

Absence of quantum Darwinism as a resource in secure quantum communication and computation

Bishal Kumar Das^{✉,*}, Sourav Manna^{✉,†} and Vaibhav Madhok^{✉,‡}

Department of Physics, Indian Institute of Technology Madras, Chennai, India, 600036

(Dated: October 6, 2025)

The emergence of classical world from underlying quantum mechanics is characterized by not only vanishing quantum correlations but also an unfolding of objectivity also known as quantum Darwinism. We show that the absence of this objectivity has a quantum advantage in cryptography and also provides the crucial missing link in efficient classical simulation of quantum circuits with zero discord. For this purpose, we consider a model of mixed state quantum computation where one is promised concordant states at all stages of the quantum circuit. A concordant quantum state has zero discord with respect to any part and there exists a basis made up of a tensor product of orthonormal local subsystem basis in which the density matrix is diagonal. Efficient classical simulation of concordant computation has surprisingly been an outstanding question in quantum information theory. We argue that a key ingredient of an efficient classical simulation algorithm, a knowledge of the local basis in which the multi-party state is diagonal, is made available by quantum Darwinism. Concordant states in the absence of quantum Darwinism cannot be efficiently simulated by existing methods and give a cryptographic advantage in communication. We show this by giving a protocol for secure quantum communication that exploits this insight. Our work also has implications for the quantum-classical border and we discuss how objectivity emerging out of Darwinism demarcates this border in three ways - empirical based on our observations and experience of objectivity, information theoretic due to the absence of any quantum correlations and lastly computational in the sense discussed above. Lastly, we show that the quantum-classical boundary as drawn by quantum Darwinism as well by what can be simulated efficiently in a mixed state quantum computation aligns with the boundary given by Hardy in his axiomatic derivation of quantum mechanics as a theory of probability.

Keywords: Efficient simulation, Quantum Algorithm, Quantum Computing, Quantum Resource theory, Cryptography

I. INTRODUCTION

The hallmark of the emergence of classical world from underlying quantum mechanics is not only the recovery of Newton's laws of motion or the vanishing of quantum coherence, but also objectivity [1–7]. The information about the system-apparatus, in a specific pointer basis, is spread across the environment and made available to multiple spatially separated observers. This phenomenon, also known as quantum Darwinism, is characterized by proliferation of information in a redundant manner, i.e, with multiple copies of information being available in any significant fraction of the environment. It is then that the multiple observers can independently access this information without disturbing the system. In order to achieve this, information about the density matrix as well as the pointer basis in which the state is diagonal is broadcast to the environment. The unique state structure compatible with the above criterion for Darwinism is called as the spectrum broadcast

structure [2, 3]. This structure is devoid of any quantum correlations and therefore has zero quantum discord across the system environment partition [8–10]. In order for the observers to acquire information without causing disturbance, not only the state should possess the unique structure mentioned above, the observers must have a knowledge of the basis in which to make these measurements. While the discussion about Darwinism has primarily focused on the emergence of zero discord spectrum broadcast states, our focus will be on the observation that an objective knowledge of the basis, the pointer basis, that enable zero disturbance measurements has profound consequences for quantum cryptography and computation. In the absence of the aforementioned objectivity, we are still in the quantum regime. In this case then, there is a quantum advantage as the state is still not completely classical. So far, the discussions about quantum Darwinism as been in the realm of quantum foundations. However, identifying the role of Darwinism or its absence as a resource in quantum computation or communication has not been addressed. The message of our work is that a state devoid of any quantum correlations like quantum discord, still have quantumness that can be harnessed for information processing.

We use mixed state quantum computation as a

* Correspondence email address: bishal.9197@physics.iitm.ac.in

† Correspondence email address: mannaphy@physics.iitm.ac.in

‡ Correspondence email address: madhok@physics.iitm.ac.in

paradigm to study these ideas. We are interested in a subset of such a computation, the concordant quantum circuits that by definition consists of states of zero discord with respect to any part at all stages of computation. This implies that there exists a basis made up of a tensor product of orthonormal local subsystem basis in which the density matrix is diagonal.

This work has three main messages. The intuition behind the quest to come up with an efficient algorithm to simulate concordant circuits is that, in the absence of any discord at any stage of computation, they only possess classical correlations and may be classified “classical like” in an information theoretic sense. However, so far one has not been able to find efficient way to classically simulate these circuits. A crucial step where these algorithms fail is the inability to find the local basis in which the concordant state at any stage of the circuit can be expressed as a diagonal density matrix. Here we argue that it is the very lack of knowledge of such a basis which is precisely the information provided by quantum Darwinism that makes these circuits quantum. On the one hand, this makes classical simulations of these circuits hard. On the other hand, we show that this lack of knowledge of the basis is a source of advantage in cryptography. This brings us to the second part of our manuscript. One is identifying the absence of Darwinism, more specifically the lack of knowledge of the local basis on each subsystem that would make the collective state diagonal, has a quantum advantage. We show this by giving a protocol for cryptography that exploits this condition to send secret messages across a bipartite channel. The cryptography protocol is intimately tied to the fact given the lack of knowledge of the local basis there is no algorithm to efficiently simulate concordant circuits. Moreover, in the absence of this knowledge, an Eve’s dropper will cause disturbance to the quantum state introducing detectable errors in message transmission. Lastly, we connect both Darwinism and its implications for efficient simulations and cryptography to fundamental axioms which makes quantum theory depart from classical probability theory.

A typical computation usually implies feeding a given initial state (n qubit) to a quantum circuit consisting of unitary gates having support over any $k(\leq n)$ qubits, updating the state at each step and at the end sample the output distribution of measurement outcomes in the computational basis. In previous work by Jozsa et al. and Vidal [11, 12], it has been shown that if any quantum circuit consists of gates which don’t increase the amount of entanglement of the initial states *unboundedly* with respect to the number of input qubits and involves pure states at any stage of the circuit, then those classes of computational processes can be efficiently simulated in a classical computer. On the one hand, it has been shown that quantum search can be performed without entanglement, but that would re-

quire an exponential number of resources than the one with entanglement [13]. Therefore, it seems entanglement acts as a resource in quantum computing. However, on the other hand, by virtue of the Gottesman-Knill theorem [14], we know the converse statement of Jozsa’s result is not true, i.e. if a circuit generates entanglement unboundedly, doesn’t always guarantee that it won’t be efficient. Clifford gates can generate entanglement but still they are efficiently simulatable in classical computer through stabilizer formalism. Therefore, we can say that for quantum computation involving only pure states, entanglement is a necessary resource for quantum advantage, but not a sufficient one [15].

However, things are different for mixed-state computations, i.e where we have mixed states as the input of the circuits [16, 17]. One would expect to simulate the mixed state computation efficiently when the state remains separable at each step of the computation. Even when entanglement is marginally present in such circuits, it is still not efficiently simulable on a classical computer and such algorithms provide an exponential speed up [18]. It was shown in [19], that in DQC1 circuits, quantum discord is the resource for quantum computational speed up for that model. Moreover, it turns out that even if the state remains discord free at each step of computation, so far there does not exist an efficient classical algorithm to simulate it [20, 21].

As mentioned above, a central question we want to address is the demarcation of the quantum-classical border. What mathematical statement will draw a definitive border between quantum states and classical probability distributions? Lucien Hardy, in his seminal work on deriving quantum theory in a finite dimension from basic axioms, showed that the only difference between quantum and classical physics, when viewed as theories of probability, is that while the former allows a continuous reversible transformation between pure states, the latter does not [22]. If the word “continuous” is dropped from the above axiom, we obtain classical probability theory. Pure states, in classical probability, are a discrete set with no continuous and reversible path between them. In quantum theory, a continuous path is not only allowed, it is fundamental to the geometry of quantum states. What are the consequences of this for bipartite or multi-partite quantum systems? For one, it hints at a way to define the quantum-classical border. Interestingly, it hints that a bipartite quantum state with no quantum correlations, as captured by entanglement or quantum discord, is still not classical. One needs a discrete set of pure states for the constituent subsystems for it to be completely classical. While this seems a corollary and a mere extension of Hardy’s program for systems of more than one party, we show it has important consequences for quantum information processing. We demonstrate that an efficient classical sim-

ulation of concordant circuits will lead to a different demarcation of the quantum-classical boundary than that dictated by quantum Darwinism as well as an axiomatic approach to quantum mechanics as given by Hardy. A consistent demarcation of the quantum-classical border requires a definitive discrete set of pure states (for finite dimensional Hilbert spaces) for the classical world, something which is a signature of quantum Darwinism as well as Hardy's axiomatic approach to quantum theory.

II. BACKGROUND

A. Quantum Darwinism

Decoherence plays a crucial role in selecting stable pointer states that survive interaction with the environment [23–26]. Superpositions rapidly decay into mixtures due to environmental entanglement, leaving only the most stable states those least perturbed by their surroundings to persist [1, 5]. Quantum Darwinism builds on this by proposing that the environment not only decoheres the system but also acts as a communication channel, redundantly encoding information about these pointer states across many environmental fragments [1, 6, 27–31]. This redundancy allows multiple observers to independently access consistent data about a system, giving rise to the appearance of objective, classical facts [4, 7, 32–35]. The proliferation of information about the pointer states into the environment results in the appearance of a definite, objective classical reality [6]. Observers perceive the system as being in one of the pointer states, even though the underlying quantum system remains in a superposition. Previous works [2, 3, 36, 37] have shown the structure of the state which can be considered as classical like states which possess both objectivity and redundancy of information.

A *spectral broadcast state* is a quantum state in which classical information about a system is redundantly and independently recorded in many parts of its environment. The canonical form of such a state is given by

$$\rho_{SE} = \sum_i p_i |i\rangle_S \langle i| \otimes \rho_{E_1}^{(i)} \otimes \rho_{E_2}^{(i)} \otimes \cdots \otimes \rho_{E_n}^{(i)} \quad (1)$$

where $\{|i\rangle\}$ is the pointer basis of the system S , $\{p_i\}$ are classical probabilities, and each $\rho_{E_k}^{(i)}$ is a state of the environment fragment E_k encoding classical information about index i . This allows multiple observers to infer the system's state without disturbing it, ensuring objectivity. A crucial element of this disturbance free measurement is an objective knowledge of the basis in which to make the measurements. Each fragment carries distinguishable, redundant records of the system's classical

state. The system itself can be a bipartite or multipartite system with zero quantum correlations across any partition also known as in a concordant state [20].

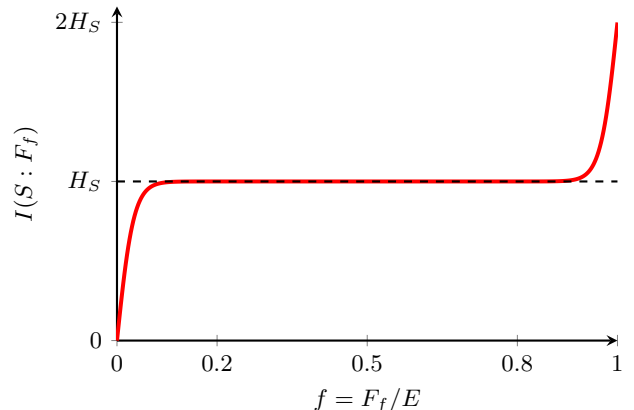


Figure 1. Information of the system S stored in the environment E . The global system-environment is in a pure state, and $I(S : F_f)$ represents the mutual information between the system and the fraction of the environment ($f = F_f/E$).

In Fig. 1, red curve depicts the mutual information between a system and fraction of environments after decoherence, for a combined system initially in a pure state. When a small fraction of the environment is considered, the mutual information I increases rapidly. This indicates that even a small portion of the environment contains significant information about the system's state. As more environmental fragments are included, the mutual information approaches a plateau at $I \approx H_s$, where H_s is the Shannon entropy of the system. This suggests that the system's state is effectively determined by the environment, and additional environmental fragments provide redundant information. In contrast, the green curve represents the behaviour of a typical random state in the Hilbert space of the system plus environment $H_s \otimes H_E$. These are highly entangled states that do not result from a decohering, information-preserving environment. Instead, the environment scrambles information due to internal interactions, like what occurs in air molecules or thermal baths. The mutual information $I(S : F_f)$ increases gradually and smoothly with f , without any plateau. There is no redundancy: small fragments of the environment do not individually contain useful information about the system. At the halfway point $f = 0.5$, there is a sharp antisymmetric jump due to entanglement entropy symmetry, and $I(S : F_f)$ reaches its maximum value of $2H_s$. This reflects the delocalization of information across the entire environment; only by accessing a substantial portion of the environment can one reconstruct the state of the system [1].

B. Quantum discord and concordant states

Quantum discord, introduced by Ollivier and Zurek, and further studied in various aspects [8, 38–41], quantifies nonclassical correlations in quantum systems by measuring the discrepancy between quantum generalizations of mutual information. It has also been shown that it is a resource for quantum communication [42, 43]. For a bipartite quantum state ρ_{AB} , the *quantum mutual information* is defined as:

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (2)$$

where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy. The *classical correlation* $J(B|A)$ is obtained by maximizing over projective measurements $\{\Pi_j^A\}$ on subsystem A :

$$J(B|A) = S(\rho_B) - \min_{\{\Pi_j^A\}} S(\rho_{B|\{\Pi_j^A\}}), \quad (3)$$

where $\rho_{B|\Pi_j^A} = \text{Tr}_A[(\Pi_j^A \otimes I_B)\rho_{AB}] / p_j$ and $p_j = \text{Tr}[(\Pi_j^A \otimes I_B)\rho_{AB}]$. The *quantum discord* is then:

$$D(B|A) = I(\rho_{AB}) - J(B|A). \quad (4)$$

Discord-zero states satisfy $D(B|A) = 0$, implying all correlations are classical. These states are diagonal in a product basis:

$$\rho_{AB} = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|, \quad (5)$$

where $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ form orthonormal bases for subsystems A and B . Zurek showed such states align with “pointer states” immune to decoherence in a preferred basis [38]. Discord zero state across any bipartition or the concordant state looks like

$$\rho = \sum_{k_1, k_2, \dots, k_N} p(k_1, k_2, \dots, k_N) \pi_{k_1}^{(1)} \otimes \pi_{k_2}^{(2)} \dots \otimes \pi_{k_N}^{(N)}, \quad (6)$$

where $\{\pi_{k_i}^{(i)}\}$ is the set of orthogonal rank one projectors for each subsystem labelled by i . The existence of discord-zero states underscores that separability alone does not guarantee classicality, as separable states may still exhibit quantum correlations.

C. Pure state computation

The foundational works of Jozsa & Linden and Vidal [11, 12] have clarified how the complexity of simulating pure-state quantum computations depends on both the number of qubits and the degree of entanglement. Jozsa & Linden demonstrated that for any

pure-state quantum algorithm to offer an exponential speed-up over classical computation, it must generate multi-partite entanglement that grows with the number of qubits; if the entanglement remains bounded as the system size increases, the quantum computation can be efficiently simulated classically. Vidal showed that the classical simulation cost of an n -qubit pure state scales as $O(\text{poly}(n) \cdot \exp(E_\chi))$, where E_χ is the entanglement entropy. The entanglement entropy, denoted as E_χ , is defined as the logarithm (base 2) of the maximum Schmidt rank across any bipartition of a quantum system. For a pure state $|\psi\rangle$, this is formally expressed as:

$$E_\chi = \log_2 \left(\max_{\text{bipartitions}} \text{Schmidt rank}(|\psi\rangle) \right),$$

where the Schmidt rank for a bipartition $A|B$ is the number of non-zero coefficients in the Schmidt decomposition $|\psi\rangle = \sum_{i=1}^m \lambda_i |u_i\rangle_A \otimes |v_i\rangle_B$. Here, E_χ quantifies the worst-case entanglement complexity by identifying the bipartition with the highest Schmidt rank. For example, a Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ has Schmidt rank 2, yielding $E_\chi = \log_2 2 = 1$. This measure distinguishes between separable states ($E_\chi = 0$) and entangled states ($E_\chi \geq 1$), and it upper-bounds the von Neumann entropy $S(\rho_A)$. Therefore the simulation becomes efficient only when E_χ grows at most logarithmically with n , but is otherwise exponential in E_χ . This establishes that the computational complexity of quantum simulation scales with the number of qubits and exponentially with the entanglement entropy, so that for algorithms like Shor’s, where entanglement scales with n , the cost of classical simulation becomes exponential. These results, presented in [11] and [12], highlight entanglement as a crucial resource distinguishing quantum from classical computation, and they show that the difficulty of simulating quantum systems is tightly linked to both n and 2^{E_χ} .

D. Concordant computation and the issue of degeneracy

Concordant State: A state ρ , with N qudit subsystems labeled by j , of arbitrary dimension d_j , is called concordant if every qudit possesses a complete set of orthogonal rank-1 projectors $\pi_{k_j}^{(j)}$ such that

$$\rho = \sum_{k_1, k_2, \dots, k_N} p(k_1, k_2, \dots, k_N) \pi_{k_1}^{(1)} \otimes \pi_{k_2}^{(2)} \dots \otimes \pi_{k_N}^{(N)}$$

where $p(k_1, k_2, \dots, k_N)$ is a probability distribution. It can also be written in terms of computational basis

as

$$\rho = \sum_{\mathbf{x}} U p(\mathbf{x}) |\mathbf{x}\rangle \langle \mathbf{x}| U^\dagger,$$

where $\mathbf{x} = \{x^{(1)}, x^{(2)}, \dots\}$ denotes computational basis and $U = U^{(1)} \otimes U^{(2)} \dots$ denotes local unitary rotation. At each step, the state involves a convex combination pure states which are nothing but a tensor product of local orthogonal projectors. The simplest example of a concordant computation is given by probabilistic classical computation using reversible gates. However, concordant computations can involve entangling gates and changes to the local basis in which the state remains diagonal. Intuitively, one approach towards an efficient simulation involves tracking each of these pure states through the set of gates through random sampling (Monte-Carlo) and collecting the statistics. However, even if the total mixture is concordant, we can still have entanglement between the subsystems. For example let ρ is an equal mixture of $|\psi_1\rangle = (|0\rangle + |1\rangle)|1\rangle$, $|\psi_2\rangle = (|0\rangle - |1\rangle)|1\rangle$ and U to be the CNOT gate. After one step the updated states will be $|\phi_1\rangle = |0\rangle|1\rangle + |1\rangle|0\rangle$, $|\phi_2\rangle = |0\rangle|1\rangle - |1\rangle|0\rangle$. But the complete density matrix will still be separable and concordant. This tells us that simply tracking a state is not an efficient option as it generates entangled states even if the total mixture is concordant.

The cause behind the generation of entanglement while tracking individual pure states as above is the degeneracy of the state which we define now.

Another way of writing the concordant states, which is very crucial to understanding the efficient simulation algorithms, is

$$\rho = \sum_k p_k \rho_k^{(a)} \otimes \pi_k^{(b)} \quad (7)$$

$$= \sum_k p_k \underbrace{\rho_k^{(a)}}_{\tilde{\rho}_k^{(a)}} \otimes \pi_k^{(b)} \quad (8)$$

Here we divide the total space into two parts a and b based on the support of the gate. The gate has the support over the qubits in b . The equation above 8 was the original definition of a zero discord state and essentially equivalent to the spectral broadcast state and $\rho_k^{(a)}$ can be regarded as the conditional density matrix on subsystem a with a probability p_k when b is a projector $\pi_k^{(b)}$.

Degeneracy: In (8), if there exist two different k 's, let's say k_1 and k_2 such that $\tilde{\rho}_{k_1}^{(a)} = \tilde{\rho}_{k_2}^{(a)}$, then the state has a degenerate subspace in part b making the decomposition in Eq.(8) non-unique. This degeneracy, as was shown in [20] and as we will discuss later, has profound consequences for the efficient simulation of concordant computation. In particular it there is no such

degeneracy, it was shown in [20] the classical simulation can proceed with a Monte-Carlo approach through probabilistic sampling of pure state trajectories evolved through the circuit. In particular, the example given above when ρ is an equal mixture of $|\psi_1\rangle = (|0\rangle + |1\rangle)|1\rangle$, $|\psi_2\rangle = (|0\rangle - |1\rangle)|1\rangle$ and U to be the CNOT gate has exactly this type of degeneracy that leads to entangled trajectories making pure state Monte-Carlo inefficient. In order to attempt efficient simulation in the presence of degeneracy, we need to first detect it and then corresponding projectors on part b are combined to get the unique decomposition, which looks like

$$\rho = \sum_k \tilde{\rho}_k^{(a)} \otimes \Pi_k^{(b)} \quad (9)$$

Eastin showed that such a scheme for concordant quantum computations involving only one- or two-qubit gates can be efficiently simulated on a classical computer [20]. His simulation protocol relies on tracking the entire history of the quantum state throughout the computation. Since degeneracy in the state can lead to the emergence of entangled trajectories, the protocol begins by identifying the degeneracy structure of the quantum state at each step. It then attempts to construct an equivalent circuit that preserves concordance and avoids generating entanglement. However, Eastin demonstrated that determining the degeneracy structure becomes an NP-hard problem when the circuit includes gates with support on more than two qubits.

Proceeding further Cable and Browne have shown in their paper [21] that a larger class of computation (circuits involving more than one and two qubit gates) can be simulated in a classical computer with some restrictions on the allowed states. The heart of their proof involved a new procedure called "Local Basis Finding (LBF)" which finds the local transformation which transforms the obtained basis after each step of computation to the computational basis. They have shown that we can't find the transformation for all of the input states. We will briefly review Eastin's simulation protocol and provide an in-depth explanation of the Local Basis Finder (LBF) protocol proposed by Cable and Browne in the next section.

E. Decomposition of concordant preserving gates and the difficulty of the computation

The core idea behind concordant computation is to ensure that the quantum state remains diagonal in some product basis (i.e., a separable basis) at every step of the computation. That is, although the computational basis may no longer be the diagonal basis of the state after a gate is applied, the state remains diagonal in some new non-entangled basis.

Let the state at time step $t - 1$ be denoted by ρ_{t-1} , and let G_t be the gate applied at time t . Then the new state is

$$\rho_t = G_t \rho_{t-1} G_t^\dagger.$$

Assuming the state at $t - 1$ is diagonal in the orthonormal product basis defined by the local unitary U_{t-1} , we can write:

$$\rho_{t-1} = U_{t-1} D_{t-1} U_{t-1}^\dagger,$$

where D_{t-1} is diagonal in the computational basis. Then the state at time t becomes:

$$\rho_t = G_t U_{t-1} D_{t-1} U_{t-1}^\dagger G_t^\dagger.$$

Now, let us assume the state ρ_t is diagonal in a new orthonormal product basis defined by the unitary U_t , so that:

$$\rho_t = U_t D_t U_t^\dagger.$$

Equating both expressions for ρ_t , we obtain:

$$D_t = U_t^\dagger G_t U_{t-1} D_{t-1} U_{t-1}^\dagger G_t^\dagger U_t. \quad (10)$$

Since D_{t-1} and D_t are both diagonal, and unitary conjugation preserves the eigenvalue spectrum, the transformation $U_t^\dagger G_t U_{t-1}$ must map a diagonal matrix to another diagonal matrix. The only unitaries that preserve diagonality in this way are permutation matrices. Therefore,

$$G_t = U_t P_t U_{t-1}^\dagger, \quad (11)$$

where P_t is a classical permutation matrix in the computational basis. This decomposition must hold for any unitary that preserves separability. The additional requirement of concordance further constrains U_t and U_{t-1} to be tensor products of local unitaries.

Given this structure, we can express the full state after s time steps as:

$$\rho^{(s)} = \sum_{\vec{i}} p_{\vec{i}}^{(0)} U_s \left(\prod_{t=s}^1 P_t \right) |\vec{i}\rangle \langle \vec{i}| \left(\prod_{t=1}^s P_t^\dagger \right) U_s^\dagger,$$

where U_s is the local unitary diagonalizing the final state, and $p_{\vec{i}}^{(0)}$ are the initial probabilities in the computational basis. This structure allows for a classical simulation of concordant computation using a probabilistic method: sample an initial basis state $|\vec{i}\rangle$ according to $p_{\vec{i}}^{(0)}$, apply the sequence of permutations $P = \prod_{t=1}^s P_t$, followed by the local unitary U_s , and then measure in the computational basis. This procedure resembles Monte Carlo sampling and enables classical

simulation of concordant quantum circuits [20, 44]. If the initial state is not degenerate, one can simply track the state not worrying about the entangled trajectories being generated and it has been shown [20, 44] that such a simulation is efficient.

The key insight to simulate this process efficiently, is to identify the decomposition of each gate G_t as in Eq. (11), i.e., in terms of local unitaries and a permutation. That is, we want to construct a circuit made only of local unitaries and permutations that acts identically to the original quantum circuit on a concordant input. This reduces to the condition:

$$G_t \Pi_k^{(b)} G_t^\dagger = U_t^{(b)} P_t U_{t-1}^{(b)\dagger} \Pi_k^{(b)} U_{t-1}^{(b)} P_t^\dagger U_t^{(b)\dagger}, \quad \forall k,$$

where $\Pi_k^{(b)}$ are projectors in the computational basis over the support b of the gate.

In [20], Eastin proposed an algorithm to find such decompositions for gates acting on one or two qubits. The first step in his protocol is to determine the degeneracy structure of the state. This is done by testing whether the state remains invariant under specific two-qubit permutation operators Q . For a given computational basis permutation Q , Eastin checks whether

$$[P^\dagger Q P, \rho_0] = 0,$$

i.e., whether the transformed permutation commutes with the initial state. If so, the state is considered degenerate under that permutation. This test is efficient for gates supported on at most two qubits, where Clifford circuit techniques can be employed to evaluate the commutation efficiently [14]. However, when the gate acts on more than two qubits, determining degeneracy becomes computationally hard. Eastin shows that the problem can be mapped to a 3-SAT problem, which is NP-hard in general.

We can derive the more general decomposition from Eq. (10) by noting that, while permutation matrices are the only unitaries that preserve a non-degenerate diagonal spectrum, this is no longer the case when degeneracies are present. In the presence of degenerate eigenvalues, the most general unitary that preserves the spectrum is a block-diagonal unitary acting non-trivially within the degenerate subspaces, followed by a classical permutation. Therefore, the decomposition generalizes to:

$$G_t = U_t P_t B_t U_{t-1}^\dagger, \quad (12)$$

where B_t is block-diagonal with respect to the eigenbasis defined by U_{t-1} , and acts only within degenerate eigenspaces. This decomposition captures the full structure of concordance-preserving gates, as formalized in [21].

In the work by Cable and Browne, they exploited the fact that any concordance preserving gate can be decomposed as Eq.(12). They called their algorithm Local Basis Finder. *Local Basis Finder (LBF) Algorithm*

and Degeneracy-Induced Ambiguity [21]: To simulate concordant quantum computations involving gates of arbitrary size, Cable and Browne propose the *Local Basis Finder (LBF)* algorithm. Unlike Eastin’s approach, which becomes intractable for large gates due to NP-hard symmetry diagnostics, LBF analyzes each gate locally to extract the required basis updates.

The LBF attempts to construct the final unitary U_t or the orthogonal basis of the decomposition in (12) by examining how projectors in the computational basis evolve under conjugation by G_t and U_{t-1} . Specifically, it considers operators of the form

$$\tilde{X}_k = G_t U_{t-1} X_k U_{t-1}^\dagger G_t^\dagger,$$

where X_k are standard computational basis projectors of any rank on the gate’s support. The algorithm checks whether each $\tilde{X}_k$ can be written as a product of local, rank-one, Hermitian, orthogonal projectors. If so, it extracts the corresponding local basis vectors using exact algebraic methods such as symbolic Gaussian elimination and eigen-decomposition.

The LBF algorithm succeeds only when all transformed projectors $\tilde{X}_k$ admit a *compatible* local basis—that is, when there exists a unique local unitary U_t that simultaneously diagonalizes all of them. However, the algorithm may fail even if the state remains concordant. These failures can be categorised into two related types: basis incompatibility and degeneracy-induced ambiguity (see appendix A for the LBF which was given in [21]).

Remark. *It is often sufficient to apply the Local Basis Finder (LBF) algorithm only to the set of rank-one computational basis projectors $X_k = |\vec{x}_k\rangle\langle\vec{x}_k|$. This sufficiency follows from two key mathematical properties: the commutator is a linear operator, and the partial trace is additive. Specifically, if a local rank-one projector $\rho^{(j)}$ satisfies*

$$[\mathbb{I}^{(b/j)} \otimes \rho^{(j)}, \tilde{X}_k] = 0 \quad (13)$$

for all rank-one projectors X_k , then it will also commute with any operator formed as a linear combination of these, including higher-rank projectors. That is, if each $\tilde{X}_k = G_t U_{t-1} X_k U_{t-1}^\dagger G_t^\dagger$ yields a compatible local basis, then so will any operator of the form $\sum_k \alpha_k \tilde{X}_k$. Consequently, if the transformed rank-one projectors are simultaneously diagonalizable in a unique product basis, then this basis must also diagonalize all higher-rank projectors, which are simply sums of the rank-one ones. On the other hand, if the rank-one projectors yield multiple local basis solutions, then higher-rank projectors must be considered to determine whether a consistent product basis exists. These higher-rank projectors, due to their nontrivial support across degenerate subspaces, can impose additional spectral constraints that

may eliminate ambiguity. Therefore, rank-one projectors provide a sufficient criterion for LBF success when they identify a unique local basis, while higher-rank projectors are required only in cases where ambiguity persists.

Both forms of failure originate from a common structural feature: *degenerate eigenspaces*. When different X_k are conjugated under the gate, the resulting $\tilde{X}_k$ may project onto degenerate subspaces. While each projector remains concordant, the degeneracy allows many valid orthonormal bases within the subspace. Since quantum mechanics permits any unitary transformation between pure states in a degenerate subspace, different projectors may suggest distinct local bases with no common refinement. In the absence of a unique, consistent set of local rank-one projectors, the LBF cannot determine a well-defined local basis U_t (see appendix B for an example where it fails).

This degeneracy-induced ambiguity reflects a uniquely quantum limitation: although the global state has no entanglement or discord, its classical representation—as a product basis probability distribution—becomes ill-defined. Classical simulation, which relies on a unique diagonal basis, cannot resolve this structural ambiguity. Meanwhile, the quantum system evolves unitarily within the degenerate subspace, preserving coherence. This disconnect highlights a form of quantum computational hardness that arises not from correlations, but from the continuous basis freedom inherent to degenerate quantum systems.

Operationally, this phenomenon motivates the introduction of block-diagonal unitaries B_t in the decomposition of concordant gate operations:

$$G_t = U_t P_t B_t U_{t-1}^\dagger.$$

As discussed in Eq. (10), when the pre-gate state has degeneracy, the most general unitary that maps one diagonal state to another includes a block-diagonal component B_t that acts nontrivially within degenerate subspaces, along with a classical permutation D_t . If the spectrum is non-degenerate, $B_t = I$, and no ambiguity arises.

Thus, the presence of degeneracy in the pre-gate state introduces representational freedom, which undermines the LBF’s ability to recover a unique local basis. As shown in [21], this degeneracy renders the simulation problem hard in general, as it leads to infinitely many equivalent product bases. To resolve this, one must first identify the degeneracy structure. But this itself is computationally difficult, especially for large systems. Furthermore, degeneracy enables trajectories through entangled subspaces, increasing simulation complexity. In contrast, when the state is non-degenerate and the gate set is efficiently representable (e.g., sparse with

support scaling as $\text{poly}(n)$, classical simulation remains efficient.

III. SECURE COMMUNICATION USING CONCORDANT STATES

A. Protocol Description

A concordant quantum state has zero discord with respect to any part and there exists a basis made up of a tensor product of orthonormal local subsystem basis in which the density matrix is diagonal. However, under a lack of knowledge of this basis implies the absence of quantum Darwinism and a transition to full classicality. The state, despite the absence of quantum correlations still possesses "quantum-ness" which can be used as a resource in quantum communication. In this section we demonstrate secure quantum communication exploiting the quantum nature of concordant states. The key motivation of developing these protocols is to exploit the "quantum-ness" of concordant states in the absence of Darwinism.

To start with for the purpose of illustration, consider a noisy version of the BB84 protocol [45, 46] where the act of measuring the quantum state by an eavesdropper causes disturbance to the quantum state and introduces errors in the transmission. In this case, Alice will transmit concordant states to the receiver Bob. The basis in which the state is diagonal is eventually revealed by Alice. Any eavesdropper without the knowledge of the basis will cause disturbance to the state which can reveal that the transmission line was compromised.

In the original BB84 protocol, Alice encodes a random classical bit string into quantum states. For each qubit, she decides a basis - The computational basis states: $\{|0\rangle, |1\rangle\}$ or the Hadamard basis states $\{|+\rangle, |-\rangle\}$ and a bit value. Thus Alice prepares one of the four possible states

$$|0\rangle, |1\rangle, |+\rangle, |-\rangle,$$

and sends this qubit to Bob. For each received qubit, Bob randomly chooses a measurement basis:

Computational basis $\{|0\rangle, |1\rangle\}$, or Hadamard basis $\{|+\rangle, |-\rangle\}$. He records the outcome of his measurement. After the transmission, Alice and Bob publicly announce (over a classical channel) which bases they used for each qubit, but not the actual results. They discard all outcomes where their bases differ. The remaining sequence of bits is called the sifted key. Alice and Bob publicly compare a randomly chosen subset of the sifted key. If the observed error rate is below a fixed threshold, they proceed. If the error rate is too high, they abort the protocol (since this indicates eavesdropping). A "noisy" version of the above protocol where

Alice encodes the information in the computational or Hadamard basis states as done previously. However, this time she uses mixed states $p_1|0\rangle\langle 0| + p_2|1\rangle\langle 1|$ or $p_1|+\rangle\langle +| + p_2|-\rangle\langle -|$ for each qubit, with $p_1 \gg p_2$ if bit value 0 is being transmitted and vice-versa for the transmission of bit value 1. In other words, there is a small amount of noise giving rise to a departure from using pure states. Then, the joint state of the string of qubits at Alice's end is a concordant state of the form

$$\rho = \sum_{k_1, k_2, \dots, k_N} p(k_1, k_2, \dots, k_N) \pi_{k_1}^{(1)} \otimes \pi_{k_2}^{(2)} \dots \otimes \pi_{k_N}^{(N)}$$

where $p(k_1, k_2, \dots, k_N)$ is a probability distribution and $\pi_{k_i}^{(i)}$ are projectors in the computational or the Hadamard basis. The protocol proceeds the standard way and Eve's inability to know the basis in which each qubit is encoded guarantees secure communication despite the absence of quantum correlations.

Concordant states—those that remain diagonal in a product basis—can be efficiently simulated under the action of specific quantum gates that preserve concordance. This property can be leveraged to design a secure communication protocol that can be used to send a quantum state securely across a channel or can be adapted for quantum cryptography - both in the presence of a possible eavesdropper. In case of cryptography, Alice sends a message to Bob using a concordant state. Bob needs only a measurement device (plus classical computer); Eve may have (i) only the ability to make measurements along with a classical computer and/or (ii) a quantum computer. We show Bob can always decode, while Eve cannot learn (or even measure in) the right basis without the secret key.

B. Setup: Concordant Encryption with Hidden Local Bases (CE-HLB)

The secret key is $K = \{U_t, U_{t-1}, \dots, U_0\}$, where each U_i is a tensor product of single-qubit unitaries, is shared between Alice and Bob as shown in Fig: 2.

1. Encoding (Alice)

1. Embed the message in a concordant state ρ_{in} not necessarily in the computational basis. The state $\rho_0 = U_0 \rho_{in} U_0^\dagger$ is a diagonal matrix.
2. For $i = 1, \dots, t$, concordant evolution is performed by gate set $G_i := U_i P_i U_{i-1}^\dagger$ with chosen permutations P_i , such that $\rho_i := G_i \rho_{i-1} G_i^\dagger$.

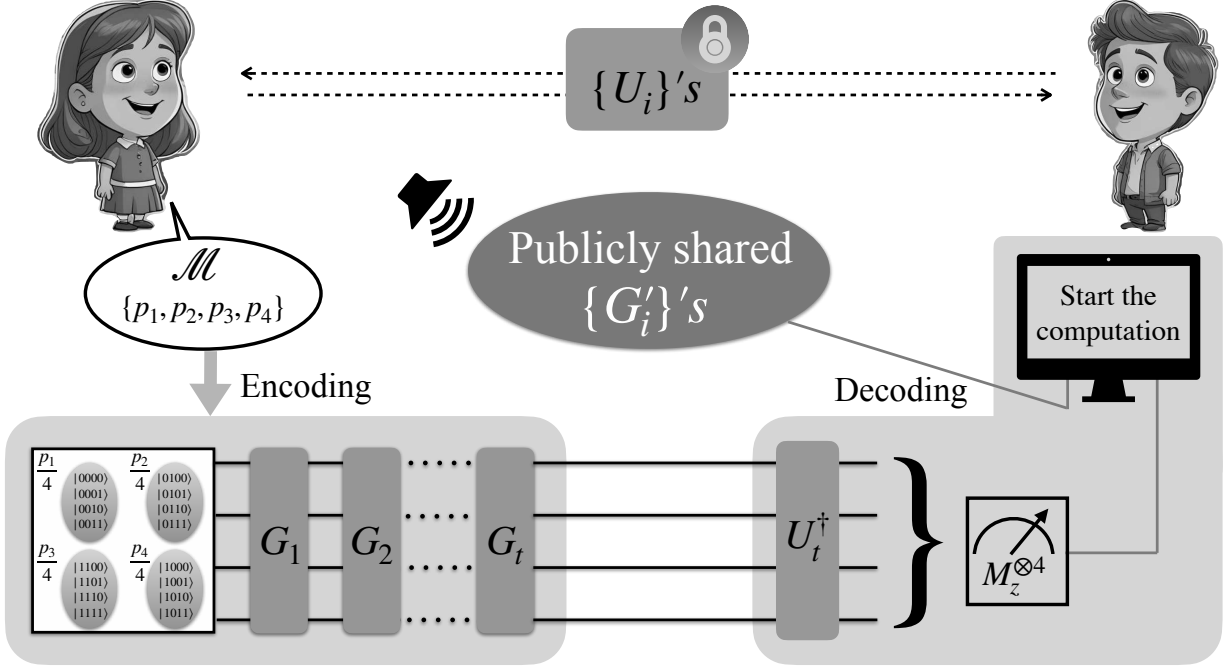


Figure 2. Graphical representation of an example of the protocol we proposed; Let's say Alice wants to communicate a discrete probability distribution $\mathcal{M}$. She encodes her message in a degenerate concordant state as shown inside the box. Then applied the sequence of gates $\{G_i\}$'s and publicly announced $\{G'_i\}$'s. Here $G_i = U_i P_i U_{i-1}^\dagger$ and $G'_i = U_i P_i B_i U_{i-1}^\dagger$ such that LBF fails for this gate set. They previously decided the set of haar random local unitaries $\{U_i\}$'s. Therefore Bob can get back the actual distribution by simulating the inverse computation in his classical computer.

3. Public key shared is the sequence of gates, $\{g_1, G'_2, G'_3, \dots, G'_t\}$, where $g_1 = U_1 P_1$ is the first gate, and $G'_i = U_i P_i B_i U_{i-1}^\dagger$ for $i > 1$ are the subsequent gates in the same order in which Alice applies G_i 's. This version includes B_k , which hides the true structure of G_i , and the initial U_0 is not broadcast (to stop the quantum attack as Eve does not know the initial basis in which the state is a diagonal). Now the final state ρ_t is shared with Bob.

2. Decoding (Bob; Measurement only + classical computing)

1. Using the secret key U_t , Bob measures ρ_t in the U_t basis (equivalently, applies $U_t^\dagger$ and measures $Z^{\otimes n}$). He then transfers the measurement outcomes to a classical computer for post-processing.
2. From the public transcript, we have the identity

$$g_1^\dagger G'_1{}^\dagger \dots G'_t{}^\dagger U_t = B_1 P_1 B_2 P_2 \dots B_t P_t.$$

Bob therefore applies the corresponding *classical*

inverse map to his outcome distribution to obtain ρ_0 . If he wishes to reconstruct the initial state, he then forms

$$\rho_{\text{in}} = U_0 \rho_0 U_0^\dagger,$$

where ρ_0 is the diagonal density matrix consistent the message Alice wants to send.

Alternatively, Bob can be empowered to perform unitaries $K = \{U_t, U_{t-1}, \dots, U_0\}$ as well as the gates $\{g_1, G'_2, G'_3, \dots, G'_t\}$ and reconstruct ρ_{in} coherently at his end. No measurements are required in this case. It is the absence of Darwinism, or the knowledge of the basis that diagonalizes the state, that prevents Eve from making these coherent operations.

3. Security Against Eve

Suppose Eve has a measurement device and a classical computer: Eve does not know U_t . Any projective measurement in a mismatched basis will (i) disturb the state and (ii) produce outcomes that destroy the permutation structure needed for classical inversion.

Moreover, without the correct basis, recovering the hidden local product unitaries from the public circuit (i.e., decomposing each G_i into $U_i B_i P_i U_{i-1}^\dagger$) is precisely the assumed hard *decomposition problem* [20].

Eve has measurement device, Quantum Computer: The public transcript ($g_1 = U_1 B_1 P_1$ instead of $G_1 = U_1 P_1 U_0^\dagger$) prevents Eve from reconstructing the true global inverse from public data. Even if she uses a quantum computer to apply the sequence $\tilde{G}_1^\dagger G_2^\dagger \dots G_t^\dagger$ to the state ρ_t , she ends up with $\rho_0 = U_0 \rho_{\text{in}} U_0^\dagger$, which remains encrypted by the unknown local basis U_0 (basis-hiding, analogous in spirit to BB84). Thus the classical information stays confined to the diagonal of an *unknown* product basis: any measurement in the wrong basis both degrades decodability and yields no useful information about Alice's message.

Here we are utilizing two aspects of quantum theory, which leverage the possibility of continuous basis transformations between pure states, either directly or indirectly. (1) the inability to perfectly measure the state without knowledge of the basis, which ensures that the intruder cannot accurately determine the probability values (analogous to BB84), and (2) the inability to efficiently decompose concordance-preserving gates, which prevents a classical intruder from determining the permutations at each step and thus from reconstructing the exact sequence of probabilities that Alice intends to send.

IV. IDENTIFYING THE QUANTUM RESOURCE FOR CRYPTOGRAPHY

Determining the decomposition of a concordant-preserving quantum operation into local unitaries, permutation gates, and block-diagonal components is a computationally nontrivial task. The difficulty becomes particularly pronounced when the pre-gate quantum state exhibits high degeneracy. In such cases, the FRASE (Full-Rank Subsystem Eigenprojector) decomposition of the state includes projectors of rank greater than one, corresponding to degenerate eigenspaces [21]. A fundamental property of quantum mechanics is that a density matrix with degenerate eigenvalues admits infinitely many possible decompositions into pure states. This non-uniqueness is a direct consequence of the superposition principle: the same mixed state can arise from distinct ensembles of pure states. Hence, degeneracy is not merely a numerical coincidence but a manifestation of the underlying structure of quantum theory itself.

This ambiguity in decomposition reflects a deeper conceptual difference between classical and quantum theories. As identified by Hardy and also by Barrett [22, 47], one of the defining features of quantum theory

is its allowance for *continuous transformations between pure states*. While classical theories admit only discrete probability distributions over distinct ontic states, quantum theory permits the interpolation of one pure state into another via unitary evolution. This ability to continuously navigate the space of pure states underpins quantum coherence and is intimately tied to the existence of degenerate mixed states. The degeneracy, therefore, signals the presence of coherent structure that is invisible to classical probabilistic descriptions.

Lemma 1. *In classical theory, all pure states are mutually orthogonal.*

Heuristic Argument. In classical theory, a pure state represents a complete specification of the system. Two distinct pure states must therefore be perfectly distinguishable. If they were not orthogonal, there would exist some overlap between them, implying a nonzero probability of confusion. This contradicts the classical requirement of perfect distinguishability. Hence, classical pure states must be mutually orthogonal. $\square$

Corollary 1.1. *In classical theory, the set of basis states is unique.*

Proof. Assume that in classical theory all pure states are mutually orthogonal. Suppose there exist two distinct basis sets $\{a_i\}$ and $\{b_j\}$. The identity operator can be expressed in both bases as

$$I = \sum_i |a_i\rangle\langle a_i| = \sum_j |b_j\rangle\langle b_j|.$$

Taking the inner product between these two representations of the identity gives

$$\langle I, I \rangle = \sum_{i,j} \langle a_i | b_j \rangle \langle b_j | a_i \rangle.$$

On the right-hand side, since the a_i and b_j are assumed to be orthogonal, we obtain 0, while on the left-hand side, the identity must satisfy $\langle I, I \rangle = 1$. This contradiction shows that two distinct orthogonal basis sets cannot coexist. Hence, the basis set in classical theory is unique. $\square$

In the framework of generalized probabilistic theories (GPTs) [47], a classical system is represented by a *simplex*. The pure states of the system correspond to the vertices of the simplex, while the mixed states are convex combinations of these vertices. For instance, a classical bit has two pure states, represented by the vertices of a line segment (the 1-simplex), and a classical trit has three pure states, represented by the vertices of a triangle (the 2-simplex). In general, an n -level classical system has n pure states corresponding to the standard basis vectors $\{e_i\}$ of $\mathbb{R}^n$, and the full state space is the

convex hull of these vectors. A key feature of classical theory is that these pure states are *mutually orthogonal*: in the vector-space embedding one has $e_i \cdot e_j = \delta_{ij}$. This orthogonality reflects the fact that classical pure states are perfectly distinguishable by a single-shot measurement. As a consequence, any mixed state in a classical theory admits a *unique* decomposition into pure states: the coordinates of the probability vector specify uniquely the mixing weights with respect to the orthogonal vertices of the simplex. In contrast, the quantum state space is not a simplex. Pure quantum states are represented by rays in Hilbert space, and nonorthogonal pure states exist in abundance. This nonorthogonality implies that quantum pure states need not be perfectly distinguishable and that a mixed quantum state may admit many different decompositions into pure states. For example, a maximally mixed state $\rho = \frac{1}{d}I$ can be written as an equal mixture of any orthonormal basis of pure states, or even as a mixture of nonorthogonal states, leading to an inherent *ambiguity of decomposition*. This distinction between the simplex structure of classical theories and the convex geometry of quantum state space underlies why degeneracy in a density matrix leads to unique decompositions in classical systems but to non-unique, ambiguous decompositions in quantum systems.

This structural richness complicates the task of identifying local bases in which the state remains concordant after the application of a quantum gate. In the Local Basis Finder (LBF) algorithm, one must explore the action of the gate on all computational basis projectors—including those of arbitrary rank—to determine whether a consistent local basis exists. This involves solving a set of nonlinear algebraic equations, whose complexity increases sharply with the size of the degenerate subspaces. Even in the non-degenerate case, where the decomposition is unique and rank-one projectors suffice, the algorithm requires matrix operations that scale polynomially in 2^k , where k is the number of qubits in the gate’s support. Although this remains tractable for small gate support, the simulation becomes intractable for gates acting on many qubits.

Crucially, the hardness of decomposition in the presence of degeneracy is not just a technical inconvenience, but a manifestation of quantum superposition as a computational resource. The fact that a degenerate state can be represented by an infinite family of pure-state ensembles—enabled by the continuous geometry of Hilbert space—exemplifies the power of quantum theory to encode ambiguity and structure simultaneously. This ambiguity becomes intrinsically hard to disentangle, especially under unknown or nonlocal transformations. Consequently, the complexity of recovering the decomposition from a black-box unitary can be repurposed as a cryptographic primitive. In the next section, we explore how this decomposition problem—rooted in

quantum degeneracy, superposition, and continuous basis freedom—can serve as a foundation for secure communication protocols based on the hardness of simulating concordant computations.

In the context of pure state computation, we know any quantum circuit can be decomposed into a combination of Clifford gates and T gates. Among them only *CNOT* is the non local gate. For pure state computation we already know entanglement is the necessary resource [11, 12]. The only gate that can generate entanglement is the *CNOT* gate, however it can do that only when the control is in superposition state of the computational basis. Though efficient techniques can be found to simulate particular cases of computations where entanglement or degeneracy (for concordant computation) is present, based on the symmetries of the states and gates involved [14, 21], but these techniques are not valid in general. Therefore, for both cases, pure state computation and mixed state computation, superposition lies at the heart of quantum supremacy.

V. QUANTUM DARWINISM AS THE BOUNDARY BETWEEN CLASSICAL AND QUANTUM COMPUTATION

Previous analysis established that identifying the local basis at each computational step remains classically intractable even for concordant states, with quantum superposition identified as the fundamental source of this complexity. Quantum Darwinism provides a framework to resolve this challenge: in classical regimes, redundant proliferation of system information across environmental degrees of freedom induces the emergence of a preferred pointer basis. This process diagonalizes the system’s density matrix, effectively encoding classical objectivity through environmental monitoring.

In the context of concordant computation, knowledge of this emergent local basis at every step circumvents the computational bottleneck of basis identification. Such apriori basis information enables efficient classical simulation by avoiding the need for LBF. Thus, Quantum Darwinism delineates the boundary between classical and quantum simulability. When environmental interactions redundantly encode pointer observables, computational dynamics become efficiently tractable, mirroring classicality. Conversely, the absence of such redundancy preservation aligns with quantum complexity, where superposition and entanglement preclude efficient classical simulation. This dichotomy positions Quantum Darwinism as a critical mechanism governing the transition between classical and quantum computational regimes.

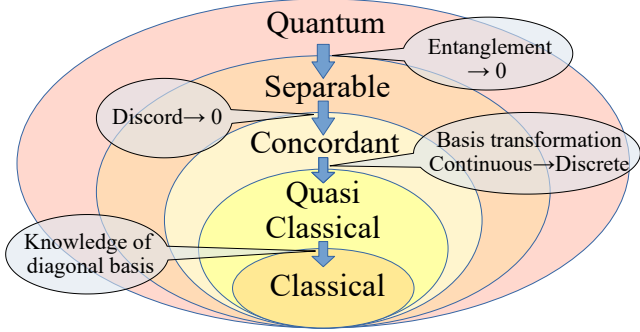


Figure 3. Hierarchy of computational models illustrating the transition from fully quantum regimes to classical computation. Quantum computations that exhibit an advantage over classical models occupy the topmost regime, enabled by entanglement and discord as genuine quantum resources. As these resources diminish, one approaches intermediate regimes between quantum and classical descriptions. In particular, the quasi-classical regime arises as an intermediate abstraction, analogous to Generalized Local Theories (GLTs) in the framework of Generalized Probabilistic Theories (GPTs) [47]. This regime should be regarded as a purely mathematical construct without physical realization. The progressive restrictions are marked by vanishing entanglement ($\text{Entanglement} \rightarrow 0$), vanishing discord ($\text{Discord} \rightarrow 0$), and the discretization of basis transformations (continuous $\rightarrow$ discrete). The final step, corresponding to full knowledge of the diagonal basis, represents the onset of Darwinism, where classical objective reality emerges.

VI. DISCUSSION AND CONCLUSION

Is there something in the structure of quantum theory as opposed to classical theory that can help us get more insight into the above discussion? In his axiomatic approach to the derivation of quantum theory from five reasonable axioms, Hardy [22] showed that the only difference between quantum theory and classical probability is that the former allows continuous reversible transformation between pure states unlike the latter. Therefore, there must be a breakdown of this very feature of quantum theory during quantum-to-classical transition. Indeed, the emergence of objectivity, and a special pointer basis as discussed above, can be viewed in light of Hardy’s work. A consistent demarcation of the quantum-classical border and unfolding of objective classicality requires a definitive discrete set of pure states (for finite dimensional Hilbert spaces) for the classical world, something which is a signature of quantum Darwinism as well as Hardy’s axiomatic approach to quantum theory. The density matrix is then viewed as a classical probability distribution in this basis as a convex set of objective pure states which are the extremal points of this set. Message of our work is

that objectivity is not only an empirical observation. It is embedded inside the very structure and axioms that makes a classical probability theory depart from quantum mechanics.

We have demonstrated that a specific task can be accomplished using a zero-discord quantum state—something no known classical algorithm can replicate. This result shows that quantum advantage does not necessarily require quantum correlations. Instead, we identify continuous transformations between pure states as the essential resource enabling quantum computational power.

A concordant quantum state has zero discord with respect to any part. A concordant computation is the one in which the state of the quantum computer at each stage is a concordant state. Efficient classical simulation of concordant computation has been an outstanding question in quantum information theory. We argue that an efficient classical simulation of concordant circuits will lead to a different demarcation of the quantum-classical boundary than that dictated by quantum Darwinism as well as an axiomatic approach to quantum mechanics as given by Hardy. A consistent demarcation of the quantum-classical border requires a definitive discrete set of pure states (for finite dimensional Hilbert spaces) for the classical world, something which is a signature of quantum Darwinism as well as Hardy’s axiomatic approach to quantum theory. We show it is the key ingredient for efficient simulations of concordant computations.

The central question we want to address is the demarcation of the quantum-classical border. What mathematical statement will draw a definitive border between quantum states and classical probability distributions? Lucien Hardy, in his seminal work on deriving quantum theory from basic axioms showed that the only difference between quantum and classical physics, when viewed as theories of probability, is that while the former allows a continuous reversible transformation between pure states, the latter does not. If the word “continuous” is dropped from the above axiom, we obtain classical probability theory. Pure states, in classical probability, are a discrete set with no continuous and reversible path between them. In quantum theory, a continuous path is not only allowed, it is fundamental to the geometry of quantum states. What are the consequences of this for bipartite or multi-partite quantum systems? For one, it hints at a way to define the quantum-classical border. Interestingly, it hints that a bipartite quantum state with no quantum correlations, as captured by entanglement or quantum discord, is still not classical. One needs a discrete set of pure states for the constituent subsystems for it to be completely classical. While this seems a corollary and a mere extension of Hardy’s program for systems of more than one party, it has profound consequences.

VII. ACKNOWLEDGEMENTS

This work was supported in part by grant DST/ICPS/QusT/Theme-3/2019/Q69 and New fac-

ulty Seed Grant from IIT Madras. The authors were supported, in part, by a grant from Mphasis to the Centre for Quantum Information, Communication, and Computing (CQuICC) at IIT Madras.

-
- [1] Wojciech Hubert Zurek. Quantum darwinism. *Nature Physics*, 5(3):181–188, Mar 2009.
 - [2] J. K. Korbicz, P. Horodecki, and R. Horodecki. Objectivity through state broadcasting: The origins of quantum darwinism, 2013.
 - [3] J. K. Korbicz. Roads to objectivity: Quantum Darwinism, Spectrum Broadcast Structures, and Strong quantum Darwinism – a review. *Quantum*, 5:571, November 2021.
 - [4] R Horodecki, J K Korbicz, and P Horodecki. Quantum origins of objectivity. *Phys. Rev. A*, 91(3), March 2015.
 - [5] Wojciech H. Zurek. Quantum darwinism, classical reality, and the randomness of quantum jumps. *Physics Today*, 67(10):44–50, Oct 2014.
 - [6] Fernando G. S. L. Brandão, Marco Piani, and Pawel Horodecki. Generic emergence of classical features in quantum darwinism. *Nature Communications*, 6(1):7908, Aug 2015.
 - [7] Harold Ollivier, David Poulin, and Wojciech H Zurek. Objective properties from subjective quantum states: environment as a witness. *Phys. Rev. Lett.*, 93(22):220401, November 2004.
 - [8] Harold Ollivier and Wojciech H. Zurek. Quantum discord: A measure of the quantumness of correlations. *Phys. Rev. Lett.*, 88:017901, Dec 2001.
 - [9] Michael Zwolak and Wojciech H. Zurek. Complementarity of quantum discord and classically accessible information. *Scientific Reports*, 3(1):1729, May 2013.
 - [10] Wojciech Hubert Zurek. Quantum discord and maxwell’s demons. *Phys. Rev. A*, 67:012320, Jan 2003.
 - [11] Richard Jozsa and Noah Linden. On the role of entanglement in quantum-computational speed-up. *Proceedings: Mathematical, Physical and Engineering Sciences*, 459(2036):2011–2032, 2003.
 - [12] Guifré Vidal. Efficient classical simulation of slightly entangled quantum computations. *Phys. Rev. Lett.*, 91:147902, Oct 2003.
 - [13] Seth Lloyd. Quantum search without entanglement. *Phys. Rev. A*, 61:010301, Dec 1999.
 - [14] Daniel Gottesman. The heisenberg representation of quantum computers, 1998.
 - [15] Román Orús and José I. Latorre. Universality of entanglement and quantum-computation complexity. *Physical Review A*, 69(5), May 2004.
 - [16] E. Knill and R. Laflamme. Power of one bit of quantum information. *Phys. Rev. Lett.*, 81:5672–5675, Dec 1998.
 - [17] Andris Ambainis, Leonard J. Schulman, and Umesh Vazirani. Computing with highly mixed states, 2000.
 - [18] Animesh Datta and Guifre Vidal. Role of entanglement and correlations in mixed-state quantum computation. *Physical Review A*, 75(4), April 2007.
 - [19] Animesh Datta, Anil Shaji, and Carlton M. Caves. Quantum discord and the power of one qubit. *Phys. Rev. Lett.*, 100:050502, Feb 2008.
 - [20] Bryan Eastin. Simulating concordant computations, 2010.
 - [21] Hugo Cable and Daniel E Browne. Exact and efficient simulation of concordant computation. *New Journal of Physics*, 17(11):113049, nov 2015.
 - [22] Lucien Hardy. Quantum theory from five reasonable axioms, 2001.
 - [23] W. H. Zurek. Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse? *Phys. Rev. D*, 24:1516–1525, Sep 1981.
 - [24] W. H. Zurek. Environment-induced superselection rules. *Phys. Rev. D*, 26:1862–1880, Oct 1982.
 - [25] Wojciech H. Zurek. Decoherence and the transition from quantum to classical – revisited, 2003.
 - [26] Wojciech H. Zurek. Decoherence, einselection, and the quantum origins of the classical. *Rev. Mod. Phys.*, 75:715–775, 2003.
 - [27] C. Jess Riedel and Wojciech H. Zurek. Quantum darwinism in an everyday environment: Huge redundancy in scattered photons. *Phys. Rev. Lett.*, 105:020404, Jul 2010.
 - [28] C Jess Riedel and Wojciech H Zurek. Redundant information from thermal illumination: quantum darwinism in scattered photons. *New Journal of Physics*, 13(7):073038, jul 2011.
 - [29] Michael Zwolak, H. T. Quan, and Wojciech H. Zurek. Quantum darwinism in a mixed environment. *Phys. Rev. Lett.*, 103:110402, Sep 2009.
 - [30] Michael Zwolak, C. Jess Riedel, and Wojciech H. Zurek. Amplification, decoherence and the acquisition of information by spin environments. *Scientific Reports*, 6(1):25277, May 2016.
 - [31] C Jess Riedel and Wojciech H Zurek. Redundant information from thermal illumination: quantum darwinism in scattered photons. *New Journal of Physics*, 13(7):073038, jul 2011.
 - [32] Robin Blume-Kohout and Wojciech H Zurek. Quantum darwinism: Entanglement, branches, and the emergent classicality of redundantly stored quantum information. *Phys. Rev. A*, 73(6), June 2006.
 - [33] Harold Ollivier, David Poulin, and Wojciech H. Zurek. Quantum darwinism in an everyday environment. *Phys. Rev. Lett.*, 95:230502, 2005.
 - [34] Michael Zwolak and Wojciech H. Zurek. Redundancy of einselected information in quantum darwinism: The irrelevance of irrelevant environment bits. *Phys. Rev. A*, 95:030101, Mar 2017.
 - [35] C Jess Riedel, Wojciech H Zurek, and Michael Zwolak. The rise and fall of redundancy in decoher-

- ence and quantum darwinism. *New Journal of Physics*, 14(8):083010, aug 2012.
- [36] Lin Chen, Eric Chitambar, Kavan Modi, and Giovanni Vacanti. Detecting multipartite classical states and their resemblances. *Phys. Rev. A*, 83:020101, Feb 2011.
 - [37] Thao P Le and Alexandra Olaya-Castro. Strong quantum darwinism and strong independence are equivalent to spectrum broadcast structure. *Phys. Rev. Lett.*, 122(1):010403, January 2019.
 - [38] Wojciech Hubert Zurek. Decoherence, einselection, and the quantum origins of the classical. *Rev. Mod. Phys.*, 75:715–775, May 2003.
 - [39] Kavan Modi, Aharon Brodutch, Hugo Cable, Tomasz Paterek, and Vlatko Vedral. The classical-quantum boundary for correlations: Discord and related measures. *Rev. Mod. Phys.*, 84:1655–1707, Nov 2012.
 - [40] Vaibhav Madhok and Animesh Datta. Interpreting quantum discord through quantum state merging. *Phys. Rev. A*, 83:032323, Mar 2011.
 - [41] Vaibhav Madhok and Animesh Datta. Quantum discord in quantum information theory – from strong subadditivity to the mother protocol. In Dave Bacon, Miguel Martin-Delgado, and Martin Roetteler, editors, *Theory of Quantum Computation, Communication, and Cryptography*, pages 188–197, Berlin, Heidelberg, 2014. Springer Berlin Heidelberg.
 - [42] Vaibhav Madhok and Animesh Datta. Quantum discord as a resource in quantum communication. *International Journal of Modern Physics B*, 27(01n03):1345041, 2013.
 - [43] Eric Chitambar and Gilad Gour. Quantum resource theories. *Reviews of modern physics*, 91(2):025001, 2019.
 - [44] M. Van den Nest. Simulating quantum computers with probabilistic methods, 2010.
 - [45] Charles H Bennett and Gilles Brassard. Quantum cryptography: Public key distribution and coin tossing. *Theoretical computer science*, 560:7–11, 2014.
 - [46] Charles H Bennett and Gilles Brassard. Quantum cryptography and its application to provably secure key expansion, public-key distribution, and coin-tossing. In *IEEE International Symposium on Information Theory*, volume 95. St-Jovite: Qebec Press, 1983.
 - [47] Jonathan Barrett. Information processing in generalized probabilistic theories. *Phys. Rev. A*, 75:032304, Mar 2007.
 - [48] Yichen Huang. Computing quantum discord is np-complete. *New Journal of Physics*, 16(3):033027, mar 2014.

Appendix A: Local Basis Finder Algorithm [21]

Let $b \subseteq \{1, \dots, n\}$ be the set of qubits on which a gate G_t acts nontrivially, and let X_k denote computational basis projectors of arbitrary rank acting on these qubits. Assume that L_{t-1} is the local basis at time step $t-1$ (initially the identity). Define the transformed operator:

$$\tilde{X}_k = G_t L_{t-1} X_k L_{t-1}^\dagger G_t^\dagger.$$

The Local Basis Finder (LBF) attempts to identify product of local rank-one projectors that diagonalizes each $\tilde{X}_k$ via the following steps:

a. Step 1: Solve the Commutator Condition For each $\tilde{X}_k$, and for each qubit $j \in b$, find all rank-one projectors $\rho^{(j)}$ satisfying:

$$[\mathbb{I}^{(b/j)} \otimes \rho^{(j)}, \tilde{X}_k] = 0.$$

These projectors form the set of candidate local basis elements on qubit j compatible with $\tilde{X}_k$.

b. Step 2: Determine the X_k -Unique Local Basis Let LB_k be the collection of local rank-one projectors (on each qubit in b) that are common to all solutions of the commutator condition for the given $\tilde{X}_k$.

That is,

$$LB_k^{(j)} = \bigcap \left\{ \text{local basis solutions of } [\mathbb{I}^{(b/j)} \otimes \rho^{(j)}, \tilde{X}_k] = 0 \right\},$$

for each $j \in b$.

c. Step 3: Select the Set χ Define the subset $\chi \subseteq \{X_k\}$ of projectors for which a unique local basis solution LB_k exists (i.e., for which LB_k is nonempty and consists of a single orthonormal basis of rank-one projectors).

d. Step 4: Compute the Compatible Local Basis If such a set χ exists, define the new local basis L_t as the intersection of the LB_k for all $X_k \in \chi$:

$$L_t = \bigcap_{X_k \in \chi} LB_k.$$

If this intersection yields a consistent product basis (i.e., a complete orthonormal basis of rank-one projectors on each qubit), then the LBF succeeds and the new local basis is updated.

e. Step 5: Failure Condition If the intersection in Step 4 is empty or contains incompatible local projectors (e.g., non-commuting sets), then the LBF fails. This occurs when:

$$\bigcap_{X_k \in \chi} LB_k = \emptyset,$$

or when the local bases inferred from different projectors are not simultaneously diagonalizable.

f. Conclusion When the LBF succeeds, it yields a product of local rank-one projectors that define the unique local basis at time t in which the state remains concordant. When it fails, no such unique basis exists due to degeneracy or basis ambiguity introduced by the gate G_t .

Appendix B: Explicit Example of LBF Failure for a Two-Qubit Gate

In this appendix, we provide a concrete example of a two-qubit gate of the form $G = VPBU$ for which the Local Basis Finder (LBF) algorithm fails, even when applied to all computational-basis projectors of all ranks. This failure arises due to basis ambiguity on one qubit: no single orthonormal basis diagonalizes the reduced states resulting from all transformed projectors.

a. Gate Construction. We define each component of the gate as follows:

- $U = I_4$ (identity),

- B = block-diagonal unitary acting nontrivially on the subspace $\text{span}\{|10\rangle, |11\rangle\}$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{bmatrix}, \quad \theta = \frac{\pi}{4}$$

- P is the classical permutation that swaps $|10\rangle \leftrightarrow |11\rangle$,
- $V = R_z(\pi/4) \otimes H$, where

$$R_z(\phi) = \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The total gate is then $G = VPB$.

b. Procedure. We compute the transformed projectors $\tilde{X}_k = GX_kG^\dagger$ for all computational-basis projectors X_k of ranks 1 to 4 (15 total). For each transformed projector, we compute the reduced density matrix on qubit B , denoted $\rho_B^{(k)}$, and extract its eigenbasis. The LBF is said to fail if there is no common orthonormal basis in which all $\rho_B^{(k)}$ are diagonal.

c. Results for Rank-1 Projectors.

- $X_1 = |00\rangle\langle 00| \Rightarrow \rho_B^{(1)} = |+\rangle\langle +|$ (diagonal in X -basis),
- $X_2 = |01\rangle\langle 01| \Rightarrow \rho_B^{(2)} = |-\rangle\langle -|$ (diagonal in X -basis),
- $X_3 = |10\rangle\langle 10| \Rightarrow \rho_B^{(3)} = |0\rangle\langle 0|$ (diagonal in Z -basis),
- $X_4 = |11\rangle\langle 11| \Rightarrow \rho_B^{(4)} = |1\rangle\langle 1|$ (diagonal in Z -basis).

These projectors yield reduced density matrices that are diagonal in *different and incompatible bases* (e.g., X vs Z basis), already demonstrating basis ambiguity.

d. Conclusion. There is no single orthonormal basis on qubit B in which all $\rho_B^{(k)}$ are diagonal. Therefore, the LBF algorithm fails in this instance due to basis ambiguity, despite the gate being concordance-preserving. This illustrates that LBF failure can persist even when all possible computational-basis projectors are considered.

e. Remark. If the block-diagonal unitary B is removed (i.e., $B = I$), then all reduced states $\rho_B^{(k)}$ become diagonal in a common basis—the X -basis in this example—and the LBF algorithm successfully identifies the local basis. Hence, the presence of B introduces degeneracy and basis ambiguity, which are responsible for the failure.

Appendix C: Intractability of the Local Basis Finder with Unknown Initial Basis

The Local Basis Finder (LBF) algorithm developed by Cable and Browne provides a classical simulation method for quantum circuits acting on concordant states—states that are diagonal in a local product basis. A crucial assumption in this simulation framework is that the initial local basis L_0 is known. This knowledge is used to track how the local basis changes under each gate in the circuit.

In this appendix, we show that when the initial basis L_0 is *not known*, the simulation task becomes computationally intractable. Specifically, the problem of identifying a local product basis in which a given density matrix is diagonal is known to be NP-hard [48]. Therefore, even if the input state is guaranteed to be concordant, the inability to identify L_0 renders the LBF unusable.

LBF Requires the Initial Basis

At each time step t , for a gate G_t acting on a set of qubits b , the LBF attempts to decompose the gate as

$$G_t = L_t^{(b)} P_t B_t L_{t-1}^{(b)\dagger},$$

where:

- $L_{t-1}^{(b)}$ is the local basis before the gate (assumed known),
- $L_t^{(b)}$ is the updated basis after the gate,
- P_t is a classical reversible permutation in the local basis,
- B_t is a block-diagonal unitary that preserves concordance.

This decomposition is constructed by conjugating computational basis projectors X_k with $G_t L_{t-1}$ and checking whether the resulting operators

$$\tilde{X}_k = G_t L_{t-1} X_k L_{t-1}^\dagger G_t^\dagger$$

can be written as products of local, rank-one projectors. The consistency of the resulting bases then defines L_t . This entire procedure critically depends on the knowledge of L_{t-1} , and therefore on the initial basis L_0 .

Unknown Initial Basis Implies NP-Hardness

Suppose that the simulation begins with a known concordant density matrix ρ_0 , but the basis L_0 in which it is diagonal is not given. Then, the first task is to recover L_0 from ρ_0 , i.e., to find a local product basis $\{|\ell_j\rangle\}$ such that:

$$\rho_0 = \sum_j p_j |\ell_j\rangle\langle\ell_j|, \quad \text{with each } |\ell_j\rangle = \bigotimes_i |\ell_j^{(i)}\rangle.$$

This problem is known to be NP-complete. In particular, It was shown that the problem of deciding whether a density matrix is diagonal in any local product basis and also calculating the discord of it is NP-complete, even when promised that the matrix is concordant [48].

The computational difficulty arises because:

- The space of local product bases is exponentially large in the number of qubits.
- Diagonalization in a given product basis is efficient, but finding such a basis requires a global search over non-convex constraints.
- There is no efficient certificate or algorithm that allows basis reconstruction without performing such a search.

Hence, the moment the initial basis L_0 is unknown, the LBF cannot be initialized, and the simulation cannot proceed.

Quantum-Theoretic Perspective

This intractability has a fundamental quantum origin. In quantum mechanics, diagonalization of a density matrix does not determine a unique local product basis unless the eigenbasis is itself a product basis. The mere knowledge that a state is diagonal in *some* product basis does not reveal which one. This is in contrast to classical probability distributions, where the basis (i.e., the sample space) is part of the structure.

This structural non-uniqueness is a manifestation of the quantum feature that any unitary transformation between pure states is allowed as long as it preserves inner products. In particular, within a degenerate eigenspace, quantum theory places no restriction on which orthonormal basis is chosen. Thus, the identity of the product basis underlying a concordant state is hidden unless it is externally specified.

Consequences for Simulation

The implications of an unknown initial basis are severe. Even with a promise that the state is concordant and the circuit preserves concordance at every step, the inability to identify L_0 leads to a breakdown in the classical simulation method. Specifically:

- The LBF cannot simulate the first gate without L_0 .
- Any attempt to reconstruct L_0 from ρ_0 is computationally intractable in general.
- Therefore, the entire simulation becomes classically intractable.

1. Summary

The LBF algorithm provides an efficient classical simulation method for concordant computation, provided that the initial local basis is known. If this basis is not known, the simulation problem becomes equivalent to solving a classically hard inverse problem: identifying the product structure of a concordant state. This is known to be NP-complete, highlighting a deep connection between classical intractability and the structural freedom of basis representations in quantum theory.

Scenario	Complexity of Simulation
Known initial basis L_0	Efficient (polynomial time for fixed gate size)
Unknown initial basis L_0	Intractable (NP-complete to recover)
Given only ρ_0	Concordance verification is NP-complete
