the RHS of (4.15) explicitly in section 2.1 and matched it to the known field theory answer above. As we noted at the beginning of this section, the path integral in the bulk should in principle include a sum over all possible 5d bulk topologies X_{bulk} ending on M_4, but in the case of a trivial topological theory that sum would merely introduce an overall coefficient, which is renormalized to ensure that the vacuum is unique,

$$Z_{\text{bulk}}(\tau,\xi,\bar{\xi}) = \Psi_0(\tau,E,\xi,\bar{\xi}) \equiv \Psi_{0\ldots0}(\tau,E,\xi,\bar{\xi}). \tag{4.16}$$

We thus obtain a conventional statement of holography (4.1) for 4d Maxwell theory, which takes the form

$$Z_\tau[M_4] = Z_{\text{bulk}}(\tau, \xi, \bar{\xi}). \tag{4.17}$$

The generalization of this relation to arbitrary gauge group $\mathrm{U}(1)^g$ is straightforward.

Since the $N = 1$ theory is topologically trivial, the bulk path integral on X_{bulk} factorizes when $M_4 = \partial X_{\text{bulk}}$ is a union of several disconnected manifolds. This is no longer true for $N > 1$.

4.3 $N > 1$: holographic dual to an ensemble

Before we formulate the duality for $N > 1$, it is helpful to revisit the 3d/2d case, in which a 3d TQFT summed over all three-dimensional topologies ending on $\partial X_{\text{bulk}} = \Sigma$ evaluates the averaged partition function of an ensemble of 2d CFTs. This proposal in its current form was put forward in [26] and proved for a general 3d TQFT with a finite number of anyons in [55], which gave a precise definition of the sum over topologies and determined the weights of the boundary ensemble. An Abelian example, relevant to what follows, was discussed in detail for a torus boundary manifold in [5] and for general a general Σ in [4].

We start with a 3d TQFT in the sandwich construction, as reviewed above. Each topological boundary condition $\mathcal{C}$ in this theory gives rise to a partition function of a 2d CFT on Σ

$$Z_{\mathcal{C}} = \langle \mathcal{B} | \mathcal{C} \rangle, \tag{4.18}$$

where the appropriate conformal boundary conditions on Σ are encoded in $\langle \mathcal{B}|$. For an Abelian theory, $\mathcal{C}$ can be associated with an even self-dual code and $|\mathcal{C}\rangle$ is given by (3.7). For the AB theory (2.5) we explicitly define the state $\langle \mathcal{B}| = \langle \Omega, E, \xi, \bar{\xi}|$ such that

$$Z_{\mathcal{C}}(\tau, \xi, \bar{\xi}) = \langle \Omega, E, \xi, \bar{\xi} | \mathcal{C} \rangle = \frac{\theta(\Omega, E_{\mathcal{C}}, \xi, \bar{\xi})}{\Phi_{3d}} \tag{4.19}$$

is the genus-g path integral of a Narain theory specified by the point in the Narain moduli space $E_{\mathcal{C}}$. The relation between a code $\mathcal{C}$ and $E_{\mathcal{C}}$ is as follows. The code $\mathcal{C}$ defines a Narain lattice $\Lambda_{\mathcal{C}}$ via the so-called Construction A, see [5, 26, 39] for details, which in turn defines $E_{\mathcal{C}} = G_{\mathcal{C}} + B_{\mathcal{C}}$.

We note, however, that the statement (4.18) is more general and holds in any TQFT that admits topological and conformal boundary conditions. The boundary ensemble is an ensemble of CFTs corresponding to all possible topological boundary conditions $\mathcal{C}$,

$$\langle Z \rangle = \sum_{\mathcal{C}} \alpha_{\mathcal{C}} Z_{\mathcal{C}}, \tag{4.20}$$

with positive weights $\alpha_{\mathcal{C}}$ reflecting the size of the symmetry group of each theory [78]. In the Abelian case all the $\alpha_{\mathcal{C}}$ are equal and can be normalized to $\alpha_{\mathcal{C}} = \mathcal{N}_{\mathcal{C}}^{-1}$ where $\mathcal{N}_{\mathcal{C}}$ is the total number of topological boundary conditions.

With an appropriate overall normalization (4.20) is equal to the path integral of the original TQFT summed over all 3d manifolds X_{bulk} ending on Σ. This sum is defined in terms of the mapping class group of a Σ' of infinite genus, related to Σ by genus reduction $\Sigma' \to \Sigma$ [55]. In general, the resulting sum will include singular topologies.

As discussed in section 3.2, in the case of the 3d Abelian theory (2.5) a state on Σ, evaluated by a path integral on a given $X_{\rm bulk}$ ending on $\Sigma = \partial X_{\rm bulk}$, is a stabilizer state specified by a self-dual symplectic code $\mathcal{L}$. The resulting statement of ensemble holography can then be formulated as follows,

$$\frac{1}{\mathcal{N}_\mathcal{C}}\sum_{\mathcal{C}}|\mathcal{C}\rangle = \sum_{\mathcal{L}}\beta_\mathcal{L}|\mathcal{L}\rangle, \tag{4.21}$$

where the coefficients $\beta_\mathcal{L}$, which result from the sum over topologies, require evaluation. Comparing with (4.20) we note that $\langle\mathcal{B}|$ has been stripped from both sides of the identity, and thus the resulting mathematical statement is about states of the TQFT on Σ.

In the specific case of an AB theory (2.5) with $N = \prod_k p_k$ a square-free product of distinct primes p_k, the identity (4.21) simplifies to

$$\frac{1}{\mathcal{N}_\mathcal{C}}\sum_{\mathcal{C}}|\mathcal{C}\rangle = \frac{A}{\mathcal{N}_\mathcal{L}}\sum_{\mathcal{L}}|\mathcal{L}\rangle. \tag{4.22}$$

This identity follows from the uniqueness of the state invariant under both symplectic (2.9) and orthogonal (2.7) groups, and can be understood in terms of Howe duality [4]. The coefficient A can be evaluated to be

$$A = \frac{1}{\mathcal{N}_\mathcal{C}}\prod_k p_k^{-ng/2}\prod_{i=0}^{n-1}(p_k^i + p_k^g). \tag{4.23}$$

and $\mathcal{N}_\mathcal{L}$ is the number of symplectic self-dual codes $\mathcal{L}$ of length $2g$, evaluated in Appendix D.

When N is not square-free there is generally more than one invariant vector and the expression is more involved. Focusing on the case $N = p^2$ for prime p and taking $g = 1$, we find that there are two invariant vectors, one for each orbit S_a of $\mathrm{Sp}(2g, Z_N)$ acting on symplectic codes, and the identity (4.21) takes the form

$$\frac{1}{\mathcal{N}_\mathcal{C}}\sum_{\mathcal{C}}|\mathcal{C}\rangle = A_{p^2}\sum_{\mathcal{L}\in S_1}|\mathcal{L}\rangle + B_{p^2}\sum_{\mathcal{L}\in S_0}|\mathcal{L}\rangle. \tag{4.24}$$

The coefficients A_{p^2}, B_{p^2} are evaluated in Appendix D.3.

Now returning to 5d, the ensemble of boundary theories is defined by the set of all topological boundary conditions $\mathcal{L}$ of the 5d theory. As we discussed in section 3.1, in the case of interest this is the set of self-dual symplectic codes and the analog of (4.18) takes the form

$$Z_\mathcal{L} = \langle\mathcal{B}_{4d}|\mathcal{L}\rangle. \tag{4.25}$$

Here $\langle\mathcal{B}_{4d}| = \langle\Omega, E, \xi = z\,\Omega_2^{1/2}, \bar\xi = \bar z\,\Omega_2^{1/2}|$ and

$$Z_\mathcal{L} \equiv Z_{\Omega_\mathcal{L}}[M_4, z, \bar z] = \frac{\tilde\theta(\Omega_\mathcal{L}, E, z, \bar z)}{\Phi_{5d}} \tag{4.26}$$

is the path integral of a Maxwell theory on M_4 characterized by E, with coupling constant $\Omega_{\mathcal{L}}$ and external sources $z, \bar{z}$. The relation between a self-dual symplectic code $\mathcal{L}$ and $\Omega_{\mathcal{L}}$ is as follows. The code defines a self-dual symplectic lattice in $\mathbb{R}^{2g}$ via Construction A [57]

$$\vec{\mathrm{v}} = \mathcal{S}(\Omega) \begin{pmatrix} N\,\vec{n} + \vec{a} \\ N\,\vec{m} + \vec{b} \end{pmatrix} / \sqrt{N}, \quad \vec{n}, \vec{m} \in \mathbb{Z}^g, \quad (\vec{a}, \vec{b}) \in \mathcal{L}, \tag{4.27}$$

and $\Omega_{\mathcal{L}}$ is the associated modular parameter of this lattice.

To derive a 5d statement of holography analogous to (4.21), we follow the argument of [55] and extend it where necessary. As shown in [49] in great generality, code states $|\mathcal{L}\rangle$, interpreted as "genus n" full enumerator polynomials of self-dual classical codes $\mathcal{L}$, span the space of $\mathrm{O}(n, n, \mathbb{Z}_N)$-invariant states in $\mathcal{H}$. It follows from (3.8) that for a given M_4 of zero signature and "genus" $n = b_2^+ = b_2^-$, the overlap between two code states is given by

$$\langle \mathcal{L} | \mathcal{L}' \rangle = |\mathcal{L} \cap \mathcal{L}'|^n, \tag{4.28}$$

where $1 \le |\mathcal{L} \cap \mathcal{L}'| \le N^g$ is the number of codewords present in both $\mathcal{L}$ and $\mathcal{L}'$. Provided the codes are distinct, this number is smaller than $|\mathcal{L}| = |\mathcal{L}'| = N^g$ and therefore for $n \to \infty$ all code states $|\mathcal{L}\rangle$ become orthogonal. Therefore, in this limit,

$$\rho = \frac{1}{N^{gn} \mathcal{N}_{\mathcal{L}}} \sum_{\mathcal{L}} |\mathcal{L}\rangle\langle\mathcal{L}| \in \mathcal{H}^* \otimes \mathcal{H} \tag{4.29}$$

is a projector on the $\mathrm{O}(n, n, \mathbb{Z}_N)$-invariant subspace of $\mathcal{H}$. The same projector can be written as the group average

$$\rho = \frac{1}{|O(n, n, \mathbb{Z}_N)|} \sum_{h \in O(n, n, \mathbb{Z}_N)} U_h. \tag{4.30}$$

From here we find the identity valid in the large b limit,

$$\frac{1}{\mathcal{N}_{\mathcal{L}}} \sum_{\mathcal{L}} |\mathcal{L}\rangle = \frac{N^{gn}}{|O(n, n, \mathbb{Z}_N)|} \sum_{h \in O(n, n, \mathbb{Z}_N)} U_h |0\rangle_{5d}, \qquad n \to \infty. \tag{4.31}$$

When N is prime, the map from $\Gamma_0(N) \backslash O(n, n, Z)$ to $\Gamma_0 \backslash O(n, n, Z_N)$ is surjective in full analogy with the 3d case (where the $\Gamma_0(N) \backslash Sp(2g, Z)$ to $\Gamma_0 \backslash Sp(2g, Z_N)$ is surjective for any N), and this sum be interpreted as follows. The RHS is a sum over simple 5d geometries that generalize 3d handlebodies. The vacuum state $|0\rangle_{5d}$ is the TQFT path integral on X_5, which is a connected sum of n copies of $B^3 \times \mathbb{S}^2$, while the sum on h runs over mapping class group transformations that generate all other "handlebodies." In the case of general N, U_h may not have an interpretation as a mapping class transformation, but the resulting state $U_h|0\rangle_{5d}$ is a code state for an orthogonal even self-dual code $\mathcal{C}_h = h\,\mathcal{C}_0$ and hence is a path integral on a possibly singular 5d topology with a particular lattice of two-cohomologies.

To obtain a version with reduced "genus" n', the boundary $M_4 = \#^n(\mathbb{S}^2 \times \mathbb{S}^2)$ of a given bulk geometry can be attached to a cobordism that degenerates a nonintersecting subset of $n - n'$ boundary 2-cycles to zero, leaving a boundary isomorphic to $\#^{n-\tilde{n}}(\mathbb{S}^2 \times \mathbb{S}^2)$.

This procedure is the analog of genus reduction in 2d. Algebraically, it is represented by taking a scalar product of (4.31) with ${}_{5d}^{n-n'}\langle 0| \otimes \langle \mathcal{B}_{4d}|$,

$$\frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}} Z_{\mathcal{L}} = \lim_{n'\to\infty} \frac{N^{gn'}}{|G_O|}\sum_{h\in G_O} {}_{5d}^{n'-n}\langle 0|\langle \mathcal{B}_{4d}|U_h|0\rangle_{5d}^{n'}, \quad G_O = \Gamma_0\backslash O(n',n',\mathbb{Z}_N). \quad (4.32)$$

Now the LHS is the desired ensemble-averaged CFT partition function on M_4 while the RHS represents a sum over various 5d topologies ending on M_4. This identity is the statement of holographic correspondence between the ensemble of 4d Maxwell theories on M_4 and the dual 5d "TQFT gravity" – the 5d theory (2.1) summed over 5d topologies ending on M_4.

The Heegaard splitting theorem [79] guarantees that the analog of the sum on the RHS in 3d includes all possible topologies with the given boundary. The five-dimensional case is more nuanced [80]. We proceed assuming the sum on the RHS of (4.32) includes all possible classes of the 5d topologies ending on M_4 that can be distinguished by the Abelian theory (2.1).

All terms appearing on the RHS of (4.32) are states associated with classical even self-dual codes $\mathcal{C}$. This allows us to evaluate both sides explicitly by matching to the LHS for various N. A more general framework to perform such calculations was recently put forward in [81]. For square-free N, we recover (4.22) which can be written in a form making the 2d/4d duality manifest,

$$\frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}} |\mathcal{L}\rangle = \frac{A'}{\mathcal{N}_{\mathcal{C}}}\sum_{\mathcal{C}} |\mathcal{C}\rangle, \quad (4.33)$$

$$A' = \frac{1}{\mathcal{N}_{\mathcal{L}}}\prod_k p_k^{-ng/2}\prod_{i=1}^{g}(p_k^i + p_k^n), \qquad AA' = 1. \quad (4.34)$$

Focusing on the case of a single U(1) gauge field ($g = 1$) and prime $N = p$, this can be written as

$$\langle Z_{4d\,\mathrm{Maxwell}}\rangle = \frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}} Z_{\tau_{\mathcal{L}}}[M_4, z, \bar{z}] = \tilde{A}\sum_{h\in\Gamma_0\backslash O(n,n,\mathbb{Z}_N)} \Psi_{0\ldots 0}(\tau, E_h\,\xi_h, \bar{\xi}_h), \quad (4.35)$$

$$\xi = \sqrt{\tau_2}z,\ \bar{\xi} = \sqrt{\tau_2}\bar{z}, \quad \tilde{A} = \frac{p+p^n}{\mathcal{N}_{\mathcal{L}}\mathcal{N}_{\mathcal{C}}}, \quad (4.36)$$

$$\mathcal{N}_{\mathcal{L}} = (p+1), \qquad \mathcal{N}_{\mathcal{C}} = |O(n,n,\mathbb{Z}_N)|/|\Gamma_0| = \prod_{i=0}^{n-1}(p^i+1), \quad (4.37)$$

where the sum in the LHS of (4.35) goes over $\mathcal{N}_{\mathcal{L}} = p+1$ possible values of the coupling constant

$$\tau_0 = p\tau, \qquad \tau_{r+1} = \frac{\tau + r}{p}, \quad r = 0\ldots p-1. \quad (4.38)$$

In disguise, this is the same mathematical identity as the one in 3d discussed in [5], but with the interpretation of the LHS and RHS interchanged. The sum in the LHS of (4.35) is over an ensemble of Maxwell theories in 5d, while in 3d it was a sum over 3d topologies. Similarly, the sum over orthogonal group in the RHS is the sum over 5d "handlebodies," while in 3d this was a sum over points in the Narain moduli space. We make the relation to 3d manifest in Section 4.4 below.

We stress that the holographic identity in 5d (4.33) being mathematically the same as the 3d identity (4.22) for any g, n is a coincidence specific for square-free N. For general N the identity takes the form

$$\frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}}|\mathcal{L}\rangle = \frac{1}{\mathcal{N}_{\mathcal{C}}}\sum_{k}\alpha_k\sum_{\mathcal{C}\in S_k}|\mathcal{C}\rangle, \tag{4.39}$$

where coefficients α_k are g, n, N-dependent and S_k are distinct orbits of $\mathrm{O}(n, n, \mathbb{Z})$. It is different from its counterpart in 3d (4.21). In general for $g > n$ coefficients α_k can not be fixed uniquely because of possible degeneracy of the states in the RHS of (4.39). For $N = p^2$, $g = 1$ and arbitrary n we find coefficients A'_{p^2}, B'_{p^2} in the Appendix D.3,

$$\frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}}|\mathcal{L}\rangle = A'_{p^2}\sum_{\mathcal{C}\in S_n}|\mathcal{C}\rangle + B'_{p^2}\sum_{\mathcal{C}\in S_0}|\mathcal{C}\rangle. \tag{4.40}$$

where we made the choice to keep all $\alpha_k = 0$ except for $k = 0, n$.

4.4 Large-N limit

For square-free N, when the orthogonal and symplectic groups act transitively on the corresponding codes, the holographic identity (4.22) and (4.33) can be rewritten as an equality between linear combinations of theta functions (below we often omit $\bar{z}$ for notational simplicity)

$$\frac{1}{\mathcal{N}_{\mathcal{C}}}\sum_{\mathcal{C}}\tilde{\theta}(\Omega, E_{\mathcal{C}}, z) = \frac{A}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}}\tilde{\theta}(\Omega_{\mathcal{L}}, E, z). \tag{4.41}$$

On the LHS the theta function is averaged over a discrete set of points in the Narain moduli space; on the RHS the average is over points in the Ω moduli space $\mathcal{M}_g$. As N increases, the number of points in each set also increases (with a possible exception of $\mathcal{N}_{\mathcal{C}}$ for $n = 1$). As $N \to \infty$, the points densely and uniformly populate the corresponding moduli spaces. In the limit, depending on whether n is larger or smaller than $g+1$, either the average over $\mathcal{M}_g$ or over Narain moduli space will diverge, with the leading behavior captured by real Eisenstein series of the symplectic or orthogonal groups correspondingly. This divergence will be compensated by A that will also diverge or go to zero. The resulting identity is a version of the Siegel–Weil formula, that made an appearance in the 3d case [2, 3] as well as in studies of string loop amplitudes [82–84].

When N is not square-free, the identities for the 3d and 5d theories take a slightly different form: points in the moduli space associated with *field theories* in 2d and 4d correspondingly are weighted equally, while points in the moduli space associated with topologies acquire nontrivial weights. As we discuss below, we find that the resulting identities in the $N \to \infty$ limit still take the same general form.

To derive the picture sketched above we start with square-free N and and evaluate the scalar product of (4.33) with $\langle\mathcal{B}|$ (taking $g=1$ for simplicity),

$$\frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}}\langle\mathcal{B}|\mathcal{L}\rangle = \frac{1}{\mathcal{N}_{\mathcal{L}}\Phi}\sum_{\mathcal{L}}\tilde{\theta}(\tau_{\mathcal{L}},E,z). \tag{4.42}$$

Taking $\Phi=\Phi_{5d}$ we interpret this equation as an average over an ensemble of Maxwell theories. Another representation, and interpretation, comes from the fact that for a square-free N all codes $\mathcal{L}$ belong to the same orbit under $SL(2,\mathbb{Z})$. Combined with the relation

$$\Psi_{0\ldots 0}(\Omega,E,\xi)=\langle\mathcal{B}|0\rangle_{3d}=\frac{1}{N^{ng/2}}\langle\mathcal{B}|\mathcal{L}_0\rangle=\frac{1}{N^{ng/2}}\frac{\theta(N\tau,E,\sqrt{N}\xi)}{\Phi}. \tag{4.43}$$

this gives

$$\frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}}\langle\mathcal{B}|\mathcal{L}\rangle=\frac{N^{n/2}}{\mathcal{N}_{\mathcal{L}}}\sum_{\gamma\in G_S}\Psi_{0\ldots 0}(\tau_\gamma,E,\xi_\gamma)=\frac{1}{\mathcal{N}_{\mathcal{L}}\,\Phi}\sum_{\gamma\in G_S}\theta(N\tau_\gamma,E,\sqrt{N}\xi_\gamma), \tag{4.44}$$
$$\mathcal{N}_{\mathcal{L}}=|G_S|,\qquad G_S=\Gamma_0\backslash SL(2,\mathbb{Z}_N).$$

Here τ_γ,ξ_γ are the modular transformations of τ,ξ given by (2.24). Up to an overall normalization the sum in the middle of (4.44) has the interpretation in 3d as the sum over handlebody geometries obtained by a modular transformation γ from the handlebody X_3 used to define the basis (2.3). To obtain (4.42) from the rightmost expression in (4.44) we note that θ is modular invariant and for a square-free N there are always $\gamma',\gamma\in SL(2,\mathbb{Z})$ such that

$$\gamma'\begin{pmatrix}N&0\\0&1\end{pmatrix}\gamma=\begin{pmatrix}p&r\\0&q\end{pmatrix}, \tag{4.45}$$

with $pq=N$ and $0\le r\le q-1$. From here follows the expression for $\tau_{\mathcal{L}}=(N\tau_\gamma)_{\gamma'}$,

$$\tau_{\mathcal{L}}=\frac{p\tau+r}{q},\quad pq=N,\quad r=0\ldots q-1, \tag{4.46}$$

for all possible $p\,|\,N$. For a prime N this reduced to (4.38). A direct check shows that (4.45) also implies

$$(\sqrt{N}\xi_\gamma)_{\gamma'}=\sqrt{\frac{p}{q}}\xi \tag{4.47}$$

and therefore $z=\xi/\sqrt{\operatorname{Im}\tau}=(\sqrt{N}\xi_\gamma)_{\gamma'}/\sqrt{\operatorname{Im}\tau_{\mathcal{L}}}$ remains the same for all terms in (4.42).

The explicit values of $\tau_{\mathcal{L}}$ given by (4.38) and the construction of the sum over $\gamma\in G_S$ above readily suggests that when the sources $z=\bar{z}=0$ vanish the sum can be written in terms of the Hecke operator T_N [85] acting on the modular form of weight zero [5],

$$\sum_{\gamma\in SL(2,\mathbb{Z})\backslash M_N}\tilde{\theta}(\gamma\tau,E,0)=\sum_{\mathcal{L}}\theta(\tau_{\mathcal{L}},E,0)=N\,T_N\,\theta(\tau,E,0). \tag{4.48}$$

Here M_N is the space of integer-valued 2×2 matrices of determinant N.

The generalization to arbitrary g and N is straightforward,

$$\frac{\Phi}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}}\langle\mathcal{B}|\mathcal{L}\rangle = \frac{N^{ng/2}}{\mathcal{N}_{\mathcal{L}}}\sum_{\gamma\in G_S}\Psi_{0\ldots 0}(\Omega_\gamma, E, \xi_\gamma) = \frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\gamma\in G_S}\tilde{\theta}(N\Omega_\gamma, E, z_\gamma) = \tag{4.49}$$

$$\frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}}\tilde{\theta}(\Omega_{\mathcal{L}}, E, z), \qquad \mathcal{N}_{\mathcal{L}} = |G_S|, \qquad G_S = \Gamma_0\backslash Sp(2g, \mathbb{Z}_N). \tag{4.50}$$

And again for vanishing sources $z = \bar{z} = 0$ the sum can be represented in terms of a Hecke operator acting on $\tilde{\theta}(N\Omega, E, 0)$.

When $n > g+1$, the sum $\sum_\gamma \Psi_{0\ldots 0}(\Omega_\gamma, E, \xi_\gamma)$ converges in the $N \to \infty$ limit, but the coefficient $N^{ng/2}/\mathcal{N}_{\mathcal{L}}$ diverges indicating that average over $\mathcal{M}_{\}}$ is singular. For $N \gg 1$ and for fixed Ω only origin of the lattice contributes to $\theta(N\Omega, E, z)$, such that

$$\Psi_{0\ldots 0}(\Omega, E, \xi) \approx \det(\Omega_2)^{n/2} e^{\frac{\pi}{2}\Omega_2^{-1}(\xi^2+\bar{\xi}^2)}. \tag{4.51}$$

Summing over γ yields the generalization of the real Eisenstein series [86],

$$\sum_{\gamma\in G_S}\Psi_{0\ldots 0}(\Omega_\gamma, E, \xi_\gamma) = \det(\Omega_2)^{n/2}E_n(\Omega, z), \qquad \xi = z\,\Omega_2^{1/2}, \tag{4.52}$$

$$E_n(\Omega, z, \bar{z}) = \sum_{\gamma\in\Gamma_0\backslash Sp(2g,\mathbb{Z})}\frac{e^{\frac{\pi}{2}(z\,U^2 z^T + \bar{z}(U^\dagger)^2\bar{z}^T)}}{|\det(C\Omega+D)|^n}. \tag{4.53}$$

We remind the reader that $\mathrm{U}(\gamma, \Omega)$ is defined through the polar decomposition of $\Omega_2^{1/2}(C\Omega + D)^{-1}$. To carefully justify (4.52) one should take into account that the approximation (4.51) is only valid if Ω is not too small, which could happen in the sum over γ when matrices C, D are sufficiently large. An argument that one can always choose a representative $\gamma \in \Gamma_0\backslash SL(2, \mathbb{Z}_N)$ with $|c|, |d|$ not exceeding $\sqrt{N}$ for a prime N and thus completing the argument was given in [5]. Generalization to higher g is an open question.

When $g > n-1$ the sum over γ diverges, but the Hecke representation suggests that after normalizing by $\mathcal{N}_{\mathcal{L}}$ it is given by the average of $\tilde{\theta}$ over the fundamental domain of Ω,

$$\frac{1}{\mathcal{N}_{\mathcal{L}}}\sum_{\mathcal{L}}\tilde{\theta}(\Omega_{\mathcal{L}}, E, z, \bar{z}) = \langle\tilde{\theta}(\Omega, E, z\,U, \bar{z}\,U^\dagger)\rangle_{\Omega,U} \equiv \int\frac{dU}{V_U}\int\frac{d^{2g}\Omega}{V_g(\det\Omega_2)^2}\tilde{\theta}(\Omega, E, zU, \bar{z}U^\dagger).$$

where

$$V_U = \prod_{k=1}^{g}\frac{2\pi^k}{\Gamma(k)} \qquad \text{and} \qquad V_g = \prod_{k=1}^{g}\frac{2\Gamma(k)\zeta(2k)}{\pi^k}$$

are the Haar volumes of $\mathrm{U}(g)$ and $\mathcal{M}_g$, respectively [87]. This is because the Hecke points, e.g. (4.48) for $g = 1$ and $N \to \infty$, no matter how this limit is taken, upon modular transformation to fundamental domain of Ω, are known to densely populate it with the canonical $\mathrm{Sp}(2g, \mathbb{R})$-invariant measure [88]. The average over the unitary group $\mathrm{U}(g)$ emerges because of the pseudo-random unitaries $U(\gamma, \Omega)$ generated by these modular transformations. As a result the average is over $\mathrm{Sp}(2g, \mathbb{Z})\backslash\mathrm{Sp}(2g, \mathbb{R})$, which is the full moduli space in 4d.

The average is finite only for $n \le g$. In particular for $g = n = 1$ it is given by

$$\langle\tilde{\theta}(\tau, r, ze^{i\varphi}, \bar{z}e^{-i\varphi})\rangle_{\tau,\varphi} = r\,e^{\pi\bar{z}^T z} + r^{-1}e^{-\pi\bar{z}^T z}, \tag{4.54}$$

where $G = r^2$.

Similar considerations for orthogonal group and prime N yields

$$\frac{\Phi}{\mathcal{N}_{\mathcal{C}}}\sum_{\mathcal{C}}\langle\mathcal{B}|\mathcal{C}\rangle = \frac{N^{ng/2}}{\mathcal{N}_{\mathcal{C}}}\sum_{h\in G_O}\Psi_{0\dots0}(\Omega,E_h,z_h) = \frac{1}{\mathcal{N}_{\mathcal{C}}}\sum_{\mathcal{C}}\tilde{\theta}(\Omega,E_{\mathcal{C}},z), \tag{4.55}$$

$$G_O = \Gamma_0\backslash O(n,n,\mathbb{Z}_N), \quad \mathcal{N}_{\mathcal{C}} = |G_O|. \tag{4.56}$$

Again, the sum over $E_{\mathcal{C}}$ can be represented via Hecke operator for the orthogonal group. When $n > g+1$, in the $N \to \infty$ limit the sum converges, conjecturally leading to an average over the full Narain moduli space $\mathrm{O}(n,n,\mathbb{R})/O(n,n,\mathbb{Z})$,

$$\frac{1}{\mathcal{N}_{\mathcal{C}}}\sum_{\mathcal{C}}\tilde{\theta}(\Omega,E_{\mathcal{C}},z,\bar{z}) = \langle\tilde{\theta}(\Omega,E,O_L z,O_R\bar{z})\rangle_{E,O_L,O_R}. \tag{4.57}$$

We expect this conclusion to be true when $N \to \infty$ independently of whether N is prime. In the opposite case of $n < g+1$ when N is prime we find

$$\sum_{\gamma\in G_S}\Psi_{0\dots0}(\Omega,E_h,\xi_h) = \det(G)^{g/2}E_g^O(E,z), \qquad \xi = z\,\Omega_2^{1/2}, \tag{4.58}$$

$$E_g^O(E,z,\bar{z}) = \sum_{h\in\Gamma_0\backslash O(n,n,\mathbb{Z})}\frac{e^{\pi\bar{z}^T O_R^T O_L z}}{|\det(CE^T+D)|^g}. \tag{4.59}$$

Matrices O_L, O_R are defied in terms of h, E as discussed below (2.26). In particular for $n=1$, c.f. (4.54), $E_g^O(E,z,\bar{z}) = r\,e^{\pi\bar{z}^T z} + r^{-1}e^{-\pi\bar{z}^T z}$, where $E = G = r^2$.

Combining all together, in the $N \to \infty$ limit we find for $g > n-1$, i.e. when the 4d manifold is fixed and the central charge of Maxwell theory is sufficiently large,

$$\Phi_{5d}\langle Z_\Omega[M_4]\rangle = \langle\tilde{\theta}(\Omega,E,z\,U)\rangle_{\Omega,U} = \det(G)^{g/2}E_g^O(E,z,\bar{z}), \quad g > n-1. \tag{4.60}$$

In the same limit and when $n > g+1$ we recover the 3d result [2, 3, 86], where on the left is the average over 2d Narain theories,

$$\Phi_{3d}\langle Z_E[\Sigma]\rangle = \langle\tilde{\theta}(\Omega,E,O\,z)\rangle_{E,O} = \det(\Omega)^{n/2}E_n(\Omega,z,\bar{z}), \quad n > g+1. \tag{4.61}$$

The second equality in (4.60) and (4.61) are the versions of the Siegel-Weil formula. In the derivation of (4.60) and (4.61) we assumed that N is prime, but in both cases for any N the LHS is the average over Hecke points leading to averages over corresponding moduli spaces, and hence we expect both of these relations to hold for $N \to \infty$ independently of how this limit is taken.

The same expressions above evaluate the divergent part of the 4d and 2d ensemble average in the limit of large sqaure-free N when the genus is larger than the central charge:

$$\Phi_{5d}\langle Z_\Omega[M_4]\rangle_\Omega = N^{-gn/2}\prod_k\prod_{i=1}^{g}\frac{(p_k^i+p_k^n)}{(p_k^i+1)}\det(\Omega)^{n/2}E_n(\Omega,z,\bar{z}), \quad n > g+1, \tag{4.62}$$

and

$$\Phi_{3d}\langle Z_E[\Sigma]\rangle_E = N^{-gn/2}\prod_k\prod_{i=0}^{n-1}\frac{(p_k^i+p_k^g)}{(p_k^i+1)}\det(G)^{g/2}E_g^O(E,z,\bar{z}), \quad g > n-1. \tag{4.63}$$

To derive these expressions we used (4.22) and (4.33) which assumme $N = \prod_k p_k$ is square-free. Coefficients in the RHS are singular when $N \to \infty$ scaling as $N^{g(n-1-g)/2}$ and $N^{n(g+1-n)/2}$ correspondingly, but the proportionality coefficient depends on the way N is taken to infinity.

Finally we consider the case of non-square-free $N = p^2$ for a prime $p \to \infty$, which is discussed in detail for arbitrary n and $g = 1$ in Appendix D.3. In 5d, we find $\tau_{\mathcal{L}}$ to be given by the $\mathcal{N}_{\mathcal{L}} = p^2 + p + 1$ Hecke points

$$\tau_0 = p^2\tau, \qquad \tau_{r+1} = \frac{\tau + r}{p^2},\ r = 0 \dots p^2 - 1, \qquad \tau_{p^2+1+r} = \tau + \frac{r}{p},\ r = 0 \dots p - 1.$$

Their average in the $p \to \infty$ limit is the average over the fundamental domain of τ, which converges for $n = 1 < g + 1$. In this case we find (D.41), which in the large p limit reduces to (4.60). Similarly in the 3d case, average over $\mathcal{N}_{\mathcal{C}}(N = p^2)$ Narain theories converges converges for $n > 2$, in which case the holographic identity is given by (D.33). In the large p limit the contribution of in the RHS $\mathcal{L}'$ vanishes – that is a particular Narain theory characterized by E with a vanishing weight coefficient – while the Poincare series of vacuum converges to the Eisenstein series. Hence, again, (4.61) emerges in the $N = p^2 \to \infty$ limit. At the same time considering $g = n = 1$ in 3d, we find that in the large $N = p^2 \to \infty$ limit the average over $\mathcal{N}_{\mathcal{C}} = 3$ Narain theories diverges, the singular term is correctly given by $N^{1/2}\det(G)^{g/2}E_g^O(E, z, \bar{z})$, but the proportionality coefficient is different from the one in (4.63).

To summarize, in the large N limit, in both 2d and 4d, whenever the ensemble over filed theories converges, it is given by the Eisenstein series, which holographically can be interpreted as the sum over 3d and 5d handlebody geometries. Thus $N \to \infty$ is a "semiclassical" limit when the singular geometries can be omitted from the bulk sum.

When the field theory average diverge, the leading singularity is given by the appropriate Eisenstein series (of orthogonal group in 2d and sympathetic group in 4d), but the overall coefficient is not universal, e.g.

$$\langle Z_\Omega[M_4]\rangle_\Omega \propto N^{n(g+1-n)/2}\det(\Omega)^{n/2}E_n(\Omega, z, \bar{z}), \quad n > g + 1. \tag{4.64}$$

In the discussion above we assumed that $n > g + 1$ or $g > n - 1$. When $n = g + 1$, in the large N limit both sides of holographic identity diverges. The case of $n = g + 1 = 2$ was analyzed in detain in [5]. It would be interesting to extend this analysis to arbitrary $n = g + 1 > 2$.

4.5 Correlators of local operators

We start with a general discussion of correlators in the context of 2d/3d (R)CFT/TQFT correspondence. In this case a conformal block involving primary operators $V_i(w_i)$ characterized by quantum numbers $h_i, \bar{h}_i$ is evaluated by the 3d path integral on $\Sigma \times [0, 1]$ with the line operators ending at the points of operator insertions w_i. When boundary is a torus, this can be written explicitly as

$$\chi_c^{h_i}(\tau, w_i) = \mathrm{Tr}_{\mathcal{H}_c^{2d}}\left(e^{-\beta H}\prod_i V_i(w_i)\right). \tag{4.65}$$

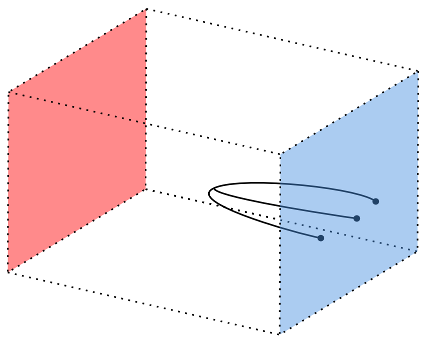

Figure 3. 2d CFT conformal block with the operator insertions, given for Abelian theory by (4.70), is given by TQFT on $\Sigma \times [0,1]$ with the line operators (Wilson lines) ending at the conformal boundary (depicted in blue).

The bulk interpretation of this expression is depicted in Fig. 3 where the TQFT state at the "inner" (pink) boundary is chosen to be $|c\rangle \in \mathcal{H}$, corresponding to the sector of the 2d CFT Hilbert space $\mathcal{H}^c_{2d}$ traced over in (4.65). Since the line operators do not end at the inner boundary, by keeping the boundary conditions there arbitrary, the bulk path integral can be understood as a "boundary" state $\langle \mathcal{B}(h_i, w_i)|$ in the dual to the Hilbert space $\mathcal{H}^g$ of the TQFT on $\Sigma \times \mathbb{R}$,

$$\chi^{h_i}_c(\Omega, w_i) = \langle \mathcal{B}(h_i, w_i)|c\rangle. \tag{4.66}$$

Choosing instead the topological boundary conditions $\mathcal{C}$ at the inner boundary would yield the correlator in the 2d CFT specified by $\mathcal{C}$,

$$\langle \prod_i V_i(w_i)\rangle_{\mathcal{C}}\, Z_{\mathcal{C}}(\Omega) = \langle \mathcal{B}(h_i, w_i)|\mathcal{C}\rangle. \tag{4.67}$$

We note that that in general $\chi^{h_i}_c(\Omega, w_i)$ and even $\langle \prod_i V_i(w_i)\rangle_{\mathcal{C}}$ are not necessarily single-valued function of the insertion points w_i. The latter would require V_i to be local with respect to c, or be present in the spectrum of $Z_{\mathcal{C}}$. In this case line operators can be easily pulled into, and back from, the inner boundary without any ambiguity.

The statement of ensemble holography discussed in section 4 ensures that the sum over topologies is equal to weighted average of topological boundary conditions

$$\sum_{\mathcal{C}} \alpha_{\mathcal{C}} |\mathcal{C}\rangle = |\Psi_{\mathrm{TQFT\,gravity}}\rangle. \tag{4.68}$$

Weights $\alpha_{\mathcal{C}}$ are positive and have probabilistic interpretation. Evaluating the scalar product of (4.68) with $\langle \mathcal{B}(h_i, w_i)|$ we immediately find that the statement of TQFT gravity being dual to the boundary ensemble extends to include correlators of primary operators, as was mentioned without explanation in [55].

The fact that the equality between bulk and boundary partition functions on any Σ (4.3) extends to include correlators is not that surprising. A CFT partition function on a higher-genus Riemann surface "knows" about the correlation functions, which can be probed at the singular points of the moduli space $\mathcal{M}_g$ when Σ factorizes. So far the holographic equality holds for any Ω, it is natural to expect that it includes the correlators of primaries.

Two comments are in order. First, the boundary ensemble includes exactly the same theories no matter we evaluate the partition function with or without the operator insertions. The operators V_i may not be local for some of the theories in the ensemble, hence the resulting correlator will have brunch cuts. Second, the sum over topologies includes exactly the same terms independently of V_i. Thus, if the sum includes only the handlebodies and hence can be written as an average over the mapping class group of Σ, which is e.g. the case for the theory (2.5) with a square-free N [4, 5], the same will be true after V_i insertions. An example of such a calculation can be found in [89]. This is different from the proposal of [90] which includes the average over the mapping class group of Σ with punctured points w_i. We discuss this difference in more detail below.

We illustrate the discussion with an example of a general Abelian 3d CS theory (2.3) assuming for simplicity $c_L = c_R$. (The result can be easily extended to any theory admitting topological boundary conditions.) The matrix K_{ij} is the Gram matrix of some lattice Λ, while the line operators (Wilson lines) are labeled by the elements of the discriminant group $c \in \Lambda^*/\Lambda$. Topological boundary conditions (even self-dual codes) $\mathcal{C}$ parameterize all possible Narain lattices $\Lambda_{\mathcal{C}}$ satisfying $\Lambda \subset \Lambda_{\mathcal{C}} \subset \Lambda^*$ [26]. Wilson lines that end at the boundary are not subject to identifications modulo elements of Λ and are parameterized by arbitrary vectors $\vec{k} = (\vec{k}_L, \vec{k}_R) \in \Lambda^*$ that play the role of quantum numbers $h, \bar{h}$. In fact corresponding vertex operators $V_i =: e^{ikX}$: have conformal left and right conformal weights $h = k_L^2/2$, $\bar{h} = k_R^2/2$.

Without any insertions the wavefunctions of the 3d theory quantized on $\Sigma \times R$ (non-analytic conformal blocks of 2d Narain CFT) are given by sums over shifted lattices Λ_c, which includes all vectors of the form $\vec{v} + \vec{c}$, for all $\vec{v} \in \Lambda$ [5, 26],

$$\chi_c(\Omega, \xi, \bar{\xi}) = \langle \Omega, \xi, \bar{\xi} | c \rangle. \tag{4.69}$$

For the AB theory (2.5) these are the wavefunctions (2.27). Focusing on the case of a torus, with the vertex operator insertions the block takes the form [91, 92]

$$\begin{aligned} \chi_c^{k_i}(\tau, z_i, \xi, \bar{\xi}) &= \langle \tau, \xi, \bar{\xi}, V_i(w_i) | c \rangle \\ &= \chi_c(\tau, \xi', \bar{\xi}') e^{-\frac{\pi}{2\tau_2}(w_L^2 + w_R^2)} \prod_{i<j} E(w_{ij}|\tau)^{k_L^i \cdot k_L^j} \overline{E(w_{ij}|\tau)}^{k_R^i \cdot k_R^j}, \end{aligned} \tag{4.70}$$

$$\xi' = \xi + w_L, \quad \bar{\xi}' = \bar{\xi} + w_R, \quad w_L \equiv \sum_i k_{L,i}\, w_i, \quad w_R \equiv \sum_i k_{R,i}\, \bar{z}_i, \quad w_{ij} \equiv w_i - w_j$$

$$\chi_c(\tau, \xi, \bar{\xi}) = \frac{\sum_{(p_L, p_R) \in \Lambda_c} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2 + 2\pi i (p_L \xi - p_R \bar{\xi}) + \pi(\xi^2 + \bar{\xi}^2)/\tau_2}}{|\eta(\tau)|^{2n}}. \tag{4.71}$$

Here w_i parameterize the points on the torus, where the vertex operators are inserted and $\sum_i k_i = 0$ lest the correlator vanishes. A simple dependence on c in (4.70) is a specific feature of the Abelian theory. The function

$$E(w|\tau) = \frac{\theta_1(w|\tau)}{\theta_1'(0|\tau)} \tag{4.72}$$

is called the prime form and its related to the Green's function of the scalar Laplacian, $G(w) = -\ln|E(w|\tau)| + \frac{2\pi(\mathrm{Im}(w))^2}{\tau_2}$ [93].

Once the ensemble-averaged correlator of $\prod_i V_i(w_i)$ is evaluated,

$$\sum_{\mathcal{C}} \langle \prod_i V_i(w_i) \rangle_{\mathcal{C}} Z_{\mathcal{C}}(\tau, \xi, \bar{\xi}) = \langle \tau, \xi, \bar{\xi}, V_i(w_i) | \Psi_{\mathrm{TQFT\,gravity}} \rangle, \tag{4.73}$$

the correlators of currents, and hence U(1) descendants, can be probed by considering a kinematic limit of w_i merging pair-wise.

We note that since V_i has positive conformal dimension, modular transformations will introduce a scalar prefactor, c.f. (2.23),

$$\langle \tau, \xi, \bar{\xi}, V_i(w_i) | U_\gamma^\dagger = (c\tau + d)^h (c\bar{\tau} + d)^{\bar{h}} \langle \tau_\gamma, \xi_\gamma, \bar{\xi}_\gamma, V_i(\gamma w_i)|, \tag{4.74}$$

$$\tau_\gamma = \frac{a\tau + b}{c\tau + d}, \quad \xi_\gamma = \frac{\xi}{c\tau + d}, \quad \bar{\xi}_\gamma = \frac{\bar{\xi}}{c\bar{\tau} + d}, \quad \gamma w_i = \frac{w_i}{c\tau + d}, \tag{4.75}$$

$$h = \frac{1}{2} \sum_i (k_L^i)^2, \quad \bar{h} = \frac{1}{2} \sum_i (k_R^i)^2. \tag{4.76}$$

Now, whenever the averaged 2d partition function is given just by the handlebody contributions, i.e. by the vacuum Poincare series,

$$\sum_{\mathcal{C}} Z_{\mathcal{C}}(\tau, \xi, \bar{\xi}) = \kappa \sum_{\gamma \in \Gamma_0 \backslash SL(2,\mathbb{Z})} \chi_0(\tau_\gamma, \xi_\gamma, \bar{\xi}_\gamma), \tag{4.77}$$

the same will apply to the correlators, with the Poincare series taking form

$$\sum_{\mathcal{C}} \langle \prod_i V_i(w_i) \rangle_{\mathcal{C}} Z_{\mathcal{C}}(\tau, \xi, \bar{\xi}) = \kappa \sum_{\gamma \in \Gamma_0 \backslash SL(2,\mathbb{Z})} (c\tau + d)^h (c\bar{\tau} + d)^{\bar{h}} \chi_0^{k_i}(\tau_\gamma, \gamma w_i, \xi_\gamma, \bar{\xi}_\gamma). \tag{4.78}$$

Note that the sum must explicitly include the pre-factor $(c\tau + d)^h (c\bar{\tau} + d)^{\bar{h}}$ because the character $\chi_0^{k_i}$ should be understood as a section of a line bundle.

If instead of the mapping class group of the torus $SL(2,\mathbb{Z})$ the vacuum character $\chi_0^{k_i}(\tau_\gamma, \gamma w_i, \xi_\gamma, \bar{\xi}_\gamma)$ were to be averaged over the mapping class group of the torus with punctured points, the result would be different. Namely, the boundary ensemble will only include theories for which all operators V_i are local. More generally, as in the case without operator insertions, the consistent procedure would be to average over the mapping class group of the punctured Riemann surface Σ, starting from an infinitely large genus and then obtaining the result for finite g via genus reduction. This would lead to an average over the boundary ensemble that includes only theories for which all V_i are local. The relative weights $\alpha_{\mathcal{C}}$, which are non-trivial in the non-Abelian case, would remain the same as in the case without insertions [78].

Extending this picture to 4d/5d is straightforward. Surface operators of the B_2 and C_2 fields in the bulk with charges k – vectors in the symplectic lattice (4.13) – ending on a closed but not necessarily connected contour $\Gamma = \cup_i \Gamma_i$ at the boundary will be calculating correlators of the Wilson lines $W_i^{\mathrm{k}_i}$ of the gauge field A_μ and its dual A_μ^D over Γ_i. The conformal blocks with Wilson operator insertions

$$\langle E, W_i^{\mathrm{k}_i}(\Gamma_i), \xi, \bar{\xi}|\mathrm{c}\rangle \tag{4.79}$$

will have a form similar to (4.70), namely, a conformal block without insertions multiplied by a c-independent factor that depends on the contour Γ and the Wilson lines charges. It can be written explicitly in terms of the Green's function for a vector Laplacian on M_4. Similarly to the kinematic limit of w_i merging pairwise, taking the boundary contour Γ_i to encircle a small region will lead to a correlation function with $F_{\mu\nu}$ or $F_{\mu\nu}^D$ in the pre-exponent.

5 4d $\mathcal{N} = 4$ SYM

In view of the ensemble holography picture developed above, an obvious question is whether similar results might apply to the case of 4d $\mathcal{N} = 4$ SYM. We postpone this discussion until section 6, while here we focus on a different question. Namely, which $\mathcal{N} = 4$ SYM theory — U(N) or SU(N) — is IIB String Theory on $AdS_5 \times \mathbb{S}^5$ dual to? This question has been extensively discussed in the literature, with strong arguments in favor of both the $su(N)$ [9, 28, 62, 94] and U(N) [11, 19] scenarios. More recent works mention that both scenarios are possible without providing details [95]. As we explain below, the choice of gauge group is specified by the boundary conditions for the bulk fields B_2, C_2.

We begin our analysis with the SymTFT construction for the family of $\mathcal{N} = 4$ field theories with gauge algebra $su(N)$ on a simply connected spin four-manifold M_4, coupled to the 5d theory (2.1) on $M_4 \times \mathbb{R}$ [9, 25, 29, 96]. This coupling defines a particular boundary state $\langle \mathcal{B}_{su(N)}|$ such that

$$Z_{ab}^{su(N)} = \langle \mathcal{B}_{su(N)}|ab\rangle_{5d} \tag{5.1}$$

are the conformal blocks of the 4d theory. These are partial sums over the 4d QFT Hilbert space, that includes only states with particular values of electric and magnetic charges. With a proper N-dependent normalization the latter are vectors in the weight lattice of $su(N)$. Values of $a, b \in \mathbb{Z}_N$ specify the charges modulo vectors of the root lattice [28, 76]. As was discussed in section 3, partition functions of all possible theories with the $su(N)$ gauge algebra and different gauge groups are given by the topological boundary conditions of the SymTFT,

$$Z_{\mathcal{L}}^{su(N)} = \langle \mathcal{B}_{su(N)}|\mathcal{L}\rangle = \sum_{\mathrm{c}\in\mathcal{L}} Z_{\mathrm{c}}^{su(N)}, \qquad \mathrm{c} \in (\mathbb{Z}_N \times \mathbb{Z}_N)^n. \tag{5.2}$$

Similarly, the SymTFT construction for $\mathcal{N} = 4$ $u(1)$ theory is a slight generalization of the Maxwell theory case considered above. The corresponding conformal blocks

$$Z_{ab}^{u(1)} = \langle \mathcal{B}_{u(1)}|ab\rangle_{5d} \tag{5.3}$$

are essentially the same as in Maxwell theory (3.1,2.31), modulo the contributions of the superpartners. These are sums over states with electric and magnetic charges, which after proper normalization are equal to a, b mod N. The analog of (5.2) are the partition functions of $\mathcal{N} = 4$ $U(1)$ theories with different values of coupling as given by (4.38) and its generalizations.

In what follows we will specify the gauge algebra when we want to emphasize that we consider all conformal blocks of the form (5.1,5.3). The gauge group U(1) or SU(N) will refer to a particular absolute theory, i.e. a particular combination of these conformal blocks.

The field theory with gauge group U(N) can be understood in terms of $u(1)$ and $su(N)$ theories as follows. At the group level $\mathrm{U}(N) = (\mathrm{SU}(N) \times \mathrm{U}(1))/\mathbb{Z}_N$, hence the U($N$) theory is a basically a product U(1) and SU(N) theories with the diagonal $\mathbb{Z}_N$ gauged. The resulting theory can described in terms of a SymTFT construction as follows. We consider the U(1) theory with coupling $\tau_0 = N\tau$ on M_4 and couple it to the SymTFT (2.1) with $g = 1$ and the fields B_2, C_2 living in $M_4 \times [0, 1]$. The resulting wavefunctions in the bulk are the conformal blocks of (5.3). We then consider the SU(N) theory with coupling τ' on M_4 and couple it to another SymTFT on $M_4 \times [0, 1]$ with fields B_2', C_2'. The corresponding wavefunctions are as in (5.1). So now the bulk includes two copies of $\mathbb{Z}_N$ gauge theory, with B_2, C_2 in the fundamental and B_2', C_2' in the anti-fundamental representation of $SL(2, \mathbb{Z})$. To gauge the center $\mathbb{Z}_N$ we cap the cylinder $M_4 \times [0, 1]$ with the topological boundary condition corresponding to the diagonal invariant. This is depicted in the left panel of Fig. 4. Using the (un)folding trick, an equivalent construction can be obtained by placing the $U(1)$ and SU(N) field theories on opposite ends of the cylinder, with just a single B_2, C_2 theory living inside. This arrangement is depicted in the right panel of Fig. 4, and it leads to the following expression for the partition function after gauging

$$Z_{\frac{\mathrm{U}(1)\times\mathrm{SU}(N)}{\mathbb{Z}_N}} = \sum_{a,b\in\mathbb{Z}_N^n} \langle \mathcal{B}^{\tau}_{u(1)}|a, b\rangle \langle \mathcal{B}^{\tau'}_{su(N)}|a, b\rangle = \sum_{a,b\in\mathbb{Z}_N^n} \langle \mathcal{B}^{\tau}_{u(1)}|a, b\rangle \langle a, b|\mathcal{B}^{-\bar{\tau}'}_{su(N)}\rangle. \tag{5.4}$$

In the expression above, instead of the $su(N)$ conformal blocks with coupling τ' we used the conjugate conformal blocks with coupling $-\bar{\tau}'$. Taking $\tau = -\bar{\tau}'$ will yield the conventional $\mathcal{N} = 4$ U(N) gauge theory

$$Z_{\mathrm{U}(N)} = \sum_{a,b\in\mathbb{Z}_N^n} Z^{u(1)}_{ab}\, \bar{Z}^{su(N)}_{ab}. \tag{5.5}$$

Distinct $\tau \neq -\bar{\tau}'$ corresponds to an exactly marginal deformation.

We now turn to the bulk description, and consider the low energy limit of IIB supergravity on $X_{\text{bulk}} \times \mathbb{S}^5$, where X_{bulk} behaves as AdS_5 near the boundary $M_4 = \partial X_{\text{bulk}}$. In this limit, at the level of the bulk action the topological sector described by (2.1) decouples from the other fields [9]. At the same time, the extended objects – the fundamental string F1 and the D1-brane that are holographically dual to Wilson line and 't Hooft line operators respectively [76, 97, 98] – are still charged under B_2 and C_2 correspondingly. Assuming that X_{bulk} is topologically the same as the handlebody X_5 used to define the

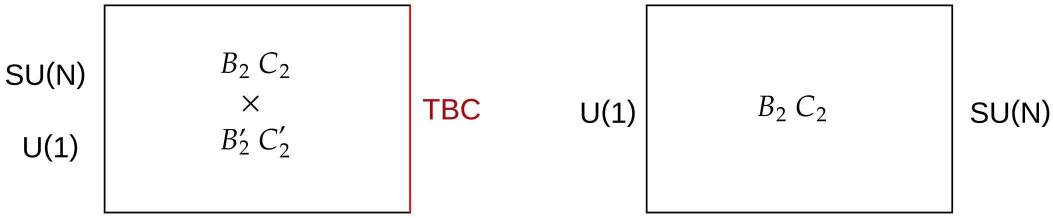

Figure 4. Gauging the center $\mathbb{Z}_N$ of $\mathrm{U}(1)\times\mathrm{SU}(N)$ theory in terms of SymTFT construction.

basis $|a,b\rangle_{5d}$, the semiclassical gravity configuration dual to the conformal block $\bar{Z}^{su(N)}_{ab}$ will include F1 and D1 branes wrapping n two-cycles of M_4 that are not shrinkable inside $X_{\rm bulk}$, such that the numbers of F1s and D1s are equal to a and b mod N. If the two-cycles of M_4 shrinkable inside $X_{\rm bulk}$ and X_5 are not the same, these configurations will contribute to a conformal block $\langle a,b|\mathcal{B}_{su(N)}\rangle$ evaluated in a different basis. The boundary partition function will include a sum over all a,b and the result will be basis-independent. If there are several semiclassical bulk geometries $X_{\rm bulk}$ satisfying the supergravity equations of motion that can end on M_4, which is often the case [99], the contributions of all such $X_{\rm bulk}$ should be summed over.

The important point, since the F1- and D1-branes are charged under B_2 and C_2, is that the semiclassical bulk calculation will automatically evaluate

$$\sum_{a,b\in\mathbb{Z}_N}\langle\mathcal{B}|ab\rangle_{5d}\,\bar{Z}^{su(N)}_{ab}, \tag{5.6}$$

where the state $\langle\mathcal{B}|$ depends on the boundary conditions (the quantization scheme) for the bulk fields B_2, C_2. This is in fact completely in parallel with the SymTFT construction discussed above. The boundary conditions for the topological sector define the boundary state $\mathcal{B}$ and the topological fields B_2, C_2 "live" in the near-boundary region, while the rest of the fields and branes inside the bulk are combined into $\mathcal{B}_{su(N)}$. The resulting picture is the same as in the right panel of Fig. 4, as discussed in detail in [100].

Assuming we quantize the topological sector via the holomorphic quantization prescription of section 2.1, e.g. by adding corresponding boundary terms as was suggested in [19], complemented by $\mathcal{N}=4$ SUSY-preserving boundary conditions for the superpartners,[7] the boundary state will be $\mathcal{B}_{u(1)}$ from (5.3). In this case, the semiclassical bulk partition function, which includes a sum over all a,b sectors,

$$Z_{\rm bulk}=\sum_{a,b\in\mathbb{Z}_N^n}\langle\mathcal{B}_{u(1)}|ab\rangle_{5d}\,\bar{Z}^{su(N)}_{ab} \tag{5.7}$$

will evaluate the partition function of the $\mathrm{U}(N)$ theory (5.4), as was pointed out in [11].

[7] The bosonic superpartners of the U(1) gauge field are 6 scalars ϕ_I. In $AdS_5\times\mathbb{S}^5$ they are dual to non-normalizable modes of the warp factor h, $ds^2_{10d}=h^{-1/2}dx_\mu^2+h^{1/2}dX_I^2$. The latter satisfies $\nabla^2_X h=-(2\pi)^4 g_2\alpha'^2 N\delta^6(X)$. The corresponding modes are due to the collective movement of the D3 branes, that shift the location of the delta-function singularity.

We note that the value of the $u(1)$ gauge field coupling constant τ will be fixed by the value of the boundary term (B.37), and should be tuned to agree with the coupling constant of the $su(N)$ gauge theory, which is determined by the value of the axion-dilaton bulk fields near the boundary [101]. Another way to look at the resulting construction is to say that the boundary terms describes a dynamical 4d U(1) theory, which is coupled to IIB supergravity via the topological B_2, C_2 fields [95].[8]

Alternatively one can impose topological boundary conditions on the bulk fields at $M_4 = \partial X_{\text{bulk}}$, as discussed in section 3.1. In such a case $\langle \mathcal{B}| = \langle \mathcal{L}|$ and (5.7) will reduce to $Z_{\text{bulk}} = Z_{\mathcal{L}}^{su(N)}$. In other words, imposing a particular topological boundary condition will result in the bulk being dual to a particular theory with $su(N)$ gauge algebra.

Thus, depending on the boundary conditions for the fields B_2, C_2 in the bulk, e.g. (2.34) or (3.18), or equivalently on the quantization scheme (i.e. the choice of polarization – holomorphic or real), IIB supergravity in the bulk will be dual to the U(N) theory or to a particular $su(N)$ theory at the boundary.

The above discussion was in the low-energy limit, in which the topological sector decouples at the level of the bulk action. If we now include corrections of higher order in α', the two-form fields B_2, C_2 will acquire kinetic terms and the corresponding sector will no longer be strictly topological. As was shown in [11], the low-energy limit of this theory (without any additional boundary terms) is equivalent to the topological theory (2.1) in *holomorphic* quantization. In this case the value of τ entering the explicit form of the bulk wavefunctions (2.19, 2.31) is specified by the kinetic terms, and will automatically be the same as the value of the axion-dilaton at the boundary. Hence the most straightforward quantization of IIB supergravity yields the gravity dual of the U(N) theory [11, 19]. Yet even if higher derivative corrections are included, or even in the full IIB String Theory, all the other scenarios we have described are still possible. We note that imposing e.g. $B_2 = 0$ at the boundary when kinetic terms are present will only enforce $\operatorname{Im}\xi = \operatorname{Im}\bar{\xi} = 0$ within holomorphic quantization. Instead, to impose topological boundary conditions, one should take into account that the basis of states $\langle \tau, E, \xi, \bar{\xi}|$ within $\mathcal{H}^*$ is over-complete; hence any state $\langle \mathcal{B}| \in \mathcal{H}^*$ imposing appropriate boundary conditions is possible. In practice this means that to impose a topological boundary condition $\mathcal{L}$, one would need to integrate the boundary values of the bulk field over $d\xi\, d\bar{\xi}$ against the kernel $\langle \mathcal{L}|\tau, E, \xi, \bar{\xi}\rangle$.

We end by noting that more exotic boundary conditions are possible. Consider for example the $\mathcal{N} = 4$ SYM theory on M_4, or equivalently IIB String Theory in the bulk ending on M_4, coupled to an *auxiliary* 5d topological theory (2.1) with fields $\tilde{B}_2, \tilde{C}_2$. This auxiliary theory will be living in an "auxiliary" 5d bulk, so the resulting construction has two 5d bulks – the conventional one and the auxilary one ending on the same 4d manifold M_4, where we require $\tilde{B}_2 = B_2, \tilde{C}_2 = C_2$.[9] Now if we sum over all possible topologies of the auxiliary bulk, thus promoting the auxiliary 5d theory to a gravitational TQFT, then as explained in section 4 the resulting state of the $\tilde{B}_2, \tilde{C}_2$-theory will be a sum $\sum_{\mathcal{L}} |\mathcal{L}\rangle$ over

[8] We thank O. Aharony for discussions on this point.

[9] A similar construction with two bulks was recently discussed in [102]. There, in contrast with our setup, the topology of the auxiliary bulk is fixed.

all topological boundary conditions. This means that our construction with two bulks is holographically dual to an ensemble of all $\mathcal{N}=4$ $su(N)$ theories

6 Conclusions

In this paper we have formulated a holographic duality between a model of 5d topological gravity — 5d Abelian TQFT (2.1) summed over all 5d topologies sharing the same boundary – and an ensemble of 4d Maxwell theories living on the boundary manifold. The duality implies the equivalence of partition functions, as well as of correlators of local operators, primary with respect to the symmetry algebra defined by the bulk TQFT, as discussed in section 4.5. This duality is a direct extension of the duality between the 3d Abelian Chern-Simons theory with compact gauge group summed over topologies and an ensemble of 2d Narain CFTs at the boundary [4, 5]. In the large N limit we find that the boundary theories densely cover the space of gauge couplings with the canonical measure. The average partition function is well-defined when the 4d central charge is sufficiently large and is given by the Eisenstein series of the orthogonal group $\mathrm{O}(n, n, \mathbb{Z})$. This is a version of the Siegel-Weil formula which made an appearance previously in the 2d/3d context [2, 3, 86] and many years earlier in the context of multiloop string amplitudes [82, 83]. The bulk sum over topologies in the $N \to \infty$ limit includes only handlebody geometries, suggesting the bulk theory becomes semiclassical. We have also shown that the holographic duality of both the 4d/5d and 2d/3d cases extends to correlators of U(1)-primaries and their descendants, as discussed in section 4.5.

The established lore suggests that ensemble holography is a feature of lower dimensions when there is no dynamical graviton. Our bulk theory is a topological theory of gravity and we see no qualitative difference between lower and higher dimensions. In our setting both scenarios, with a boundary ensemble and with a unique boundary theory, are equally valid. Which of these two scenarios is realized is determined by the properties of the bulk TQFT. If the bulk theory is topologically trivial, corresponding to level $N = 1$, the boundary ensemble includes only one theory.

To bridge the gap between topological and conventional gravity in the bulk, we need to address two different but related questions. The first question is how to extend our setup to include a dynamical graviton in the bulk. The second is to understand how our setup connects to the semiclassical gravity regime. In what follows we focus on the 2d/3d case, which is simpler.

With regard to the first question, an obvious limitation of our setup is that the classical sources J in (4.1) couple only to boundary currents of the $\mathrm{U}(1)^n \times \mathrm{U}(1)^n$ symmetry, while the stress-energy tensor is built out of these currents by the Sugawara construction. To introduce classical sources for $T_{\mu\nu}$ directly, the first step would be to extend the bulk TQFT to include a line operator for each conformal primary. The resulting Virasoro TQFT (VTQFT) could potentially describe, in the sense of the sandwich construction, any 2d conformal theory. Accordingly, a gravitational theory that sums the VTQFT over all 3d topologies will be dual to a weighted unitary ensemble of all 2d CFTs. That is schematically the same as what happens in pure 3d quantum gravity, see [103–108] for related developments, although in that case the result is divergent and requires regularization. This divergence could be more than a technical problem — different regularization schemes might lead to either scenario, with or without an ensemble, underlying the notion that both are equally valid. In order to break conformal symmetry and couple external sources to the stress tensor directly, one should go beyond VTQFT and introduce a line operator for each state in the boundary QFT Hilbert space.

Another important question to address is the semiclassical gravity limit. In 3d we expect it to emerge "automatically" from the VTQFT summed over 3d topologies, in the limit of large central charge. In fact we have already seen an avatar of this behavior in the Abelian case, where for $N \to \infty$ the bulk sum includes only handlebody geometries, as in the semiclassical gravity case [109]. This is in contrast to the trivial $N = 1$ theory that does not differentiate between topologies. The sum over handlebodies as $N \to \infty$ does not imply, however, that the gravitational bulk theory admitting a semiclassical regime is necessarily dual to an ensemble. As we discussed in [110] and section 5 of [5] the topological bulk theory dual to a single, *typical* Narain CFT can be recast in a form that includes a "semiclassical" sum over handlebodies, although this representation is far from unique.

To summarize the discussion above, at this point we do not see any qualitative difference between lower and higher dimensional models of holography. Both types of holographic duality, involving either a single boundary theory or an ensemble, can exist in higher dimensions. The ensemble interpretation seems more general; it reduces to the single-theory scenario for special choices of the bulk theory.

In addition to establishing the 4d/5d holographic duality between Maxwell theories and 5d Abelian TQFTs, in section 3 this paper develops a connection between 5d Abelian TQFTs and codes. We briefly summarize this connection here.

- Non-anomalous subgroups of the 2-form symmetry group in 5d are parameterized by classical symplectic codes. Maximal non-anomalous subgroups are in one-to-one correspondence with symplectic self-dual codes $\mathcal{L}$. The TQFT states defined by topological boundary conditions are quantum stabilizer states $|\mathcal{L}\rangle$ defined in terms of classical codes $\mathcal{L}$ via the CSS construction. This is an extension of the 3d story, where the underlying codes are even [26].

- Translating this result into the language of anyons (topological defects), and working in any number of dimensions, the Hilbert space of a theory obtained via (partial) anyon condensation in an Abelian TQFT is a quantum stabilizer code of CSS type, parameterized by a self-orthogonal classical code. The meaning of self-orthogonality, i.e. the choice of the inner product, depends on the dimension. For 3d theories the corresponding codes are even, for 5d theories they are symplectic.

- Up to an overall normalization, the path integral of the 3d Abelian TQFT (2.5) on any 3d manifold with boundary is a stabilizer state $|\mathcal{L}\rangle$ specified by a classical symplectic self-dual code $\mathcal{L}$, i.e. a state of the 5d theory (2.1) defined by a topological boundary condition. (Here we invoke the isomorphisms between the Hilbert spaces $\mathcal{H}_\Sigma$ and $\mathcal{H}_{M_4}$ of the 3d and 5d theories.) Similarly, up to an overall normalization, the path integral of the 5d theory on any topology with boundary is a stabilizer state $|\mathcal{C}\rangle$ defined by a classical even self-dual code $\mathcal{C}$, which is also a state defined by a topological boundary condition in the 3d theory. The intriguing relation between topologies in 3d/5d and topological boundary conditions in 5d/3d begs for a geometric interpretation in terms of the 7d 3-form theory.

As a spin-off development we clarified the holographic dictionary between the gauge group of 4d $\mathcal{N}=4$ SYM theory and the boundary conditions of the IIB String Theory fields B_2, C_2 in the bulk. As discussed in section 5, holomorphic quantization of these fields, or equivalently the self-dual boundary condition (2.34) and its generalization to non-zero $\xi, \bar{\xi}$, will yield the bulk dual of the U(N) theory. Imposing a topological boundary condition for B_2, C_2 instead will result in a particular $su(N)$ theory.

A Modular and orthogonal transformations

Modular transformations of $|(\alpha,\beta)\rangle_{3d}$ are specified for the generators of the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ (mapping class group of Σ), mapped to $\mathrm{Sp}(2g,\mathbb{Z}_N)$,

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z}_N) \tag{A.1}$$

that preserves the intersection matrix of one-cohomologies on Σ

$$J = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}. \tag{A.2}$$

For invertible $A \in GL(g,\mathbb{Z})$ and $D = (A^{-1})^T$, $B = C = 0$ the transformation is simple

$$U_\gamma |(\alpha,\beta)\rangle_{3d} = |A^{-1}(\alpha,\beta)\rangle_{3d}, \tag{A.3}$$

where A is acting on α_I and β_I as on fundamental vectors.

For $A = D = \mathbb{1}_g$, $C = 0$ and integer symmetric B – a generalization of T-generator of $SL(2,\mathbb{Z})$ – action on basis elements is a pure phase

$$U_\gamma |(\alpha,\beta)\rangle_{3d} = e^{-\frac{2\pi i}{N}\alpha^T B \beta} |(\alpha,\beta)\rangle_{3d}. \tag{A.4}$$

Finally, $\gamma = -J \in \mathrm{Sp}(2g,\mathbb{Z})$ acts by the Fourier transform

$$U_\gamma |(\alpha,\beta)\rangle_{3d} = \frac{1}{N^{gn/2}} \sum_{\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}_N^{gn}} e^{\frac{2\pi i}{N}(\alpha\tilde{\beta}+\tilde{\alpha}\beta)} |(\tilde{\alpha},\tilde{\beta})\rangle_{3d}. \tag{A.5}$$

Similarly generators of the orthogonal group $\mathrm{O}(n,n,\mathbb{Z})$ that preserves the intersection form η (2.4) of two-cohomologies on M_4 is mapped to

$$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{O}(n,n,\mathbb{Z}_N) \tag{A.6}$$

which acts on $|(a,b)\rangle_{5d}$ as follows. For invertible $A \in GL(g,\mathbb{Z})$ and $D = (A^{-1})^T$, $B = C = 0$ the transformation is simple

$$U_h|(a,b)\rangle_{5d} = |(a,b)A^{-1}\rangle_{5d}, \tag{A.7}$$

where A is acting on a_i and b_i as on fundamental co-vectors.

The generator $h = \eta \in \mathrm{O}(n,n,\mathbb{Z})$ acts by the Fourier transform

$$U_h|(a,b)\rangle_{5d} = \frac{1}{N^{gn/2}} \sum_{a',b' \in \mathbb{Z}_N^{gn}} e^{\frac{2\pi i}{N}\mathrm{Tr}(a^T b' - b^T a')}|(a',b')\rangle_{5d}. \tag{A.8}$$

For $A = D = \mathbb{1}_n$, $C = 0$ and integer antisymmetric B action on the basis elements is a pure phase

$$U_h|(a,b)\rangle_{5d} = e^{-\frac{2\pi i}{N}\mathrm{Tr}(a^T B b)}|(a,b)\rangle_{5d}. \tag{A.9}$$

B Quantization and the dimensional reduction of 7d theory

B.1 2d geometry and 3d Chern-Simons theory

We start with geometric preliminaries. A Riemann surface Σ of genus g admits a basis of real-valued one-forms $\omega_I^{(1)}$, $I = 1\dots 2g$ with the canonical intersection form

$$\int_\Sigma \omega_I^{(1)} \wedge \omega_J^{(1)} = J_{IJ}, \qquad J = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}, \tag{B.1}$$

such that first g forms $\omega_I^{(1)}$ are dual to "a"-cycles, and $\omega_{I+g}^{(1)}$ are dual to "b"-cycles. Next we introduce the metric

$$\int_\Sigma \omega_I^{(1)} \wedge \star\omega_J^{(1)} = \mathbb{G}_{IJ}, \qquad \mathbb{G} = \mathbb{e}^T\mathbb{e}, \qquad \mathbb{e} = \Omega_2^{-1/2}\begin{pmatrix} -\Omega_1 & 1 \\ \Omega_2 & 0 \end{pmatrix}, \tag{B.2}$$

$$\mathbb{G} = \left(\begin{array}{c|c} \Omega_1\Omega_2^{-1}\Omega_1 + \Omega_2 & -\Omega_1\Omega_2^{-1} \\ \hline -\Omega_2^{-1}\Omega_1 & \Omega_2^{-1} \end{array}\right), \tag{B.3}$$

defined in terms of the modular parameter $\Omega = \Omega_1 + i\,\Omega_2$ of Σ. Here the Hodge star $\star$ is defined such that $\int A \wedge \star A$ is positive-definite for any one-form A, i.e. the holomorphic differentials on Σ will be the $-i$ eigenvectors of $\star$. The metric (B.2) is compatible with the intersection form, i.e. $J^{-1}\mathbb{G} = -\mathbb{G}^{-1}J$.

The holomorphic differentials can be written explicitly as follows,

$$\omega_I = \omega_I^{(1)} + \Omega_{IJ}\,\omega_{g+J}^{(1)}, \quad I,J = 1\dots g. \tag{B.4}$$

A straightforward calculation gives

$$\int_\Sigma \omega_I \wedge \omega_J^* = -2i\,(\Omega_2)_{IJ}. \tag{B.5}$$

The modular group

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}) \tag{B.6}$$

that acts fundamentally on the vector of cohomologies $(\omega_I^{(1)}, -\omega_{g+I}^{(1)})$ acts on Ω in the standard way

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}. \tag{B.7}$$

Now we consider chiral level-N Chern-Simons theory

$$\frac{N}{4\pi} \int \mathcal{A} \wedge d\mathcal{A} \tag{B.8}$$

on $\Sigma \times \mathbb{R}$. This is an auxiliary theory not related to (2.3). To implement the holomorphic quantization we add the boundary term [5]

$$\frac{N}{4\pi} \int_\Sigma \mathcal{A} \wedge \star \mathcal{A} \tag{B.9}$$

and decompose $\mathcal{A}$ into harmonic part and fluctuations [13]

$$\mathcal{A} = \frac{i\pi}{\sqrt{N}} \zeta_I (\Omega_2^{-1})^{IJ} \omega_J^* + \text{c.c.} + \partial\chi. \tag{B.10}$$

With the boundary term (B.9) ζ is fixed at the boundary while ζ^* will be fluctuating freely. The wavefunctions of the model, which are holomorphic functions of ζ (B.8) are well known [11, 13, 16]. Up to a multiplicative factor due to small fluctuations, their explicit form is given by (we only consider trivial spin structure on Σ)

$$\Theta_{c_1 \ldots c_g}(\Omega, \zeta) \;=\; \det(\Omega_2) \sum_{v_1 \ldots v_g} e^{i\pi\, v^T \Omega v + 2\pi i\, v^T \zeta + \pi\, \Omega_2^{-1} \zeta^2/2}. \tag{B.11}$$

Here the sum goes over $v_I = (n_I N + c_I)/\sqrt{N}$, where $n_I \in \mathbb{Z}$ and $c_I \in \mathbb{Z}_N$ parameterize the wavefunction. These wavefunctions are orthogonal with the measure $e^{-\pi\, \Omega_2^{-1} |\zeta|^2}$ inherited from (B.9), and accordingly at the quantum level

$$\zeta^* \to \frac{\Omega_2}{\pi} \frac{\partial}{\partial \zeta}. \tag{B.12}$$

The wavefunctions (B.11) form a representation of the group of Wilson line operators wrapping cycles $\Gamma(n,m)$ of Σ, defined as dual to $\Gamma^\vee(n,m) = n^I \omega_I^{(1)} + m^I \omega_{g+I}^{(1)}$,

$$W_\Gamma \;=\; e^{i \oint_\Gamma A} = e^{N^{-1/2}(n+\Omega m)^T \partial/\partial\zeta - \pi N^{-1/2}(n+\Omega^* m)^T \Omega_2^{-1} \zeta}, \tag{B.13}$$

$$W_\Gamma W_{\Gamma'} \;=\; W_{\Gamma+\Gamma'} e^{\frac{i\pi}{N}(nm' - n'm)}. \tag{B.14}$$

and

$$W_\Gamma \Theta_{c_1 \ldots c_g}(\Omega, \zeta) = \Theta_{c'_1 \ldots c'_g}(\Omega, \zeta) e^{\frac{2\pi i}{N} c \cdot n + \frac{i\pi}{N} n \cdot m}, \tag{B.15}$$

where $c'_I = c_I + m_I \bmod N$.

B.2 4d geometry

Similarly to 2d, a spin 4-manifold M_4 of zero signature admits a basis or real-valued two-forms with the canonical intersection form

$$\int_{M_4} \omega_i^{(2)} \wedge \omega_j^{(2)} = \eta_{ij}, \qquad i,j = 1 \dots 2n, \tag{B.16}$$

where η is defined in (2.4). The metric

$$\int_{M_4} \omega_i^{(2)} \wedge \star\, \omega_j^{(2)} = \mathcal{G}_{ij}, \qquad \mathcal{G} = \Lambda^T \Lambda, \qquad \Lambda = G^{-1/2} \begin{pmatrix} B & 1 \\ G & 0 \end{pmatrix} \tag{B.17}$$

$$\mathcal{G} = \left(\begin{array}{c|c} G - BG^{-1}B & -BG^{-1} \\ \hline G^{-1}B & G^{-1} \end{array} \right), \tag{B.18}$$

can be written in terms of $E = G + B$, the analog of the modular parameter in 2d. In the context of Narain theories positive-definite symmetric G is the metric on the n-dimensional torus, while antisymmetric B is the B-field. The metric (B.17) is compatible with η in the sense that $\eta^{-1}\mathcal{G} = \mathcal{G}^{-1}\eta$, and hence both can be diagonalized simultaneously in terms of the (anti)-self-dual forms $\star\,\omega_i \pm = \pm \omega_i^{\pm}$,

$$\int_{M_4} \omega_i^{\pm} \wedge \omega_j^{\pm} = \pm \delta_{ij}, \qquad i,j, = 1 \dots n. \tag{B.19}$$

Comparing this with (B.5) we find that $\omega_i^{\pm}$ are normalized a bit differently, they diagonalize $\mathcal{G}$ while ω_I does not diagonalize $\mathbb{G}$. They are related to $\omega_i^{(2)}$ as follows, c.f. (B.4),

$$\begin{aligned} \sqrt{2}\, G_{ij}^{1/2} \omega_j^{+} &= \omega_i^{(2)} + E_{ij}\, \omega_{n+j}^{(2)}, \\ \sqrt{2}\, G_{ij}^{1/2} \omega_j^{-} &= \omega_i^{(2)} - E_{ji}\, \omega_{n+j}^{(2)}. \end{aligned} \tag{B.20}$$

B.3 6d geometry and 7d "Chern-Simons" theory

The story in six dimensions is similar to 2d. In what follows we focus on $M_6 = \Sigma \times M_4$, with M_4 a simply-connected spin 4-manifold of signature zero. There are $4gn$ real-valued three-forms with the canonical intersection form (3.6) of size $4gn$,

$$\omega_A^{(3)} = \{\omega_I^{(1)} \wedge \eta_{ij} \omega_j^{(2)},\, \omega_{g+I}^{(1)} \wedge \omega_i^{(2)}\}, \qquad I = 1 \dots g, \quad i,j = 1 \dots 2n. \tag{B.21}$$

These three-forms are analogs of $\omega_I^{(1)}$ in 2d. Next, we introduce a basis of the "anti-self-dual" three-forms, the eigenvectors of the Hodge star analogous to ω_I,

$$\omega_I \wedge \omega_i^{+}, \qquad \omega_I^{*} \wedge \omega_i^{-}, \qquad I = 1 \dots g,\; i = 1 \dots n. \tag{B.22}$$

Comparing with (B.4) we find that correct linear combinations are

$$\omega_A = \left\{ \frac{(G^{-1/2})_{ij}}{\sqrt{2}} (\omega_I \wedge \omega_j^{+} - \omega_I^{*} \wedge \omega_j^{-}),\, \frac{(G^{1/2} - BG^{-1/2})_{ij}\, \omega_I \wedge \omega_i^{+}}{\sqrt{2}} + \frac{(G^{1/2} + BG^{-1/2})_{ij}\, \omega_I^{*} \wedge \omega_i^{-}}{\sqrt{2}} \right\},$$

such that the relation between ω_A and $\omega_A^{(3)}$ is given by (B.4) with the 6d "modular parameter"

$$\begin{aligned} \mathbf{\Omega} &= \Omega_1\, \eta^{-1} + i\, \Omega_2\, \mathcal{G}^{-1}, & \text{(B.23)} \\ \omega_A &= \omega_A^{(3)} + \mathbf{\Omega}_{AB} \omega_{B+\mathrm{g}}^{(3)}, \qquad \mathrm{g} = 2gn. & \text{(B.24)} \end{aligned}$$

The matrix $\mathbf{\Omega}$ determines the metric

$$\int_{M_6} \omega_A^{(3)} \wedge \star\, \omega_B^{(3)} = \mathbf{G}_{AB}, \quad \mathbf{G} = \mathbf{\Lambda}^T \mathbf{\Lambda}, \qquad \mathbf{\Lambda} = \mathbf{\Omega}_2^{-1/2} \begin{pmatrix} -\mathbf{\Omega}_1 & 1 \\ \mathbf{\Omega_2} & 0 \end{pmatrix}, \tag{B.25}$$

which is a straightforward generalization of (B.2).

To quantize the 7d theory (2.2) we add the boundary term (2.11) and introduce holomorphic variables ζ_A via

$$H_3 = \frac{i\pi}{\sqrt{N}} \sum_A \zeta_A (\mathbf{\Omega}_2^{-1})^{AB} \omega_B^* + \text{c.c.} + \partial\chi. \tag{B.26}$$

Then the wavefunction (up to contribution of the fluctuating modes) is given by (B.11),

$$\Theta_{c_1 \dots c_{\mathrm{g}}}(\mathbf{\Omega}, \zeta) \;=\; \det(\mathbf{\Omega}_2) \sum_{v_1 \dots v_{\mathrm{g}}} e^{i\pi\, v^T \mathbf{\Omega}\, v + 2\pi i\, v^T \zeta + \pi\, \mathbf{\Omega}_2^{-1} \zeta^2/2}, \tag{B.27}$$

where $v_A = (N n_A + c_A)/\sqrt{N}$ and the sum is over all $n_A \in \mathbb{Z}$.

To obtain the wavefunctions of section 2.1 we need to change the variables from ζ defined in (B.26) to $\xi, \bar{\xi}$ defined in (2.13),

$$P\zeta = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}, \qquad P = \frac{G^{-1/2}}{\sqrt{2}} \left(\begin{array}{c|c} G + B & 1 \\ \hline -G + B & 1 \end{array} \right). \tag{B.28}$$

Here ζ is a $2n$ by g matrix while $\xi, \bar{\xi}$ are matrices n by g. It is then straightforward to check that $P^T P = \mathcal{G}$, $\mathcal{O} = \Lambda\, \eta$, $u_I = \mathcal{O} v_I$ and (B.27) reduces to (2.19).

One can instead reshuffle (B.21) to

$$\tilde{\omega}_A^{(3)} = \{ -J_{IJ} \omega_J^{(1)} \wedge \omega_i^{(2)}, \omega_I^{(1)} \wedge \omega_{n+i}^{(2)} \}, \qquad I, J = 1 \dots 2g, \quad i = 1 \dots n, \tag{B.29}$$

with corresponding holomorphic differentials being

$$\tilde{\omega}_A = \left\{ \frac{i(\Omega_2^{-1})^{IJ} (G^{1/2})_{ij}}{2\sqrt{2}} (\omega_J \wedge \omega_j^+ - \omega_J^* \wedge \omega_j^-), \frac{i(\Omega_2^{-1})^{JK} (G^{1/2})_{ij}}{2\sqrt{2}} (\Omega_{IJ}^* \omega_K \wedge \omega_j^+ - \Omega_{IJ} \omega_K^* \wedge \omega_j^-) \right\}$$

such that

$$\begin{aligned} \tilde{\mathbf{\Omega}} &= B J^{-1} + i\, G\, \mathbb{G}^{-1}, & \text{(B.30)} \\ \tilde{\omega}_A &= \tilde{\omega}_A^{(3)} + \tilde{\mathbf{\Omega}}_{AB} \tilde{\omega}_{B+\mathrm{g}}^{(3)}, \qquad \mathrm{g} = 2gn. & \text{(B.31)} \end{aligned}$$

Comparing (B.26) with $\mathbf{\Omega}$ substituted by $\tilde{\mathbf{\Omega}}$ and (2.13) we readily find

$$\tilde{P}\zeta = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}, \qquad \tilde{P} = \frac{G^{-1/2}}{\sqrt{2}} \left(\begin{array}{c|c} -\Omega & 1 \\ \hline -\Omega^* & 1 \end{array} \right). \tag{B.32}$$

In particular $\tilde{P}^\dagger \Omega_2^{-1} \tilde{P} = G^{-1}\mathbb{G}$ and

$$2\tilde{P}^T \begin{pmatrix} 0 & \Omega_2^{-1} \\ \Omega_2^{-1} & 0 \end{pmatrix} \tilde{P} = \tilde{\boldsymbol{\Omega}}_2^{-1} = G^{-1}\mathbb{G}, \tag{B.33}$$

such that the wavefunction (B.27) reduces to (2.31).

Finally we discuss how to obtain the 3d theory (2.5) and the 5d theory (2.1) directly from the 7d theory (2.2). To that end one introduces one-forms A^i, B^i and expand

$$H_3 = \sum_{i=1}^{n} A^i \wedge \omega_i^{(2)} + B^i \wedge \omega_{n+i}^{(2)}. \tag{B.34}$$

This yields (2.5) upon the subsitution into (2.2). Accordingly, the boundary term (2.11) will be given by

$$\frac{N}{4\pi} \int_\Sigma \mathcal{G}_{ij} \, (A, B)^i \wedge \star (A, B)^j, \tag{B.35}$$

where (A^i, B^i) is a vector with $2n$ components.

Similarly, the 5d theory (2.1), with an additional boundary term $\frac{N}{4\pi} \int C_2 \wedge B_2$, will follow from the 7d after the substitution

$$H_3 = \sum_{I=1}^{g} B_2^I \, \omega_I^{(1)} + C_2^I \, \omega_{n+I}^{(1)}, \tag{B.36}$$

while the boundary term (2.11) becomes

$$\frac{N}{4\pi} \int_{M_4} \mathbb{G}_{IJ} (B, C)^I \wedge \star (B, C)^J = \frac{N}{4\pi} \int_{M_4} \left| \Omega_2^{-1/2} (C_2 - \Omega \, B_2) \right|^2. \tag{B.37}$$

Comparing (B.34) and (B.36) with (2.13) we readily find (2.15) and (2.16).

C Proof that L is self-dual

Here we show that additive symplectic code L defined in (3.23) is self-dual. In what follows it will be convenient to slightly change the notations of section 3.2 and make the split of codes of length g into $g' = g - \tilde{g}$ and $\tilde{g}$ explicit

$$(a_1, \dots, a_g, b_1, \dots, b_g) \to (a_1, \dots, a_{g'}, b_1, \dots, b_{g'} | a_{g'+1}, \dots, a_g, b_{g'+1}, \dots, b_g).$$

With these notations we introduce the following sets L_0, L_1:

$$(a_1, \dots a_{g'}, b_1, \dots b_{g'}) \in L_0, \quad \text{iff} \quad (a_1, \dots a_{g'}, b_1, \dots b_{g'} | 0, \dots, 0) \in \mathcal{L}_\gamma, \tag{C.1}$$

$$(a_1, \dots a_{g'}, b_1, \dots b_{g'}) \in L_1, \quad \text{iff} \quad (a_1, \dots a_{g'}, b_1, \dots b_{g'} | *, \dots, *) \in \mathcal{L}_\gamma. \tag{C.2}$$

Here $*$ means corresponding element could be arbitrary. Clearly both L_0, L_1 are closed under addition, hence these are additive codes. Trivially, $L_0 \subset L_1$. Since $\mathcal{L}_\gamma$ is symplectic, $L_0 \subset L_1^\perp$. In fact any element $\mathrm{x} \in L_1^\perp$, if completed by zeros to be of the length g, will be orthogonal to any element of $\mathcal{L}_\gamma$. Hence it belongs to $\mathcal{L}_\gamma$, and consequently $\mathrm{x} \in L_0$. This means $L_0 = L_1^\perp$ and L_0 is a symplectic code (i.e. self-orthogonal with respect to symplectic inner product (3.5)).

We similarly introduce R_0, R_1,

$$(a_1, \dots a_{\tilde{g}}, b_1, \dots b_{\tilde{g}}) \in R_0, \quad \text{iff} \quad (0, \dots, 0|a_1, \dots a_{\tilde{g}}, b_1, \dots b_{\tilde{g}}) \in \mathcal{L}_\gamma, \tag{C.3}$$

$$(a_1, \dots a_{\tilde{g}}, b_1, \dots b_{\tilde{g}}) \in R_1, \quad \text{iff} \quad (*, \dots, *|a_1, \dots a_{\tilde{g}}, b_1, \dots b_{\tilde{g}}) \in \mathcal{L}_\gamma, \tag{C.4}$$

and conclude that $R_0 = R_1^\perp$ is symplectic.

Finally we introduce $M_0 \subset R_0, M_1 \subset R_1$ that include all codewords of the form

$$(a_1, \dots, a_{\tilde{g}}, 0, \dots, 0) \in M_0 \quad \text{iff} \quad (0, \dots, 0|a_1, \dots, a_{\tilde{g}}, 0, \dots, 0) \in \mathcal{L}_\gamma, \tag{C.5}$$

$$(a_1, \dots, a_{\tilde{g}}, 0, \dots, 0) \in M_1 \quad \text{iff} \quad (*, \dots, *|a_1, \dots, a_{\tilde{g}}, 0, \dots, 0) \in \mathcal{L}_\gamma. \tag{C.6}$$

Now consider an arbitrary element

$$(\mathrm{x}|\mathrm{y}) \in \mathcal{L}_\gamma. \tag{C.7}$$

It will contribute to the sum in (3.25) if any only if $\mathrm{y} \in M_1$. Furthermore each x can "pair" with exactly $|M_0|$ different elements $\mathrm{y} \in M_1$. Hence $m(\mathrm{x}) = |M_0|$ for any x.

Additive codes M_0, M_1 can be defined as

$$M_0 = \tilde{\mathcal{L}}_0 \cap R_0, \qquad M_1 = \tilde{\mathcal{L}}_0 \cap R_1, \tag{C.8}$$

where $\tilde{\mathcal{L}}_0$ is the self-dual symplectic code that defines $|0\rangle^{\tilde{g}}$,

$$(a_1, \dots, a_{\tilde{g}}, 0 \dots, 0) \in \tilde{\mathcal{L}}_0, \qquad a_i \in \mathbb{Z}_N. \tag{C.9}$$

Let us now consider a bilinear form defined in terms of the canonical symplectic product on $(\mathbb{Z}_N \times \mathbb{Z}_N)^{\tilde{g}}$,

$$(\mathrm{y}_1, \mathrm{y}_0), \tag{C.10}$$

where $\mathrm{y}_1 \in R_1, \mathrm{y}_0 \in \tilde{\mathcal{L}}_0$. To calculate the rank of this form, we note that iff $\mathrm{y}_1 \in \tilde{\mathcal{L}}_0^\perp = \tilde{\mathcal{L}}_0$ the bilinear product is zero for any $\mathrm{y}_0 \in \tilde{\mathcal{L}}_0$, and therefore the rank is $r = |R_1|/|M_1|$. Alternatively, iff $\mathrm{y}_0 \in R_1^\perp = R_0$ the bilinear product is zero for any $\mathrm{y}_1 \in R_1$. As a result

$$r = \frac{|R_1|}{|M_1|} = \frac{|\tilde{\mathcal{L}}_0|}{|M_0|}. \tag{C.11}$$

Our goal now is to calculate the size of L. It consists of all distinct $\mathrm{x} \in L_1$ such that there are $\mathrm{y} \in M_1$ and $(\mathrm{x}|\mathrm{y}) \in \mathcal{L}_\gamma$. For each element of M_1 there are $|L_0|$ different elements x (related by a shift by an arbitrary element from L_0), but different $\mathrm{y}, \mathrm{y}' \in M_1$ related by a shift $\mathrm{y} - \mathrm{y}' \in M_0$ will yield the same x. Hence

$$|L| = \frac{|L_0||M_1|}{|M_0|}. \tag{C.12}$$

Taking into account (C.11) and that $|\tilde{\mathcal{L}}_0| = N^{\tilde{g}}$ we find

$$|L| = \frac{|L_0||R_1|}{N^{\tilde{g}}}. \tag{C.13}$$

Now we go back to $\mathcal{L}_\gamma$ and readily conclude that for each $\mathrm{x} \in L_1$ there are $|R_0|$ possible y (all related by shifts from R_0) such that $(\mathrm{x}, \mathrm{y}) \in \mathcal{L}_\gamma$. Similarly, for any $\mathrm{y} \in R_0$ there are $|L_0|$ different x such that $(\mathrm{x}, \mathrm{y}) \in \mathcal{L}_\gamma$. And therefore

$$|L_1||R_0| = |L_0||R_1| = |\mathcal{L}_\gamma| = N^g. \tag{C.14}$$

Together with (C.13) this means $|L| = N^{g-\tilde{g}}$, which means L is a symplectic code of maximal size, i.e. it is self-dual.

D Codes over $\mathbb{Z}_N$ for $N = p$ and $N = p^2$

D.1 Counting "orthogonal" codes over $\mathbb{Z}_N \times \mathbb{Z}_N$

Let us first calculate the number of codes over $\mathbb{Z}_p \times \mathbb{Z}_p$ (for a prime p) of length n and with k generators, which are even in the sense of (3.4). For $k = 0$ there is a unique such code, consisting of the zero codeword. For $k = 1$ the number is determined by the total number of even non-zero codewords of length n, denoted $\mathcal{N}(n)$ and calculated in [4] see Appendix F there,

$$\mathcal{N}(n) = (p^n - 1)(p^{n-1} + 1) + 1. \tag{D.1}$$

Accordingly the number of codes with $k = 1$ generator is

$$N(k = 1, n, p) = \frac{\mathcal{N}(n) - 1}{p - 1} = \frac{(p^n - 1)(p^{n-1} + 1)}{p - 1}, \tag{D.2}$$

where the denominator takes into account that the collinear vectors would generate the same code.

For $k = 2$ we first can choose one of $\mathcal{N}(n) - 1$ non-zero even vectors, and then supplement it by one of $P(n) - p + 1$ even vectors, that are orthogonal to the first one (and not collinear with the first one). Here $P(n)$ is the number of even non-zero vectors orthogonal to any given non-zero even codeword, see the Appendix F of [4],

$$P(n) - p + 1 = (p^n - p)(p^{n-2} + 1). \tag{D.3}$$

We therefore find for

$$N(k = 2, n, p) = \frac{(\mathcal{N}(n) - 1)(P(n) - p + 1)}{(p - 1)p(p^2 - 1)}. \tag{D.4}$$

The denominator here is exactly the size of $GL(2, \mathbb{Z}_p)$ which counts the number of ways the same code can be generated by pairs of different generators.

Generalizing this to arbitrary k we find the following formula for the number of even codes of length n and exactly k generators

$$N(k,n,p)=\prod_{i=1}^{k}\frac{(p^{n-i}+1)(p^{n+1-i}-1)}{p^i-1}. \tag{D.5}$$

This formula agrees with the total number of even self-dual codes $\mathcal{C}$ when $k=n$ [5]

$$\mathcal{N}_{\mathcal{C}}(n,p)=N(n,n,p)=\frac{|O(n,n,\mathbb{Z}_p)|}{|\Gamma_0|}=\prod_{i=0}^{n-1}(p^i+1). \tag{D.6}$$

When N is square-free $N=\prod_k p_k$, the total number of codes is simply the product of (D.6)

$$\mathcal{N}_{\mathcal{C}}(n,N)=\prod_k \mathcal{N}_{\mathcal{C}}(n,p_k). \tag{D.7}$$

Now we are ready to calculate the number of even self-dual codes of length n over $\mathbb{Z}_N\times\mathbb{Z}_N$ where $N=p^2$ and p is a prime. Such codes fall into different orbits O_a based on their group structure, $\mathbb{Z}_N^a(\mathbb{Z}_p\times\mathbb{Z}_p)^{(n-a)}$, parameterized by $a=0\dots n$. To calculate the number of codes in each orbit we can consider a map mod p such that an even self-dual code of type $\mathbb{Z}_N^a(\mathbb{Z}_p\times\mathbb{Z}_p)^{n-a}$ will become an even code over $\mathbb{Z}_p\times\mathbb{Z}_p$ with $k=a$ generators. There are $N(k,n,p)$ codes of this type. Besides, the map has a non-trivial kernel. It is clear that if we consider a code with the group structure $\mathbb{Z}_N^m$ of the form $(a_1\dots a_m,0\dots 0)$ for $a_i\in\mathbb{Z}_N$, then any orthogonal transformation of the form

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \tag{D.8}$$

with any antisymmetric matrix $X=0 \bmod p$ will not change the resulting code over $\mathbb{Z}_p$. There are $p^{a(a-1)/2}$ such matrices X. This fully describes the degeneracy of the mod p map for this given code. The size of the kernel mapping even self-dual codes over $\mathbb{Z}_N\times\mathbb{Z}_N$ to even codes over $\mathbb{Z}_p\times\mathbb{Z}_p$ is the same for all codes with the same group structure. Hence we find for the number of codes of type $\mathbb{Z}_N^a(\mathbb{Z}_p\times\mathbb{Z}_p)^{(n-a)}$ is

$$|O_a|\equiv\mathcal{N}_{\mathcal{C}}(a,n,p^2)=p^{a(a-1)/2}N(a,n,p). \tag{D.9}$$

This formula matches previously known results for particular p and n. The total number of even self-dual codes over $\mathbb{Z}_N\times\mathbb{Z}_N$ for $N=p^2$ is

$$\mathcal{N}_{\mathcal{C}}(n,p^2)=\sum_{a=0}^{n}\mathcal{N}_{\mathcal{C}}(a,n,p^2). \tag{D.10}$$

D.2 Counting "symplectic" codes over $\mathbb{Z}_N\times\mathbb{Z}_N$

We start by counting the number of codes over $\mathbb{Z}_p\times\mathbb{Z}_p$ of length g symplectic in the sense of (3.5) with exactly k generators. For $k=1$ there are

$$N_s(k=1,g,p)=\frac{p^{2g}-1}{p-1} \tag{D.11}$$

such codes, where the numerator evaluates the number of non-zero vectors of length $2n$ and the denominator is the same as in (D.2). For $k = 2$, there are

$$N_s(k = 2, g, p) = \frac{(p^{2g} - 1)(p^{2g-2} - 1)p}{(p - 1)p(p^2 - 1)} \tag{D.12}$$

such codes etc. In the end we find for arbitrary k

$$N_s(k, g, p) = \prod_{i=1}^{k} \frac{(p^{2(g+1-i)} - 1)}{(p^i - 1)}. \tag{D.13}$$

When $k = g$ the number of self-dual symplectic codes $\mathcal{L}$ is [4]

$$\mathcal{N}_{\mathcal{L}}(g, p) = N_s(g, g, p) = \frac{|Sp(2g, \mathbb{Z}_p)|}{|\Gamma_0|} = \prod_{i=1}^{g} (p^i + 1). \tag{D.14}$$

Again, when $N = \prod_k p_k$ is square-free, the total number of codes is

$$\mathcal{N}_{\mathcal{L}}(n, N) = \prod_k \mathcal{N}_{\mathcal{L}}(n, p_k). \tag{D.15}$$

To calculate the number of symplectic codes over $\mathbb{Z}_N \times \mathbb{Z}_N$ for $N = p^2$, we notice that such codes split into orbits S_a based on their group structure $\mathbb{Z}_N^a(\mathbb{Z}_p \times \mathbb{Z}_p)^{(g-a)}$, parameterized by $a = 0 \dots g$. To calculate the number of codes in each orbit we can consider a map mod p such that a symplectic self-dual code of type $\mathbb{Z}_N^a(\mathbb{Z}_p \times \mathbb{Z}_p)^{g-a}$ will become symplectic code over $\mathbb{Z}_p \times \mathbb{Z}_p$ with $k = a$ generators. There are $N_s(k, n, p)$ codes of this type. Besides, the map has a non-trivial kernel. It is clear that if we consider a code with the group structure $\mathbb{Z}_N^m$ of the form $(a_1 \dots a_m, 0 \dots 0)$ for $a_i \in \mathbb{Z}_N$, then any orthogonal transformation of the form

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \tag{D.16}$$

with any symmetric matrix $X = 0 \bmod p$ will not change the resulting code over $\mathbb{Z}_p$. There are $p^{a(a+1)/2}$ such matrices X. This fully describes the degeneracy of the mod p map for this given code. The size of the kernel mapping symplectic self-dual codes over $\mathbb{Z}_N \times \mathbb{Z}_N$ to symplectic codes over $\mathbb{Z}_p \times \mathbb{Z}_p$ is the same for all codes with the same group structure. Hence we find for the number of codes of type $\mathbb{Z}_N^a(\mathbb{Z}_p \times \mathbb{Z}_p)^{(g-a)}$ is

$$|S_a| \equiv \mathcal{N}_{\mathcal{L}}(a, g, p^2) = p^{a(a+1)/2} N_s(a, g, p). \tag{D.17}$$

As a consistency check we note that

$$|S_g| = \frac{|\mathrm{Sp}(2g, \mathbb{Z}_p^2)|}{|\Gamma_0|}. \tag{D.18}$$

The total number of even self-dual codes over $\mathbb{Z}_N \times \mathbb{Z}_N$ for $N = p^2$ is

$$\mathcal{N}_{\mathcal{L}}(g, p^2) = \sum_{a=0}^{n} \mathcal{N}_{\mathcal{L}}(a, g, p^2). \tag{D.19}$$

D.3 Sum over topologies for $g = 1$

In what follows we focus on the case $g = 1$ and evaluate the sum over 3d topologies ending on a torus for the AB theory (2.5) with $N = p^2$ by *matching* the LHS of (4.24). In other words, instead of performing a genuine sum using genus reduction, we note that the resulting sum at genus $g = 1$ includes only states $|\mathcal{L}\rangle$ (3.8) for the symplectic self-dual codes $\mathcal{L}$ of length $g = 1$ over $\mathbb{Z}_N \times \mathbb{Z}_N$, $N = p^2$, and match corresponding coefficients. We leave the task of evaluating the sum over 3d manifolds from the first principles for the future.

Symplectic codes of length $g = 1$ over $\mathbb{Z}_N \times \mathbb{Z}_N$ with $N = p^2$ split under the action of $SL(2,\mathbb{Z})$ into two orbits, S_1 and S_0. First has the size $|S_1| = p(p+1)$ and includes codes of the form $(a, r\,a)$ for any $a \in \mathbb{Z}_N$ and $r = 0 \dots N-1$, as well as (rpa, a) where $r = 0 \dots p-1$. The orbit S_0 includes just a single code of the form $\mathcal{L}' = (pa, pb)$, where $a, b \in \mathbb{Z}_p$. Obviously this code is invariant under $SL(2,\mathbb{Z})$ by itself.

We want to find coefficients α, β such that

$$\sum_{a=0}^{n} \sum_{\mathcal{C}\in O_a} |\mathcal{C}\rangle = \frac{\alpha}{N^{n/2}} \sum_{\mathcal{L}\in S_1} |\mathcal{L}\rangle + \frac{\beta}{N^{n/2}} \sum_{\mathcal{L}\in S_0} |\mathcal{L}\rangle = \alpha \sum_{\gamma\in\Gamma_0\backslash SL(2,\mathbb{Z}_N)} U_\gamma |0\rangle_{3d} + \beta|p\rangle. \tag{D.20}$$

Here we used that the vacuum state of the 3d theory (2.5) on a torus

$$|0\rangle_{3d} = \frac{1}{p^n} \sum_{a_i\in\mathbb{Z}_k} |a_1, 0, \dots a_n, 0\rangle_{5d} \tag{D.21}$$

is the n-th tensor power of the stabilizer state $|\mathcal{L}_0\rangle = \sum_{\mathrm{c}\in\mathcal{L}_0} |\mathrm{c}\rangle_{5d}$ for $\mathcal{L}_0 = (a, 0)$ up to an overall normalization factor $N^{n/2}$. We also introduced the state

$$|p\rangle = \frac{1}{N^{n/2}} |\mathcal{L}'\rangle = \frac{1}{p^n} \sum_{a_i,b_i\in\mathbb{Z}_p} |pa_1, pb_1, \dots pa_n, pb_n\rangle_{5d} \tag{D.22}$$

such that $\langle p|p\rangle = 1$.

To calculate the coefficients α, β we can evaluate them first for each O_a:

$$\frac{1}{\mathcal{N}_{\mathcal{C}}(a, n, p^2)} \sum_{\mathcal{C}\in O_a} |\mathcal{C}\rangle = \alpha_a \sum_{\gamma\in\Gamma_0\backslash SL(2,\mathbb{Z}_N)} U_\gamma |0\rangle + \beta_a |p\rangle. \tag{D.23}$$

We obtain the first equation for α_a, β_a by evaluating the scalar product of (D.23) with $\langle 0|$:

$$1 = \alpha_a \left(1 + \frac{(p-1)}{p^n} + \frac{p^2}{p^{2n}}\right) + \frac{\beta_a}{p^n} \tag{D.24}$$

where $\langle 0|p\rangle = 1/p^n$ follows from (D.21) and (D.22) and we used explicit expression for all $p^2 + p$ codes in the orbit of $(a, 0)$ to evaluate their scalar product with $(a, 0)$.

To obtain second equation, we evaluate the scalar product with $\langle p|$. In this case the RHS is simple. Since $\langle p|$ is modular-invariant all $p^2 + p$ states in the $SL(2, Z)$ orbit of $|0\rangle$ have the same scalar product. Evaluation of the LHS can proceed as follows. All codes

$\mathcal{C}$ in O_a are mapped to each other by the orthogonal group. State $|p\rangle$ is invariant under this group, hence we can evaluate the scalar product between $|\mathcal{L}'\rangle$ and $|\mathcal{C}\rangle$ for any chosen $\mathcal{C} \in S_0$. It is convenient to choose it in the form

$$\mathcal{C} = (\alpha_1, 0, \dots \alpha_a, 0, p\alpha_{a+1}, p\beta_{a+1}, \dots p\alpha_n, p\beta_n). \tag{D.25}$$

Here $\alpha_i \in \mathbb{Z}_N$ for $i \le n-a$ and $\alpha_i, \beta_i \in \mathbb{Z}_p$ for $i > n-a$. We stress, these are orthogonal codes, not symplectic ones. A beautiful thing is that the code $\mathcal{L}'$ is both symplectic and orthogonal. Hence the scalar product between corresponding states is

$$\frac{\langle \mathcal{L}'|\mathcal{C}\rangle}{N^n} = \frac{p^{2n-a}}{p^{2n}}. \tag{D.26}$$

Combining all together we find

$$p^{n-a} = \alpha_a \frac{p^2+p}{p^n} + \beta_a, \tag{D.27}$$

and

$$\alpha_a = \frac{(p^a-1)\, p^{2n-a}}{(p^n-1)\,(p^n+p)}, \tag{D.28}$$

$$\beta_a = \frac{p^{n-a}\left(p\,(p-(p+1)p^a) + p^{2n} + (p-1)p^n\right)}{(p^n-1)\,(p^n+p)}. \tag{D.29}$$

Eventually we get for the coefficients in (D.20) and (4.24)

$$A_{p^2} = \frac{\alpha}{p^n \mathcal{N}_{\mathcal{C}}(n,p^2)}, \qquad \alpha = \sum_{a=0}^{n} \alpha_a |O_a|, \tag{D.30}$$

$$B_{p^2} = \frac{\beta}{p^n \mathcal{N}_{\mathcal{C}}(n,p^2)}, \qquad \beta = \sum_{a=0}^{n} \beta_a |O_a|. \tag{D.31}$$

In the large $p \to \infty$ limit we find for $n > 2$,

$$\alpha_a = 1 - \delta_{a,0}, \qquad \beta_a = p^{n-a}, \qquad |O_a| \approx (1+\delta_{a,n}) p^{a(2n-1-a)}, \tag{D.32}$$

and the sum in (D.31) is saturated for $a = n-1$,

$$\frac{1}{\mathcal{N}_{\mathcal{C}}} \sum_{\mathcal{C}} |\mathcal{C}\rangle = \sum_{\gamma \in \Gamma_0 \backslash SL(2,\mathbb{Z}_{p^2})} U_\gamma |0\rangle + \frac{p^{1-n}}{3} |\mathcal{L}'\rangle, \qquad \mathcal{N}_{\mathcal{C}} \approx 3p^{n(n-1)}. \tag{D.33}$$

We note that $|\mathcal{L}'\rangle^{\otimes n} = |\mathcal{C}'\rangle^{\otimes g}$ where $\mathcal{C}' \in O_0$ is the unique code in the orbit O_0 of the form $(p\alpha, p\beta) \in \mathcal{C}'$, $\alpha, \beta \in \mathbb{Z}_p^n$.

The expression (D.33) is valid also for finite p and $n \gg 1$. It is interesting to note that in the large central charge limit $n \to \infty$ only handlebody contributions survive, confirming the expectation of [81].

In the reminder of this section we switch to 5d theory (still for $g = 1$) and evaluate the sum over 5d topologies by matching the LHS of (4.39),

$$\sum_{\mathcal{L}\in S_0} |\mathcal{L}\rangle + \sum_{\mathcal{L}\in S_1} |\mathcal{L}\rangle = \sum_{\mathcal{C}\in O_a} \frac{\delta_a}{|O_a|}|\mathcal{C}\rangle. \tag{D.34}$$

On the LHS all codes (Maxwell theories) enter with the same coefficient. On the RHS coefficients are ambiguous because different states $|\mathcal{C}\rangle$ are linearly-dependent. Terms on the RHS have the interpretation of 5d topologies, but there is no "first principles" way to fix δ_a because of this ambiguity. Similarly to (D.24) and (D.27) we obtain

$$\sum_{a=0}^{n} \delta_a = p^n \left(1 + \frac{p-1}{p^n} + \frac{p^2}{p^{2n}} + \frac{1}{p^n}\right), \tag{D.35}$$

$$\sum_{a=0}^{n} \delta_a p^{n-a} = p^n \left(\frac{p^2+p}{p^n} + 1\right). \tag{D.36}$$

For $g = 1$, there are two states invariant under both orthogonal and symplectic groups, hence any two orbits will suffice. We find most convenient to keep only $a = n$ and $a = 0$, while all other δ_k are taken to be zero,

$$A'_{p^2} = \frac{\delta_n}{|O_n|\mathcal{N}_{\mathcal{L}}}, \quad \delta_n = p^n + p - p^{2-n}, \qquad \delta_1 = \cdots = \delta_{n-1} = 0, \tag{D.37}$$

$$B'_{p^2} = \frac{\delta_0}{\mathcal{N}_{\mathcal{L}}}, \qquad \delta_0 = p^{2-n} + p^{2(1-n)}, \qquad \mathcal{N}_{\mathcal{L}} = p^2 + p + 1. \tag{D.38}$$

In the large p limit and $n > 2$ we find

$$\delta_0 = p^{2-n}, \quad \delta_n = p^n, \tag{D.39}$$

and

$$\frac{1}{\mathcal{N}_{\mathcal{L}}} \sum_{\mathcal{L}} |\mathcal{L}\rangle = \frac{p^{2(n-1)}}{|O_n|} \sum_{h\in\Gamma_0\backslash O(n,n,\mathbb{Z}_{p^2})} U_h|0\rangle_{5d} + p^{-n}|\mathcal{C}'\rangle. \tag{D.40}$$

When $n = 1$ we find for arbitrary p,

$$\frac{1}{\mathcal{N}_{\mathcal{L}}} \sum_{\mathcal{L}} |\mathcal{L}\rangle = \frac{p^2}{p^2+p+1} \sum_{h\in\Gamma_0\backslash O(n,n,\mathbb{Z}_{p^2})} U_h|0\rangle_{5d} + \frac{p}{p^2+p+1}|\mathcal{C}'\rangle. \tag{D.41}$$

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