

Efficient Optimized Degree Realization: Minimum Dominating Set & Maximum Matching*

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October 28, 2025

Abstract

The DEGREE REALIZATION problem requires, given a sequence d of n positive integers, to decide whether there exists a graph whose degrees correspond to d , and to construct such a graph if it exists. A more challenging variant of the problem arises when d has many different realizations, and some of them may be more desirable than others. We study *optimized realization* problems in which the goal is to compute a realization that optimizes some quality measure. Efficient algorithms are known for the problems of finding a realization with the maximum clique, the maximum independent set, or the minimum vertex cover. In this paper, we focus on two problems for which such algorithms were not known. The first is the DEGREE REALIZATION with MINIMUM DOMINATING SET problem, where the goal is to find a realization whose minimum dominating set is minimized among all the realizations of the given sequence d . The second is the DEGREE REALIZATION with MAXIMUM MATCHING problem, where the goal is to find a realization with the largest matching among all the realizations of d . We present polynomial time realization algorithms for these two open problems.

A related problem of interest and importance is *characterizing* the sequences with a given value of the optimized function. This leads to an efficient computation of the optimized value without providing the realization that achieves that value. For the MAXIMUM MATCHING problem, a succinct characterization of degree sequences with a maximum matching of a given size was known. This paper provides a succinct characterization of sequences with minimum dominating set of a given size.

*The results have appeared earlier in “Igor Kalinichev, Efficient Optimized Degree Realization: Minimum Dominating Set & Maximum Matching, M.Sc. Thesis, The Weizmann Institute of Science, August 2025.”

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1 Introduction

Given a non-increasing sequence $d = (d_1, \dots, d_n)$ of positive integers, the DEGREE REALIZATION (DR) problem requires to decide if d is the degree sequence of some n -vertex (simple undirected) graph $G = (V, E)$, that is, $\deg_G(i) = d_i$ for every $i \in [1, n]$, and to construct such a graph if it exists. In such a case, we say that d is *graphic*. For instance, the sequence $d = (4, 3, 2, 1, 1)$ cannot be realized since $\sum_i d_i$ is odd, and $d' = (4, 3, 1, 1, 1)$ cannot be realized despite the fact that $\sum_i d'_i$ is even. In contrast, $d'' = (4, 3, 2, 2, 1)$ is graphic.

The two key questions studied extensively in the past concern identifying characterizations (or, necessary and sufficient conditions) for a sequence to be graphic, and developing effective and efficient algorithms for finding a realizing graph for a given sequence if it exists. A necessary and sufficient condition for a given sequence of integers to be graphic (also implying an $O(n)$ decision algorithm) was presented by Erdős and Gallai in [9]. (For alternative proofs see [1, 6, 8, 30, 31, 32].) Havel [19] and Hakimi [18] described an $O(\sum_i d_i)$ -time algorithm that given a sequence d of integers proves that the given sequence is not graphic or computes an m -edge graph realizing it, where $m = \frac{1}{2} \sum_{i=1}^n d_i$.

A more challenging variant of the problem arises when the given sequence has many possible realizing graphs, but some realizations are more desirable than others in various ways. In such a case, it is of interest to look for a realizing graph that also optimizes some quality measure. Hereafter, we refer to such problems as *optimized realization* problems.

For example, let us consider the classical MAXIMUM CLIQUE (MC) problem. For a graph $G = (V, E)$, a clique is a vertex set $Q \subseteq V$ such that $(v, w) \in E$ for every $v, w \in Q$. The size of the maximum clique in G is denoted $\text{MC}(G) = \max\{|Q| \mid Q \text{ is a clique in } G\}$. It may be desirable to find, for a given graphic sequence d , a realizing graph G with the maximum possible clique. The resulting optimized realization problem is formally defined as follows. Letting $\text{MC}(d) = \max\{\text{MC}(G) \mid G \text{ is a realization of } d\}$, the MAXIMUM CLIQUE DEGREE REALIZATION (MC-DR) problem requires to find a realizing graph G attaining $\text{MC}(d)$.

Similarly, one may define optimized realization problems corresponding to other graph optimization problems, including MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MAXIMUM MATCHING, and MINIMUM DOMINATING SET. Their optimal values on a given graph G are defined as $\text{MIS}(G)$, $\text{MVC}(G)$, $\text{MM}(G)$, and $\text{MDS}(G)$, respectively. The corresponding optimized realization problems on degree sequences are named MIS DEGREE REALIZATION (MIS-DR), MVC DEGREE REALIZATION (MVC-DR), MM DEGREE REALIZATION (MM-DR), and MDS DEGREE REALIZATION (MDS-DR), and their optimal values on a given sequence d are denoted by $\text{MIS}(d)$, $\text{MVC}(d)$, $\text{MM}(d)$, and $\text{MDS}(d)$.

As is well known, the graph versions of most of these optimization problems are NP-hard. In contrast, the optimized realization versions of the Maximum Clique, Minimum Vertex Cover and Maximum Independent Set problems have been shown to be polynomial-time solvable. Rao [26] gave a characterization of sequences which can be realized by a graph containing K_ℓ (i.e., a clique of size ℓ). This result was based on a phenomenon collectively known as the *prefix lemma*, which for MC says that if d has a realization containing K_ℓ , then it has a realization such that the ℓ vertices of maximum degree induce a clique K_ℓ . This result implies a polynomial-time algorithm for MC-DR. Recalling that MIS is equivalent to MC in the complement graph, and since the set of all the vertices that are not in an independent set are a vertex cover, it follows that polynomial-time algorithms also exist for MIS-DR and MVC-DR. Note that a prefix lemma also applies to MVC. This prefix property is satisfied also by MDS [16] and MM [17]. However, so far it was not known how to exploit this property in order to derive a polynomial time algorithm for MM-DR or for MDS-DR. These two problems were handled in some special cases in [5] and in [16] respectively, but the general problems were left open.

Our results. This paper provides polynomial time realization algorithms for both MDS-DR and MM-DR. The algorithm for MDS-DR makes use of an existing prefix lemma [16] while the algorithm for MM-DR makes use of a stronger version of an existing prefix lemma [17], established in the current paper.

In addition, we develop Erdős-Gallai like characterizations for MDS-DR. These characterizations can be used to efficiently compute the size of the minimum dominating set of a sequence without providing a realization by searching for the size of the minimum dominating set. Interestingly, our characterizations for the MDS-DR problem are based on the construction and correctness of our realization algorithm, and in turn, the characterizations help us reducing the complexity of the realization algorithm, because the realization algorithm does not need to search for the size of the minimum dominating set. A similar complexity reduction is possible also for the MM-DR problem for which characterizations are already known [11].

As mentioned above, there are several known algorithms for finding a graph G realizing d , including the well-known Havel–Hakimi algorithm [19, 18]. For our purposes, though, we need to use a somewhat lesser-known but highly versatile algorithm due to Fulkerson, Hoffman and McAndrew [14], hereafter named the FHM algorithm, which is based on (i) realizing (d, d) by a *bipartite* graph $\hat{G}$, (ii) converting $\hat{G}$ to a half integral general (non-bipartite) realization G^ω for d , and (iii) rounding G^ω to an integral general realization G . Our approach is based on modifying step (i) so that the bipartite realization $\hat{G}$ has optimal MDS (or MM), and preserving this property during the transformations of steps (ii) and (iii). The challenging obstacle is that the rounding process of step (iii) is oblivious to the issue of MDS (or MM) size, so a bipartite $\hat{G}$ with small MDS might be transformed into a general G with large MDS. Hence, it is necessary to modify the FHM algorithm in non-trivial ways in order to solve the MDS-DR and MM-DR optimized realization problems¹.

Related work. One related direction involves network realization with a given subgraph. Kundu [22] showed that the sequences d and the component-wise difference $d - d'$, such that $d'_i \in \{k, k + 1\}$ and $d' \leq d$ (component-wise), are graphic only if there exists a graph G realizing d that has a subgraph G' realizing d' . This result can be used to decide whether a given sequences has a perfect matching by assigning $d'_i = 1$ for every i . Kleitman and Wang [21] gave an algorithm for computing a realization of d that contains a subgraph which realizes d' . Their algorithm can be used to compute a realization of d that contains a perfect matching, if one exists. Extensions of the above result were presented in [21, 24].

Rao and Rao [27] and Kundu [22] gave a characterization of sequences that can be realized by a Hamiltonian graph. Chungphaisan [7] gave an algorithm, that given a sequence d , constructs a realization with a Hamiltonian cycle (or path), if one exists. Rao [26] characterized sequences that can be realized by a graph containing K_ℓ (a clique of size ℓ). This result was based on the Prefix Lemma for MC. An alternative proof was given in [20], and a constructive proof was presented in [34]. It follows that MC-DR can be solved in polynomial time by checking if d has a realization containing K_ℓ , for $\ell \in [2, n]$, where $[i, j] = \{i, \dots, j\}$, for $i \leq j$. In contrast, it is NP-hard to approximate MC within a ratio of $O(n^{1-\varepsilon})$, for any $\varepsilon > 0$ [35]. Gould, Jackson and Lehel [17] extended Rao’s result by showing that if d has a realization containing H as a subgraph (but not necessarily an induced sub-graph), then there exists a realization of d containing H such that the vertices of H have the $|V(H)|$ largest degrees. Yin [33] used this result to further extend the result of Rao by giving a characterization of graphic sequences that contains a split graph $S_{r,s}$ composed of a clique of size r and an independent set of size s . (Observe that $S_{r,1}$ is a clique of size $r + 1$.)

Gentner, Henning and Rautenbach [16] proved the prefix lemma for MDS-DR and gave a realization algorithm for MDS-DR on sequences with $d_1 = O(1)$, leaving the general case open. They also provided characterizations for MIS-DR and MDS-DR in forests. Gentner, Henning and Rautenbach [15] gave characterizations to realizations that minimize the maximum independent set and that maximize the minimum dominating set in forests. Note that MDS is not approximable within $\alpha \log n$, for some $\alpha > 0$, unless $P = NP$ [29], and within $(1 - \varepsilon) \log n$, for any $\varepsilon > 0$, unless $NP \subseteq DTIME(n^{\log \log n})$ [12].

¹The same approach can provide an alternative algorithm for MC-DR and for MVC-DR (or MIS-DR). We omit the details.

Bock and Rautenbach [5] studied MM-DR in trees and bipartite graphs, where the partition is given. The result on bipartite graphs is based on a stronger version of the prefix lemma that focuses on a specific matching (see Lemma 4.3). Recently, Erdős et al. [11] studied MM-DR. Given a sequence d and an integer ν , they presented an Erdős-Gallai type characterization, based on a system of $O(n)$ inequalities, which is satisfied if and only if d has a realization with a matching of size ν . Applying these characterizations to a given degree sequence, the maximum size of a matching can be computed efficiently, although the specific realization is not provided.

Fulkerson, Hoffman and McAndrew [14] obtained conditions for the existence of an f -factor that are applicable only to the family of multi-graphs that satisfy the so called *odd cycle condition*. Kundu [23] used this result to provide simplified conditions for the factorization of graphs that satisfy the odd cycle condition. Rao [28] and Yin [33] also used the result of [14]. Anstee used the technique of [14] to provide an algorithmic proof of the f -factor theorem [2] and a simplified characterization for the existence of a (g, f) -factor for special cases [3].

2 Realization with Minimum Dominating Set

In this section, we describe an algorithm for constructing a realization of a given graphic sequence d , which in addition has a dominating set D of the minimum size γ among all the possible realizations. For this, we employ the Prefix Lemma 2.1 [16] and a suitable modification of the FHM realization algorithm [14].

Our algorithm proceeds in several steps, presented in the coming subsections. First, the MDS-DR problem over general graph is reduced to the same problem over bipartite graphs. It is done by a modification of the FHM realization algorithm that ensures preservation of a dominating set. Then, the MDS-DR problem over bipartite graphs is reduced to a maximum flow problem. To be more specific, given a candidate size γ for the minimum dominating set and a degree sequence pair (d, d) , we construct a bipartite flow graph $G_{d,\gamma}$, such that if the maximum flow attained in $G_{d,\gamma}$ equals $\sum_{i=1}^n d_i$, then it corresponds to a realization $\hat{G}$ of (d, d) with a dominating set of size 2γ . The Prefix Lemma 2.1 narrows down the search of the dominating set to a polynomial number of candidates, which allows solving MDS-DR in polynomial time.

2.1 Prefix Lemma

Define a γ -*prefix-dominated realization* of the sequence d to be a realization, where the vertices with the γ highest degrees (i.e., $d_1, d_2, \dots, d_\gamma$) form a dominating set.

Lemma 2.1. (Prefix Lemma for MDS) [16] *If a sequence d has a realization with a minimum dominating set of size γ , then d has γ -prefix-dominated realization.*

Although the prefix lemma states that if $\gamma = \text{MDS}(d)$ then there is a realization with dominating set on γ vertices of highest degrees, it is not immediately clear how to select edges so as to obtain this realization, hence additional ideas are needed.

2.2 Reduction to a Bipartite Sequence Pair

A bipartite graph $\hat{G} = (V, W, \hat{E})$, where $V = \{v_1, v_2, \dots, v_n\}$ and $W = \{w_1, w_2, \dots, w_n\}$, is a γ -*prefix-dominated realization* for the sequence pair (d, d) if it satisfies the following properties.

- (D1) $\hat{G}$ realizes the sequence pair (d, d) .
- (D2) $(v_i, w_i) \notin \hat{E}$ for every $i \in [1, n]$.
- (D3) $\hat{D} = \hat{D}_V \cup \hat{D}_W$ is a dominating set in $\hat{G}$, where $\hat{D}_V = \{v_1, \dots, v_\gamma\}$ and $\hat{D}_W = \{w_1, \dots, w_\gamma\}$ are prefixes of V and W , respectively.

We describe a polynomial time algorithm based on the FHM algorithm [14], that given a γ -prefix dominated realization $\hat{G}$ for the sequence pair (d, d) produces a γ -prefix dominated realization G for d .

Step 1: Compute a half-integral solution.

- a. For all $i, j \in [1, n]$, let $y_{ij} = \begin{cases} 1, & \{v_i, w_j\} \in \hat{E}, \\ 0, & \text{otherwise,} \end{cases}$ and $\omega(i, j) = \frac{1}{2}(y_{ij} + y_{ji})$.
- b. Define a weighted graph $G^\omega = (V^\omega, E^\omega, \omega)$ with vertex set $V^\omega = [1, n]$ and an edge $e = (i, j)$ of weight $\omega(e) = \omega(i, j)$ for every $i, j \in V^\omega$. Clearly, ω is half-integral.
- c. Define the *weighted degree* of a vertex $i \in V^\omega$ to be $d^\omega(i) = \sum_{j \in V^\omega} \omega(i, j)$. Note that G^ω realizes d in the *weighted* sense, namely,
$$d^\omega(i) = \sum_{j \in V^\omega} \omega(i, j) = \frac{1}{2} \left(\sum_{j=1}^n y_{ij} + \sum_{j=1}^n y_{ji} \right) = d_i, \text{ for any } i \in V^\omega.$$
- d. Partition the vertex set V^ω of G^ω into $D = [1, \gamma]$ and $S = [\gamma + 1, n]$. Note that by construction, D is a dominating set for G^ω , and moreover, the total weight of the edges connecting any nondominating vertex in S with its dominating neighbours in D is at least one. Indeed, for any $s \in S$, by property (D3),

$$\sum_{x \in D} \omega(s, x) = \sum_{j \in [1, \gamma]} \omega(s, j) = \frac{1}{2} \sum_{j \in [1, \gamma]} y_{sj} + \frac{1}{2} \sum_{j \in [1, \gamma]} y_{js} \geq \frac{1}{2} + \frac{1}{2} = 1. \quad (1)$$

Step 2: Preparing for discarding non-integral weights while keeping the degrees.

Construct a graph $G^{1/2} = (V^{1/2}, E^{1/2})$ by removing from G^ω the edges of integral weight and keeping only those of weight $1/2$. Formally, $V^{1/2} = V^\omega$ and $E^{1/2} = \{e \in E^\omega \mid \omega(e) = 1/2\}$.

Observation 2.2. *The graph $G^{1/2}$ is even (namely, all its vertex degrees are even).*

Proof. Each vertex $i \in V^\omega$ has an integral weighted degree $d^\omega(i)$, so the number of edges of weight $1/2$ incident to i must be even. $\square$

The first two steps are similar to the FHM algorithm [14], and the main changes w.r.t. that algorithm occur in the subsequent steps, and pertain to the process of modifying G^ω and getting rid of non-integral weights while keeping the degrees unchanged without violating Inequality (1). The modifications happen concurrently on the graphs G^ω and $G^{1/2}$ (i.e., whenever changing the weight of some edge e in G^ω from 0 or 1 to $1/2$ or vice versa, $G^{1/2}$ is modified accordingly, adding or removing the edge e).

Specifying the modifications require the following definition. A 4-vertex path $P[a, s, b, c]$ in $G^{1/2}$ is a *2-dom path* if $s \in S$ and $a, b \in D$, i.e., the nondominating s has two neighboring dominators.

Step 3: Eliminate 2-dom paths.

The next step in the algorithm transforms the weights in G^ω until it is free of 2-dom paths. This is done as follows. While there is a 2-dom path in $G^{1/2}$, apply one of the following three *modification rules* to the edge weights in G^ω , according to a weight of the edge (a, c) in G^ω . The different possible situations are visualized in Figure 1. All figures in this section maintain the convention that black nodes are dominating, white nodes are nondominating and gray nodes can be either. Furthermore, solid lines represent edges of positive weight and dashed lines represent edges of weight 0.

(MR1) If $\omega(a, c) = 0$, then set $\omega(a, s) \leftarrow 0$, $\omega(s, b) \leftarrow 1$, $\omega(b, c) \leftarrow 0$ and $\omega(a, c) \leftarrow 1/2$.

(MR2) If $\omega(a, c) = 1/2$, then set $\omega(a, s) \leftarrow 1$, $\omega(s, b) \leftarrow 0$, $\omega(b, c) \leftarrow 1$ and $\omega(a, c) \leftarrow 0$.

(MR3) If $\omega(a, c) = 1$, then set $\omega(a, s) \leftarrow 1$, $\omega(s, b) \leftarrow 0$, $\omega(b, c) \leftarrow 1$ and $\omega(a, c) \leftarrow 1/2$.

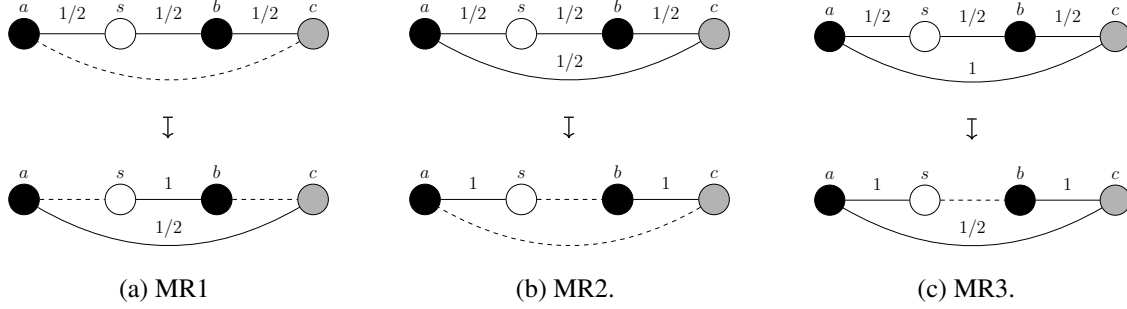


Figure 1: Illustration to the modification rules.

Note that the modifications preserve the weighted degree of every vertex in G^ω and the total weight of the edges between any nondominating vertex and its dominating neighbors. Moreover, each time a modification is applied, the number of edges in $G^{1/2}$ decreases. Therefore, the 2-dom paths are eliminated from $G^{1/2}$ (with the corresponding paths eliminated from G^ω) within a polynomial number of steps.

Step 4: Separate special cycles.

At the start of this step, $G^{1/2}$ is free of 2-dom paths. Let S' be the set of all the nondominating vertices in G^ω that are connected to a dominator vertex with at least one edge $\hat{e}$ with $\omega(\hat{e}) = 1$. Choose such an edge $\hat{e}_s$ arbitrarily for each vertex $s \in S'$ and denote the collection of these edges by $E' = \{\hat{e}_s \mid s \in S'\}$.

Next consider the set $S^\Delta = S \setminus S'$ of the remaining nondominating vertices. Since every $s \in S^\Delta$ is not connected to any vertex in D with an edge of weight one in G^ω , it has at least two dominating neighbours in $G^{1/2}$ by Inequality (1). Choose arbitrarily two such neighbours $a_s, b_s \in D$ for every $s \in S^\Delta$.

Observation 2.3. *For every $s \in S^\Delta$, a_s and b_s are not connected to any vertices in $G^{1/2}$ besides possibly each other and s .*

Proof. Having another vertex c neighboring, say, b_s , in $G^{1/2}$ would imply the existence of a 2-dom path $P[a_s, s, b_s, c]$ in $G^{1/2}$, leading to a contradiction. $\square$

In all modifications performed in subsequent steps of the algorithm, edges are only removed from (but never added to) $G^{1/2}$, so Lemma 2.3 continues to hold until the end of the algorithm's execution.

Since a_s and b_s have integral weighted degrees in G^ω , Lemma 2.3 implies that the edge (a_s, b_s) must exist in $G^{1/2}$. It follows that there is a cycle $C[a_s, s, b_s]$ in $G^{1/2}$ for each $s \in S^\Delta$ (see Figure 2). Let $\mathcal{C}^\Delta = \{C[a_s, s, b_s] \mid s \in S^\Delta\}$ be a set of all such cycles.

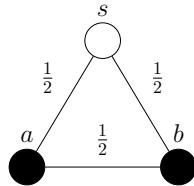


Figure 2: A cycle in $\mathcal{C}^\Delta$.

Observation 2.4. *Any two different cycles $C, C' \in \mathcal{C}^\Delta$ have disjoint vertices.*

Proof. Consider cycles $C = C[s, a_s, b_s]$ and $C' = C[s', a'_s, b'_s]$. By construction, $s \neq s'$, so C and C' can only intersect in a_s or b_s , but this cannot happen by Lemma 2.3. $\square$

Step 5: Partition into cycles.

Denote the set of all the edges of the cycles in $\mathcal{C}^\Delta$ by

$$E(\mathcal{C}^\Delta) = \{(a_s, b_s), (a_s, s), (s, b_s) \mid C[a_s, s, b_s] \in \mathcal{C}^\Delta\}$$

Consider a subgraph H of $G^{1/2}$ with the same vertices, $V(H) = V^{1/2}$, and edges $E(H) = E^{1/2} \setminus E(\mathcal{C}^\Delta)$. Since $G^{1/2}$ is an even graph by Observation 2.2 and the cycles in $\mathcal{C}^\Delta$ are disjoint by Observation 2.4, it follows that H is an even graph. So there is a polynomial time algorithm for partitioning edge set of H into disjoint cycles, such that each cycle contains an entire connected component. Denote the set of these cycles by $\mathcal{C}'$ and let $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}^\Delta$. We call a cycle *even* (resp., *odd*) if it has an even (resp., odd) number of edges.

Observation 2.5. *The number of odd cycles in $\mathcal{C}$ is even.*

Proof. Observe that $\sum_{e \in E^\omega} \omega(e) = \frac{1}{2} \sum_{i=1}^n d_i = m$, where m is the number of edges, which is an integer. Therefore, the number of edges with weight $1/2$ must be even. Since the cycles in $\mathcal{C}$ cover all of $E^{1/2}$ and are disjoint, the observation follows. $\square$

Observation 2.6. *If cycles $C \in \mathcal{C}^\Delta$ and $C' \in \mathcal{C}'$ have a non-empty intersection, then $C \cap C' = \{s\}$ for $s \in S^\Delta$ corresponding to the cycle C .*

Proof. Let $C = C[a_s, s, b_s]$ with $a_s, b_s \in D$ and $s \in S^\Delta$. By Lemma 2.3, a_s and b_s have degree zero in H , so they do not belong to any cycle in $\mathcal{C}'$. $\square$

Step 6: Eliminate even cycles.

For every even cycle $C \in \mathcal{C}$ do the following.

1. Traverse C starting from an arbitrary vertex $x \in C$ and continuing along the cycle until returning to x . Denote the resulting sequence of edges by $E(C) = (e_1, e_2, \dots, e_\ell)$.
2. Increase (respectively, decrease) the weights of the edges on even (resp., odd) positions in the sequence $E(C)$ by $1/2$. That is, for every $i \in [1, \ell]$ set

$$\omega(e_i) \leftarrow \begin{cases} 1, & i \text{ is even,} \\ 0, & i \text{ is odd.} \end{cases}$$

Note that this procedure does not change the weighted degrees in G^ω or the weights of the edges in E' and does not affect other cycles in $\mathcal{C}$. Hereafter, we refer to such modifications as *neutral*.

Step 7: Eliminate odd cycles.

Finally, arrange the odd cycles in $\mathcal{C}$ (whose number is even by Lemma 2.5) in pairs. For every pair (C, C') , proceed according to Case 1 below if the cycles intersect and according to Case 2 otherwise.

Case 1: There is a vertex $x \in C \cap C'$:

1. As before, traverse both cycles starting at x . Denote the resulting sequences of edges in C and C' by $E(C) = (e_1, e_2, \dots, e_\ell)$ and $E(C') = (e'_1, e'_2, \dots, e'_k)$, respectively.
2. Cycles in $\mathcal{C}'$ are disjoint by definition and cycles in $\mathcal{C}^\Delta$ are disjoint by Lemma 2.4. Thus one of the cycles (C, C') belongs to $\mathcal{C}'$ and the other to $\mathcal{C}^\Delta$. Redefine them so $C \in \mathcal{C}^\Delta$ and $C' \in \mathcal{C}'$.
3. For every $i \in [1, \ell]$ and $j \in [1, k]$, modify the edge weights in the cycles as follows:

$$\omega(e_i) \leftarrow \begin{cases} 0, & i \text{ is even,} \\ 1, & i \text{ is odd.} \end{cases} \quad \omega(e'_j) \leftarrow \begin{cases} 1, & j \text{ is even,} \\ 0, & j \text{ is odd.} \end{cases}$$

Note that this modification is neutral.

By Lemma 2.6, $x \in S^\Delta$ and $C = C[x, a_x, b_x]$ for some $a_x, b_x \in D$. Note that x is connected to a_x and b_x in G^ω with edges of weight one after the modification. Neither a_x nor b_x belongs to any other cycles in $\mathcal{C}$ by Lemma 2.3, so the weights $\omega(x, a_x)$ and $\omega(x, b_x)$ are not modified further by the algorithm. Let E_1^Δ contain an edge (x, a_x) for every cycle $C \in \mathcal{C}^\Delta$ that was processed in this case.

Case 2: $C \cap C' = \emptyset$:

First, we have the following lemma.

Lemma 2.7. *For any two disjoint odd cycles $C, C' \in \mathcal{C}$, there exist $x \in C$ and $y \in C'$, such that $(x, y) \notin E'$, $\omega(x, y) \neq 1/2$, and if C or C' belongs to $\mathcal{C}^\Delta$, then the corresponding vertex does not belong to S^Δ .*

Proof. Recall that each cycle in $\mathcal{C}^\Delta$ contains exactly one vertex from S^Δ . Choose vertices $x \in C$ and $y' \in C'$, so if $C \in \mathcal{C}^\Delta$ (respectively, $C' \in \mathcal{C}^\Delta$), then $x \notin S^\Delta$ (respectively, $y' \notin S^\Delta$). If $(x, y') \notin E'$, then the chosen x and $y = y'$ satisfy the Lemma (the condition $\omega(x, y) \neq 1/2$ is proved later). Otherwise, one of the vertices x, y' must be dominating and the other nondominating. Without loss of generality assume $y' \in D$ and $x \in S$. Replace y' with $y \in C'$, such that $y \neq y'$ and if $C' \in \mathcal{C}^\Delta$, then $y \notin S^\Delta$. Since each cycle has at least three vertices, this is always possible. Each edge in E' corresponds to a unique nondominating vertex and (x, y') corresponds to x , so E' does not contain (x, y) .

Next we prove that $\omega(x, y) \neq 1/2$. First, assume that $C, C' \in \mathcal{C}'$. Then by construction, they are not connected by any edge in $E^{1/2} \setminus E(\mathcal{C}^\Delta)$. Assume, towards contradiction, that they are connected by an edge $e \in E(\mathcal{C}^\Delta)$, so e belongs to a cycle $C[s, a_s, b_s]$ with $a_s, b_s \in D$ and $s \in S^\Delta$. It follows that either C or C' contains either a_s or b_s . However it is not possible by Lemma 2.6, contradiction. Now assume that $C = C[s, a_s, b_s] \in \mathcal{C}^\Delta$ with $a_s, b_s \in D$ and $s \in S^\Delta$. Since $x \in C \setminus S^\Delta$, x is either a_s or b_s . However a_s and b_s do not have any edges outside C in $G^{1/2}$ by Lemma 2.3, contradiction. The lemma follows. $\square$

Next we describe how to modify a pair (C, C') .

1. Choose vertices $x \in C$ and $y \in C'$ according to Lemma 2.7.
2. Traverse both cycles starting in x and y accordingly and denote the resulting sequences of edges in C and C' by $E(C) = (e_1, e_2, \dots, e_\ell)$ and $E(C') = (e'_1, e'_2, \dots, e'_k)$ respectively.
3. It follows that $\omega(x, y) \in \{0, 1\}$. Let $\xi = \omega(x, y)$ and perform the following. Set $\omega(x, y) \leftarrow 1 - \xi$ and for every $i \in [1, \ell]$ and $j \in [1, k]$ modify the edge weights in the cycles as follows

$$\omega(e_i) \leftarrow \begin{cases} 1 - \xi, & i \text{ is even,} \\ \xi, & i \text{ is odd,} \end{cases} \quad \omega(e'_j) \leftarrow \begin{cases} 1 - \xi, & j \text{ is even,} \\ \xi, & j \text{ is odd.} \end{cases}$$

Note that this modification is neutral.

If $\mathcal{C}^\Delta$ contains C or C' , then the corresponding $s \in S^\Delta$ is connected to a dominator vertex with an edge $\hat{e}_s$, such that $\omega(\hat{e}_s) = 1$ after applying the modification. Indeed, s was not chosen as a starting vertex x or y , so it is connected to one of its two neighbors a_s or b_s in the corresponding cycle with an edge of weight one. But both its neighbors are dominating vertices, so $\hat{e}_s$ exists. Note that neither a_s nor b_s belongs to any other cycles in $\mathcal{C}$ by Observation 2.3, so the weight $\omega(\hat{e}_s)$ is not modified further by the algorithm. Let E_2^Δ contain all edges $\hat{e}_s$ for $s \in S^\Delta$ that were processed in this case.

Step 8: Generate the output G .

Denote $E^\Delta = E_1^\Delta \cup E_2^\Delta$. For each cycle $C_s \in \mathcal{C}^\Delta$, the corresponding $s \in S^\Delta$ is connected to a dominator vertex with an edge $\hat{e}_s \in E^\Delta$ of weight one. Since there is one-to-one correspondence between $\mathcal{C}^\Delta$ and S^Δ , E^Δ covers all the vertices of S^Δ , in the sense that for every $s \in S^\Delta$ exists the corresponding edge $\hat{e}_s \in E^\Delta$, such that $s \in \hat{e}_s$.

After applying Steps 3–8, each edge in G^ω has weight either one or zero. On the other hand, each step preserves the weighted degrees in G^ω , so G^ω is still a weighted realization of d . It follows that G^ω transforms into a simple graph G with an edge (i, j) whenever $\omega(i, j) = 1$ in G^ω . Clearly, G realizes d .

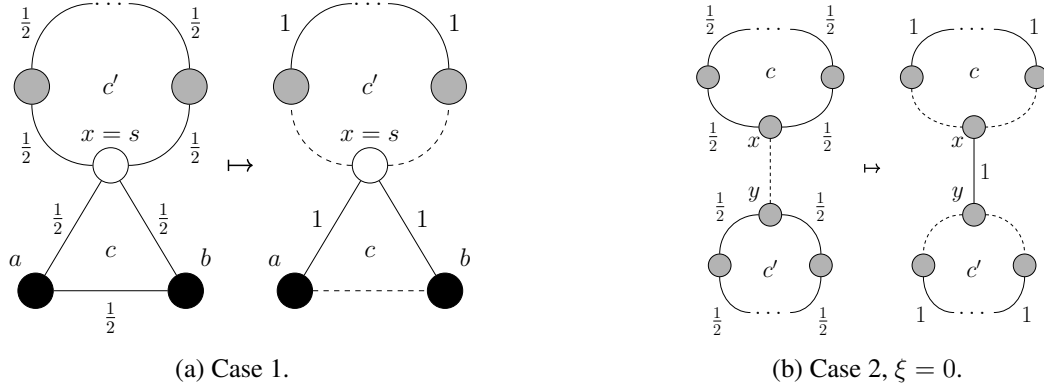


Figure 3: Illustrations of Step 7.

Finally, since $S = S' \cup S^\Delta$, every nondominating vertex $s \in S$ is connecting to the dominating set by some edge in $E' \cup E^\Delta$. These edges have weight one in G^ω , so G contains them all. Therefore, D is a dominating set in G .

Lemma 2.8. *There is a γ -prefix-dominated realization $\hat{G}$ of (d, d) if and only if there is a γ -prefix-dominated realization G of d .*

Proof. ($\Rightarrow$) If there γ -prefix-dominated realization of (d, d) , then the aforementioned algorithm produces a γ -prefix-dominated realization of d .

($\Leftarrow$) Let $G = (V, E)$ with $V = [1, n]$ be a realization of d , where $\deg(i) = d_i$ for $i \in [1, n]$, with a dominating set $D = [1, \gamma]$. Consider the graph $\hat{G} = (\hat{V}, \hat{W}, \hat{E})$ with $\hat{V} = \{v_1, \dots, v_n\}$, $\hat{W} = \{w_1, \dots, w_n\}$ and $(v_i, w_j) \in \hat{E}$ if and only if $(i, j) \in E$. Clearly, $\hat{G}$ realizes (d, d) (satisfying property (D1)) and does not have edges (v_i, w_i) for every $i \in [1, n]$ (satisfying (D2)). Let $\hat{D} = \hat{D}_V \cup \hat{D}_W$, where $\hat{D}_V = \{v_1, \dots, v_\gamma\}$ and $\hat{D}_W = \{w_1, \dots, w_\gamma\}$ are prefixes of $\hat{V}$ and $\hat{W}$, respectively. Since D is a dominating set in G , every vertex in $\hat{W} \setminus \hat{D}_W$ is connected to some dominator vertex in $\hat{D}_V$ and every vertex in $\hat{V} \setminus \hat{D}_V$ is connected to some dominator vertex in $\hat{D}_W$. Thus, $\hat{D}$ is a dominating set in $\hat{G}$ (satisfying (D3)). $\square$

2.3 Reduction to Flow

We have reduced the initial problem of finding a general realization with a dominating set of a certain type to the same problem in a bipartite setting. Hence, we are left with the task of constructing, for a given sequence d with $\text{MDS}(d) = \gamma$, a γ -prefix-dominated realization $\hat{G} = (V, W, \hat{E})$ for the sequence pair (d, d) .

To find the desired realization minimizing the dominating set, we go through every possible value of γ and check if a suitable realizing graph $\hat{G}$ exists by reducing the problem to a flow problem. Specifically, given d and a value γ , we construct a directed bipartite flow graph $G_{d,\gamma}$ with edge set $\mathcal{E}$ as follows. Corresponding to V and W , we have the sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. The node set of the flow graph is $V = X \cup Y \cup X'_S \cup Y'_S \cup \{s, t\}$, where the nodes are organized into the following categories.

- Candidate dominators: $X_D = \{x_i \mid i \in [1, \gamma]\}$ and $Y_D = \{y_j \mid j \in [1, \gamma]\}$,
- Nondominating nodes: $X_S = \{x_i \mid i \in [\gamma + 1, n]\}$ and $Y_S = \{y_j \mid j \in [\gamma + 1, n]\}$,
- Supplementary vertices: $X'_S = \{x'_i \mid i \in [\gamma + 1, n]\}$ and $Y'_S = \{y'_j \mid j \in [\gamma + 1, n]\}$.
- Source and sink: $\{s, t\}$.

The source s is connected to the nodes of X . Similarly, the nodes of Y are connected to the sink t . In addition, the supplementary nodes of X'_S and Y'_S are used to create the connections and flow capacities and enforce the degree and domination constraints. See Figure 4, in which the flow from s to t goes left-to-right.

The edges of $\mathcal{E}$ are capacitated and directed (from left to right). We use the notation (α, β, δ) for an edge that leads from the node α to the node β and can carry up to δ units of flow. The source s has edges leading to every node $x_i \in X$, with capacity d_i , to enforce the degree constraint for the corresponding vertex v_i in the realizing graph G . Similarly, there are edges leading from every node $y_j \in Y$ to the sink t , with capacity d_j , to enforce the degree constraint for the vertices w_j in G .

Additional edges are employed to ensure the correctness of the reduction. We write $\tilde{X} \tilde{\bowtie} \tilde{Y}$ to indicate that the vertices $\tilde{X}$ and $\tilde{Y}$ induce an almost complete bipartite graph in $G_{d,\gamma}$, missing only the edges mentioned in property (D2) (i.e., $\{(x_i, y_i) \mid i \in [1, n]\}$ and $\{(x'_i, y'_i) \mid i \in [\gamma + 1, n]\}$, which are not present in $G_{d,\gamma}$). The various groups of V are connected by $\mathcal{E}$ in the following way: $X_D \tilde{\bowtie} Y_D$, $X_D \tilde{\bowtie} Y_S$, $X_S \tilde{\bowtie} Y_D$ and $X'_S \tilde{\bowtie} Y'_S$ with each edge having capacity one. Finally, every $x_i \in X_S$ is connected to the corresponding $x'_i \in X'_S$ with capacity $d_i - 1$, and similarly, every $y'_j \in Y'_S$ is connected to the corresponding $y_j \in Y_S$ with capacity $d_j - 1$. Denote these connections by $X_S \equiv X'_S$ and $Y_S \equiv Y'_S$. The connections are depicted schematically in Figure 4 and all the edges are listed below.

$$\begin{aligned} \mathcal{E} = & \{(s, x_i, d_i) \mid i \in [1, n]\} \cup \{(y_j, t, d_j) \mid j \in [1, n]\} \cup \{(x_i, y_j, 1) \mid i, j \in [1, \gamma], i \neq j\} \\ & \cup \{(x_i, y_j, 1) \mid i \in [1, \gamma], j \in [\gamma + 1, n]\} \cup \{(x_i, y_j, 1) \mid i \in [\gamma + 1, n], j \in [1, \gamma]\} \\ & \cup \{(x_i, x'_i, d_i - 1) \mid i \in [\gamma + 1, n]\} \cup \{(y'_j, y_j, d_j - 1) \mid j \in [\gamma + 1, n]\} \\ & \cup \{(x'_i, y'_j, 1) \mid i, j \in [\gamma + 1, n], i \neq j\}. \end{aligned}$$

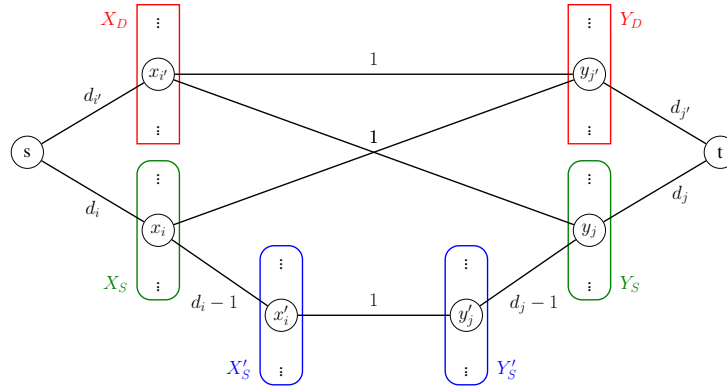


Figure 4: The flow graph.

Next, we compute a maximum flow from the source s to the sink t in $G_{d,\gamma}$. Since all the capacities are integral, there is an integral maximum flow, which can be computed with corresponding algorithms. Hence, we may assume that the computed maximum flow is integral. Let $D = D_A \cup D_B$ with $D_A = \{v_1, \dots, v_\gamma\}$ and $D_B = \{w_1, \dots, w_\gamma\}$. Note that by definition, $D_A = \{v_i \mid x_i \in X_D\}$ and $D_B = \{w_j \mid y_j \in Y_D\}$.

Lemma 2.9. *There exists a bipartite graph $\hat{G} = (V, W, \hat{E})$ that realizes the degree sequence pair (d, d) such that D is a dominating set in $\hat{G}$ if and only if the maximum $s - t$ flow in $G_{d,\gamma}$ is $\sum_{i=1}^n d_i$.*

Proof. ($\Leftarrow$) Suppose the maximum $s-t$ flow in $G_{d,\gamma}$ is $\sum_{i=1}^n d_i$, and let flow be an integral maximum flow. Define the bipartite graph $\hat{G} = (V, W, \hat{E})$, where $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots, w_n\}$, by adding an edge (v_i, w_j) to $\hat{E}$, for every $i, j \in [1, n]$ such that there is an edge e in $G_{d,\gamma}$ of the form (x_i, y_j) or (x'_i, y'_j) with $\text{flow}(e) = 1$. Since the flow is integral, its values on the edges (x_i, y_j) and (x'_i, y'_j) are either 0 or 1.

To verify that $\hat{G}$ correctly realizes (d, d) , we first observe that the selected edges form a graph (rather than a multigraph). Indeed, for any pair $i, j \in [1, n]$, the flow graph $G_{d,\gamma}$ contains either only the edge (x'_i, y'_j) (in case $x_i \in X_S$ and $y_j \in Y_S$) or only the edge (x_i, y_j) (otherwise), but not both.

Next we show that $\deg_G(v_i) = d_i$ for every $i \in [1, n]$ and $\deg_G(w_j) = d_j$ for every $j \in [1, n]$. Note that the total flow from s to t is $\sum_{i=1}^n d_i$, which means that all edges from vertices y_j to t and from s to x_i are saturated. Four cases should be considered.

Case A: If $v_i \in D_A$, then the degree $\deg_{\hat{G}}(v_i)$ of v_i in $\hat{G}$ is set by the number of edges (x_i, y_j) in $\mathcal{E}$ with $\text{flow}(x_i, y_j) = 1$ for $j \in [1, n]$. Since x_i receives a flow of d_i from the source s , by flow conservation it outputs the same amount (in one unit flows) through edges (x_i, y_j) , hence its degree is d_i .

Case B: If $w_j \in D_B$, then $\deg_{\hat{G}}(w_j)$ is determined by the number of edges (x_i, y_j) in $\mathcal{E}$ with $\text{flow}(x_i, y_j) = 1$ for $i \in [1, n]$, and its analysis is similar to Case A.

Case C: If $v_i \in V \setminus D_A$, then $\deg_{\hat{G}}(v_i)$ is determined by the number of edges (x_i, y_j) and (x'_i, y'_j) in $\mathcal{E}$ carrying a flow unit for $j \in [1, n]$. We divide the output flow of v_i into two terms: $\text{flow}(s, x_i) = \sum_{j \in D_B} \text{flow}(x_i, y_j) + \text{flow}(x_i, x'_i)$. The first term increases the degree by the corresponding amount of flow, because each edge (x_i, y_j) has capacity one. The rest of the flow goes to x'_i . Since x'_i outputs flow through the edges (x'_i, y'_j) with unit capacity, it increases the degree of v_i by $\text{flow}(x_i, x'_i)$. Hence, the degree of v_i is $\deg_{\hat{G}}(v_i) = \text{flow}(s, x_i) = d_i$.

Case D: If $w_j \in W \setminus D_B$, then $\deg_{\hat{G}}(w_j)$ is determined by the number of edges (x_i, y_j) and (x'_i, y'_j) in $\mathcal{E}$ carrying a flow unit for $i \in [1, n]$, and its analysis is similar to Case C.

Therefore, $\hat{G}$ correctly realizes the degree sequence pair (d, d) , so property (D1) holds. Also, since $G_{d, \gamma}$ contains neither $\{(x_i, y_i) \mid i \in [1, n]\}$ nor $\{(x'_i, y'_i) \mid i \in [\gamma + 1, n]\}$, it follows that $\hat{G}$ does not have edges $\{(v_i, w_i) \mid i \in [1, n]\}$, hence it satisfies property (D2).

It remains to show property (D3), namely, that D is a dominating set for $\hat{G}$. Since $y_j \in Y_S$ conserves flow and $\text{flow}(y'_j, y_j) \leq d_j - 1$, there is at least one $x_i \in X_D$ with $\text{flow}(x_i, y_j) = 1$. Similarly, for every $x_i \in X_S$ there exists at least one $y_j \in Y_D$ such that $\text{flow}(x_i, y_j) = 1$. It follows that at least one vertex from D is connected to each $v_i \in V \setminus D$ and $w_j \in W \setminus D$.

($\Rightarrow$) Conversely, suppose that $\hat{G} = (V, W, \hat{E})$, where $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots, w_n\}$, is a bipartite graph realizing (d, d) , such that D is a dominating set. We need to define a flow function on $G_{d, \gamma}$ and prove that the maximum $s - t$ flow equals to $\sum_{i=1}^n d_i$. Obviously, the flow cannot exceed $\sum_{i=1}^n d_i$, so it suffices to prove that $\sum_{i=1}^n d_i$ is achievable. The flow is defined as follows.

1. Saturating all edges from the source and to the sink:

Set $\text{flow}(s, x_i) \leftarrow d_i$ and $\text{flow}(y_j, t) \leftarrow d_j$ for every $i, j \in [1, n]$.

2. Flow to supplementary X vertices:

For every $i \in [\gamma + 1, n]$, set $N_i \leftarrow |\{(v_i, w_j) \in \hat{E} \mid j \in [\gamma + 1, n]\}|$.

Set $\text{flow}(x_i, x'_i) \leftarrow N_i$ for every $x_i \in X_S$

(Note that since v_i is connected with at least one dominating vertex, $N_i \leq d_i - 1$, so the flows satisfy the capacity constraints.)

3. Flow from supplementary Y vertices:

For every $j \in [\gamma + 1, n]$, set $M_j \leftarrow |\{(v_i, w_j) \in \hat{E} \mid i \in [\gamma + 1, n]\}|$.

Set $\text{flow}(y'_j, y_j) \leftarrow M_j$ for every $y_j \in Y_S$.

(Note that again the flows satisfy the capacity constraints.)

4. Flow through edges:

For every pair $i, j \in [1, n]$ such that $(v_i, w_j) \in \hat{E}$:

(a) If $x_i \in X_S$ and $y_j \in Y_S$, then set $\text{flow}(x'_i, y'_j) \leftarrow 1$,

(b) If either $x_i \in X_D$ or $y_j \in Y_D$, then set $\text{flow}(x_i, y_j) \leftarrow 1$.

5. Completing the flow function:

Set the flows of all other edges to 0.

Proof of Flow Legality. Note that by construction, the total flow out of the source s and into the sink t is $\sum_{i=1}^n d_i$. It remains to verify that all other vertices conserve flow.

Node x_i : By construction, the incoming flow at x_i is $\text{flow}(s, x_i) = d_i$. As for the outgoing flow, recall that there are exactly d_i indices j for which $(v_i, w_j) \in \hat{E}$.

- If $i \in [1, \gamma]$, then the flows emanating from x_i are $\text{flow}(x_i, y_j) = 1$ for precisely those indices. Thus, the outgoing flow is $\sum_{j=1}^n \text{flow}(x_i, y_j) = d_i$.
- If $i \in [\gamma + 1, n]$, then the outgoing flows from x_i are $\text{flow}(x_i, y_j) = 1$ for $j \in [1, \gamma]$ and $\text{flow}(x_i, x'_i) = N_i = |\{(v_i, w_j) \in \hat{E} \mid j \in [\gamma + 1, n]\}|$. Thus, the outgoing flow is $\sum_{j \in [1, \gamma]} \text{flow}(x_i, y_j) + \text{flow}(x_i, x'_i) = d_i$.

Node y_j : By construction, the outgoing flow at y_j is $\text{flow}(y_j, t) = d_j$. As for the incoming flow, recall that there are exactly d_j indices i for which $(v_i, w_j) \in \hat{E}$.

- If $j \in D_B$ then the flows incoming in y_j are $\text{flow}(x_i, y_j) = 1$ for precisely those indices. Thus, the outgoing flow is $\sum_{i=1}^n \text{flow}(x_i, y_j) = d_j$.
- If $j \notin D_B$ then the incoming flows to y_j are $\text{flow}(x_i, y_j) = 1$ for $i \in D_A$ and $\text{flow}(y'_j, y_j) = M_j = |\{(v_i, w_j) \in \hat{E} \mid i \in V \setminus D_A\}|$. Thus, the outgoing flow is $\sum_{i \in D_A} \text{flow}(x_i, y_j) + \text{flow}(y'_j, y_j) = d_j$.

Node x'_i : The incoming flow at x'_i is N_i by construction. The outgoing flow is $\sum_{j \in [\gamma+1, n]} \text{flow}(x'_i, y'_j) = N_i$ by definition.

Node y'_j : The outgoing flow from y'_j is M_j by construction. The incoming flow is $\sum_{i \in [\gamma+1, n]} \text{flow}(x'_i, y'_j) = M_j$ by definition.

Hence, the flow function defined above is valid, and the maximum $s - t$ flow in $G_{d, \gamma}$ equals to $\sum_{i=1}^n d_i$, implying the existence of a bipartite graph G with D as a dominating set. $\square$

The next theorem states the main result of this section.

Theorem 2.10. *There exists a polynomial-time algorithm for constructing a realization of a given graphic sequence d that also has a dominating set D of minimum size (among all possible realizations of d).*

In the next section we derive a more efficient algorithm by providing a succinct characterization for the problem and using it for a faster search.

3 An Erdős–Gallai Type

In this section we complement the previous result by providing an Erdős–Gallai Type characterization for the degree sequences d that have a realization with a dominating set of size γ .

It was proven in the previous section that a degree sequence d has a realization with a minimum dominating set of size γ if and only if the flow graph $G_{d, \gamma}$ admits a flow of size $\sum_{i=1}^n d_i$. Therefore, by the max–flow min–cut theorem, d has a realization with a minimum dominating set of size γ if and only if the capacity of every cut in $G_{d, \gamma}$ is at least $\sum_{i=1}^n d_i$. This can serve as a basis for the sought characterization. Unfortunately, a naive implementation yields a characterization of length exponential in the input size. To overcome that difficulty, we apply the following strategy. As a first step, we show that a minimum cut belongs to one of three families of cuts $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$. Each cut in the above families induces a constraint whose size is at least $\sum_{i=1}^n d_i$. Next, we decrease the number of constraints by showing that it is sufficient to consider $O(n)$ constraints per family.

3.1 The characterization

We prove the following theorem.

Theorem 3.1. *A sequence d has a realization with a dominating set of size γ if and only if the following systems of constraints are satisfied.*

$$\sum_{i=\gamma+1}^{\gamma+k} (d_i - 1) \leq k(k-1) - (n - \gamma) + \sum_{i=1}^{\gamma} d_i + \sum_{i=\gamma+k+1}^n \min\{k, d_i - 1\}, \text{ for every } k \in [0, n - \gamma] \quad (2)$$

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min\{k, d_i\}, \text{ for every } k \in [0, n] \quad (3)$$

$$\sum_{i=\gamma+1}^{\gamma+k} d_i \leq k(k-1) + \sum_{i=1}^{\gamma} \min\{k, d_i\} + \sum_{i=\gamma+k+1}^n \min\{k, d_i - 1\}, \text{ for every } k \in [0, d_1] \quad (4)$$

Notice that System (3) coincides with the Erdős-Gallai conditions. The rest of this subsection is devoted to proving the above theorem.

Let $c(S, T)$ be the capacity of an s - t cut (S, T) of $G_{d, \gamma}$. Let $\mathcal{S}$ be the set of s - t cuts of the graph $G_{d, \gamma}$. Clearly, $\mathcal{S} = \mathcal{S}_{00} \cup \mathcal{S}_{01} \cup \mathcal{S}_{10} \cup \mathcal{S}_{11}$, where

$$\begin{aligned} \mathcal{S}_{00} &= \{(S, T) \mid S \cap X_D = \emptyset, T \cap Y_D = \emptyset\} \\ \mathcal{S}_{01} &= \{(S, T) \mid S \cap X_D = \emptyset, T \cap Y_D \neq \emptyset\} \\ \mathcal{S}_{10} &= \{(S, T) \mid S \cap X_D \neq \emptyset, T \cap Y_D = \emptyset\} \\ \mathcal{S}_{11} &= \{(S, T) \mid S \cap X_D \neq \emptyset, T \cap Y_D \neq \emptyset\} \end{aligned}$$

We show that the capacities of minimum cuts in $\mathcal{S}_{01}$ and in $\mathcal{S}_{10}$ are the same.

Lemma 3.2. $\min \{c(S, T) \mid (S, T) \in \mathcal{S}_{01}\} = \min \{c(S, T) \mid (S, T) \in \mathcal{S}_{10}\}.$

Proof. Let $(S, T) \in \mathcal{S}_{01}$ and define (S', T') as follows:

$$\begin{aligned} S' &= \{x_i \mid y_i \in T\} \cup \{x'_i \mid y'_i \in T\} \cup \{y_i \mid x_i \in T\} \cup \{y'_i \mid x'_i \in T\}, \\ T' &= \{x_i \mid y_i \in S\} \cup \{x'_i \mid y'_i \in S\} \cup \{y_i \mid x_i \in S\} \cup \{y'_i \mid x'_i \in S\}. \end{aligned}$$

Observe that $(S', T') \in \mathcal{S}_{10}$ by construction. Moreover, due to the symmetric structure of $G_{d, \gamma}$, there is a capacity preserving bijection between the edges in (S, T) and the edges in (S', T') . More specifically, (x_i, y_j) maps to (x_j, y_i) , (x'_i, y'_j) maps to (x'_j, y'_i) , (x_i, x'_i) maps to (y_i, y'_i) , (y_i, y'_i) maps to (x_i, x'_i) , (s, x_i) maps to (y_i, t) , and finally (y_i, t) maps to (s, x_i) . Therefore, $c(S', T') = c(S, T)$. $\square$

Our next goal is to consider smaller cut families. The following lemmas facilitate this goal.

Lemma 3.3. *Let (S, T) be a cut of $G_{d, \gamma}$ such that $T \cap Y_D = \emptyset$. Also, let $S' = S \cup X_S$ and $T' = T \setminus X_S$. Then $c(S', T') \leq c(S, T)$.*

Proof. Let $x_i \in X_S \setminus S$. By adding x_i to S' we remove the edge (s, x_i) from the cut, but we may add the edge (x_i, x'_i) to the cut (see Figure 4). Hence, $c(S, T) - c(S', T') \geq \sum_{x_i \in X_S \setminus S} (d_i - (d_i - 1)) \geq 0$. $\square$

Symmetrical arguments imply that

Lemma 3.4. *Let (S, T) be a cut of $G_{d, \gamma}$ such that $S \cap X_D = \emptyset$. Also, let $T' = T \cup Y_S$ and $S' = S \setminus Y_S$. Then $c(S', T') \leq c(S, T)$.*

Lemma 3.5. Let (S, T) be a cut of $G_{d, \gamma}$ such that $T \cap Y_D \neq \emptyset$. Also, let $S' = S \setminus (Z \cup Z')$ and $T' = T \cup (Z \cup Z')$, where $Z = \{x_i \in S \mid x'_i \notin S, i \in [\gamma+1, n]\}$ and $Z' = \{x'_i \in S \mid x_i \notin S, i \in [\gamma+1, n]\}$. Then, $c(S', T') \leq c(S, T)$.

Proof. Let $x_i \in Z$, where $i \in [\gamma+1, n]$. The removal of x_i adds the edge (s, x_i) to the cut, but removes the edge (x_i, x'_i) and at least one edge of the form (x_i, y_j) , where $j \in [1, \gamma]$. Next let $x'_i \in Z'$, where $i \in [\gamma+1, n]$. The removal of x'_i removes edges of the form (x'_i, y'_j) , where $y'_j \in Y'_S \cap T$, for $i \neq j$. (See Figure 4.) It follows that

$$c(S, T) - c(S', T') = \sum_{x_i \in Z} (d_i - 1 + |T \cap Y_D| - d_i) + \sum_{x'_i \in Z'} |T \cap Y'_S \setminus \{y'_i\}| \geq |Z|(|T \cap Y_D| - 1) \geq 0. \quad \square$$

Symmetrical arguments imply that

Lemma 3.6. Let (S, T) be a cut of $G_{d, \gamma}$ such that $S \cap X_D \neq \emptyset$. Also, let $S' = S \cup (Z \cup Z')$ and $T' = T \setminus (Z \cup Z')$, where $Z = \{y_i \mid y'_i \notin S, i \in [\gamma+1, n]\}$ and $Z' = \{y'_i \mid y_i \notin S, i \in [\gamma+1, n]\}$. Then, $c(S', T') \leq c(S, T)$.

Define the following cut families:

- $\mathcal{F}_1 = \{(S_{I,J}^1, V \setminus S_{I,J}^1) \mid I, J \subseteq [\gamma+1, n]\}$,
where $S_{I,J}^1 = \{x'_i \mid i \in I\} \cup \{y'_j \mid j \in J\} \cup X_S \cup Y_D$.
- $\mathcal{F}_2 = \{(S_{I,J}^2, V \setminus S_{I,J}^2) \mid I, J \subseteq [1, n]\}$,
where $S_{I,J}^2 = \{x_i \mid i \in I\} \cup \{y_j \mid j \in J\} \cup \{x'_i \mid i \in I \cap [\gamma+1, n]\} \cup \{y'_j \mid j \in J \cap [\gamma+1, n]\}$.
- $\mathcal{F}_3 = \{(S_{I,J}^3, V \setminus S_{I,J}^3) \mid I \subseteq [\gamma+1, n], J \subseteq [1, n]\}$,
where $S_{I,J}^3 = \{x_i, x'_i \mid i \in I\} \cup \{y_j \mid j \in J \cap [1, \gamma]\} \cup \{y'_j \mid j \in J \cap [\gamma+1, n]\}$.

We obtain the following.

Corollary 3.7. We have that

- $\min \{c(S, T) \mid (S, T) \in \mathcal{F}_1\} \leq \min \{c(S, T) \mid (S, T) \in \mathcal{S}_{00}\}$.
- $\min \{c(S, T) \mid (S, T) \in \mathcal{F}_2\} \leq \min \{c(S, T) \mid (S, T) \in \mathcal{S}_{11}\}$.
- $\min \{c(S, T) \mid (S, T) \in \mathcal{F}_3\} \leq \min \{c(S, T) \mid (S, T) \in \mathcal{S}_{10}\}$.

Proof. Let (S, T) be a minimum cut in $G_{\gamma, d}$. We consider the following three cases:

Case 1: If $(S, T) \in \mathcal{S}_{00}$, then by Lemmas 3.3 and 3.4, there exists a cut $(S', T') \in \mathcal{F}_1$ such that $c(S', T') \leq c(S, T)$.

Case 2: If $(S, T) \in \mathcal{S}_{11}$, then by Lemmas 3.5 and 3.6, there exists a cut $(S', T') \in \mathcal{F}_2$ such that $c(S', T') \leq c(S, T)$.

Case 3: If $(S, T) \in \mathcal{S}_{10}$, then by Lemmas 3.4 and 3.5, there exists a cut $(S', T') \in \mathcal{F}_3$ such that $c(S', T') \leq c(S, T)$. $\square$

The next lemma will helps us to reduce the number of constraints in each family.

Lemma 3.8. For a non-decreasing sequence of integers $\{a_i\}_{i=1}^n$ and a non-decreasing function $f : [0, m] \rightarrow \mathbb{R}$ the following two constraint systems are equivalent

$$\sum_{i \in I} a_i \leq f(|I|) + \sum_{i \in I} \min\{|I| - 1, a_i\} + \sum_{i \in [1, n] \setminus I} \min\{|I|, a_i\}, \quad I \subseteq [1, n], |I| \leq m \quad (5)$$

$$\sum_{i=1}^k a_i \leq f(k) + k(k-1) + \sum_{i=k+1}^n \min\{k, a_i\}, \quad k \in [0, m] \quad (6)$$

Proof. First we prove that System (5) is equivalent to

$$\sum_{i \in I} a_i \leq f(|I|) + |I|(|I| - 1) + \sum_{i \in [1, n] \setminus I} \min\{|I|, a_i\}, \quad I \subseteq [1, n], |I| \leq m. \quad (7)$$

Since $\min\{|I| - 1, a_i\} \leq |I| - 1$, it follows that System (5) implies System (7). On the other hand, fix I and consider $I' = \{i \in I \mid |I| - 1 \leq a_i\}$. Clearly, $|I'| \leq |I|$. The inequality for I in System (5) can be rewritten as

$$\sum_{i \in I'} a_i \leq f(|I|) + (|I| - 1)|I'| + \sum_{i \in [1, n] \setminus I} \min\{|I|, a_i\}. \quad (8)$$

Now consider the inequality for I' in System (7):

$$\sum_{i \in I'} a_i \leq f(|I'|) + (|I'| - 1)|I'| + \sum_{i \in [1, n] \setminus I'} \min\{|I'|, a_i\}.$$

Since f is non-decreasing we have that

$$\begin{aligned} \sum_{i \in I'} a_i &\leq f(|I|) + (|I'| - 1)|I'| + (|I| - |I'|)|I'| + \sum_{i \in [1, n] \setminus I} \min\{|I'|, a_i\} \\ &= f(|I|) + (|I| - 1)|I'| + \sum_{i \in [1, n] \setminus I} \min\{|I|, a_i\}, \end{aligned}$$

namely (8) is satisfied.

Clearly, System (6) is implied by System (7) by choosing $I = [1, k]$. On the other hand, let $k = |I|$. We have that

$$\sum_{i \in I} a_i \leq \sum_{i=1}^k a_i \leq f(k) + k(k-1) + \sum_{i=k+1}^n \min\{k, a_i\} \leq f(|I|) + |I|(|I| - 1) + \sum_{i \in [1, n] \setminus I} \min\{|I|, a_i\},$$

which means that System (7) follows from System (6). $\square$

In the next three lemmas we use Lemma 3.8 to reduce the number of constraints in each family. Next figures follow the convention that yellow blocks belong to S and blue blocks belong to T in a s - t cut (S, T) . Multicolored blocks may contain vertices from both parts of the cut.

Lemma 3.9. $c(S, T) \geq \sum_{i=1}^k d_i$, for every $(S, T) \in \mathcal{F}_1$ if and only if System (2) is satisfied.

Proof. Consider a cut $(S_{I,J}^1, V \setminus S_{I,J}^1) \in \mathcal{F}_1$, where $I, J \subseteq [\gamma + 1, n]$, and recall that

$$S_{I,J}^1 = \{x'_i \mid i \in I\} \cup \{y'_j \mid j \in J\} \cup X_S \cup Y_D.$$

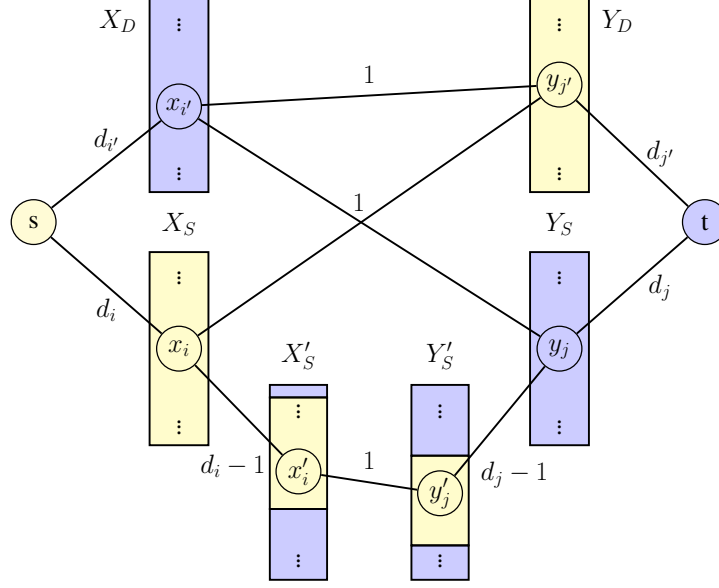


Figure 5: s - t cut from $\mathcal{F}_1$ in the flow graph.

We have that

$$\begin{aligned}
 c(S_{I,J}^1, V \setminus S_{I,J}^1) &= \sum_{i=1}^{\gamma} d_i + \sum_{i \in [\gamma+1, n] \setminus I} (d_i - 1) + \sum_{i \in I, j \in [\gamma+1, n] \setminus J, i \neq j} 1 + \sum_{j \in J} (d_j - 1) + \sum_{j=1}^{\gamma} d_j \\
 &= 2 \sum_{i=1}^{\gamma} d_i + \sum_{i \in [\gamma+1, n] \setminus I} (d_i - 1) + \sum_{j \in [\gamma+1, n] \setminus (I \cup J)} |I| + \sum_{j \in I \setminus J} (|I| - 1) + \sum_{j \in J} (d_j - 1).
 \end{aligned}$$

(See example in Figure 5.) Hence the capacity of all cuts in $\mathcal{F}_1$ is least $\sum_{i=1}^n d_i$ if and only if for every $I, J \subseteq [\gamma + 1, n]$

$$\sum_{i \in I} d_i \leq \sum_{i=1}^{\gamma} d_i - (n - \gamma - |I|) + \sum_{j \in [\gamma+1, n] \setminus (I \cup J)} |I| + \sum_{j \in I \setminus J} (|I| - 1) + \sum_{j \in J} (d_j - 1).$$

It is not hard to verify that this system is satisfied if and only if for every $I \subseteq [\gamma + 1, n]$

$$\sum_{i \in I} (d_i - 1) \leq \sum_{i=1}^{\gamma} d_i - (n - \gamma) + \sum_{j \in [\gamma+1, n] \setminus I} \min\{|I|, d_j - 1\} + \sum_{j \in I} \min\{|I| - 1, d_j - 1\}.$$

The lemma follows due to Lemma 3.8 applied with $a_i = d_{\gamma+i} - 1$, for $i \in [1, n - \gamma]$, $f(|I|) = \sum_{i=1}^{\gamma} d_i - (n - \gamma)$ and $m = n$. $\square$

Lemma 3.10. $c(S, T) \geq \sum_{i=1}^k d_i$, for every $(S, T) \in \mathcal{F}_2$ if and only if System (3) is satisfied.

Proof. Consider a cut $(S_{I,J}^2, V \setminus S_{I,J}^2) \in \mathcal{F}_2$, where $I, J \subseteq [1, n]$, and recall that

$$S_{I,J}^2 = \{x_i \mid i \in I\} \cup \{y_j \mid j \in J\} \cup \{x'_i \mid i \in I \cap [\gamma + 1, n]\} \cup \{y'_j \mid j \in J \cap [\gamma + 1, n]\}.$$

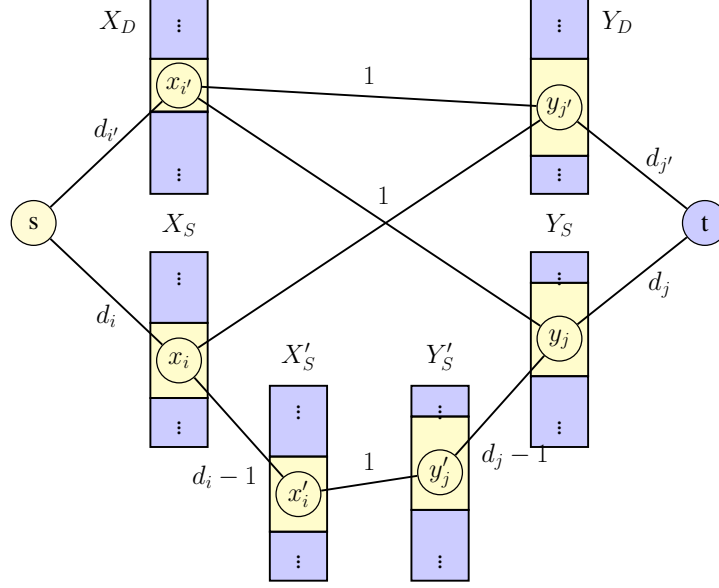


Figure 6: s - t cut from $\mathcal{F}_2$ in the flow graph.

Observe that

$$\begin{aligned} c(S_{I,J}^2, V \setminus S_{I,J}^2) &= \sum_{i \in [1,n] \setminus I} d_i + \sum_{i \in I, j \in [1,n] \setminus J, i \neq j} 1 + \sum_{j \in J} d_j \\ &= \sum_{i \in [1,n] \setminus I} d_i + \sum_{j \in [1,n] \setminus (I \cup J)} |I| + \sum_{j \in I \setminus J} (|I| - 1) + \sum_{j \in J} d_j. \end{aligned}$$

(See example in Figure 6.) It follows that the capacity of all cuts in $\mathcal{F}_2$ is at least $\sum_{i=1}^n d_i$ if and only if for every $I, J \subseteq [1, n]$

$$\sum_{i \in I} d_i \leq \sum_{j \in [1,n] \setminus (I \cup J)} |I| + \sum_{j \in I \setminus J} (|I| - 1) + \sum_{j \in J} d_j.$$

It is not hard to verify that this system is satisfied if and only if for every $I \subseteq [1, n]$

$$\sum_{i \in I} d_i \leq \sum_{i \in [1,n] \setminus I} \min\{|I|, d_i\} + \sum_{i \in I} \min\{|I| - 1, d_i\}.$$

The lemma follows due to Lemma 3.8 applied with $a_i = d_i$, for $i \in [1, n]$, $f(|I|) = 0$ and $m = n$. $\square$

Lemma 3.11. $c(S, T) \geq \sum_{i=1}^k d_i$, for every $(S, T) \in \mathcal{F}_3$ if and only if $\sum_{i=1}^\gamma d_i \geq n - \gamma$ and System (4) is satisfied.

Proof. Consider a cut $(S_{I,J}^3, V \setminus S_{I,J}^3) \in \mathcal{F}_3$, where $I \subseteq [\gamma + 1, n]$ and $J \subseteq [1, n]$, and recall that

$$S_{I,J}^3 = \{x_i, x'_i \mid i \in I\} \cup \{y_j \mid j \in J \cap [1, \gamma]\} \cup \{y'_j \mid j \in J \cap [\gamma + 1, n]\}.$$

Observe that

$$\begin{aligned}
c(S_{I,J}^3, V \setminus S_{I,J}^3) &= \sum_{i=1}^{\gamma} d_i + \sum_{i \in [\gamma+1, n] \setminus I} d_i + \sum_{i \in I, j \in [1, n] \setminus J, i \neq j} 1 + \sum_{j \in [1, \gamma] \cap J} d_j + \sum_{j \in [\gamma+1, n] \cap J} (d_j - 1) \\
&= \sum_{i=1}^{\gamma} d_i + \sum_{i \in [\gamma+1, n] \setminus I} d_i + \sum_{j \in [1, n] \setminus (I \cup J)} |I| + \sum_{j \in I \setminus J} (|I| - 1) + \sum_{j \in [1, \gamma] \cap J} d_j + \sum_{j \in [\gamma+1, n] \cap J} (d_j - 1).
\end{aligned}$$

(See example in Figure 7.) It follows that the capacity of all cuts in $\mathcal{F}_3$ is at least $\sum_{i=1}^n d_i$ if and only if for every $I \subseteq [\gamma + 1, n]$ and $J \subseteq [1, n]$

$$\sum_{i \in I} d_i \leq \sum_{j \in [1, n] \setminus (I \cup J)} |I| + \sum_{j \in I \setminus J} (|I| - 1) + \sum_{j \in [1, \gamma] \cap J} d_j + \sum_{j \in [\gamma+1, n] \cap J} (d_j - 1).$$

It is not hard to verify that this system is satisfied if and only if for every $I \subseteq [\gamma + 1, n]$

$$\sum_{i \in I} d_i \leq \sum_{j \in [1, \gamma]} \min\{|I|, d_j\} + \sum_{j \in [\gamma+1, n] \setminus I} \min\{|I|, d_j - 1\} + \sum_{j \in I} \min\{|I| - 1, d_j - 1\}.$$

If $|I| > d_1$, then the system transforms to

$$n - \gamma \leq \sum_{j \in [1, \gamma]} d_j + \sum_{j \in [\gamma+1, n] \setminus I} d_j.$$

Notice that all the equations for $|I| > d_1$ follow from the single one $\sum_{i=1}^{\gamma} d_i \geq n - \gamma$ (which corresponds to the case where $I = [\gamma + 1, n]$). Otherwise, if $|I| \leq d_1$, then rewrite the system as

$$\sum_{i \in I} (d_i - 1) \leq -|I| + \sum_{j \in [1, \gamma]} \min\{|I|, d_j\} + \sum_{j \in [\gamma+1, n] \setminus I} \min\{|I|, d_j - 1\} + \sum_{j \in I} \min\{|I| - 1, d_j - 1\}.$$

Observe that $f(|I|) = -|I| + \sum_{j \in [1, \gamma]} \min\{|I|, d_j\}$ is a non-decreasing function of $|I|$ in the range (i.e., $|I| \leq d_1$). The lemma follows due to Lemma 3.8 with $a_i = d_{\gamma+i} - 1$ for $i \in [1, n - \gamma]$, and $m = d_1$. $\square$

Since the equation for $k = 0$ in System (2) implies $\sum_{i=1}^{\gamma} d_i \geq n - \gamma$, Lemma 3.1 is implied by Lemmas 3.9 to 3.11.

3.2 Complexity analysis and a faster realization algorithm

This section discusses the running time of two algorithms: (i) an algorithm to compute $\text{MDS}(d)$, which is implied by the characterizations given in Lemma 3.1, and (ii) our realization algorithm for MDS-DR .

Before addressing the running time of an algorithm for computing $\text{MDS}(d)$, consider first a decision algorithm. Since the three systems in Lemma 3.1 contain $O(n)$ constraints, it follows that a naive implementation of a decision algorithm would result in a $O(n^2)$ running time. However, it is well known that the Erdős-Gallai characterization (i.e., System (3)) can be computed in $O(n)$ time, since the k th constraint can be computed from the $(k - 1)$ th constraint in $O(1)$ time. A similar idea can be used for the other two systems. Hence, the decision algorithm can be implemented with a $O(n)$ running time. Finally, Using a binary search an algorithm to compute $\text{MDS}(d)$ whose running time is $O(n \log n)$ is obtained.

As for the realization algorithm, given $\gamma = \text{MDS}(d)$, to find a γ -prefix dominated realization of d we first build the flow graph $G_{d, \gamma}$ and find a maximum s - t flow in it. Since $G_{d, \gamma}$ has $O(n)$ vertices and $O(n^2)$ edges, a maximum flow can be computed in $O(n^3)$ time using the maximum flow algorithm by

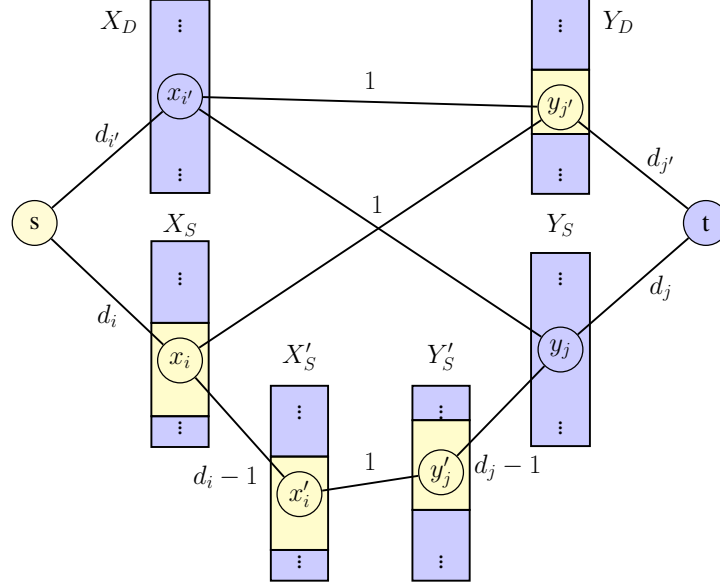


Figure 7: s - t cut from $\mathcal{F}_3$ in the flow graph.

Orlin [25]. Given a maximum flow, adjacency matrix for γ -prefix-dominated realization $\hat{G}$ for the sequence pair (d, d) can be computed in $O(n^2)$ time. Next we go through all the steps of transformation of $\hat{G}$ into γ -prefix dominated realization G for d and show that each step takes $O(n^3)$ time. We need the following observations.

Observation 3.12. *If $s \in S$ is adjacent to at least three vertices in D in $G^{1/2}$, then there is a 2-dom path $P[a, s, b, c]$.*

Proof. Let $a, a', b \in D$ be three dominator vertices adjacent to s . Since $G^{1/2}$ is an even graph, b must be adjacent to some vertex $x \in V^{1/2}$ besides s . If $x \neq a$, then $P[a, s, b, x]$ is 2-dom path. Otherwise, $P[a', s, b, a]$ is 2-dom path. $\square$

Observation 3.13. *For a vertex $s \in S$ if there is no 2-dom paths $P[a, s, b, c]$ in $G^{1/2}$, then there will not be any such 2-dom paths after applying MR1, MR2 or MR3 to any 2-dom path in $G^{1/2}$.*

Proof. By Lemma 3.12 we can assume that s is connected to at most two dominator vertices in $G^{1/2}$. Note that neither of the modifications can increase the number of dominator vertices adjacent to s in $G^{1/2}$. Therefore, if s is connected to at most one dominator vertex, the Observation follows.

Assume that s is connected to exactly two dominator vertices denoted a and b . Note that if a, b or s participated in any 2-dom path, it would imply the existence of $c \notin \{a, s, b\}$ adjacent to a or to b in $G^{1/2}$. But then there exists a 2-dom path $P[a, s, b, c]$ or $P[b, s, a, c]$. It follows that modifications of 2-dom paths cannot create any new edges in $G^{1/2}$ going through a, b or s and therefore cannot create any new 2-dom paths containing s . The observation follows. $\square$

Observation 3.14. *Given $s \in S$ it is possible to find a 2-dom path $P[a, s, b, c]$ in $G^{1/2}$ or make sure there is no such paths in $O(n)$ time.*

Proof. First, find the dominator vertices adjacent to s in $O(n)$ time. If there is at most one such dominator vertex, then there is no 2-dom path $P[a, s, b, c]$ in $G^{1/2}$. If there is at least three dominator vertices, it is possible to find 2-dom paths in $O(n)$ time following the proof of Lemma 3.12.

The remaining case is when s is connected to exactly two dominator vertices a and b . If a and b are connected only to each other and s in the graph $G^{1/2}$, then there is no 2-dom path going through s . Otherwise, if a or b is adjacent to $x \notin \{a, b, s\}$ in the graph $G^{1/2}$, then the required 2-dom path is easily obtained. Also, notice that the neighbors of a and b can be examined in $O(n)$ time. The observation follows. $\square$

Step 1: The adjacency matrix of the weighted graph G^ω can be computed from the adjacency matrix of $\hat{G}$ in $O(n^2)$.

Step 2: It takes $O(n^2)$ to compute the adjacency matrix of $G^{1/2}$ using the adjacency matrix of G^ω .

Step 3: Examine the vertices in S one by one. For $s \in S$ find a 2-dom path $P[a, s, b, c]$ in $G^{1/2}$ and apply a corresponding modification to it. Since any modification decreases the number of neighbors of s in $G^{1/2}$ by 2, it follows that after $O(n)$ such steps no 2-dom paths $P[a, s, b, c]$ remain in $G^{1/2}$. By Lemma 3.13 and 3.14, all the 2-dom paths can be removed this way in $O(n^3)$ time.

Step 4: S' , E' , S^Δ and $\mathcal{C}^\Delta$ can be found straightforwardly in $O(n^2)$ time. Note that it takes only $O(1)$ time to check if a vertex is a dominator or not, since $D = [1, \gamma]$.

Step 5: The adjacency matrix of H can be computed in $O(n^2)$ time from $\mathcal{C}^\Delta$ and the adjacency matrix of $G^{1/2}$. Since an Eulerian cycle can be found by linear time in the number of edges, a partition of H into disjoint cycles $\mathcal{C}'$ takes $O(n^2)$ time.

Step 6: Separating even cycles in $\mathcal{C}$ and applying the corresponding modification to them takes $O(n^2)$ time.

Step 7: Since the vertices are enumerated, one can determine the intersection of the cycles C and C' in $O(|C| \log |C| + |C'| \log |C'|)$ time by sorting their vertices and then computing the intersection. If they intersect, it takes $O(|C| + |C'|)$ time to apply the corresponding modification. If they do not intersect, the proof of Lemma 2.7 implies that suitable x and y can be found in $O(n)$ time and $O(|C| + |C'|)$ time is needed to apply the modification. The complexity for all pairs is $O(n^3)$.

Step 8: At this step G^ω does not have edges of weight $1/2$, so it can be transformed into G in $O(n^2)$ time.

It follows that the algorithm can be implemented in $O(n^3)$ time.

We conclude the section with the following result.

Theorem 3.15. *There exists an $O(n^3)$ time algorithm for constructing a realization of a given graphic sequence d that also has a dominating set D of minimum size (among all possible realizations of d).*

4 Realization with Maximum Matching

This section presents an algorithm for MM-DR, based on the Inverted Prefix Lemma 4.3 and a modification of the FHM realization algorithm². The algorithm follows the same two stages of the algorithm for MDS-DR described in Section 2, first reducing the problem to the bipartite setting and then reducing it to a maximum flow problem. Finally, the Inverted Prefix Lemma for maximum matching narrows the search to a polynomial number of cases.

²An alternative algorithm the Inverted Prefix Lemma and the techniques of [22]. Here we present the FHM-based solution for uniformity of presentation.

4.1 Prefix Lemma

For MM-DR, there is a general prefix lemma by Gould, Jacobson and Lehel [17].

Lemma 4.1. (Arbitrary Prefix Lemma for MM) [17] *If a sequence d has a realization with a matching of size ν , then d has a realization with a matching of the same size such that 2ν vertices participating in the matching have the highest 2ν degrees in d , i.e., $d_1, d_2, \dots, d_{2\nu}$.*

For our purposes, however, we need a stronger type of prefix lemma, similar to the one used in [5] for solving MM-DR over bipartite graphs. Towards proving this stronger claim, we introduce some notation. Consider a realization $G = (V, E)$ of a degree sequence d , with four vertices $x, y, u, v \in V$ such that $(x, u), (y, v) \in E$ and $(x, y), (u, v) \notin E$. Then the *FLIP operation* transforms G into a graph $G' = (V, E')$ by replacing the former two edges with the latter two, i.e., setting $E' = E \cup \{(x, y), (u, v)\} \setminus \{(x, u), (y, v)\}$. This operation preserves the degrees of individual vertices, but could lead to non-isomorphic realization. It was widely studied and used in different contexts, see [13, 10, 4].

Claim 4.2. *Let $G = (V, E)$ be a realization of d with $V = \{v_1, v_2, \dots, v_n\}$, such that $\deg_G(v_i) = d_i$ for every $i \in [1, n]$. If there is a matching $M \subseteq E$ on the first 2ν vertices $v_1, \dots, v_{2\nu}$, then there is a realization $G' = (V, E')$ of d with the "inverted" matching $M^{inv} = \{(v_i, v_{2\nu-i+1}) \mid i \in [1, \nu]\}$, $M^{inv} \subseteq E'$, and the same degrees $\deg_{G'}(v_i) = d_i$ for every $i \in [1, n]$.*

Proof. Call the pair (G, M) *valid* if $G = (V, E)$ is defined on $V = \{v_1, \dots, v_n\}$ with $\deg(v_i) = d_i$, $M \subseteq E$, and M is on $\{v_1, v_2, \dots, v_{2\nu}\}$. Consider a valid pair (G, M) and let $I(M) = \{i \mid (v_i, v_{2\nu-i+1}) \notin M, i \in [1, \nu]\}$ and $f(M) = \min I(M)$. If $I(M) = \emptyset$, then $M = M^{inv}$, and we are done.

We describe procedure **Invert** that given a valid pair (G, M) violating the claim produces a new valid pair (G', M') , such that either (G', M') satisfies the claim or $f(M') > f(M)$. Repeatedly applying **Invert** gradually modifies (G, M) while preserving validity and strictly increasing $f(M)$. Since f is bounded above by ν , the process terminates in at most ν steps and the obtained pair (G', M') satisfies the claim.

Given a pair (G, M) not satisfying the claim, let $i = f(M)$ and $j = 2\nu - i + 1$. Since $(v_i, v_j) \notin M$, there are edges $(v_i, v_x), (v_y, v_j) \in M$, and the involved vertices satisfy $\deg_G(v_i) \geq \deg_G(v_y)$ and $\deg_G(v_j) \leq \deg_G(v_x)$ by the definition of f . Let $M' = M \cup \{(v_i, v_j), (v_x, v_y)\} \setminus \{(v_i, v_x), (v_y, v_j)\}$. If there is a realization G' such that $M' \subseteq G'$, then the pair (G', M') is valid and either satisfies the claim or increases f . Proceed according to one of the following cases to obtain G' . See Figure 8.

Case 1: $(v_i, v_j), (v_x, v_y) \in E(G)$. Then simply take $G' = G$.

Case 2: $(v_i, v_j), (v_x, v_y) \notin E(G)$. Then perform a FLIP operation on G replacing $(v_i, v_x), (v_y, v_j)$ with $(v_i, v_j), (v_x, v_y)$. G' obtained in this way satisfies $M' \subseteq G'$.

Case 3: $(v_i, v_j) \in E(G)$ and $(v_x, v_y) \notin E(G)$. Then there is a vertex $v_z \in V(G) \setminus \{v_i, v_j, v_x, v_y\}$, such that $(v_x, v_z) \in E(G)$ and $(v_j, v_z) \notin E(G)$. Indeed, $\deg_G(v_x) \geq \deg_G(v_j)$ and v_y is connected to v_j , but not to v_x , so there exists v_z compensating for this. Obtain G' by doing a FLIP replacing $(v_x, v_z), (v_j, v_y)$ with $(v_x, v_y), (v_j, v_z)$.

Case 4: $(v_i, v_j) \notin E(G)$ and $(v_x, v_y) \in E(G)$. Then there is a vertex $v_z \in V(G) \setminus \{v_i, v_j, v_x, v_y\}$, such that $(v_i, v_z) \in E(G)$ and $(v_y, v_z) \notin E(G)$. Similarly to the previous case, this is because $\deg_G(v_i) \geq \deg_G(v_y)$. Obtain G' by doing a FLIP replacing $(v_i, v_z), (v_j, v_y)$ with $(v_i, v_j), (v_y, v_z)$. $\square$

Combining Lemma 4.1 and Lemma 4.2 gives a more specific Prefix Lemma for MAXIMUM MATCHING.

Lemma 4.3. (Inverted Prefix Lemma for MM) *If a sequence d has a realization $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_n\}$ such that $\deg_G(v_i) = d_i$ for every $i \in [1, n]$, with a matching M of size ν , then d has a realization $G' = (V, E')$ with the matching $M' = \{(v_i, v_{2\nu-i+1}) \mid i \in [1, \nu]\}$ and the same degrees $\deg_{G'}(v_i) = d_i$ for every $i \in [1, n]$.*

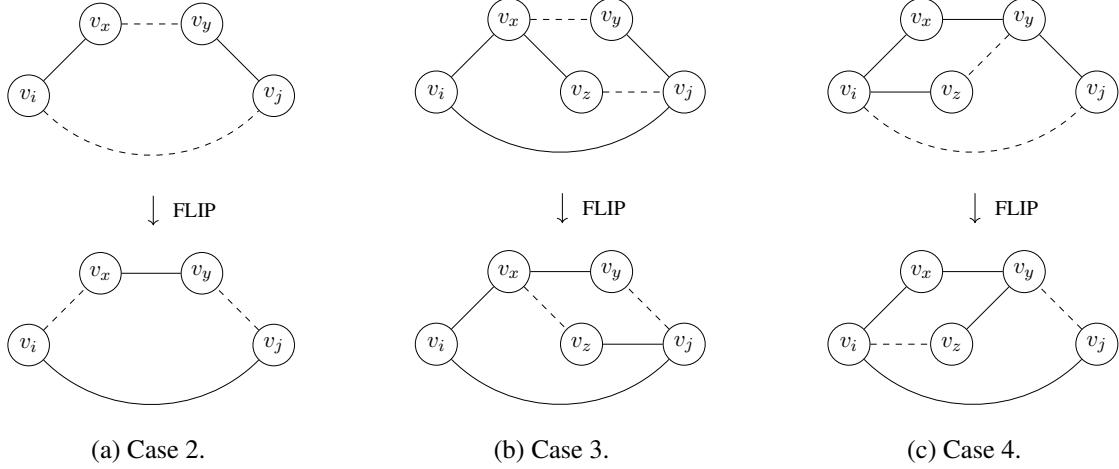


Figure 8: Illustration of the cases in the proof of Claim 3. All the illustrating figures maintain the convention that solid lines represent edges existing in the graph and dashed lines represent edges not in the graph.

4.2 Reduction to a Bipartite Sequence Pair

A bipartite graph $\hat{G} = (V, W, \hat{E})$, where $V = \{v_1, v_2, \dots, v_n\}$ and $W = \{w_1, w_2, \dots, w_n\}$, is a ν -matched realization for the sequence pair (d, d) if it satisfies the following properties.

(M1) $\hat{G}$ realizes the sequence pair (d, d) .

(M2) $(v_i, w_i) \notin \hat{E}$ for every $i \in [1, n]$.

(M3) $\hat{M} = \{(v_i, w_{2\nu-i+1}) \mid i \in [1, 2\nu] \subseteq \hat{E}$.

First, we describe an algorithm that given a ν -matched realization $\hat{G}$ of (d, d) , produces a graph G realizing d with a matching of size ν .

1. Compute a half-integral solution.

(a) For all $i, j \in [1, n]$, let

$$y_{ij} = \begin{cases} 1, & \{v_i, w_j\} \in \hat{E}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \omega(i, j) = \frac{1}{2}(y_{ij} + y_{ji}).$$

(b) Define a weighted graph $G^\omega = (V^\omega, E^\omega, \omega)$ with vertex set $V^\omega = [1, n]$ and an edge $e = (i, j)$ of weight $\omega(e) = \omega(i, j)$ for every $i, j \in V^\omega$. Clearly, ω is half-integral.

(c) Let $M^\omega = \{(i, 2\nu - i + 1) \mid i \in [1, \nu]\}$ be the inverted matching in G^ω . According to (M3)

$$\omega(e) = 1, \text{ for every } e \in M^\omega. \quad (9)$$

(d) Define the *weighted degree* of a vertex $i \in V^\omega$ to be $d^\omega(i) = \sum_{j \in V^\omega} \omega(i, j)$. Note that G^ω realizes d in the *weighted* sense, namely,

$$d^\omega(i) = \sum_{j \in V^\omega} \omega(i, j) = \frac{1}{2} \left(\sum_{j=1}^n y_{ij} + \sum_{j=1}^n y_{ji} \right) = d_i, \text{ for any } i \in V^\omega.$$

2. Preparing for discarding non-integral weights while keeping the degrees unchanged.

Construct a graph $G^{1/2} = (V^{1/2}, E^{1/2})$ obtained by removing from G^ω the edges of integral weight and keeping only those of weight $1/2$. Formally, $V^{1/2} = V^\omega$ and $E^{1/2} = \{e \in E^\omega \mid \omega(e) = 1/2\}$.

In later stages of the construction, whenever G^ω is modified (by changing the weight of some edge e from $1/2$ to 0 or 1), $G^{1/2}$ is modified accordingly (by removing the edge e).

3. Partition into cycles.

- (a) Partition the edge set of $G^{1/2}$ into disjoint (not necessarily simple) cycles each covering an entire connected component of the graph. Since $G^{1/2}$ is an even graph by Lemma 4.4, this can be done in polynomial number of steps.
- (b) Let $\mathcal{C}$ be a set of cycles in the partition.
- (c) Let $\mathcal{C}^{even} \leftarrow \{C \in \mathcal{C} \mid C \text{ is of even length}\}$, $\mathcal{C}^{odd} \leftarrow \mathcal{C} \setminus \mathcal{C}^{even}$.

4. Eliminate even cycles.

For every cycle $C \in \mathcal{C}^{even}$, do the following.

- (a) Traverse C starting from an arbitrary vertex $x \in C$ and continuing along the cycle until returning to x . Denote the resulting sequence of edges by $E(C) = (e_1, e_2, \dots, e_\ell)$.
- (b) Increase (resp., decrease) edge weights on even (resp., odd) positions in the sequence $E(C)$ by $1/2$. That is, for every $i \in [1, \ell]$, $\omega(e_i)$ is set to 1 if i is even and 0 otherwise. (This does not change the weighted degrees in G^ω and does not affect other cycles in $\mathcal{C}$ or edges in M^ω .)

5. Eliminate odd cycles.

Arrange the cycles in $\mathcal{C}^{odd}$ in pairs (recall that by Lemma 4.5 their number is even). For every pair of cycles (C, C') choose vertices $x \in C, y \in C'$ according to Obs. 4.6. Then do the following.

- (a) Starting from $x \in C$, traverse C via edges $E(C) = (e_1, e_2, \dots, e_\ell)$. Starting from vertex $y \in C'$, traverse C' via edges $E(C') = (e'_1, e'_2, \dots, e'_k)$.
- (b) Let $\xi = \omega(x, y)$ (by Lemma 4.6, this weight must be either 0 or 1).
- (c) Set $\omega(x, y) \leftarrow 1 - \xi$.
- (d) For every $i \in [1, \ell]$ and $j \in [1, k]$, modify the edge weights in the cycles C and C' as follows:

$$\omega(e_i) \leftarrow \begin{cases} 1 - \xi, & i \text{ is even,} \\ \xi, & i \text{ is odd,} \end{cases} \quad \omega(e'_j) \leftarrow \begin{cases} 1 - \xi, & j \text{ is even,} \\ \xi, & j \text{ is odd.} \end{cases}$$

By Lemma 4.7, at this stage each edge in G^ω has weight 1 or 0, G^ω realizes d , and Eq. (9) holds.

6. Generate the output G .

Transform G^ω into a simple graph G , with an edge (i, j) whenever $\omega(i, j) = 1$ in G^ω . Note that $M = M^\omega$ forms a matching in G by Eq. (9).

The algorithm is justified by the following Observations.

Observation 4.4. *The graph $G^{1/2}$ is even (namely, all its vertex degrees are even).*

A connected even graph has an Euler cycle, implying the following.

Observation 4.5. *The number of cycles in $\mathcal{C}^{odd}$ is even.*

Proof. Observe that $\sum_{e \in E^\omega} \omega(e) = \frac{1}{2} \sum_{i=1}^n d_i = m$, where m is the number of edges, which is an integer. Therefore, the number of edges with weight $1/2$ must be even. Since the cycles in $\mathcal{C}$ cover all of $E^{1/2}$ and are disjoint, the observation follows. $\square$

Observation 4.6. *For any pair of disjoint odd cycles $C, C' \in \mathcal{C}^{odd}$, there exist $x \in C$ and $y \in C'$, such that $(x, y) \notin M^\omega$ and $\omega(x, y) \neq 1/2$.*

Proof. Let $x \in C$, $y' \in C'$ be arbitrary vertices. If $(x, y') \in M^\omega$, let $y \neq y'$ be any other vertex in C' . Otherwise, let $y = y'$. Since M^ω is a matching, it follows that $(x, y) \notin M^\omega$.

Assume, towards contradiction, that $\omega(x, y) = 1/2$. Then the edge (x, y) belongs to $G^{1/2}$, implying that x and y belong to the same connected component in $G^{1/2}$. But this contradicts the fact that $x \in C$ and $y \in C'$ where C and C' cover different connected components of $G^{1/2}$. $\square$

Observation 4.7. *At the end of Step 5, each edge in G^ω has weight 1 or 0, G^ω realizes d and Eq. (9) holds.*

Proof. After Step 5, G^ω contains no more edges of weight $1/2$. Moreover, neither of the modifications performed in Step 5 change the weighted degrees in G^ω , affect other cycles in $\mathcal{C}$ or edges in M^ω . The observation follows. $\square$

The reduction correctness follows from the next lemma.

Lemma 4.8. *There is a ν -matched realization of (d, d) if and only if $\text{MM}(d) \geq \nu$.*

Proof. $(\Rightarrow)$ If there ν -matched realization of (d, d) , then the aforementioned algorithm produces a realization of d with matching of size ν .

$(\Leftarrow)$ Given a realization $G = (V, E)$ of d with vertices $V = [1, n]$ and a matching $M \subseteq E$ of size ν , there is a realization $G' = (V, E')$ with a matching $M' = \{(v_i, v_{2\nu-i+1}) \mid i \in [1, \nu]\}$ by the Inverted Prefix Lemma (Lemma 4.3). Consider the following graph $\hat{G} = (\hat{V}, \hat{W}, \hat{E})$ with $\hat{V} = \{v_1, \dots, v_n\}$, $\hat{W} = \{w_1, \dots, w_n\}$ and $(v_i, w_j) \in \hat{E}$ if and only if $(i, j) \in E'$. Clearly, $\hat{G}$ realizes (d, d) and $(v_i, w_i) \notin \hat{E}$ for every $i \in [1, n]$, so it satisfies (M1) and (M2). Moreover, the set $\hat{M} = \{(v_i, w_j) \mid (i, j) \in M'\}$ is a matching of size 2ν in $\hat{G}$ satisfying (M3). Indeed, since M' is a matching, the edges in $\hat{M}$ form a matching too and each edge $(i, j) \in M'$ leads to two edges $(v_i, w_j), (v_j, w_i) \in \hat{M}$. The Lemma follows. $\square$

4.3 Reduction to Flow

According to the previous section (in particular, Lemma 4.8, given a degree sequence d all we need is to find a bipartite ν -matched realization of (d, d) for maximum possible ν . Next we describe how to construct a flow graph $G_{d,\nu}$ for any ν , such that ν -prefix realization of (d, d) exists if and only if the maximum flow in $G_{d,\nu}$ is $\sum_{i=1}^n d_i - 2\nu$. Moreover, given an integer flow of this value we describe how to construct a desired realization. To find a solution to MM-DR one simply needs to iterate over $\nu \in [1, n]$ and find the maximum one with the aforementioned maximum flow.

First we describe construction of $G_{d,\nu}$. The source s is connected to nodes $x_i \in X$ (for $i \in [1, n]$) that correspond to the vertices of V . Similarly, the node set Y contains nodes y_j (for $j \in [1, n]$), corresponding to the vertices in W , and these are connected to the sink t . The nodes are organized into the following sets:

- Candidates for the matching: $X_M = \{x_i \mid i \in [1, 2\nu]\}$ and $Y_M = \{y_j \mid j \in [1, 2\nu]\}$,
- Rest of the nodes: $X_R = \{x_i \mid i \in [2\nu + 1, n]\}$ and $Y_R = \{y_j \mid j \in [2\nu + 1, n]\}$,
- Source and sink: $\{s, t\}$.

The edges are capacitated and directed from left to right, i.e., an edge (α, β, γ) leads from the node α to the node β and can carry up to γ units of flow. The source s has edges leading to every node x_i , with capacity $d_i - 1$ for $x_i \in X_M$ and d_i for $x_i \in X_R$. Similarly, there are edges leading from every node y_j to the sink t , with capacity $d_j - 1$ for $y_j \in Y_M$ and d_j for $y_j \in Y_R$. This enforces the degree constraint for the vertices v_i and w_i in G . Finally, there's a unit capacity edge $(x_i, y_j, 1)$ for every $i, j \in [1, n]$, such that $i \neq j$ and $j \neq 2\nu - i + 1$. Overall, the edge set of the flow graph is defined as follows:

$$\begin{aligned} \tilde{E} = & \{(s, x_i, d_i - 1) \mid i \in [1, 2\nu]\} \cup \{(s, x_i, d_i) \mid i \in [2\nu + 1, n]\} \\ & \cup \{(x_i, y_j, 1) \mid i, j \in [1, n], i \neq j, j \neq 2\nu - i + 1\} \\ & \cup \{(y_j, t, d_j - 1) \mid j \in [1, 2\nu]\} \cup \{(y_j, t, d_j) \mid j \in [2\nu + 1, n]\} . \end{aligned}$$

The construction is justified by the following lemma.

Lemma 4.9. *There exists a ν -matched bipartite graph $\hat{G} = (V, W, \hat{E})$ that realizes the (d, d) if and only if the value of the maximum $s - t$ flow in $G_{d,\gamma}$ is equal to $\sum_{i=1}^n d_i - 2\nu$.*

Proof. Suppose the value of the maximum $s - t$ flow in $G_{d,2\nu}$ is $\sum_{i=1}^n d_i - 2\nu$. Define the bipartite graph $\hat{G} = (V, W, \hat{E})$ as follows:

1. For each node $x_i \in X$, create a corresponding node $v_i \in V$.
2. For each node $y_j \in Y$, create a corresponding node $w_j \in W$.
3. For every $1 \leq i, j \leq n$, if $\text{flow}(x_i, y_j) = 1$, then create an edge (v_i, w_j) and add it to $\hat{E}$.
4. For every $1 \leq i \leq 2\nu$ create an edge $(v_i, w_{2\nu-i+1})$ and add it to $\hat{E}$.

It remains to verify that $\hat{G}$ correctly realizes (d, d) and is ν -matched. The degree of v_i in $\hat{G}$ equals to the number of edges (x_i, y_j) that carry flow if $2\nu \leq i \leq n$ and one more than the corresponding flow if $1 \leq i \leq 2\nu$. Since x_i conserves flow, and since all the edges (s, x_i) are saturated, $\deg(v_i) = d_i$. Similarly, $\deg(w_j) = d_j$ for every $j \in [1, n]$, so $\hat{G}$ satisfies (M1). Since $G_{d,\gamma}$ does not have edges (x_i, y_i) for any $i \in [1, n]$, it follows that $\hat{G}$ does not have edges (v_i, w_i) satisfying (M2). Finally, the matching $\hat{M} = \{(v_i, w_{2\nu-i+1}) \mid 1 \leq i \leq 2\nu\}$ was added in step 4, so $\hat{G}$ satisfies (M3).

Conversely, suppose $\hat{G} = (V, W, \hat{E})$, where $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots, w_n\}$, is a 2ν -matched bipartite graph realizing (d, d) with the matching $\hat{M} = \{(v_i, w_{2\nu-i+1}) \mid 1 \leq i \leq 2\nu\}$. We need to define a flow function on $G_{d,\nu}$ and prove that its value equals $\sum_{i=1}^n d_i - 2\nu$. Obviously, maximum flow cannot exceed this sum, thus it is enough to construct flow with aforementioned value.

1. Saturate Edges from Source and to Sink:

$$\text{flow}(s, x_i) \leftarrow \begin{cases} d_i - 1 & i \in [1, 2\nu], \\ d_i & i \in [2\nu + 1, n]. \end{cases} \quad \text{flow}(y_j, t) \leftarrow \begin{cases} d_j - 1 & j \in [1, 2\nu], \\ d_j & j \in [2\nu + 1, n]. \end{cases}$$

2. Flow Through Edges:

For every $i \in [1, n]$ and $j \in [1, n]$ such that $(v_i, w_j) \in \hat{E} \setminus \hat{M}$, set $\text{flow}(x_i, y_j) \leftarrow 1$,

3. Completing the flow function:

Set the flows of all other edges to 0.

We need to argue that the defined flow is legal. Note that by construction, the total flow out of the source s is $\sum_{i=1}^n d_i - 2\nu$ and the total flow into the sink t is $\sum_{j=1}^n d_j - 2\nu$. It remains to verify that for every vertex except s and t , the incoming and outgoing flows are equal.

- **Node x_i :** By construction, the incoming flow at x_i is $\text{flow}(s, x_i) = d_i - 1$ if $x_i \in X_M$ and $\text{flow}(s, x_i) = d_i$ if $x_i \in X_R$. As for the outgoing flow, recall that there are exactly $d_i - 1$ indices j for which $(v_i, w_j) \in \hat{E} \setminus \hat{M}$ if $i \in [1, 2\nu]$ and d_i such indices if $i \in [2\nu + 1, n]$. Hence, x_i conserves flow.
- **Node y_j :** By construction, the outgoing flow at y_j is $\text{flow}(y_j, t) = d_j - 1$ if $y_j \in X_M$ and $\text{flow}(y_j, t) = d_j$ if $y_j \in X_R$. As for the incoming flow, recall that there are exactly $d_j - 1$ indices i for which $(v_i, w_j) \in \hat{E} \setminus \hat{M}$ if $j \in [1, 2\nu]$ and d_j such indices if $j \in [2\nu + 1, n]$. Hence, y_j conserves flow.

The claim follows. $\square$

The running time analysis of the realization algorithm for MM-DR is similar to the one for MDS-DR. We conclude the section with the following result.

Theorem 4.10. *There exists an $O(n^3)$ time algorithm for constructing a realization of a given graphic sequence d that also has a matching of maximum size (among all possible realizations of d).*

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