Very ample sheaves on weighted projective spaces and weighted blowups

Stevell Muller, Erik Paemurru

3rd October 2025

Abstract

We consider graded rings R generated by n homogeneous elements of positive integer degrees $w_1, \ldots, w_n$ that have least common multiple d. We show that for every integer $k \geq \max(1, n-2)$, the kdth Veronese subring $R^{(kd)}$ is generated in degree 1, which implies that the line bundle $\mathcal{O}(kd)$ is very ample on the scheme $\operatorname{Proj}(R)$. This statement is sharp for every $n \geq 4$. We show that if all the weights w_i are less than 15, then $R^{(d)}$ is generated in degree 1. This bound is sharp for all $n \geq 4$. We prove that if the weights are pairwise coprime, then $R^{(d)}$ is always generated in degree 1. We show that for almost all of the vectors $(w_1, \ldots, w_n)$, $R^{(d)}$ is generated in degree 1. Finally, we show that there exist 14 fundamental vectors such that if all the weights are less than 42 and $R^{(d)}$ is not generated in degree 1, then up to permutation, a subsequence of $(w_1, \ldots, w_n)$ is equal to a fundamental vector. We prove similar statements for Rees rings and the line bundle $\mathcal{O}(kd)$ on the weighted blowup of the affine n-space with weights $(w_1, \ldots, w_n)$, with the inequality $k \geq \max(1, n-2)$ replaced by $k \geq \max(1, n-1)$.

Contents

2020 *Mathematics Subject Classification.* 05E14, 13A02, 13A30, 14A25, 14A15.

Keywords. Divisors, invertible sheaves, generation in degree 1, projective normality, projectively normal.

This work was supported by SFB 195 No. 286237555 of DFG. The second author is also supported by the Simons Foundation, grant SFI-MPS-T-Institutes-00007697, and the Ministry of Education and Science of the Republic of Bulgaria, grant DO1-239/10.12.2024.

1 Introduction

All rings are commutative and unital. A *graded ring* is a $\mathbb{Z}_{\geq 0}$-graded ring. Given a graded ring R, let R_m denote its mth homogeneous component. For every positive integer r, the *rth Veronese subring* of a graded ring R is the graded ring $R^{(r)} := \bigoplus_{m\in\mathbb{Z}_{\geq 0}} R_{rm}$, where R_{rm} is the mth graded component of $R^{(r)}$. We say that R is generated by elements $f_1, f_2, \ldots, f_n \in R$ that are homogeneous of positive degrees if $R = R_0[f_1, \ldots, f_n]$. If moreover $f_1, \ldots f_n$ are algebraically independent over R_0, then the scheme $\operatorname{Proj} R$ is called a *weighted projective space* over R_0, denoted $\mathbb{P}_{R_0}(\boldsymbol{w})$, where $\boldsymbol{w}$ is the n-tuple of the degrees of $f_1, \ldots, f_n$. In any case, $\operatorname{Proj} R$ is a subscheme of the weighted projective space $\mathbb{P}_{R_0}(\boldsymbol{w})$. We say that R is generated in degree 1 if $R = R_0 \oplus \bigoplus_{m\in\mathbb{Z}_{\geq 1}} R_1^m$.

In this paper, given a graded ring R with a finite generating sequence of homogeneous elements of positive integer degrees $(w_1, \ldots, w_n)$ where n is a positive integer, we give precise answers to the following questions in various settings:

- When is $R^{(r)}$ generated in degree 1?
- When is $\mathcal{O}_{\operatorname{Proj} R}(r)$ very ample over $\operatorname{Spec} R_0$?

While the most interesting case is when the degree zero part R_0 is the field of complex numbers $\mathbb{C}$, it is no more difficult to work over an arbitrary ring.

Denote

$$d_{\boldsymbol{w}} := \operatorname{lcm}(w_1, w_2, \ldots, w_n).$$

By Remark 1.1, we may assume r is divisible by $d_{\boldsymbol{w}}$.

Remark 1.1. Let k be a positive integer. Then:

(a) $\mathcal{O}_{\operatorname{Proj} R}(kd_{\boldsymbol{w}})$ is invertible since on each $D_+(f_i)$, it is given by $(f_i^{kd_{\boldsymbol{w}}/w_i})$,

(b) $\mathcal{O}_{\operatorname{Proj} R}(kd_{\boldsymbol{w}})$ is basepoint-free as the common zero locus of $f_1^{kd_{\boldsymbol{w}}/w_1}, \ldots, f_n^{kd_{\boldsymbol{w}}/w_n}$ is empty,

(c) $\mathcal{O}_{\operatorname{Proj} R}(kd_{\boldsymbol{w}})$ is ample by Proposition 2.8(a).

Conversely, if $R = R_0[x_1, \ldots, x_n]$ is a graded polynomial ring over a nonzero ring R_0, $i \in \{1, \ldots, n\}$ and r is a positive integer not divisible by the degree of x_i, then $\mathcal{O}_{\operatorname{Proj} R}(r)$ is not basepoint-free since its base locus contains the nonempty set $V(\{x_j \mid j \neq i\})$.

Moreover, if the vector of the degrees of $x_1, \dots, x_n$ is well-formed (Definition 4.2), then $\mathcal{O}_{\operatorname{Proj} R}(r)$ is not invertible. To see this, note that $\mathcal{O}_{\operatorname{Proj} R}(r)$ is generated at the local ring of $V(\{x_j \mid j \neq i\})$ by terms of weight r of the form $x_i^{a_i} \prod_{j \neq i} x_j^{b_j}$, where a_i is an integer and the b_j are nonnegative integers, and note that for every $k \neq j$, there exist a_i and b_j with $b_k = 0$.

Remark 1.2. Generation in degree 1 has two immediate consequences:

(a) By Corollary 2.9, if $R^{(r)}$ is generated in degree 1, then the line bundle $\mathcal{O}_{\operatorname{Proj} R}(r)$ is very ample over $\operatorname{Spec} R_0$.

(b) By Proposition 2.10 and Lemma 2.11, if R_0 is an algebraically closed field and R is a graded polynomial ring such that $R^{(r)}$ is generated in degree 1, then the global sections of $\mathcal{O}_{\operatorname{Proj} R}(r)$ define a projectively normal embedding.

In all the examples found in this paper, if $\mathcal{O}_{\operatorname{Proj} R}(r)$ is very ample, then $R^{(r)}$ is generated in degree 1. On the other hand, it is not clear whether this holds for all weighted projective spaces.

Remark 1.3. In [BR86, Lemma 4B.6], it is erroneously claimed that for every projective algebraic $\mathbb{k}$-scheme X and ample Cartier divisor Y, the conditions of Y being very ample and the $\mathbb{k}$-algebra $\bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(mY))$ being generated in degree 1 are equivalent. A counterexample is given by any normal projective variety over an algebraically closed field that is linearly normal but not projectively normal.

It is important to note that the $d_{\boldsymbol{w}}$th Veronese subring is not always generated in degree 1:

Example 1.4 ([Del75, Remark 2.6])**.** Let $R := Q[x_1, x_2, x_3, x_4]$ be the graded polynomial ring with an arbitrary nonzero coefficient ring Q where the variables x_1, x_2, x_3, x_4 have degrees $\boldsymbol{w} := (1, 6, 10, 15)$. In this case, $R^{(60)}$ is generated in degree 1 but $R^{(30)}$ is not. The reason is that denoting $\boldsymbol{u} := (1, 4, 2, 1)$, the monomial $\boldsymbol{x}^{\boldsymbol{u}} := x_1 x_2^4 x_3^2 x_4$ with weighted-degree $\boldsymbol{w} \cdot \boldsymbol{u} = 60$ is not the product of two monomials of weighted-degree 30.

Remark 1.5. In Example 1.4, if Q is an integral domain, then $\mathcal{O}_{\operatorname{Proj} R}(30)$ is not very ample over $\operatorname{Spec} R_0$ since the monomial $\boldsymbol{x}^{\boldsymbol{u}}/x_2^{10}$ does not belong to the coordinate ring of the affine open $D_+(x_2^5)$ of $Y := \operatorname{Proj} H^0(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R}(30))$ but does belong to its integral closure. In particular, if Q is an integral domain, then Y is not normal. If Q is an algebraically closed field, then $\mathcal{O}_{\operatorname{Proj} R}(30)$ separates points since $a x_i^{30/w_i} - b x_j^{30/w_j}$ is a global section for all indices i, j and elements $a, b \in Q$ but $\mathcal{O}_{\operatorname{Proj} R}(30)$ does not separate tangent vectors since there is a tangent vector $\operatorname{Spec} Q[\varepsilon] \to D_+(x_2)$ of $\operatorname{Proj} R$ given by $\boldsymbol{x}^{\boldsymbol{u}}/x_2^{10} \mapsto \varepsilon$ and $h/x_2^5 \mapsto 0$ for all $h \in R_{30}$.

The problem of determining the smallest positive integer r such that $R^{(r)}$ is generated in degree 1 or $\mathcal{O}_{\operatorname{Proj} R}(r)$ is very ample over $\operatorname{Spec} R_0$ turns out to be very subtle. Note that Castelnuovo–Mumford regularity is defined with respect to a very ample line bundle, a fundamental invariant useful for computing cohomology. It is surprising that the answers to many basic questions concerning this least integer r were previously unknown. This paper fills this gap in the literature.

Previously, the state of the art in determining which divisors on weighted projective spaces are very ample was the following:

Proposition 1.6 ([Del75, Proposition 2.3], see [BR86, Theorem 4B.7] for a translation)**.** *If $n = 1$, then define $G(\boldsymbol{w}) := -w_1$, and otherwise define*

$$G(\boldsymbol{w}) := -\sum_{i=1}^{n} w_i + \frac{1}{n-1}\sum_{v=2}^{n}\binom{n-2}{v-2}^{-1}\sum_{\substack{J\subseteq\{1,\dots,n\}\\ |J|=v}} m_J,$$

where for a subset J of $\{1,\dots,n\}$, we denote $m_J := \operatorname{lcm}\{w_i \mid i \in J\}$. If k is a positive integer satisfying $kd_{\boldsymbol{w}} > G(\boldsymbol{w})$, then $R^{(kd_{\boldsymbol{w}})}$ is generated in degree 1.

In this paper, we improve on it in several ways:

(1) Instead of a one-way implication, we give a combinatorial condition equivalent to generation in degree 1 (see Proposition 2.5) and a combinatorial condition equivalent to very ampleness (see Proposition 2.17).

(2) Instead of a criterion which is sometimes inconclusive, we give a combinatorial algorithm that guarantees to determine generation in degree 1 (see appendix A). The algorithm is very fast, with an average runtime of 3.305 µs on a single CPU core for the roughly 2 trillion weight vectors that we checked (see Remark 5.4).

(3) The combinatorial criteria that we give are often stronger than Proposition 1.6. For a concrete comparison, looking at strictly increasing vectors $\boldsymbol{w} = (w_1,\dots,w_n)$ of positive integers at most 15 where $4 \le n \le 15$, Corollary 3.18 alone gives a bound $\boldsymbol{w}\cdot\boldsymbol{v} - d_{\boldsymbol{w}}$ (see Proposition 2.4 for how this bound is related to generation in degree 1) that is strictly smaller than $G(\boldsymbol{w})$ from Proposition 1.6 roughly 98.9% of the time. The stronger bound $\boldsymbol{w}\cdot\boldsymbol{\gamma} - d_{\boldsymbol{w}}$ from Algorithm 2 is strictly smaller than $G(\boldsymbol{w})$ roughly 99.997% of the time.

(4) We give several criteria of generation in degree 1 that are easier to use and compute by hand than Proposition 1.6.

(5) In contrast to Proposition 1.6, even if some of our criteria are not conclusive, they still restrict the number of cases to check.

(6) We describe what happens for uniformly randomly chosen weights (see Theorem 1.25).

(7) In addition to graded rings and the corresponding weighted projective spaces, we also treat Rees rings and the corresponding weighted blowups.

Our first result is the following:

Theorem 1.7 (Propositions 2.4 and 4.1)**.** *Let $w_1,\dots,w_n$ be pairwise coprime positive integers. Let R be a graded ring generated by homogeneous elements of degrees $w_1,\dots,w_n$. Then, the $d_{\boldsymbol{w}}$th Veronese subring of R is generated in degree 1.*

As mentioned above, this gives the immediate corollary:

Corollary 1.8. *Let $\boldsymbol{w} := (w_1,\dots,w_n)$ be pairwise coprime positive integers. Let $\mathbb{P}(\boldsymbol{w})$ be the weighted projective space over an arbitrary ring Q. Then, $\mathcal{O}_{\mathbb{P}(\boldsymbol{w})}(d_{\boldsymbol{w}})$ is very ample over* $\operatorname{Spec} Q$*. Moreover, if Q is an algebraically closed field, then the global sections of $\mathcal{O}_{\mathbb{P}(\boldsymbol{w})}(d_{\boldsymbol{w}})$ define a projectively normal embedding.*

Remark 1.9. We give two applications of Corollary 1.8.

(a) In the special case of Corollary 1.8 where $n = 3$, we recover the result [Del75, Remark 2.6] (see [BR86, Remark 3 in section 4B] for a translation). Namely, since every weighted projective plane is isomorphic to a well-formed (Definition 4.2) weighted projective plane, we find that every ample line bundle on a weighted projective plane is very ample and moreover, defines a projectively normal embedding if the base ring is an algebraically closed field.

(b) Let $n \geq 4$ be an integer, let $\boldsymbol{w} := (w_1, \ldots, w_n)$ be a vector of pairwise coprime positive integers and let X be a smooth hypersurface of $\mathbb{P}_{\mathbb{C}}(\boldsymbol{w})$ of weighted-degree divisible by $d_{\boldsymbol{w}}$. Note that by [PS, Proposition 3.8.1], every smooth well-formed weighted hypersurface is of the form above. By Corollary 1.8, X is a very ample Cartier divisor of $\mathbb{P}_{\mathbb{C}}(\boldsymbol{w})$ and defines a projectively normal embedding. Therefore, by the Lefschetz hyperplane theorem for quasiprojective varieties ([HL85, Theorem 1.1.3] or [HL20, Theorem 9.3.1]), a general X is simply connected.

In this paper, we also study generation in degree 1 and very ampleness for Rees rings and weighted blowups. A *Rees ring* is a graded subring $R = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} t^m I_m$ of the graded polynomial ring $A[t]$ where A is a ring, $\deg(t) = 1$ and where $A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ is a filtration of ideals of A. In this case, we denote $R_0 := A$. Given a Rees ring R, a positive integer n, elements $f_1, \ldots, f_n \in R_0$ and a vector $\boldsymbol{w} = (w_1, \ldots, w_n)$ of positive integers, we say that R *is* $\boldsymbol{w}$*-generated over* R_0 *by* $f_1, \ldots, f_n$ if

$$R = R_0\Big[\Big\{t^d f_i \;\Big|\; i \in \{1, \ldots, n\},\, d \in \{0, \ldots, w_i\}\Big\}\Big].$$

We say a Rees ring R is $(w_1, \ldots, w_n)$*-generated* if there exist $f_1, \ldots, f_n \in R_0$ such that R is $(w_1, \ldots, w_n)$-generated over R_0 by $f_1, \ldots, f_n$.

If Q is a nonzero ring and $R = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} t^m I_m$ is the Rees ring $(w_1, \ldots, w_n)$-generated over the polynomial ring $R_0 = Q[x_1, \ldots, x_n]$ by its variables $x_1, \ldots, x_n$, then the natural morphism $\varphi\colon \operatorname{Proj} R \to \operatorname{Spec} R_0$ is called the *weighted blowup* of the affine n-space $\mathbb{A}^n_Q := \operatorname{Spec} R_0$ with weights $(w_1, \ldots, w_n)$. If $R^{(r)}$ is generated in degree 1 for some positive integer r, then φ is the blowup along I_r, which is an ideal in R_0. If Q is an integral domain, then R coincides with the integral closure of $R_0[t^{w_1} x_1, \ldots, t^{w_n} x_n]$ in $R_0[t]$. If Q is the field of complex numbers $\mathbb{C}$, then φ coincides with the toric morphism in [KM92, Proposition-Definition 10.3].

Example 1.10. The weighted blowup with weights $(2, 3)$ of the affine plane $\mathbb{A}^2_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x_1, x_2]$ is the morphism $\varphi\colon \operatorname{Proj} R \to \mathbb{A}^2_{\mathbb{C}}$, where R is given by

$$R = \mathbb{C}[x_1, tx_1, t^2 x_1, x_2, tx_2, t^2 x_2, t^3 x_2].$$

In this case, the graded ring

$$R^{(6)} = \mathbb{C}[x_1, x_2, t^6 x_1^3, t^6 x_1^2 x_2, t^6 x_2^2]$$

is generated in degree 1 (or equivalently, the rings $R_0 \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 1}} R_6^m$ and $\bigoplus_{m \in \mathbb{Z}_{\geq 0}} R_{6m}$ are equal) and therefore φ coincides with the blowup along the ideal $(x_1^3, x_1^2 x_2, x_2^2)$ of $\mathbb{C}[x_1, x_2]$, which is the integral closure of the ideal (x_1^3, x_2^2).

Note that every Rees ring is a graded ring. Moreover, every $\boldsymbol{w}$-generated Rees ring is generated as a graded ring by homogeneous elements with degrees in the set $\{1, 2, \ldots, \max(\boldsymbol{w})\}$. Since $\operatorname{lcm}(1, 2, \ldots, \max(\boldsymbol{w}))$ is usually much bigger than $d_{\boldsymbol{w}} := \operatorname{lcm}(\boldsymbol{w})$, we have the following weak bound:

Proposition 1.11 (Propositions 2.4 and 5.12)**.** *Let R be a $\boldsymbol{w}$-generated Rees ring. Let $N := \operatorname{lcm}(1, 2, \dots, \max(\boldsymbol{w}))$. Then, $R^{(N)}$ is generated in degree 1.*

Remark 1.12. Similarly to Remark 1.1, if R is a $(w_1, \dots, w_n)$-generated Rees ring, then for every positive integer k, the sheaf $\mathcal{O}_{\operatorname{Proj} R}(kd_{\boldsymbol{w}})$ is an ample basepoint-free invertible sheaf. Conversely, if A is a polynomial ring with n variables over a nonzero ring Q, R is the Rees ring $(w_1, \dots, w_n)$-generated over A by its variables and r is a positive integer such that $\mathcal{O}_{\operatorname{Proj} R}(r)$ is basepoint-free, then $d_{\boldsymbol{w}}$ divides r.

Similarly to Example 1.4, the $d_{\boldsymbol{w}}$th Veronese subring is not necessarily generated in degree 1:

Example 1.13. Let $R_0 := Q[x_1, x_2, x_3, x_4]$ be the polynomial ring with an arbitrary nonzero coefficient ring Q. Denote $\boldsymbol{w} := (1, 6, 10, 15)$. Let R be the Rees ring $\boldsymbol{w}$-generated over R_0 by x_1, x_2, x_3, x_4. Then, $R^{(60)}$ is generated in degree 1 but $R^{(30)}$ is not. If Q is an integral domain, then $\mathcal{O}_{\operatorname{Proj} R}(30)$ is not very ample over $\operatorname{Spec} R_0$.

The result analogous to Theorem 1.7 for Rees rings is the following:

Theorem 1.14 (Propositions 2.4 and 4.1)**.** *Let $\boldsymbol{w} := (w_1, \dots, w_n)$ be pairwise coprime positive integers. Let R be a $\boldsymbol{w}$-generated Rees ring. Then, $R^{(d_{\boldsymbol{w}})}$ is generated in degree* 1*.*

Similarly, we have an immediate corollary for very ampleness:

Corollary 1.15. *Let $\boldsymbol{w} := (w_1, \dots, w_n)$ be pairwise coprime positive integers. Let $W \to \mathbb{A}^n$ be the weighted blowup with weights $\boldsymbol{w}$ of the affine n-space over an arbitrary ring. Then, $\mathcal{O}_W(d_{\boldsymbol{w}})$ is very ample over $\mathbb{A}^n$.*

Below, we give results on generation in degree 1 and very ampleness in various settings.

1.1 Sharp upper bound

In this section, Q is an integral domain and k and n are positive integers.

We have the following sharp result:

Theorem 1.16 (Proposition 2.4 and Corollary 4.6)**.** *Let R be a graded ring generated by homogeneous elements of degrees $w_1, \dots, w_n$. Then, for every integer $k \geq \max(1, n-2)$, $R^{(kd_{\boldsymbol{w}})}$ is generated in degree* 1*.*

The sharpness is due to the following:

Example 1.17 (Proposition 2.17, Remark 2.19, Corollary 7.3, Theorem 7.5)**.** Let $n \geq 4$. There exist infinitely many vectors $\boldsymbol{p} := (p_1, \dots, p_{n-1})$ of pairwise different prime numbers such that $k_{\boldsymbol{p}} = 1$ (see Lemma 7.1 for the definition of $k_{\boldsymbol{p}}$). For each such vector $\boldsymbol{p}$, define

$$\boldsymbol{w}_{\boldsymbol{p}} := \left(1, \frac{\prod p_i}{p_1}, \dots, \frac{\prod p_i}{p_{n-1}}\right) \in \mathbb{Z}^n_{\geq 1},$$

where every product is over $i \in \{1, \dots, n-1\}$. Note that the vector $\boldsymbol{w}_{\boldsymbol{p}}$ is well-formed. Then, for every $\boldsymbol{p}$ with $k_{\boldsymbol{p}} = 1$, the line bundle $\mathcal{O}_{\operatorname{Proj} R}((n-3)d_{\boldsymbol{w}_{\boldsymbol{p}}})$ is not very ample over $\operatorname{Spec} Q$, where R is the graded polynomial ring R over Q with n variables of degrees $\boldsymbol{w}_{\boldsymbol{p}}$. Some explicit examples of suitable $\boldsymbol{p}$ are given in Table 2 for $n \leq 15$.

To the best knowledge of the authors, this provides the first example of a graded ring R generated by homogeneous elements of positive integer degrees $w_1, \dots, w_n$ such that the $2d_{\boldsymbol{w}}$th Veronese subring of R is not generated in degree 1.

For Rees rings, the corresponding result is slightly weaker, with $n-2$ replaced by $n-1$:

Theorem 1.18 (Proposition 2.4 and Corollary 3.2)**.** *Let R be a $(w_1, \dots, w_n)$-generated Rees ring. Then, for every integer $k \geq \max(1, n-1)$, $R^{(kd_{\boldsymbol{w}})}$ is generated in degree* 1.

Again, the result is sharp:

Example 1.19 (Proposition 2.18, Remark 2.19, Corollary 7.3, Theorem 7.6)**.** Let $n \geq 3$. There exist infinitely many vectors $\boldsymbol{p} := (p_1, \dots, p_n)$ of pairwise different prime numbers such that $k_{\boldsymbol{p}} = n-1$ (see Lemma 7.1 for the definition of $k_{\boldsymbol{p}}$). For each such vector $\boldsymbol{p}$, define

$$\boldsymbol{v_p} := \left(\frac{\prod p_i}{p_1}, \dots, \frac{\prod p_i}{p_n} \right) \in \mathbb{Z}^n_{\geq 1},$$

where every product is over $i \in \{1, \dots, n\}$. Then, for every $\boldsymbol{p}$ with $k_{\boldsymbol{p}} = n-1$, the line bundle $\mathcal{O}_{\operatorname{Proj} R}((n-2)d_{\boldsymbol{v_p}})$ is not very ample over $\operatorname{Spec} R_0$, where R is the Rees ring $\boldsymbol{w}$-generated over the n-variable polynomial ring over Q by its variables. Some explicit examples of suitable $\boldsymbol{p}$ are given in Table 3 for $n < 15$.

On the other hand, for weighted blowups with respect to well-formed (see Definition 4.2) weights, the result is as strong as for graded rings:

Theorem 1.20 (Proposition 2.4 and Theorem 4.3)**.** *Let $\boldsymbol{w} := (w_1, \dots, w_n)$ be a well-formed vector of positive integers and let R be a $\boldsymbol{w}$-generated Rees ring. Then, for every integer $k \geq \max(1, n-2)$, $R^{(kd_{\boldsymbol{w}})}$ is generated in degree* 1.

The result is sharp by the following:

Example 1.21 (Proposition 2.18, Remark 2.19, Corollary 7.3, Theorem 7.5)**.** Let $n \geq 4$ and define $\boldsymbol{w_p}$ as in Example 1.17. Then, for every $\boldsymbol{p}$ with $k_{\boldsymbol{p}} = 1$, the line bundle $\mathcal{O}_{\operatorname{Proj} R}((n-3)d_{\boldsymbol{w_p}})$ is not very ample over $\operatorname{Spec} Q$, where R is the Rees ring $\boldsymbol{w}$-generated over the n-variable polynomial ring over Q by its variables.

1.2 Small weights

In this section, we discuss generation in degree 1 and very ampleness for small weights. First, we introduce some notation in order to give a compact answer. We say that a vector $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$ is in the *expansion* of a vector $\boldsymbol{w}' \in \mathbb{Z}^m_{\geq 1}$ if $\boldsymbol{w}$ and $\boldsymbol{w}'$ are the same vector up to reordering the entries of $\boldsymbol{w}$ and removing any number of entries of $\boldsymbol{w}$ that divide $\operatorname{lcm}(\boldsymbol{w}')$. For example, $(1, 2, 6, 10, 15)$ is in the expansion of $(1, 6, 10, 15)$. We say that a vector $\boldsymbol{w}$ is in the expansion of a set if it is in the expansion of some element in the set.

Theorem 1.22 (Propositions 2.4, 2.5, 2.17 and 5.10, Corollary 2.15, and Theorem 5.7)**.** *Let n be a positive integer and let $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$ be such that $\max(\boldsymbol{w}) < 42$. Let R be a graded polynomial ring with a nonzero coefficient ring and with variables of degrees $\boldsymbol{w}$. Then, all of the following hold:*

(a) $R^{(d_{\boldsymbol{w}})}$ *is not generated in degree* 1 *if and only if* $\boldsymbol{w}$ *is in the expansion of the set*

$$\left\{\begin{matrix} (1,6,10,15),\ (1,6,22,33),\ (1,12,20,30), \\ (1,12,21,28),\ (1,12,22,33),\ (1,15,21,35), \\ (1,21,28,36),\ (1,21,30,35),\ (2,12,15,20), \\ (2,12,20,30),\ (2,21,30,35),\ (4,15,24,40), \\ (4,24,30,40),\ (1,3,24,30,40) \end{matrix}\right\},$$

(b) *if* R_0 *is a normal domain, then* $R^{(d_{\boldsymbol{w}})}$ *is generated in degree* 1 *if and only if* $\mathcal{O}_{\operatorname{Proj} R}(d_{\boldsymbol{w}})$ *is very ample over* R_0*,*

(c) $R^{(2d_{\boldsymbol{w}})}$ *is generated in degree* 1*.*

Remark 1.23*.* Theorem 1.22 can be used to describe very ample divisors on some known lists of weighted hypersurfaces.

(a) By [Rei80, Theorem 4.5] and [IF00, 13.3], there is a list of 95 families of canonical K3 surfaces that are well-formed quasismooth hypersurfaces in weighted projective spaces over the complex numbers. Since the largest weight that appears in the list is 33, it is easy to check from Theorem 1.22 that $\mathcal{O}_{\mathbb{P}(\boldsymbol{w})}(d_{\boldsymbol{w}})$ is very ample for every vector $\boldsymbol{w} := (w_1, w_2, w_3, w_4)$ in that list. Since very ample divisors pull back to very ample divisors under embeddings, $\mathcal{O}_X(d_{\boldsymbol{w}})$ is very ample also on the K3 surface X. Moreover, by Proposition 2.10, the global sections of both $\mathcal{O}_{\mathbb{P}(\boldsymbol{w})}(d_{\boldsymbol{w}})$ and $\mathcal{O}_X(d_{\boldsymbol{w}})$ define projectively normal embeddings.

(b) Similarly, in [IF00, 16.6] there is a list of 95 families of quasismooth terminal Fano varieties X of dimension 3 that are well-formed quasismooth hypersurfaces in weighted projective spaces over the complex numbers with anticanonical class $\mathcal{O}_X(1)$ that are not linear cones. Its completeness was conjectured in [IF00, 15.2] and proved in [CCC11, Theorem 6.1], see [Oka23, Theorem 2.11] for an alternative proof. For all weight vectors $\boldsymbol{w}$, the line bundles $\mathcal{O}_{\mathbb{P}(\boldsymbol{w})}(d_{\boldsymbol{w}})$ and $\mathcal{O}_X(d_{\boldsymbol{w}})$ are very ample and their global sections define projectively normal embeddings.

Now, we discuss generation in degree 1 and very ampleness for Rees rings and weighted blowups.

Theorem 1.24 (Propositions 2.4, 2.6, 2.18 and 5.10 and Theorem 5.7)**.** *Let* n *be a positive integer and let* $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$ *be such that* $\max(\boldsymbol{w}) < 42$*. Denote* $d := \operatorname{lcm}(\boldsymbol{w})$*. Let* Q *be a nonzero ring and let* R *be the Rees ring* $\boldsymbol{w}$*-generated over the polynomial ring* $Q[x_1, \dots, x_n]$ *by the elements* $x_1, \dots, x_n$*. Then, all of the following hold:*

(a) $R^{(d)}$ *is not generated in degree* 1 *if and only if* $\boldsymbol{w}$ *is in the expansion of the set*

$$\left\{\begin{matrix} (6,14,21),\ (6,26,39),\ (10,14,35),\ (12,15,20), \\ (12,26,39),\ (14,20,35),\ (15,20,24),\ (15,24,40), \\ (20,28,35),\ (24,30,40),\ (28,35,40), \\ (1,6,10,15),\ (1,6,22,33),\ (1,12,20,30), \\ (1,12,21,28),\ (1,12,22,33),\ (1,15,21,35), \\ (1,21,28,36),\ (1,21,30,35),\ (2,12,20,30), \\ (2,21,30,35),\ (3,12,20,30) \end{matrix}\right\},$$

(b) *if Q is a normal domain, then $R^{(d)}$ is generated in degree 1 if and only if $\mathcal{O}_{\operatorname{Proj} R}(d)$ is very ample over Q,*

(c) *$R^{(2d)}$ is generated in degree 1.*

Note that in Theorem 1.24, the morphism $\operatorname{Proj} R \to \operatorname{Spec} R_0$ is precisely the weighted blowup with weights $\boldsymbol{w}$ of the affine n-space over Q.

1.3 Density

By Example 1.17, there exist infinitely many well-formed vectors $\boldsymbol{w}$ such that $R^{(d_{\boldsymbol{w}})}$ is not generated in degree 1 for some graded ring generated by homogeneous elements of degrees $\boldsymbol{w}$. On the other hand, Theorem 1.22 suggests that for most weight vectors $\boldsymbol{w}$, $R^{(d_{\boldsymbol{w}})}$ is generated in degree 1. Below, we make this precise. We let $\{1, 2, \ldots, M\}^n$ denote the set of integer vectors $(w_1, w_2, \ldots, w_n)$ where for all $i \in \{1, \ldots, n\}$, we have $1 \leq w_i \leq M$.

Theorem 1.25 (Proposition 2.4 and Theorem 6.3)**.** *For every nonempty vector $\boldsymbol{w}$ of positive integers, we define the following condition:*

† $R^{(d_{\boldsymbol{w}})}$ is generated in degree 1 for every $\boldsymbol{w}$-generated Rees ring R and for every graded ring R generated by homogeneous elements of degrees $\boldsymbol{w}$.

For all positive integers n and M, we define

$$P(n, M) := \frac{|\{\boldsymbol{w} \in \{1, 2, \ldots, M\}^n \mid \boldsymbol{w} \textit{ satisfies condition } \dagger\}|}{M^n}.$$

Then, for every positive integer n,

$$\lim_{M \to +\infty} P(n, M) = 1.$$

1.4 Overview of the paper

In section 2, we introduce the notion of a *bubble*. An example of a bubble is the vector $\boldsymbol{u}$ in Example 1.4. We prove equivalence of generation in degree 1 and the existence of certain bubbles, as well as very ampleness and the existence of bubbles satisfying Condition 2.2, in appropriate settings. In section 3, we give obstructions to the existence of bubbles. In sections 4 to 6, we use these obstructions to prove generation in degree 1, and we construct the sharp examples of bubbles satisfying Condition 2.2 that are needed in sections 1.1 and 7.

In appendix A, we describe the algorithms used to compute the finite lists of vectors of weights less than 42 in Section 1.2. In appendix B, we give the tables of vectors of weights less than 42 that admit a bubble with positive coordinates, as well as the lists of vectors of prime numbers that give examples of weights $\boldsymbol{w}$ such that $R^{(n-3)d_{\boldsymbol{w}}}$ or $R^{(n-2)d_{\boldsymbol{w}}}$ is not generated in degree 1, as mentioned in Examples 1.17 and 1.19.

2 Relating bubbles to rings and very ampleness

In this section, we reduce the problem of determining whether the line bundles we are interested in are very ample to a combinatorial problem. For this, we introduce the notion of *bubbles*; the rest of the paper is dedicated to thoroughly studying these bubbles, determining when they exist, and under which circumstances the existence of such bubbles is an obstruction to very ampleness.

2.1 Notation

We use the notation below throughout the paper.

Let n and $\boldsymbol{w} := (w_1, \ldots, w_n)$ be positive integers. The least common multiple of $w_1, \ldots, w_n$ is denoted by $d_{\boldsymbol{w}}$. For a real number x, the greatest integer not greater than x is denoted by $\lfloor x \rfloor$ and the least integer not smaller than x is denoted by $\lceil x \rceil$. Given a finite set S, the number of elements of S is denoted $|S|$. Denote

$$\mathbb{Z}^n_{\geq 0} := \{(v_1, \ldots, v_n) \mid \forall i \in \{1, \ldots, n\},\, v_i \in \mathbb{Z}_{\geq 0}\}.$$

Given $\boldsymbol{a} := (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ and $\boldsymbol{b} := (b_1, \ldots, b_n) \in \mathbb{Z}^n_{\geq 0}$, we denote by

$$\boldsymbol{a} \cdot \boldsymbol{b} := \sum_{i \in \{1, \ldots, n\}} a_i b_i$$

the usual scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$. Let '$\preceq$' be the product order on $\mathbb{Z}^n_{\geq 0}$, meaning that $\boldsymbol{a} \preceq \boldsymbol{b}$ if the inequality $a_i \leq b_i$ holds for every $i \in \{1, \ldots, n\}$. For all $i \in \{1, \ldots, n\}$, we define $\boldsymbol{e}_i \in \mathbb{Z}^n_{\geq 0}$ to be the unit vector with 1 in the ith coordinate. For all $i \in \{1, \ldots, n\}$,

$$\pi_i \colon \mathbb{Z}^n_{\geq 0} \to \mathbb{Z}^{n-1}_{\geq 0}$$

denotes the projection which drops the ith coordinate and

$$\kappa_i \colon \mathbb{Z}^{n-1}_{\geq 0} \to \mathbb{Z}^n_{\geq 0}$$

denotes the inverse of π_i restricted to $\{\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0} \mid u_i = 0\}$.

For every nonnegative integer m, we define:

$$V_{\boldsymbol{w}}(m) := \left\{\boldsymbol{v} \in \mathbb{Z}^n_{\geq 0} \;\middle|\; \boldsymbol{w} \cdot \boldsymbol{v} = m\right\},$$
$$V^+_{\boldsymbol{w}}(m) := \left\{\boldsymbol{v} \in \mathbb{Z}^n_{\geq 0} \;\middle|\; \boldsymbol{w} \cdot \boldsymbol{v} \geq m\right\}.$$

For every set $S \subseteq \mathbb{Z}^n_{\geq 0}$ and positive integer m, we define mS to be the Minkowski sum

$$mS := \left\{\boldsymbol{s}_1 + \ldots + \boldsymbol{s}_m \in \mathbb{Z}^n_{\geq 0} \;\middle|\; \forall i \in \{1, \ldots, n\} \colon s_i \in S\right\}.$$

Definition 2.1. A **$\boldsymbol{w}$-bubble** is an element $\boldsymbol{u}$ of $\mathbb{Z}^n_{\geq 0}$ such that the inequality $\boldsymbol{w} \cdot \boldsymbol{u} \geq 2d_{\boldsymbol{w}}$ holds and there is no vector $\boldsymbol{v} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{v} = d_{\boldsymbol{w}}$ and $\boldsymbol{v} \preceq \boldsymbol{u}$.

We define the following condition for every triple $(\boldsymbol{w}, \boldsymbol{u}, l)$ where $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble and l is a positive integer:

Condition 2.2. There exists $i \in \{1, \ldots, n\}$ such that for all integers m satisfying the inequality $lm \geq k$, where

$$k := \left\lfloor \frac{\boldsymbol{w} \cdot \boldsymbol{u}}{d_{\boldsymbol{w}}} \right\rfloor,$$

we have the following:

$$\boldsymbol{u} + \frac{(lm - k)d_{\boldsymbol{w}}}{w_i} \boldsymbol{e}_i \notin mV^+_{\boldsymbol{w}}(ld_{\boldsymbol{w}}).$$

We will see in Propositions 2.17 and 2.18 that in appropriate settings, Condition 2.2 is equivalent to $\mathcal{O}(ld_{\boldsymbol{w}})$ not being very ample.

Remark 2.3. If Condition 2.2 holds for $(\boldsymbol{w}, \boldsymbol{u}, l)$, then $l < \lfloor \frac{\boldsymbol{w} \cdot \boldsymbol{u}}{d_{\boldsymbol{w}}} \rfloor$.

Lemma 5.9 will give a criterion for satisfying Condition 2.2 when $l = 1$, the most common case.

2.2 Relating generation in degree 1 to existence of bubbles

In the following list of results, we relate the existence of bubbles to a Veronese subring being generated in degree 1.

Proposition 2.4. *Let $n, w_1, \dots, w_n$ be positive integers. Let R be one of the following:*

(1) *a graded ring generated over R_0 by n homogeneous elements of degrees respectively $w_1, \dots, w_n$, or*

(2) *a $\boldsymbol{w}$-generated Rees ring.*

For every integer $k \geq 2$, if $R^{((k-1)d_{\boldsymbol{w}})}$ is not generated over R_0 in degree 1, then there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} \geq kd_{\boldsymbol{w}}$. In particular, if there are no $\boldsymbol{w}$-bubbles, then $R^{(d_{\boldsymbol{w}})}$ is generated over R_0 in degree 1

Proof. Let $f_1, \dots, f_n$ be n elements as in (1) or (2) of the statement. For every vector $\boldsymbol{v} \in \mathbb{Z}^n_{\geq 0}$, denote

$$\boldsymbol{f}^{\boldsymbol{v}} := f_1^{v_1} \cdot \ldots \cdot f_n^{v_n}.$$

First, we prove that in both cases there exists a vector $\boldsymbol{a} \in \mathbb{Z}^n_{\geq 0}$ that satisfies the inequality $\boldsymbol{w} \cdot \boldsymbol{a} \geq m(k-1)d_{\boldsymbol{w}}$ for some integer $m \geq 2$ such that there is no vector $\boldsymbol{b} \in \mathbb{Z}^n_{\geq 0}$ satisfying both $\boldsymbol{w} \cdot \boldsymbol{b} = (k-1)d_{\boldsymbol{w}}$ and $\boldsymbol{b} \preceq \boldsymbol{a}$.

(1) If $R^{((k-1)d_{\boldsymbol{w}})}$ is not generated over R_0 in degree 1, then there exists a vector $\boldsymbol{a} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{a} = m(k-1)d_{\boldsymbol{w}}$ for some integer $m \geq 2$ such that $\boldsymbol{f}^{\boldsymbol{a}} \notin R^m_{(k-1)d_{\boldsymbol{w}}}$. We can choose $\boldsymbol{a}$ such that $\boldsymbol{f}^{\boldsymbol{a}}$ is not divisible by any element of $R_{(k-1)d_{\boldsymbol{w}}}$.

(2) Let $I_0, I_1, \dots$ be ideals of R_0 such that we have the equality $R = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} t^m I_m$. If $R^{((k-1)d_{\boldsymbol{w}})}$ is not generated over R_0 in degree 1, then there exists a vector $\boldsymbol{a} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{a} \geq m(k-1)d_{\boldsymbol{w}}$ for some integer $m \geq 2$ such that $\boldsymbol{f}^{\boldsymbol{a}} \notin I^m_{(k-1)d_{\boldsymbol{w}}}$. We can choose $\boldsymbol{a}$ such that there is no vector $\boldsymbol{b} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{b} \preceq \boldsymbol{a}$ and $\boldsymbol{w} \cdot \boldsymbol{b} = (k-1)d_{\boldsymbol{w}}$.

Now, let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_N$ be a sequence of maximal length such that for every $i \in \{1, \dots, N\}$, we have that $\boldsymbol{v}_i \preceq \boldsymbol{a} - \sum_{j \in \{i+1, \dots, N\}} \boldsymbol{v}_j$ and $\boldsymbol{w} \cdot \boldsymbol{v}_i = d_{\boldsymbol{w}}$ hold. By the choice of $\boldsymbol{a}$, we necessarily have the inequality $N \leq k-2$. Define $\boldsymbol{u} := \boldsymbol{a} - \sum_{i \in \{1, \dots, N\}} \boldsymbol{v}_i$. We find that the following inequalities hold:

$$\boldsymbol{w} \cdot \boldsymbol{u} \geq (m(k-1) - (k-2))d_{\boldsymbol{w}} \geq kd_{\boldsymbol{w}}.$$

By construction, there is no vector $\boldsymbol{v} \in \mathbb{Z}^n_{\geq 0}$ satisfying both $\boldsymbol{w} \cdot \boldsymbol{v} = d_{\boldsymbol{w}}$ and $\boldsymbol{v} \preceq \boldsymbol{u}$. Therefore, $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble. □

Propositions 2.5, 2.6, 2.17 and 2.18 are partial converses to Proposition 2.4 for polynomial rings.

Proposition 2.5. *Let $n, w_1, \dots, w_n$ be positive integers. Let Q be a nonzero ring and let R be the graded polynomial ring $Q[x_1, \dots, x_n]$, where $\deg(x_i) = w_i$ for every $i \in \{1, \dots, n\}$. Then, $R^{(d_{\boldsymbol{w}})}$ is generated over R_0 in degree 1 if and only if no $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $d_{\boldsymbol{w}} \mid \boldsymbol{w} \cdot \boldsymbol{u}$.*

Proof. For every $\boldsymbol{v} \in \mathbb{Z}^n_{\geq 0}$, define

$$\boldsymbol{x}^{\boldsymbol{v}} := x_1^{v_1} \cdot \ldots \cdot x_n^{v_n}.$$

The ring $R^{(d_{\boldsymbol{w}})}$ is not generated in degree 1 if and only if there exists a vector $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} = kd_{\boldsymbol{w}}$ for some integer $k \geq 2$ such that the monomial $\boldsymbol{x}^{\boldsymbol{u}}$ is not a product of k monomials of degree $d_{\boldsymbol{w}}$. Equivalently, there exists a vector $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} = kd_{\boldsymbol{w}}$ for some integer $k \geq 2$ such that the monomial $\boldsymbol{x}^{\boldsymbol{u}}$ is not divisible by any monomial of degree $d_{\boldsymbol{w}}$. This is equivalent to the existence of a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $d_{\boldsymbol{w}} \mid \boldsymbol{w} \cdot \boldsymbol{u}$. □

Proposition 2.6. *Let $n, w_1, \dots, w_n$ be positive integers. Let Q be a nonzero ring and let R be the Rees ring $\boldsymbol{w}$-generated over the polynomial ring $Q[x_1, \dots, x_n]$ by the elements $x_1, \dots, x_n$. Then, $R^{(d_{\boldsymbol{w}})}$ is generated over R_0 in degree 1 if and only if every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{u} \in kV^+_{\boldsymbol{w}}(d_{\boldsymbol{w}})$ where $k := \lfloor \frac{\boldsymbol{w}\cdot\boldsymbol{u}}{d_{\boldsymbol{w}}} \rfloor$. In particular, if there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} \in \{ld_{\boldsymbol{w}}, ld_{\boldsymbol{w}} + 1, \dots, ld_{\boldsymbol{w}} + l - 1\}$ for some integer $l \geq 2$, then $R^{(d_{\boldsymbol{w}})}$ is not generated over R_0 in degree 1.*

Proof. Define $\boldsymbol{x}^{\boldsymbol{v}}$ as in the proof of Proposition 2.5.

The ring $R^{(d_{\boldsymbol{w}})}$ is not generated over R_0 in degree 1 if and only if there exists a vector $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} \geq kd_{\boldsymbol{w}}$ for some integer $k \geq 2$ such that the monomial $\boldsymbol{x}^{\boldsymbol{u}}$ is not a product of k monomials of weight at least $d_{\boldsymbol{w}}$. Equivalently, there exists a vector $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} \geq kd_{\boldsymbol{w}}$ for some integer $k \geq 2$ such that the monomial $\boldsymbol{x}^{\boldsymbol{u}}$ is not a product of k monomials of weight at least $d_{\boldsymbol{w}}$ and $\boldsymbol{x}^{\boldsymbol{u}}$ is not divisible by any monomial of weight $d_{\boldsymbol{w}}$. Equivalently, there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} \geq kd_{\boldsymbol{w}}$ and $\boldsymbol{u} \notin kV^+_{\boldsymbol{w}}(d_{\boldsymbol{w}})$ for some integer $k \geq 2$. This is equivalent to the existence of a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $\boldsymbol{u} \notin kV^+_{\boldsymbol{w}}(d_{\boldsymbol{w}})$, where $k := \lfloor \frac{\boldsymbol{w}\cdot\boldsymbol{u}}{d_{\boldsymbol{w}}} \rfloor$.

If a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u} \in \{ld_{\boldsymbol{w}}, \dots, ld_{\boldsymbol{w}} + l - 1\}$ for some integer $l \geq 2$, then necessarily $k := \lfloor \frac{\boldsymbol{w}\cdot\boldsymbol{u}}{d_{\boldsymbol{w}}} \rfloor \geq l$ and $\boldsymbol{u} \notin kV^+_{\boldsymbol{w}}(d_{\boldsymbol{w}})$. By the above, $R^{(d_{\boldsymbol{w}})}$ is not generated over R_0 in degree 1. □

In this section, we connected generation in degree 1 and the existence of bubbles. Before we relate very ampleness and bubbles, we state some general results on very ampleness.

2.3 Very ampleness and projective normality

We give some basic results on very ampleness and projective normality in a general setting.

Remark 2.7. In this paper, we use the definition of relative very ampleness of an invertible sheaf given in [Sta25, Definition 01VM]. Note that this differs from [Har77, definition above Remark II.5.16.1]. In particular, by [Har77, Exercise II.7.14], Proposition 2.8 would not hold using the definition of relative very ampleness in [Har77]. The two definitions coincide when the base is affine and the schemes are Noetherian.

Proposition 2.8 gives a characterisation of very ampleness that is useful for our purposes, namely relating very ampleness to the generation in degree 1 of a certain graded algebra.

Proposition 2.8. *Let $\varphi\colon Y \to X$ be a proper morphism of schemes and let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Then, all of the following hold:*

(a) *$\mathcal{L}$ is φ-ample if and only if Y is isomorphic over X to $\operatorname{Proj}_X \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \varphi_*(\mathcal{L}^{\otimes m})$,*

(b) *$\mathcal{L}$ is φ-very ample if and only if Y is isomorphic over X to $\operatorname{Proj}_X \mathcal{A}$, where $\mathcal{A} = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathcal{A}_m$ is the $\mathbb{Z}_{\geq 0}$-graded $\mathcal{O}_X$-subalgebra of $\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \varphi_*(\mathcal{L}^{\otimes m})$ generated over $\mathcal{O}_X$ by $\varphi_*\mathcal{L}$,*

(c) *if $\mathcal{L}$ is an $\mathcal{O}_Y$-submodule of the sheaf of rational functions $\mathcal{K}_Y$, then we have a natural isomorphism of $\mathbb{Z}_{\geq 0}$-graded $\mathcal{O}_X$-algebras*

$$\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \varphi_*(\mathcal{L}^{\otimes m}) \to \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \varphi_*(\mathcal{L}^m),$$

where the product on the right-hand side is the product in $\mathcal{K}_Y$ and where $\mathcal{L}^0$ is defined to be $\mathcal{O}_Y$.

Proof. (a) "$\Longrightarrow$". This is [Sta25, Lemma 0C6J].

"$\Longleftarrow$". It suffices to prove the case where X is affine. By [Sta25, Lemma 01K4], Y is quasi-compact over X. The scheme Y is covered by affine opens $D_+(f)$ where

$$f \in H^0(X, \varphi_*(\mathcal{L}^{\otimes m})) = H^0(Y, \mathcal{L}^{\otimes m})$$

and m is a positive integer. By [GW20, Remark 13.46(4)], for each such f we have an equality $D_+(f) = Y_f$. By [Sta25, Definition 01PS], it follows that $\mathcal{L}$ is φ-ample.

(b) "$\Longrightarrow$". Let ψ denote the natural map $\varphi^*\varphi_*\mathcal{L} \to \mathcal{L}$. Since ψ is surjective, by [Sta25, Lemma 01O9] we have a canonical morphism $r_{\mathcal{L},\psi} \colon Y \to \mathbb{P}(\varphi_*\mathcal{L})$ over X. Since φ is proper, $r_{\mathcal{L},\psi}$ is proper, therefore with closed image. The $\mathbb{Z}_{\geq 0}$-graded $\mathcal{O}_X$-algebra $\mathcal{A} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathcal{A}_m$ can be equivalently defined as the image of $\mathrm{Sym}(\varphi_*\mathcal{L})$ in $\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \varphi_*(\mathcal{L}^{\otimes m})$. So, the image of $r_{\mathcal{L},\psi}$ is precisely the closed subscheme given by $\mathrm{Proj}_X \mathcal{A}$. Now, since $\mathcal{L}$ is φ-very ample, by [Sta25, Lemma 01VR] the morphism $r_{\mathcal{L},\psi}$ is a closed embedding. Therefore, Y is isomorphic over X to $\mathrm{Proj}_X \mathcal{A}$.

"$\Longleftarrow$". It suffices to prove the case where X is affine. The surjective graded $\mathcal{O}_X$-algebra homomorphism $\mathrm{Sym}(\varphi_*\mathcal{L}) \to \mathcal{A}$ induces a closed embedding $\iota \colon \mathrm{Proj}_X(\mathcal{A}) \to \mathbb{P}(\varphi_*\mathcal{L})$ over X. Let A denote the graded $\mathcal{O}_X(X)$-algebra $\mathrm{Sym}\, H^0(Y, \mathcal{L})$. There exists a quasi-coherent saturated homogeneous ideal I of A such that $\mathrm{Proj}_X \mathcal{A}$ is isomorphic over X to $\mathrm{Proj}(A/I)$. By [Sta25, Lemma 01N1], the equality $\iota^*(\mathcal{O}_{\mathrm{Proj}(A)}(1)) = \mathcal{O}_{\mathrm{Proj}(A/I)}(1)$ holds. Since $Y = \mathrm{Proj}_X(\mathcal{A})$ is covered by affine opens $D_+(f)$ for $f \in H^0(Y, \mathcal{L})$, the invertible sheaf $\mathcal{L}$ is basepoint-free, therefore generated by its global sections. The invertible sheaf $\mathcal{O}_Y(1)$ is also generated by its global sections, which coincide with the global sections of $\mathcal{L}$, showing that $\mathcal{O}_Y(1) = \mathcal{L}$. So, $\mathcal{L}$ is φ-very ample.

(c) Since $\mathcal{L}$ is locally principal, for every nonnegative integer m, every stalk $(\mathcal{L}^{\otimes m})_P$ of $\mathcal{L}^{\otimes m}$ is generated by $f \cdot 1^{\otimes m}$ over $\mathcal{O}_{Y,P}$ for some $f \in \mathcal{K}_{Y,P}$. So, we have a natural map between the stalks of $\mathcal{L}^{\otimes m}$ and $\mathcal{L}^m$, sending $f \cdot 1^{\otimes m}$ to f. This induces the map $\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \varphi_*(\mathcal{L}^{\otimes m}) \to \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \varphi_*(\mathcal{L}^m)$. ☐

The following useful corollary shows that generation in degree 1 implies very ampleness.

Corollary 2.9. *Let $\varphi \colon Y \to X$ be a proper morphism of schemes and let $\mathcal{L}$ be a φ-ample invertible $\mathcal{O}_Y$-module such that $\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \varphi_*(\mathcal{L}^{\otimes m})$ is generated in degree 1. Then, $\mathcal{L}$ is φ-very ample.*

Proof. Follows by applying Proposition 2.8 parts (a) and (b) and by using that the sheaves of algebras $\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \varphi_*(\mathcal{L}^{\otimes m})$ and $\mathcal{A}$ coincide. ☐

Moreover, we show that under certain conditions, generation in degree 1 is equivalent to projective normality. In Proposition 2.10, we use the following definition: for every field $\Bbbk$ and all nonnegative integers n and k, a closed embedding $X \hookrightarrow \mathbb{P}^n_{\Bbbk}$ is called **k-normal** if the natural map $H^0(\mathbb{P}^n_{\Bbbk}, \mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(k)) \to H^0(X, \mathcal{O}_X(k))$ is surjective.

Proposition 2.10. *Let $\Bbbk$ be a field. Let X be a geometrically integral projective $\Bbbk$-scheme and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Then, the following conditions are equivalent:*

(i) *$\bigoplus_{m\in\mathbb{Z}_{\geq 0}} H^0(X, \mathcal{L}^{\otimes m})$ is generated in degree 1 and $H^0(X, \mathcal{L}^{\otimes k}) = H^0(X, \mathcal{O}_X(k))$ for every nonnegative integer k,*

(ii) *$\mathcal{L}$ is very ample over $\Bbbk$ and its global sections define a closed embedding which is k-normal for every nonnegative integer k.*

Moreover, if X is normal and $\Bbbk$ is an algebraically closed field, then the above two conditions are equivalent to the following:

(iii) *$\mathcal{L}$ is very ample over $\Bbbk$ and its global sections define a projectively normal embedding.*

Proof. Since X is geometrically integral, [Sta25, Lemma 0BUG] implies $H^0(X, \mathcal{O}_X) = \Bbbk$. Denote $R := \bigoplus_{m\in\mathbb{Z}_{\geq 0}} H^0(X, \mathcal{L}^{\otimes m})$. See [GW20, section 13.1 "Graded localization" and section 13.4] for the notations $R(k)$ and $S_{(f)}$. The sheaf $\mathcal{O}_X(k)$ is by definition $\widetilde{R(k)}$, therefore $H^0(D_+(f), \mathcal{O}_X(k)) = R(k)_{(f)}$ for every element $f \in R$ homogeneous of positive degree.

For every positive integer k, we have the inclusions

$$R_1^k \subseteq R_k \subseteq H^0(X, \mathcal{O}_X(k)).$$

To see the second inclusion, define $\mathcal{F}$ to be the sheaf given on the affine opens $D_+(f)$ by R_k, where $f \in R$ is a homogeneous positive degree element which is not a zero divisor. We have a natural injection $H^0(D_+(f), \mathcal{F}) = R_k \to R(k)_{(f)} = H^0(D_+(f), \mathcal{O}_X(k))$, which induces the injection $R_k = H^0(X, \mathcal{F}) \to H^0(X, \mathcal{O}_X(k))$.

Let $f_0, \dots, f_n$ be a basis of R_1 over $\Bbbk$. Let $\mathbb{P}^n_\Bbbk := \operatorname{Proj} \Bbbk[x_0, \dots, x_n]$. We have the natural map

$$\bigoplus_{k\in\mathbb{Z}_{\geq 0}} H^0(\mathbb{P}^n_\Bbbk, \mathcal{O}_{\mathbb{P}^n_\Bbbk}(k)) \to \bigoplus_{k\in\mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(k))$$
$$\forall i \in \{0, \dots, n\} \colon x_i \mapsto f_i,$$

with image $R_0 \oplus \bigoplus_{k\in\mathbb{Z}_{\geq 1}} R_1^k$.

(i) $\Longrightarrow$ (ii). Since R is generated in degree 1, for every nonnegative integer k, we have a surjection $H^0(\mathbb{P}^n_\Bbbk, \mathcal{O}_{\mathbb{P}^n_\Bbbk}(k)) \to R_k = H^0(X, \mathcal{O}_X(k))$.

(ii) $\Longrightarrow$ (i). Surjectivity of the map $\bigoplus_{k\in\mathbb{Z}_{\geq 0}} H^0(\mathbb{P}^n_\Bbbk, \mathcal{O}_{\mathbb{P}^n_\Bbbk}(k)) \to \bigoplus_{k\in\mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(k))$ implies that $R = R_0 \oplus \bigoplus_{m\in\mathbb{Z}_{\geq 1}} R_1^m$, meaning that R is generated in degree 1.

(ii) $\Longleftrightarrow$ (iii). Follows from [Har77, Exercise II.5.14]. □

The extra condition "$H^0(X, \mathcal{L}^{\otimes k}) = H^0(X, \mathcal{O}_X(k))$" in Proposition 2.10(i) is always satisfied for graded polynomial rings:

Lemma 2.11. *Let Q be a ring and let R be the graded polynomial ring $Q[x_1, \dots, x_n]$ with positive integer degrees for $x_1, \dots, x_n$. Let N be a positive integer such that $R^{(N)}$ is generated in degree 1 and denote $X := \operatorname{Proj} R^{(N)}$. Then, $H^0(X, \mathcal{O}_X(k)) = R_{kN}$ for every nonnegative integer k.*

Proof. As noted already in the proof of Proposition 2.10, we have $R_k \subseteq H^0(X, \mathcal{O}_X(k))$. The sheaf $\mathcal{O}_X(k)$ is by definition $\widetilde{R^{(N)}(k)}$. For every nonnegative integer k, we have

$$H^0(X, \mathcal{O}_X(k)) = \bigcap_{f\in R_N} R^{(N)}(k)_{(f)} = \bigcap_{f\in R_N} \Big((R^{(N)})_{(f)} \cdot R_{kN}\Big),$$

where the intersections are taken in the field of fractions $\operatorname{Frac} R$. Let $w_1, \dots, w_n$ be the degrees of $f_1, \dots, f_n$. Note that N is necessarily divisible by the least common multiple of $w_1, \dots, w_n$. We find

$$H^0(X, \mathcal{O}_X(k)) = \bigcap_{i\in\{1,\dots,n\}} \Big((R^{(N)})_{(x_i^{N/w_i})} \cdot R_{kN}\Big).$$

Let $h \in \bigcap_{i\in\{1,\dots,n\}}\left((R^{(N)})_{(x_i^{N/w_i})} \cdot R_{kN}\right)$. It suffices to show that $h \in R_{kN}$. There exist a nonnegative integer r and elements $g_1, \dots, g_n \in R_{rN}$ and $h_1, \dots, h_n \in R_{kN}$ such that

$$h = \frac{g_1}{x_1^{rN/w_1}} h_1 = \dots = \frac{g_n}{x_n^{rN/w_n}} h_n.$$

Therefore, for all $i, j \in \{1, \dots, n\}$, we have

$$g_i h_i x_j^{rN/w_j} = g_j h_j x_i^{rN/w_i}.$$

It follows that x_i^{rN/w_i} divides $g_i h_i$. Therefore, $h = \frac{g_1}{x_1^{rN/w_1}} h_1 \in R_{kN}$. □

2.4 Relating very ampleness to existence of bubbles

We summarise this section. We study rings R, where R is either a graded polynomial ring with variables of degrees $w_1, \dots, w_n$ or a Rees ring $\boldsymbol{w}$-generated over a polynomial ring by its variables. Since we use normality, we require the coefficient ring of the polynomial ring to be a normal domain. Propositions 2.17 and 2.18 give combinatorial criteria of very ampleness and non-very ampleness of the sheaves $\mathcal{O}_{\mathrm{Proj}(R)}(r)$ where r is a positive integer divisible by $d_{\boldsymbol{w}}$. If $\mathcal{O}_{\mathrm{Proj}(R)}(r)$ is not very ample, then $R^{(r)}$ is not generated in degree 1.

One step in the proof of Proposition 2.17 requires that the fraction fields of $S_{(x_i^{(ld_{\boldsymbol{w}}/w_i)})}$ and $Q[x_1, \dots, x_n]_{(x_i)}$ are equal. We introduce Condition 2.12 on $\boldsymbol{w}$ that guarantees this.

Condition 2.12. There exist vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}_1, \dots, \boldsymbol{c}_n \in \mathbb{Z}^n_{\geq 0}$ satisfying both of the following:

(1) $\boldsymbol{w} \cdot \boldsymbol{a} - \boldsymbol{w} \cdot \boldsymbol{b} = \delta$, and

(2) for every $j \in \{1, \dots, n\}$, there exists a positive integer m_j such that

$$\frac{w_j}{\delta}\boldsymbol{a} + \boldsymbol{c}_j \in m_j V_{\boldsymbol{w}}(d_{\boldsymbol{w}}) \quad \text{and}$$
$$\boldsymbol{e}_j + \frac{w_j}{\delta}\boldsymbol{b} + \boldsymbol{c}_j \in m_j V_{\boldsymbol{w}}(d_{\boldsymbol{w}}),$$

where $\delta := \gcd(w_1, \dots, w_n)$.

In the cases that we consider, Condition 2.12 is satisfied by Corollary 2.15, whose proof relies on the two following lemmas.

Lemma 2.13. *Suppose that there exists $i \in \{1, \dots, n\}$ such that for every nonempty sequence $\alpha_1, \dots, \alpha_N$ of positive integers less than or equal to n such that $w_i \mid w_{\alpha_1} + \dots + w_{\alpha_N}$, there exists a nonempty subsequence $\beta_1, \dots, \beta_M$ such that $w_i \mid w_{\beta_1} + \dots + w_{\beta_M}$ and $w_{\beta_1} + \dots + w_{\beta_M} \leq d_{\boldsymbol{w}}$ hold. Then, the vector $\boldsymbol{w}$ satisfies Condition 2.12.*

Proof. Choose any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n_{\geq 0}$ such that $\boldsymbol{w} \cdot \boldsymbol{a} - \boldsymbol{w} \cdot \boldsymbol{b} = \delta$. For every $j \in \{1, \dots, n\}$, let $\boldsymbol{q}_j \in \mathbb{Z}^n_{\geq 0}$ be any vector such that w_i divides $\boldsymbol{w} \cdot \left(\frac{w_j}{\delta}\boldsymbol{a} + \boldsymbol{q}_j\right)$. Note that

$$\boldsymbol{w} \cdot \left(\frac{w_j}{\delta}\boldsymbol{a} + \boldsymbol{q}_j\right) = \boldsymbol{w} \cdot \left(\boldsymbol{e}_j + \frac{w_j}{\delta}\boldsymbol{b} + \boldsymbol{q}_j\right).$$

By iteratively applying the statement hypothesis for the pair $(\boldsymbol{w}, i)$, one can find two positive integers k and l and some vectors $\boldsymbol{r}_1, \dots, \boldsymbol{r}_k \in \mathbb{Z}^n_{\geq 0}$ and $\boldsymbol{s}_1, \dots, \boldsymbol{s}_l \in \mathbb{Z}^n_{\geq 0}$, all depending on j, such that all of the following are satisfied:

$$\frac{w_j}{\delta}\boldsymbol{a} + \boldsymbol{q}_j = \boldsymbol{r}_1 + \dots + \boldsymbol{r}_k,$$
$$\boldsymbol{e}_j + \frac{w_j}{\delta}\boldsymbol{b} + \boldsymbol{q}_j = \boldsymbol{s}_1 + \dots + \boldsymbol{s}_l,$$

$$\forall \gamma \in \{1, \dots, k\} \colon w_i \mid \boldsymbol{w} \cdot \boldsymbol{r}_\gamma \text{ and } \boldsymbol{w} \cdot \boldsymbol{r}_\gamma \leq d_{\boldsymbol{w}},$$
$$\forall \gamma \in \{1, \dots, l\} \colon w_i \mid \boldsymbol{w} \cdot \boldsymbol{s}_\gamma \text{ and } \boldsymbol{w} \cdot \boldsymbol{s}_\gamma \leq d_{\boldsymbol{w}}.$$

Define $m_j := \max(k, l)$ and let

$$\boldsymbol{c}_j := \boldsymbol{q}_j + \frac{m_j d_{\boldsymbol{w}} - \boldsymbol{w} \cdot (\boldsymbol{r}_1 + \ldots + \boldsymbol{r}_k)}{w_i} \boldsymbol{e}_i.$$

It is left to show that Condition 2.12 holds with the chosen $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}_1, \dots, \boldsymbol{c}_n$ and $m_1, \dots, m_n$. Fix $j \in \{1, \dots, n\}$. For every $\gamma \in \{1, \dots, m_j\}$, define

$$\boldsymbol{t}_\gamma := \begin{cases} \boldsymbol{r}_\gamma + \dfrac{d_{\boldsymbol{w}} - \boldsymbol{w} \cdot \boldsymbol{r}_\gamma}{w_i} \boldsymbol{e}_i & \text{if } \gamma \leq k, \\ \dfrac{d_{\boldsymbol{w}}}{w_i} \boldsymbol{e}_i & \text{otherwise,} \end{cases}$$
$$\boldsymbol{u}_\gamma := \begin{cases} \boldsymbol{s}_\gamma + \dfrac{d_{\boldsymbol{w}} - \boldsymbol{w} \cdot \boldsymbol{s}_\gamma}{w_i} \boldsymbol{e}_i & \text{if } \gamma \leq l, \\ \dfrac{d_{\boldsymbol{w}}}{w_i} \boldsymbol{e}_i & \text{otherwise.} \end{cases}$$

Note that for every $\gamma \in \{1, \dots, m_j\}$, we have

$$\boldsymbol{w} \cdot t_\gamma = \boldsymbol{w} \cdot u_\gamma = d_{\boldsymbol{w}}.$$

Moreover, the equations

$$\frac{w_j}{\delta} \boldsymbol{a} + \boldsymbol{c}_j = \boldsymbol{t}_1 + \ldots + \boldsymbol{t}_{m_j} \quad \text{and}$$
$$\boldsymbol{e}_j + \frac{w_j}{\delta} \boldsymbol{b} + \boldsymbol{c}_j = \boldsymbol{u}_1 + \ldots + \boldsymbol{u}_{m_j}$$

hold. This proves the lemma. □

We use of the following well-known lemma in the proofs of Corollary 2.15 and Lemma 3.4.

Lemma 2.14. *Let l, m be positive integers. Then, both of the following hold:*

(a) *for every sequence $a_1, \dots, a_m$ of positive integers, there exists a positive integer $k \leq m$ and a k-element subset I of $\{1, \dots, m\}$ such that m divides $\sum_{i \in I} a_i$, and*

(b) *for every sequence $a_1, \dots, a_l$ of positive integers, there exists a (possibly empty) subsequence $b_1, \dots, b_k$, where $k \geq l - m + 1$, such that m divides $b_1 + \ldots + b_k$.*

Corollary 2.15. *Suppose that there exists an index $k \in \{1, \dots, n\}$, a nonnegative integer l and prime number p such that $w_k = p^l \gcd(\boldsymbol{w})$. Then, the vector $\boldsymbol{w}$ satisfies Condition 2.12.*

Proof. Without loss of generality, $\gcd(\boldsymbol{w}) = 1$. By Lemma 2.13, it suffices to show that $w_{\alpha_1} + \ldots + w_{\alpha_N} \leq d_{\boldsymbol{w}}$ holds for every nonempty sequence $\alpha_1, \dots, \alpha_N$ of positive integers less than or equal to n that contains no nonempty proper subsequence $\beta_1, \dots, \beta_M$ with $w_k \mid w_{\beta_1} + \ldots + w_{\beta_M}$. Note that $w_{\alpha_i} \leq d_{\boldsymbol{w}} / p^{l - \delta_i}$ holds for every $i \in \{1, \dots, N\}$, where δ_i is the exponent of p in the prime factorisation of w_{α_i}. We prove a slightly more general statement below.

Let $q_1, \dots, q_N$ be any nonempty sequence of positive integers satisfying both of the following:

(1) there is no nonempty proper subsequence $r_1, \dots, r_M$ such that $w_k \mid r_1 + \ldots + r_M$,
(2) for every $j \in \{1, \dots, N\}$, we have that $q_j \leq d_{\boldsymbol{w}}/p^{l-\gamma_j}$ holds, where γ_j is the exponent of p in the prime factorisation of q_j.

By Lemma 2.13, it suffices to show that $q_1 + \ldots + q_N \leq d_{\boldsymbol{w}}$.

For every nonnegative integer γ, let $A(\gamma)$ denote the number of indices j such that $\gamma_j = \gamma$. If $i_1, i_2, \dots, i_p \in \{1, \dots, N\}$ are pairwise different indices such that $\gamma_{i_1} = \ldots = \gamma_{i_p}$, then by Lemma 2.14, there exists a nonempty subsequence $j_1, \dots, j_m$ such that $p^{\gamma_{i_1}+1}$ divides $q_{j_1} + \ldots + q_{j_m}$. We may replace the terms $q_{j_1}, \dots, q_{j_m}$ by the single term $q_{j_1} + \ldots + q_{j_m}$. So, without loss of generality, for all $\gamma \in \{0, \dots, l-1\}$, we have an inequality $A(\gamma) \leq p - 1$.

The statement is clear for $N = 1$. Now we consider $N \geq 2$. By (1), there is no $j \in \{1, \dots, N\}$ such that $w_k = p^l$ divides r_j. So, $A(\gamma) = 0$ for all $\gamma \geq l$. We compute:

$$\begin{aligned}
\sum_{j \in \{1,\dots,N\}} q_j &\leq \sum_{\gamma \in \{0,\dots,l-1\}} A(\gamma) \frac{d_{\boldsymbol{w}}}{p^{l-\gamma}} \\
&\leq \sum_{\gamma \in \{0,\dots,l-1\}} \left(\frac{1}{p^{l-\gamma-1}} - \frac{1}{p^{l-\gamma}} \right) d_{\boldsymbol{w}} \\
&= \left(1 - \frac{1}{p^l} \right) d_{\boldsymbol{w}}.
\end{aligned}$$

This proves the corollary. □

Condition 2.12 was introduced so that the conclusion of the following lemma holds.

Lemma 2.16. *Let Q be a ring. Suppose that $\boldsymbol{w}$ satisfies Condition 2.12, and let R be the graded polynomial ring $Q[x_1, \dots, x_n]$, where $\deg(x_i) = w_i$ for every $i \in \{1, \dots, n\}$. For a fixed $l \in \mathbb{Z}_{\geq 1}$, we define $S := \sum_{m \in \mathbb{Z}_{\geq 0}} R^m_{ld_{\boldsymbol{w}}}$ to be the graded polynomial ring with mth homogeneous component $R^m_{ld_{\boldsymbol{w}}}$. Then, for every $i \in \{1, \dots, n\}$, the natural monomorphism $S_{(x_i^{ld_{\boldsymbol{w}}/w_i})} \to Q[x_1, \dots, x_n]_{(x_i)}$ induces an isomorphism of the total rings of fractions of $S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$ and $Q[x_1, \dots, x_n]_{(x_i)}$.*

Proof. Without loss of generality, Q is not the zero ring. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}_1, \dots, \boldsymbol{c}_n \in \mathbb{Z}^n_{\geq 0}$ be defined as in Condition 2.12, and let

$$f := \frac{\boldsymbol{x}^{\boldsymbol{a}}}{\boldsymbol{x}^{\boldsymbol{b}}}.$$

Let $i \in \{1, \dots, n\}$. The ring $\operatorname{Quot} S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$ is generated over $\operatorname{Quot} Q$ by quotients of weighted-homogeneous polynomials, both of weighted-degree $ld_{\boldsymbol{w}}$. Note that $\frac{f^{w_j/\delta}}{x_j}$ can be written as the quotient

$$\frac{f^{w_j/\delta}}{x_j} = \frac{\boldsymbol{x}^{(w_j/\delta)\boldsymbol{a} + \boldsymbol{c}_j} x_i^{\alpha_j}}{x_j \boldsymbol{x}^{(w_j/\delta)\boldsymbol{b} + \boldsymbol{c}_j} x_i^{\alpha_j}},$$

where we define:

$$\alpha_j := \frac{\left(\left\lceil \frac{m_j}{l} \right\rceil l - m_j \right) d_{\boldsymbol{w}}}{w_i}.$$

Therefore, for every $j \in \{1, \dots, n\}$, we have that

$$\frac{f^{w_j/\delta}}{x_j} \in \operatorname{Quot} S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$$

holds.

Let $g \in Q[x_1, \ldots, x_n]$ be any monomial with weighted-degree divisible by w_i. Then, g can be written as a product $g = \prod_{j \in \{1,\ldots,N\}} x_{\beta_j}$, where $\beta_1, \ldots, \beta_N \in \{1, \ldots, n\}$. We have

$$\frac{g}{x_i^{(\deg g)/w_i}} = \left(\frac{f^{w_i/\delta}}{x_i}\right)^{(\deg g)/w_i} \prod_{j \in \{1,\ldots,N\}} \frac{x_{\beta_j}}{f^{w_{\beta_j}/\delta}} \in \operatorname{Frac} S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}.$$

It follows that $\operatorname{Quot}(Q[x_1, \ldots, x_n]_{(x_i)}) = \operatorname{Quot} S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$. □

Proposition 2.17. *Let Q be a normal domain. Suppose that $\boldsymbol{w}$ satisfies Condition 2.12, and let R be the graded polynomial ring $Q[x_1, \ldots, x_n]$, where $\deg(x_i) = w_i$ for every $i \in \{1, \ldots, n\}$. Then, for every $l \in \mathbb{Z}_{\geq 1}$, the sheaf $\mathcal{O}_{\operatorname{Proj} R}(ld_{\boldsymbol{w}})$ is very ample over $\operatorname{Spec} R_0$ if and only if $(\boldsymbol{w}, \boldsymbol{u}, l)$ does not satisfy Condition 2.2 for any $\boldsymbol{w}$-bubble $\boldsymbol{u}$ with $d_{\boldsymbol{w}} \mid \boldsymbol{w} \cdot \boldsymbol{u}$.*

Proof. Define $Y := \operatorname{Proj} R$, $X := \operatorname{Spec} R_0$ and $\mathcal{L} := \mathcal{O}_Y(ld_{\boldsymbol{w}})$. Let $\varphi \colon Y \to X$ be the natural morphism. Define the graded ring

$$S := H^0(X, \mathcal{O}_X) \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 1}} H^0(X, \varphi_* \mathcal{L})^m,$$

with the homogeneous degree zero part given by $S_0 := H^0(X, \mathcal{O}_X) = R_0$ and for every positive integer m, the homogeneous degree m part given by $S_m := H^0(X, \varphi_* \mathcal{L})^m$. For every positive integer m, the ideal S_m consists of the polynomials in $Q[x_1, \ldots, x_n]$ that are the products of m homogeneous polynomials of degree $ld_{\boldsymbol{w}}$. By Proposition 2.8, $\mathcal{L}$ is φ-very ample if and only if $\operatorname{Proj} S$ and Y are isomorphic over X.

We have a natural monomorphism $S_{(x_i^{ld_{\boldsymbol{w}}/w_i})} \to Q[x_1, \ldots, x_n]_{(x_i)}$. As an abuse of notation, we identify $S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$ with a subring of $Q[x_1, \ldots, x_n]_{(x_i)}$. By Lemma 2.16, the fraction fields of $Q[x_1, \ldots, x_n]_{(x_i)}$ and $S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$ are equal for every $i \in \{1, \ldots, n\}$.

We show that every element of $Q[x_1, \ldots, x_n]_{(x_j)}$ is integral over $S_{(x_j^{ld_{\boldsymbol{w}}/w_j})}$ for every $j \in \{1, \ldots, n\}$. For every monomial $g \in Q[x_1, \ldots, x_n]$, we have $g^{ld_{\boldsymbol{w}}} \in S_{\deg g}$. Therefore, for every monomial $g \in Q[x_1, \ldots, x_n]$ with degree divisible by w_j, the element $g/x_j^{(\deg g)/w_j}$ is integral over $S_{(x_j^{ld_{\boldsymbol{w}}/w_j})}$. Since the sum of integral elements is integral, every element of $Q[x_1, \ldots, x_n]_{(x_j)}$ is integral over $S_{(x_j^{ld_{\boldsymbol{w}}/w_j})}$.

Since Q is a normal domain, Y is a normal scheme. We find that $\mathcal{L}$ is φ-very ample if and only if $Q[x_1, \ldots, x_n]_{(x_i)}$ is equal to $S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$ for every $i \in \{1, \ldots, n\}$. Equivalently, for every integer $i \in \{1, \ldots, n\}$ and vector $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} = cw_i$ for some nonnegative integer c, we have

$$\frac{\boldsymbol{x}^{\boldsymbol{u}}}{x_i^c} \in S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}.$$

Equivalently, for every integer $i \in \{1, \ldots, n\}$ and vector $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} = cw_i$ for some nonnegative integer c, there exists a nonnegative integer m satisfying $lmd_{\boldsymbol{w}} \geq \boldsymbol{w} \cdot \boldsymbol{u}$ such that

$$\boldsymbol{u} + \left(lm\frac{d_{\boldsymbol{w}}}{w_i} - c\right)\boldsymbol{e}_i \in mV_{\boldsymbol{w}}(ld_{\boldsymbol{w}}).$$

It suffices to consider vectors $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$, not identically zero, satisfying $d_{\boldsymbol{w}} \mid \boldsymbol{w} \cdot \boldsymbol{u}$. So, $\mathcal{L}$ is φ-very ample if and only if the following condition is satisfied for every vector $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$:

‡ if $\boldsymbol{w}\cdot\boldsymbol{u} = kd_{\boldsymbol{w}}$ for some positive integer k, then for every integer $i \in \{1,\dots,n\}$, there exists a nonnegative integer m satisfying $lm \geq k$ such that

$$\boldsymbol{u} + \frac{(lm-k)d_{\boldsymbol{w}}}{w_i}\boldsymbol{e}_i \in mV_{\boldsymbol{w}}(ld_{\boldsymbol{w}}).$$

If a vector $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$ does not satisfy ‡ and is not a $\boldsymbol{w}$-bubble, then there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}'$ with $\boldsymbol{u}' \preceq \boldsymbol{u}$ and moreover, no such $\boldsymbol{u}'$ satisfies ‡. Therefore, $\mathcal{L}$ is φ-very ample if and only if for every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ with $d_{\boldsymbol{w}} \mid \boldsymbol{w}\cdot\boldsymbol{u}$ and for every integer $i \in \{1,\dots,n\}$, there exists a nonnegative integer m satisfying $lm \geq k$ such that

$$\boldsymbol{u} + \frac{(lm-k)d_{\boldsymbol{w}}}{w_i}\boldsymbol{e}_i \in mV_{\boldsymbol{w}}(ld_{\boldsymbol{w}}),$$

where the integer k is defined by $\boldsymbol{w}\cdot\boldsymbol{u} = kd_{\boldsymbol{w}}$. We may replace $V_{\boldsymbol{w}}(ld_{\boldsymbol{w}})$ by $V^+_{\boldsymbol{w}}(ld_{\boldsymbol{w}})$ without altering the statement. It follows that $\mathcal{L}$ is φ-very ample if and only if for every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $d_{\boldsymbol{w}} \mid \boldsymbol{w}\cdot\boldsymbol{u}$, the triple $(\boldsymbol{w},\boldsymbol{u},l)$ does not satisfy Condition 2.2. ☐

Proposition 2.18. *Let Q be a normal domain. Let l be a positive integer. Let R be the Rees ring $\boldsymbol{w}$-generated over the polynomial ring $Q[x_1,\dots,x_n]$ by the elements $x_1,\dots,x_n$. Then, $\mathcal{O}_{\operatorname{Proj} R}(ld_{\boldsymbol{w}})$ is very ample over $\operatorname{Spec} R_0$ if and only if $(\boldsymbol{w},\boldsymbol{u},l)$ does not satisfy Condition 2.2 for any $\boldsymbol{w}$-bubble $\boldsymbol{u}$.*

Proof. Define $X, Y, \mathcal{L}, \varphi$ and S as in the proof of Proposition 2.17. By Proposition 2.8, $\mathcal{L}$ is φ-very ample if and only if $\operatorname{Proj} S$ and Y are isomorphic over X. For every $i \in \{1,\dots,n\}$, similarly to the proof of Proposition 2.17, we identify $S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$ with a subring of $Q[x_1,\dots,x_n]_{(x_i)}$ and we see that every element of $Q[x_1,\dots,x_n]_{(x_i)}$ is integral over $S_{(x_i^{ld_{\boldsymbol{w}}/w_j})}$. Moreover, since $R_0 = Q[x_1,\dots,x_n] = S_0$, the fraction fields of $Q[x_1,\dots,x_n]_{(x_i)}$ and $S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$ are both equal to $\operatorname{Frac}(Q[x_1,\dots,x_n])$.

Since Y is normal, we find that $\mathcal{L}$ is φ-very ample if and only if $Q[x_1,\dots,x_n]_{(x_i)}$ is equal to $S_{(x_i^{ld_{\boldsymbol{w}}/w_i})}$ for every $i \in \{1,\dots,n\}$. Equivalently, for every integer $i \in \{1,\dots,n\}$ and vector $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w}\cdot\boldsymbol{u} \geq cw_i$ for some nonnegative integer c, there exists a nonnegative integer m satisfying $lmd_{\boldsymbol{w}} \geq cw_i$ such that

$$\boldsymbol{u} + \left(lm\frac{d_{\boldsymbol{w}}}{w_i} - c\right)\boldsymbol{e}_i \in mV^+_{\boldsymbol{w}}(ld_{\boldsymbol{w}}).$$

Equivalently, for every integer $i \in \{1,\dots,n\}$ and $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $k \geq ld_{\boldsymbol{w}}$, where $k := \lfloor \frac{\boldsymbol{w}\cdot\boldsymbol{u}}{d_{\boldsymbol{w}}} \rfloor$, there exists a nonnegative integer m satisfying $lm \geq k$ such that

$$\boldsymbol{u} + \frac{(lm-k)d_{\boldsymbol{w}}}{w_i}\boldsymbol{e}_i \in mV^+_{\boldsymbol{w}}(ld_{\boldsymbol{w}}).$$

Equivalently, for every $\boldsymbol{w}$-bubble $\boldsymbol{u}$, the triple $(\boldsymbol{w},\boldsymbol{u},l)$ does not satisfy Condition 2.2. ☐

Remark 2.19. If Q is only an integral domain in Propositions 2.17 and 2.18 and $\mathcal{O}_{\operatorname{Proj} R}(ld_{\boldsymbol{w}})$ is very ample over $\operatorname{Spec} R_0$, then $(\boldsymbol{w},\boldsymbol{u},l)$ does not satisfy Condition 2.2 for any $\boldsymbol{w}$-bubble $\boldsymbol{u}$. To see this, note that very ampleness is equivalent to an isomorphism of schemes by Proposition 2.8, so we can apply Propositions 2.17 and 2.18 after a base change to the normalisation of $\operatorname{Spec} R_0$.

In this paper, all of the results on generation, respectively nongeneration, in degree 1 rely on Proposition 2.4, respectively Propositions 2.5, 2.6, 2.17 and 2.18. Moreover, we have shown how to relate the obstruction of generation in degree 1 with the combinatorics of bubbles. In Sections 3.1 to 3.3 we give tools to prove nonexistence of bubbles.

3 Tools for showing nonexistence of bubbles

Our goal is to provide an as accurate as possible answer to when the line bundles of the previous sections are very ample. In particular, we are interested in determining when a given weight vector has no bubbles, meaning there are no obstructions for generation in degree 1 for the Veronese subrings we have defined. For what follows, we fix $n \geq 1$ a positive integer and $\boldsymbol{w} := (w_1, \dots, w_n) \in \mathbb{Z}_{\geq 1}$.

3.1 Nonexistence of big bubbles

The main results of this section are Corollaries 3.2 and 3.5. We show that the coordinates of a $\boldsymbol{w}$-bubble are actually bounded, and we deduce upper bounds for the scalar product of $\boldsymbol{w}$ and any $\boldsymbol{w}$-bubble. We start by observing the following.

Lemma 3.1. *If $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble, then $u_i \leq \dfrac{d_{\boldsymbol{w}}}{w_i} - 1$ holds for every $i \in \{1, \dots, n\}$.*

Proof. Let $\boldsymbol{u}$ be a $\boldsymbol{w}$-bubble and suppose that $u_i \geq \dfrac{d_{\boldsymbol{w}}}{w_i}$ holds for some $1 \leq i \leq n$. It gives the inequality $w_i u_i \geq d_{\boldsymbol{w}}$, where both sides are divisible by w_i. Hence there exists $\gamma \in \{1, \dots u_i\}$ such that $w_i \gamma = d_{\boldsymbol{w}}$, giving rise to an element $\boldsymbol{v} := \gamma \boldsymbol{e}_i \preceq \boldsymbol{u}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{v} = d_{\boldsymbol{w}}$. This is a contradiction. Therefore, $u_i \leq \dfrac{d_{\boldsymbol{w}}}{w_i} - 1$ holds for every $1 \leq i \leq n$. $\square$

This gives us an upper bound on the scalar product between $\boldsymbol{w}$ and any $\boldsymbol{w}$-bubble.

Corollary 3.2. *Every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u} < n d_{\boldsymbol{w}}$.*

Proof. Follows from Lemma 3.1. $\square$

We show in Theorems 7.5 and 7.6 that the bound in Corollary 3.2 is sharp.

Lemma 3.3 is a generalisation of Lemma 3.1 that we use for proving Theorem 4.3.

Lemma 3.3. *Let $\boldsymbol{u}$ be a $\boldsymbol{w}$-bubble and let I be a nonempty subset of $\{1, \dots, n\}$. Then, the following holds:*

$$\sum_{i \in I} w_i u_i \leq d_{\boldsymbol{w}} + (|I| - 1) \operatorname{lcm}(\{w_i \mid i \in I\}) - \sum_{i \in I} w_i.$$

Proof. Denote $g := \operatorname{lcm}(\{w_i \mid i \in I\})$. For all $i \in I$, let a_i, b_i be the unique nonnegative integers such that

$$u_i = a_i \frac{g}{w_i} + b_i \quad \text{and} \quad b_i \leq \frac{g}{w_i} - 1$$

hold. Consider the finite set V of vectors $\boldsymbol{v} \in \mathbb{Z}^n_{\geq 0}$ with $\boldsymbol{v} \preceq \boldsymbol{u}$ such that $v_j = 0$ if $j \notin I$ and $v_i w_i$ is divisible by g for every $i \in I$. Then,

$$\{\boldsymbol{w} \cdot \boldsymbol{v} \mid \boldsymbol{v} \in V\} = \{0,\, g,\, 2g,\, \dots,\, \sum_{i \in I} a_i g\}.$$

Since $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble, it follows that $\sum_{i \in I} a_i g \leq d_{\boldsymbol{w}} - g$ holds. Therefore, we obtain the inequality

$$\sum_{i \in I} w_i u_i = \sum_{i \in I} a_i g + \sum_{i \in I} w_i b_i \leq d_{\boldsymbol{w}} - g + |I|\, g - \sum_{i \in I} w_i,$$

proving the lemma. $\square$

Lemma 3.4. *Assume that $n \geq 2$ and that $w_1 \leq \ldots \leq w_n$ hold. If $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$ is a $\boldsymbol{w}$-bubble, then for all $i \in \{1, \ldots, n\}$ the following holds:*

$$u_i \leq \begin{cases} w_n - 2 & \text{if } i \neq n, \\ w_{n-1} - 2 & \text{if } i = n. \end{cases}$$

Proof. Let us fix $i \in \{1, \ldots, n\}$. Define

$$\alpha_i := \begin{cases} w_n & \text{if } i \neq n, \\ w_{n-1} & \text{if } i = n. \end{cases}$$

Let $\boldsymbol{u}' := \boldsymbol{u} - u_i \boldsymbol{e}_i$. Our strategy to prove the lemma consists of constructing an element $\boldsymbol{v}' \in \mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{v}' \preceq \boldsymbol{u}'$ such that the inequalities

$$d_{\boldsymbol{w}} - w_i(\alpha_i - 1) \leq \boldsymbol{w} \cdot \boldsymbol{v}' < d_{\boldsymbol{w}} + w_i$$

hold and $\boldsymbol{w} \cdot \boldsymbol{v}'$ is divisible by w_i. If such an element $\boldsymbol{v}'$ exists, then the element

$$\boldsymbol{v} := \boldsymbol{v}' + \left(\frac{d_{\boldsymbol{w}} - \boldsymbol{w} \cdot \boldsymbol{v}'}{w_i} \right) \boldsymbol{e}_i \in \mathbb{Z}^n_{\geq 0}$$

satisfies $\boldsymbol{w} \cdot \boldsymbol{v} = d_{\boldsymbol{w}}$. Thus, since $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble, we cannot have that $\boldsymbol{v} \preceq \boldsymbol{u}$. This implies the following

$$u_i \leq \frac{d_{\boldsymbol{w}} - \boldsymbol{w} \cdot \boldsymbol{v}'}{w_i} - 1 \leq \frac{w_i(\alpha_i - 1)}{w_i} - 1,$$

giving the desired inequality. Let us show now that such an element $\boldsymbol{v}'$ exists to conclude the proof.

Let $s := \sum_{1 \leq j \leq n} u'_j$ and let $j_1, \ldots, j_s \in \{1, \ldots, n\}$ be such that $\boldsymbol{u}' = \sum_{1 \leq k \leq s} \boldsymbol{e}_{j_k}$. Since $\boldsymbol{w} \cdot \boldsymbol{u} \geq 2d_{\boldsymbol{w}}$ and $w_i u_i \leq d_{\boldsymbol{w}} - w_i$ hold respectively by definition and by Lemma 3.1, we find that the inequality $\boldsymbol{w} \cdot \boldsymbol{u}' \geq d_{\boldsymbol{w}} + w_i$ is satisfied. We therefore let r be the greatest integer such that $\boldsymbol{w} \cdot \sum_{1 \leq k \leq r} \boldsymbol{e}_{j_k} < d_{\boldsymbol{w}} + w_i$ holds. By maximality of r, we obtain the following inequalities:

$$d_{\boldsymbol{w}} - \alpha_i + w_i \leq \boldsymbol{w} \cdot \sum_{k \in \{1, \ldots, r\}} \boldsymbol{e}_{j_k} < d_{\boldsymbol{w}} + w_i.$$

By Lemma 2.14, there exists a subset J of $\{1, \ldots, r\}$ satisfying that $|J| \geq r - w_i + 1$ and such that w_i divides $\boldsymbol{w} \cdot \sum_{k \in J} \boldsymbol{e}_{j_k}$. We have the following:

$$d_{\boldsymbol{w}} - w_i(\alpha_i - 1) \leq \boldsymbol{w} \cdot \sum_{k \in J} \boldsymbol{e}_{j_k} < d_{\boldsymbol{w}} + w_i.$$

Define $\boldsymbol{v}' := \sum_{k \in J} \boldsymbol{e}_{j_k}$ and note that $\boldsymbol{v}' \preceq \boldsymbol{u}'$ holds to conclude. □

We give two corollaries that are weaker than Lemma 3.4 but slightly easier to use.

Corollary 3.5. *Every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u} \leq (\max(\boldsymbol{w}) - 2) \cdot \sum_i w_i$.*

Proof. Direct consequence of Lemma 3.4. □

If w_n is big compared to w_{n-1}, then Corollary 3.6 may give a better bound than Corollary 3.5.

Corollary 3.6. *Assume that $n \geq 2$ and that $w_1 \leq \ldots \leq w_n$ hold. Then, every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u} \leq n w_{n-1}(w_n - 2)$.*

Proof. By Lemma 3.4, the following inequality holds:

$$\boldsymbol{w} \cdot \boldsymbol{u} \leq (w_{n-1} - 2)w_n + (w_n - 2) \sum_{i \in \{1,\dots,n-1\}} w_i. \tag{3.1}$$

By assumption, we have that $(w_{n-1} - 2)w_n \leq (w_n - 2)w_{n-1}$ holds, so the corollary follows. $\square$

3.2 Divisibility chains

The main results of this section are Propositions 3.9 and 3.11, which show nonexistence of $\boldsymbol{w}$-bubbles in special cases concerning divisibility.

We use Notation 3.7 in Lemmas 3.8 and 3.10 and Proposition 3.9.

Notation 3.7. Let $G_{\boldsymbol{w}}$ be the directed graph with set of vertices $\{1, \dots, n\}$ and such that there exists an arrow from i to j if $w_i \mid w_j$.

Lemma 3.8. *Let $\boldsymbol{u}$ be a $\boldsymbol{w}$-bubble. Then for every directed path in $G_{\boldsymbol{w}}$ with vertices I, the following is satisfied:*

$$\sum_{i \in I} w_i u_i < d_{\boldsymbol{w}}.$$

Proof. Let $I = (i_1, \dots, i_N)$ be the vertex sequence of a directed path in $G_{\boldsymbol{w}}$. By definition, for all $j \in \{1, \dots, N-1\}$, we have that w_{i_j} divides $w_{i_{j+1}}$.

If there exists $j \in \{1, \dots, N-1\}$ such that $w_{i_j} u_{i_j} \geq w_{i_{j+1}}$ holds, then we define

$$\boldsymbol{u}' := \boldsymbol{u} + \boldsymbol{e}_{i_{j+1}} - \frac{w_{i_{j+1}}}{w_{i_j}} \boldsymbol{e}_{i_j}.$$

Remark that $\boldsymbol{u}'$ has nonnegative coordinates and it satisfies $\sum_{i \in I} w_i \cdot u'_i = \sum_{i \in I} w_i \cdot u_i$. Suppose that $\boldsymbol{u}'$ is not a $\boldsymbol{w}$-bubble and let $\boldsymbol{v}' \preceq \boldsymbol{u}'$ be an element of $\mathbb{Z}^n_{\geq 0}$ such that $\boldsymbol{w} \cdot \boldsymbol{v}' = d_{\boldsymbol{w}}$. We define

$$\boldsymbol{v} := \begin{cases} \boldsymbol{v}' & \text{if } v'_{i_{j+1}} \leq u_{i_{j+1}}, \\ \boldsymbol{v}' - \boldsymbol{e}_{i_{j+1}} + \dfrac{w_{i_{j+1}}}{w_{i_j}} \boldsymbol{e}_{i_j} & \text{otherwise.} \end{cases}$$

Then, $\boldsymbol{v} \preceq \boldsymbol{u}$ is an element of $\mathbb{Z}^n_{\geq 0}$ and $\boldsymbol{w} \cdot \boldsymbol{v} = \boldsymbol{w} \cdot \boldsymbol{v}' = d_{\boldsymbol{w}}$, a contradiction. Therefore, $\boldsymbol{u}'$ is a $\boldsymbol{w}$-bubble. By induction, we can thus assume that for all $j \in \{1, \dots, N-1\}$,

$$w_{i_j} u_{i_j} \leq w_{i_{j+1}} - w_{i_j}$$

holds. We deduce the following inequality:

$$\sum_{i \in I} w_i u_i \leq (u_{i_N} + 1) w_{i_N} - w_{i_1}.$$

By Lemma 3.1, we know that $(u_{i_N} + 1) w_{i_N} \leq d_{\boldsymbol{w}}$. Therefore, the inequalities

$$(u_{i_N} + 1) w_{i_N} - w_{i_1} \leq d_{\boldsymbol{w}} - w_{i_1} < d_{\boldsymbol{w}},$$

hold, proving the lemma. $\square$

Proposition 3.9. *If there exists a partition $\{I_1, \dots, I_N\}$ of $\{1, \dots, n\}$ such that each I_j is the vertex set of a directed path in $G_{\boldsymbol{w}}$, then every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u} < N d_{\boldsymbol{w}}$.*

Proof. Follows from Lemma 3.8. $\square$

Lemma 3.10. *Let $\boldsymbol{u}$ be a $\boldsymbol{w}$-bubble. If there exists a directed path in $G_{\boldsymbol{w}}$, with vertex sequence $I := (i_1, \dots, i_N)$, and integers $k, l \in \{1, \dots, n\} \setminus I$ such that $w_{i_N} \mid w_k + w_l$, then*

$$\sum_{i \in I \cup \{k,l\}} w_i u_i < 2d_{\boldsymbol{w}}.$$

Proof. Let $I := (i_1, \dots, i_N)$ be the vertex sequence of a directed path in $G_{\boldsymbol{w}}$, and suppose there exist $k, l \in \{1, \dots, n\} \setminus I$ satisfying $w_{i_N} \mid w_k + w_l$. Let moreover $\{\alpha, \beta\} = \{k, l\}$ be such that $u_\alpha \leq u_\beta$ holds . According to Lemma 3.1, we observe the inequalities:

$$0 < w_\alpha + w_\beta \leq d_{\boldsymbol{w}} - (w_\alpha + w_\beta)u_\alpha.$$

For all $j \in \{1, \dots, N\}$, we define

$$c_{i_j} := \frac{d_{\boldsymbol{w}} - (w_\alpha + w_\beta)u_\alpha - \sum_{l \in \{j+1, \dots, N\}} w_{i_l} u_{i_l}}{w_{i_j}}.$$

Suppose that there exists $j \in \{1, \dots, N\}$ such that $u_{i_j} \geq c_{i_j}$, and without loss of generality, suppose that j is the greatest satisfying this property. The existence and maximality of j imply that

$$\sum_{l \in \{j+1, \dots, N\}} w_{i_l} u_{i_l} < d_{\boldsymbol{w}} - (w_\alpha + w_\beta)u_\alpha \leq \sum_{l \in \{j, \dots, N\}} w_{i_l} u_{i_l}$$

hold, and all three are divisible by w_{i_j}. Hence, there exists $\gamma \in \{1, \dots, u_{i_j}\}$ such that

$$\gamma w_{i_j} + \sum_{l \in \{j+1, \dots, N\}} w_{i_l} u_{i_l} = d_{\boldsymbol{w}} - (w_\alpha + w_\beta)u_\alpha.$$

In particular, $\boldsymbol{c} := \gamma e_{i_j} + \sum_{l \in \{j+1, \dots, N\}} u_{i_l} \boldsymbol{e}_{i_l} + u_\alpha \boldsymbol{e}_\alpha + u_\alpha \boldsymbol{e}_\beta \preceq \boldsymbol{u}$ is an element of $\mathbb{Z}^n_{\geq 0}$ such that $\boldsymbol{w} \cdot \boldsymbol{c} = d_{\boldsymbol{w}}$. This contradicts our assumption on $\boldsymbol{u}$ being a $\boldsymbol{w}$-bubble.

Therefore, $u_{i_j} < c_{i_j}$ holds for every $1 \leq j \leq N$ and in particular, $\sum_{i \in I} w_i u_i < d_{\boldsymbol{w}} - (w_\alpha + w_\beta)u_\alpha$. Since $\sum_{i \in I} w_i u_i$, $d_{\boldsymbol{w}}$ and $w_\alpha + w_\beta$ are all divisible by w_{i_1}, we obtain the inequality:

$$\sum_{i \in I} w_i u_i \leq d_{\boldsymbol{w}} - (w_\alpha + w_\beta)u_\alpha - w_{i_1}. \tag{3.2}$$

We also have the following two inequalities:

$$w_\alpha u_\alpha \leq w_\alpha u_\alpha, \tag{3.3}$$
$$w_\beta u_\beta \leq d_{\boldsymbol{w}} - w_\beta. \tag{3.4}$$

Adding the inequalities (3.2) to (3.4) gives

$$\sum_{i \in I \cup \{k,l\}} w_i \cdot u_i \leq 2d_{\boldsymbol{w}} - w_\beta(u_\alpha + 1) - w_{i_1} < 2d_{\boldsymbol{w}},$$

which concludes the proof $\square$

Proposition 3.11. *If $n \geq 3$ and after a suitable permutation of the weights, we have a divisibility chain*

$$w_1 \mid w_2 \mid \dots \mid w_{n-2} \mid w_{n-1} + w_n,$$

then there are no $\boldsymbol{w}$-bubbles.

Proof. Follows from Lemma 3.10. □

Remark 3.12. For an example application, by [Kaw03, Theorem 1.13] and [Yam18, Theorem 2.6], every 3-dimensional divisorial contraction over $\mathbb{C}$ with centre an isolated cA_k singularity is locally analytically the restriction $Y \to X$ to a hypersurface of the weighted blowup with weights $\boldsymbol{w}$ of $\mathbb{A}_{\mathbb{C}}$, where either $\boldsymbol{w}$ is one of the following:

(1) $(r_1, r_2, a, 1)$, where a divides $r_1 + r_2$,
(2) $(1, 5, 3, 2)$, or
(3) $(4, 3, 2, 1)$.

As a corollary of Propositions 2.4 and 3.11 and Corollary 2.9, in each of the three cases, the line bundle $\mathcal{O}_Y(d_{\boldsymbol{w}})$ is very ample.

3.3 Inductive arguments

In this section, we relate the existence of $\boldsymbol{w}$-bubbles to the existence of $\pi_i(\boldsymbol{w})$-bubbles.

Lemma 3.13. *All of the following hold:*

(a) *For every $\boldsymbol{u} \in \mathbb{Z}^n_{\geq 0}$, the vector $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble if and only if $\boldsymbol{u}$ is a $\boldsymbol{w}'$-bubble, where $\boldsymbol{w}' := \left(\frac{1}{\gcd(w_1,\dots,w_n)} w_i\right)_{1 \leq i \leq n}$,*
(b) *If there exist $1 \leq i < j \leq n$ such that $w_i = w_j$, then for every integer c, there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} = c$ if and only if there exists a $\pi_i(\boldsymbol{w})$-bubble $\boldsymbol{u}'$ satisfying $\pi_i(\boldsymbol{w}) \cdot \boldsymbol{u}' = c$,*
(c) *If there exists $1 \leq i \leq n$ such that $w_i = d_{\pi_i(\boldsymbol{w})}$, then κ_i restricts to a bijection from $\pi_i(\boldsymbol{w})$-bubbles to $\boldsymbol{w}$-bubbles.*

Proof. (a) Follows by defining $\boldsymbol{u}' := \boldsymbol{u}$ and remarking that $d_{\alpha\boldsymbol{w}} = \alpha d_{\boldsymbol{w}}$ for every $\alpha \in \mathbb{Q}_{>0}$ such that $\alpha\boldsymbol{w} \in \mathbb{Z}^n_{\geq 0}$.

(b) Let $1 \leq i < j \leq n$ be such that $w_i = w_j$. By assumption, $d_{\pi_i(\boldsymbol{w})} = d_{\boldsymbol{w}}$. Let $\boldsymbol{u}$ be a $\boldsymbol{w}$-bubble and let

$$\boldsymbol{u}' := \pi_i(\boldsymbol{u} + u_i \boldsymbol{e}_j).$$

One has that $\boldsymbol{\pi}_i(\boldsymbol{w}) \cdot \boldsymbol{u}' = \boldsymbol{w} \cdot \boldsymbol{u} \geq 2d_{\boldsymbol{w}}$ hold. Suppose that $\boldsymbol{u}'$ is a not a $\pi_i(\boldsymbol{w})$-bubble and let $\boldsymbol{v}' \prec \boldsymbol{u}'$ be an element of $\mathbb{Z}^{n-1}_{\geq 0}$ such that $\pi_i(\boldsymbol{w}) \cdot \boldsymbol{v}' = d_{\boldsymbol{w}}$. There are two cases.

(1) Either $v'_j \leq u_j$ holds, in which case $\kappa_i(\boldsymbol{v}') \preceq \boldsymbol{u}$ and $\boldsymbol{w} \cdot \kappa_i(\boldsymbol{v}') = \pi_i(\boldsymbol{w}) \cdot \boldsymbol{v}' = d_{\boldsymbol{w}}$ hold too,
(2) or the element

$$\boldsymbol{v} := \kappa_i(\boldsymbol{v}') + (v'_j - u_j)\boldsymbol{e}_i - (v'_j - u_j)\boldsymbol{e}_j \in \mathbb{Z}^n_{\geq 0}$$

satisfies $\boldsymbol{v} \preceq \boldsymbol{u}$ with $\boldsymbol{w} \cdot \boldsymbol{v} = d_{\boldsymbol{w}}$.

In both cases, we arrive at a contradiction with the assumption that $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble. Conversely, if $\boldsymbol{u}'$ is a $\pi_i(\boldsymbol{w})$-bubble, then it follows directly that $\kappa_i(\boldsymbol{u}')$ is a $\boldsymbol{w}$-bubble.

(c) Let $1 \leq i \leq n$ be such that $w_i = d_{\pi_i(\boldsymbol{w})}$. By assumption, again, $d_{\pi_i(\boldsymbol{w})} = d_{\boldsymbol{w}}$. According to Lemma 3.1, any $\boldsymbol{w}$-bubble $\boldsymbol{u}$ must satisfy that $w_i u_i \leq d_{\boldsymbol{w}} - w_i = 0$, and thus $u_i = 0$. The result is therefore trivial. □

Lemma 3.14 is a version of Lemma 3.13(a) for bubbles of weight divisible by $d_{\boldsymbol{w}}$. Lemma 3.14 is a combinatorial version of the fact that the weighted projective spaces $\mathbb{P}(w_1, gw_2, \dots, gw_n)$ and $\mathbb{P}(w_1, w_2, \dots, w_n)$ are isomorphic if $\gcd(w_1, g) = 1$.

Lemma 3.14. *Assume that* $\gcd(\boldsymbol{w}) = 1$. *Let* $i \in \{1, \dots, n\}$ *and denote* $g := \gcd(\pi_i(\boldsymbol{w}))$. *Then, for every integer* c *divisible by* $d_{\boldsymbol{w}}$, *there exists a* $\boldsymbol{w}$*-bubble* $\boldsymbol{u}$ *satisfying* $\boldsymbol{w} \cdot \boldsymbol{u} = c$ *if and only if there exists a* $\boldsymbol{w}'$*-bubble* $\boldsymbol{u}'$ *satisfying* $\boldsymbol{w}' \cdot \boldsymbol{u}' = c/g$, *where* $w'_i := w_i$ *and* $w'_j := w_j/g$ *for* $j \neq i$.

Proof. Define $\tilde{\boldsymbol{w}} \in \mathbb{Z}^n_{\geq 1}$ by $\tilde{w}_i := g w_i$ and $\tilde{w}_j = w_j$ if $j \in \{1, \dots, n\} \setminus \{i\}$. If $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble such that $\boldsymbol{w} \cdot \boldsymbol{u}$ is divisible by $d_{\boldsymbol{w}}$, then necessarily $g \mid u_i$. So, we have a bijection between $\boldsymbol{w}$-bubbles $\boldsymbol{u}$ satisfying $d_{\boldsymbol{w}} \mid \boldsymbol{w} \cdot \boldsymbol{u}$ and $\tilde{\boldsymbol{w}}$-bubbles $\tilde{\boldsymbol{u}}$ satisfying $d_{\tilde{\boldsymbol{w}}} \mid \tilde{\boldsymbol{w}} \cdot \tilde{\boldsymbol{u}}$, sending $\boldsymbol{u}$ to $\tilde{\boldsymbol{u}}$, where $\tilde{u}_i := u_i/g$ and $\tilde{u}_j := u_j$ if $j \in \{1, \dots, n\} \setminus \{i\}$. Note that $\boldsymbol{w} \cdot \boldsymbol{u} = \tilde{\boldsymbol{w}} \cdot \tilde{\boldsymbol{u}}$. The lemma now follows from Lemma 3.13(a). □

Lemma 3.15 implies that if there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$, then one of the following holds:

- all coordinates of $\boldsymbol{u}$ are positive,
- there exists $i \in \{1, \dots, n\}$ and a $\pi_i(\boldsymbol{w})$-bubble such that $d_{\boldsymbol{w}} = d_{\pi_i(\boldsymbol{w})}$, or
- there exists $i \in \{1, \dots, n\}$ and a $\pi_i(\boldsymbol{w})$-bubble $\boldsymbol{u}'$ satisfying $\pi_i(\boldsymbol{w}) \cdot \boldsymbol{u}' \geq 3 d_{\pi_i(\boldsymbol{w})}$.

We show in Theorem 5.7(c) that the last is impossible if $\max(\boldsymbol{w}) < 42$.

Lemma 3.15. *Let* $i \in \{1, \dots, n\}$. *For every* $\boldsymbol{w}$*-bubble* $\boldsymbol{u}$, *either* u_i *is positive or there is a* $\pi_i(\boldsymbol{w})$*-bubble* $\boldsymbol{u}' \preceq \pi_i(\boldsymbol{u})$ *satisfying* $\pi_i(\boldsymbol{w}) \cdot \boldsymbol{u}' \geq \boldsymbol{w} \cdot \boldsymbol{u} + d_{\pi_i(\boldsymbol{w})} - d_{\boldsymbol{w}}$.

Proof. Let $\boldsymbol{u}$ be a $\boldsymbol{w}$-bubble such that $u_i = 0$. If $d_{\pi_i(\boldsymbol{w})} = d_{\boldsymbol{w}}$, then the statement is clear. For the rest of the proof, we consider the case $d_{\boldsymbol{w}} = m d_{\pi_i(\boldsymbol{w})}$ for some integer $m \geq 2$. In this case, $d_{\pi_i(\boldsymbol{w})} - d_{\boldsymbol{w}} = -(m-1) d_{\pi_i(\boldsymbol{w})}$.

Seeking a contradiction, assume there is no $\pi_i(\boldsymbol{w})$-bubble $\boldsymbol{u}' \preceq \pi_i(\boldsymbol{u})$ such that the inequality $\pi_i(\boldsymbol{w}) \cdot \boldsymbol{u}' \geq \boldsymbol{w} \cdot \boldsymbol{u} - (m-1) d_{\pi_i(\boldsymbol{w})}$ holds. Let $\boldsymbol{u}_1 := \pi_i(\boldsymbol{u})$. Since

$$\pi_i(\boldsymbol{w}) \cdot \boldsymbol{u}_1 = \boldsymbol{w} \cdot \boldsymbol{u} > \boldsymbol{w} \cdot \boldsymbol{u} - (m-1) d_{\pi_i(\boldsymbol{w})}$$

hold, there exists $\boldsymbol{v}_1 \in \mathbb{Z}^{n-1}_{\geq 0}$ such that $\boldsymbol{v}_1 \preceq \boldsymbol{u}_1$ and $\pi_i(\boldsymbol{w}) \cdot \boldsymbol{v}_1 = d_{\pi_i(\boldsymbol{w})}$. Define $\boldsymbol{u}_2 := \pi_i(\boldsymbol{u}) - \boldsymbol{v}_1$. Similarly, since

$$\pi_i(\boldsymbol{w}) \cdot \boldsymbol{u}_2 = \boldsymbol{w} \cdot \boldsymbol{u} - d_{\pi_i(\boldsymbol{w})} \geq \boldsymbol{w} \cdot \boldsymbol{u} - (m-1) d_{\pi_i(\boldsymbol{w})}$$

hold, there exists $\boldsymbol{v}_2 \in \mathbb{Z}^{n-1}_{\geq 0}$ such that $\boldsymbol{v}_2 \preceq \boldsymbol{u}_2$ and $\pi_i(\boldsymbol{w}) \cdot \boldsymbol{v}_2 = d_{\pi_i(\boldsymbol{w})}$. By induction, there exist vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbb{Z}^{n-1}_{\geq 0}$ such that $\boldsymbol{v} := \kappa_i(\boldsymbol{v}_1 + \ldots + \boldsymbol{v}_m) \preceq \boldsymbol{u}$ and $\boldsymbol{w} \cdot \boldsymbol{v} = d_{\boldsymbol{w}}$. This contradicts $\boldsymbol{u}$ being a $\boldsymbol{w}$-bubble. □

We use the following notation in Lemma 3.17.

Notation 3.16. For all $i \in \{1, \dots, n\}$ and $r \in \{0, \dots, w_i - 1\}$, define

$$\begin{aligned} B_{\boldsymbol{w},i,r} &:= \left\{ \boldsymbol{w} \cdot \boldsymbol{b} \;\middle|\; \boldsymbol{b} \in \{0,1\}^n,\ b_i = 0,\ \boldsymbol{w} \cdot \boldsymbol{b} \equiv r \mod w_i \right\}, \\ b_{\boldsymbol{w},i,r} &:= \begin{cases} \min(B_{\boldsymbol{w},i,r}) & \text{if } B_{\boldsymbol{w},i,r} \text{ is nonempty,} \\ (w_i - 1) \cdot \max(\pi_i(\boldsymbol{w})) & \text{otherwise,} \end{cases} \\ c_{\boldsymbol{w},i} &:= \max\{ b_{\boldsymbol{w},i,r} \mid r \in \{0, \dots, w_i - 1\} \}. \end{aligned}$$

Lemma 3.17. *Let* $\boldsymbol{u}$ *be a* $\boldsymbol{w}$*-bubble such that* $u_j > 0$ *holds for every* $j \in \{1, \dots, n\}$. *Then, for every* $i \in \{1, \dots, n\}$ *such that the inequality* $\sum_{j \in \{1,\dots,n\} \setminus \{i\}} w_j < d_{\boldsymbol{w}} + w_i$ *holds, we have an inequality:*

$$u_i \leq \left\lfloor \frac{\max(\pi_i(\boldsymbol{w})) + c_{\boldsymbol{w},i}}{w_i} \right\rfloor - 2.$$

Proof. Let $i \in \{1, \dots, n\}$ be such that the inequality $\sum_{j\in\{1,\dots,n\}\setminus\{i\}} w_j < d_{\boldsymbol{w}} + w_i$ holds. Define $\alpha_i := \max(\pi_i(\boldsymbol{w}))$. Let $\boldsymbol{u}' := \boldsymbol{u} - u_i\boldsymbol{e}_i$ and note that $\boldsymbol{w} \cdot \boldsymbol{u}' \geq d_{\boldsymbol{w}} + w_i$ holds since $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble. We follow a strategy similar to the one in the proof of Lemma 3.4.

Let $\boldsymbol{a}' := \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \boldsymbol{e}_j$. Note that $\boldsymbol{a}' \preceq \boldsymbol{u}'$ and the inequality

$$\boldsymbol{w} \cdot \boldsymbol{a}' < d_{\boldsymbol{w}} + w_i$$

holds by the assumption on i. Let $s := \sum_{j\in\{1,\dots,n\}}(u'_j - a'_j)$ and let $j_1, \dots, j_s \in \{1, \dots, n\}$ be such that $\boldsymbol{u}' - \boldsymbol{a}' = \sum_{k\in\{1,\dots,s\}} \boldsymbol{e}_{j_k}$. We define $t \in \{0, \dots, s\}$ to be the greatest integer such that the following holds:

$$\boldsymbol{w} \cdot (\boldsymbol{a}' + \sum_{k\in\{1,\dots,t\}} \boldsymbol{e}_{j_k}) < d_{\boldsymbol{w}} + w_i.$$

Let $\boldsymbol{a} := \boldsymbol{a}' + \sum_{k\in\{1,\dots,t\}} \boldsymbol{e}_{j_k}$. It follows that $\boldsymbol{a}' \preceq \boldsymbol{a} \preceq \boldsymbol{u}'$ and

$$d_{\boldsymbol{w}} - \alpha_i + w_i \leq \boldsymbol{w} \cdot \boldsymbol{a} < d_{\boldsymbol{w}} + w_i.$$

Let $0 \leq r \leq w_i - 1$ be such that $\boldsymbol{w} \cdot \boldsymbol{a} \equiv r \mod w_i$. There are two cases.

(1) If $B_{\boldsymbol{w},i,r}$ is nonempty, then let $\boldsymbol{b} \in \{0, 1\}^n$ with $b_i = 0$ be such that $\boldsymbol{w} \cdot \boldsymbol{b} = b_{\boldsymbol{w},i,r}$. We define $\boldsymbol{v}' := \boldsymbol{a} - \boldsymbol{b}$. Note that $\boldsymbol{v}'$ is an element of $\mathbb{Z}^n_{\geq 0}$ that satisfies both $\boldsymbol{v}' \preceq \boldsymbol{u}'$ and $w_i \mid \boldsymbol{w} \cdot \boldsymbol{v}'$. Moreover, the inequalities

$$d_{\boldsymbol{w}} - \alpha_i - b_{\boldsymbol{w},i,r} + w_i \leq \boldsymbol{w} \cdot \boldsymbol{v}' < d_{\boldsymbol{w}} + w_i$$

hold. Similarly to the proof of Lemma 3.4, we find that the following upper bound on u_i:

$$u_i \leq \frac{\alpha_i + b_{\boldsymbol{w},i,r} - w_i}{w_i} - 1.$$

Since $b_{\boldsymbol{w},i,r} \leq c_{\boldsymbol{w},i}$, we arrive at the desired inequality.

(2) If $B_{\boldsymbol{w},i,r}$ is empty, then we obtain the equality $b_{\boldsymbol{w},i,r} = (w_i - 1)\alpha_i$ and in particular $c_{\boldsymbol{w},i} \geq (w_i - 1)\alpha_i$ holds. The previous inequality and Lemma 3.4 give us that the following inequalities hold:

$$\frac{\alpha_i + c_{\boldsymbol{w},i}}{w_i} - 2 \geq \alpha_i - 2 \geq u_i.$$

This concludes the proof. □

Corollary 3.18. *Let $\boldsymbol{v} \in \mathbb{Z}^n$ be the vector with ith entry*

$$\left\lfloor \frac{\max(\pi_i(\boldsymbol{w})) + c_{\boldsymbol{w},i}}{w_i} \right\rfloor - 2$$

for every $i \in \{1, \dots, n\}$. Then, every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ with positive coordinates satisfies

$$\boldsymbol{w} \cdot \boldsymbol{u} \leq \boldsymbol{w} \cdot \boldsymbol{v}.$$

Proof. Follows from Lemma 3.17. □

Remark 3.19. It is possible for the bound in Lemma 3.17 to be negative. For example, if $\boldsymbol{w} := (1, 3, 5, 5, 12, 15, 25)$ and $i = 7$, then $c_{\boldsymbol{w},i} = 32$ and

$$\left\lfloor \frac{\max(\pi_i(\boldsymbol{w})) + c_{\boldsymbol{w},i}}{w_i} \right\rfloor - 2 = -1,$$

implying by Lemma 3.17 that there are no $\boldsymbol{w}$-bubbles with positive coordinates.

Lemma 3.20 shows that the bound in Lemma 3.17 is always at least as strong as the bound in Lemma 3.4.

Lemma 3.20. *For every $i \in \{1, \ldots, n\}$, we have the following inequality:*

$$c_{\boldsymbol{w},i} \leq (w_i - 1) \cdot \max(\pi_i(\boldsymbol{w})).$$

Proof. To prove the lemma, it suffices to show that if $B_{\boldsymbol{w},i,r}$ is nonempty, then the inequality $\min(B_{\boldsymbol{w},i,r}) \leq (w_i - 1) \cdot \max(\pi_i(\boldsymbol{w}))$ holds.

Let $\boldsymbol{b} \in \{0,1\}^n$ be such that $b_i = 0$ and $\boldsymbol{w} \cdot \boldsymbol{b} = \min(B_{\boldsymbol{w},i,r})$. Let $N, i_1, \ldots, i_N$ be positive integers such that $\boldsymbol{b} = \sum_{j \in \{1,\ldots,N\}} \boldsymbol{e}_{i_j}$. If $N \geq w_i$ holds, then let $1 \leq k < l \leq n$ be integers such that $\boldsymbol{w} \cdot \sum_{j \in \{1,\ldots,k\}} \boldsymbol{e}_{i_j}$ and $\boldsymbol{w} \cdot \sum_{j \in \{1,\ldots,l\}} \boldsymbol{e}_{i_j}$ are congruent modulo w_i. We find that the vector $\boldsymbol{b}' := \boldsymbol{b} - \sum_{j \in \{k+1,\ldots,l\}} \boldsymbol{e}_{i_j}$ satisfies $\boldsymbol{b}' \in \{0,1\}^n$, $b'_i = 0$ and $\boldsymbol{w} \cdot \boldsymbol{b}' \equiv r \mod w_i$. Since $\boldsymbol{w} \cdot \boldsymbol{b}' < \boldsymbol{w} \cdot \boldsymbol{b}$ holds by construction, this contradicts the statement that $\boldsymbol{w} \cdot \boldsymbol{b} = \min(B_{\boldsymbol{w},i,r})$. Therefore, $N < w_i$ and $\min(B_{\boldsymbol{w},i,r}) \leq (w_i - 1) \cdot \max(\pi_i(\boldsymbol{w}))$ hold. □

4 Sharp upper bound for bubbles

We have shown in Corollary 3.6 that any $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u} < n d_{\boldsymbol{w}}$. In this section we show that if $\boldsymbol{w} \cdot \boldsymbol{u}$ is divisible by $d_{\boldsymbol{w}}$ or if $\boldsymbol{w}$ is *well-formed*, which we define below, then one obtains a sharper bound on the scalar product of $\boldsymbol{w}$ and $\boldsymbol{u}$. In the process, we prove that pairwise coprime weights do not admit bubbles, which is an interesting statement on its own.

Proposition 4.1. *If n is a positive integer and $w_1, \ldots, w_n$ are pairwise coprime positive integers, then there are no $\boldsymbol{w}$-bubbles.*

Proof. By Lemma 3.13(b), it suffices to consider the case where $\boldsymbol{w}$ is strictly increasing. Moreover, by Corollary 3.2, we only have to prove the cases where $n \geq 3$ holds.

We consider the case $n = 3$. We have $d_{\boldsymbol{w}} = w_1 w_2 w_3$. There are thus two cases:

- if $w_1 = 1$, then Proposition 3.9 tells us directly that there are no $\boldsymbol{w}$-bubbles,
- if $w_1 > 1$, remark that every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies:

  $$2 w_1 w_2 w_3 \leq \boldsymbol{w} \cdot \boldsymbol{u} < 3 w_2 w_3,$$

  where the first inequality holds since $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble and the second holds by Corollary 3.6. Since $w_1 > 1$, we find that there are no $\boldsymbol{w}$-bubbles.

Next, we consider the case $n \geq 4$. By Corollary 3.6, it suffices to prove that the inequality $2 d_{\boldsymbol{w}} \geq n w_{n-1} w_n$ holds. Since $\boldsymbol{w}$ is strictly increasing, we have the inequality:

$$2 d_{\boldsymbol{w}} = 2(w_1 \cdot \ldots \cdot w_{n-2}) w_{n-1} w_n \geq 2(n-2)! w_{n-1} w_n.$$

Since $n \geq 4$ holds, we find that $2(n-2)! \geq n$, proving the proposition. □

Definition 4.2. Let $n \geq 2$ be an integer. We say that a vector $\boldsymbol{w} \in \mathbb{Z}_{\geq 1}^n$ is *well-formed* if for every $i \in \{1, \dots, n\}$, we have $\gcd(\{w_1, \dots, w_n\} \setminus \{w_i\}) = 1$.

Theorem 4.3. *Let $\boldsymbol{w} \in \mathbb{Z}_{\geq 1}^n$ be well-formed. Then, every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u} < (n-1)d_{\boldsymbol{w}}$.*

Proof. The case $n = 2$ follows by Corollary 3.2. In the case $n = 3$, the weights are pairwise coprime and the result follows from Proposition 4.1. From now on, let $n \geq 4$ and assume that there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} \geq (n-1)d_{\boldsymbol{w}}$. We prove two claims.

Claim 4.4. *For all pairwise distinct integers $i, j, k, l \in \{1, \dots, n\}$, we have $\operatorname{lcm}(w_i, w_j) = d_{\boldsymbol{w}}$ or $\operatorname{lcm}(w_k, w_l) = d_{\boldsymbol{w}}$.*

Proof. Assume that there exist pairwise distinct $i, j, k, l \in \{1, \dots, n\}$ such that both $\operatorname{lcm}(w_i, w_j)$ and $\operatorname{lcm}(w_k, w_l)$ are less than $d_{\boldsymbol{w}}$. By applying Lemma 3.1 to every $\alpha \in \{1, \dots, n\} \setminus \{i, j, k, l\}$ and by applying Lemma 3.3 twice, namely to $I = \{i, j\}$ and $I = \{k, l\}$, we find that the inequality

$$\boldsymbol{w} \cdot \boldsymbol{u} \leq (n-2)d_{\boldsymbol{w}} + \operatorname{lcm}(w_i, w_j) + \operatorname{lcm}(w_k, w_l) - \sum_{\alpha=1}^{n} w_\alpha$$

holds. Since $\operatorname{lcm}(w_i, w_j)$ and $\operatorname{lcm}(w_k, w_l)$ divide $d_{\boldsymbol{w}}$, we necessarily have

$$\operatorname{lcm}(w_i, w_j) \leq \frac{d_{\boldsymbol{w}}}{2} \quad \text{and}$$
$$\operatorname{lcm}(w_k, w_l) \leq \frac{d_{\boldsymbol{w}}}{2}.$$

But this contradicts our assumption on $\boldsymbol{u}$. This proves the claim. ☐

Claim 4.5. *For all pairwise different i, j, k, one of the equalities $\operatorname{lcm}(w_i, w_j) = d_{\boldsymbol{w}}$, $\operatorname{lcm}(w_i, w_k) = d_{\boldsymbol{w}}$ or $\operatorname{lcm}(w_j, w_k) = d_{\boldsymbol{w}}$ holds.*

Proof. Assume that the inequalities $\operatorname{lcm}(w_i, w_j) < d_{\boldsymbol{w}}$, $\operatorname{lcm}(w_i, w_k) < d_{\boldsymbol{w}}$ and $\operatorname{lcm}(w_j, w_k) < d_{\boldsymbol{w}}$ all hold. Choose any $l \in \{1, \dots, n\} \setminus \{i, j, k\}$. We find that for every $m \in \{1, \dots, n\} \setminus \{l\}$, we have $\operatorname{lcm}(w_l, w_m) = d_{\boldsymbol{w}}$. Since gcd and lcm are distributive over each other, we have $\operatorname{lcm}(w_l, \gcd(\pi_m(\boldsymbol{w}))) = d_{\boldsymbol{w}}$. Since $\gcd(\pi_m(\boldsymbol{w})) = 1$, we find $w_l = d_{\boldsymbol{w}}$. By Lemma 3.13(c), $\pi_m(\boldsymbol{u})$ is a $\pi_m(\boldsymbol{w})$-bubble with $\pi_m(\boldsymbol{w}) \cdot \pi_m(\boldsymbol{u}) = (n-1)d_{\boldsymbol{w}}$. This contradicts Corollary 3.2, proving the claim. ☐

Next, we show that there exists a unique integer $i_0 \in \{1, \dots, n\}$ such that for all $j, k \in \{1, \dots, n\} \setminus \{i_0\}$ with $j \neq k$, we have $\operatorname{lcm}(w_j, w_k) = d_{\boldsymbol{w}}$. The existence follows from the two claims above. Now we show uniqueness. It suffices to show that for every i_0 as above, we have $\operatorname{lcm}(w_{i_0}, w_j) < d_{\boldsymbol{w}}$ for all $j \in \{1, \dots, n\} \setminus \{i_0\}$. Assume that there exists $j \in \{1, \dots, n\} \setminus \{i_0\}$ such that $\operatorname{lcm}(w_{i_0}, w_j) = d_{\boldsymbol{w}}$. Then $\operatorname{lcm}(w_j, w_k) = d_{\boldsymbol{w}}$ for all $k \in \{1, \dots, n\} \setminus \{j\}$. Similarly to the proof of Claim 4.5, we find a contradiction, proving the claim.

Denote $J := \{1, \dots, n\} \setminus \{i_0\}$ and for every $j \in J$, denote $q_j := \frac{d_{\boldsymbol{w}}}{w_j}$.

We show that the integers q_j with $j \in J$ are pairwise coprime and with product $d_{\boldsymbol{w}}$. Choose different $j, k \in J$. Since $\operatorname{lcm}(w_j, w_k) = d_{\boldsymbol{w}}$, we find that $\gcd(\frac{d_{\boldsymbol{w}}}{w_j}, \frac{d_{\boldsymbol{w}}}{w_k}) = 1$. Now, let m denote the least common multiple of the elements $\frac{d_{\boldsymbol{w}}}{w_j}$ where $j \in J$. Then, there exist positive integers $a_1, \dots, a_n$ satisfying $\gcd(a_1, \dots, a_n) = 1$ and $mw_j = d_{\boldsymbol{w}} a_j$ for all j. Since $\pi_{i_0}(\boldsymbol{w}) = 1$, we find

$$m = \gcd(\{mw_j \mid j \in J\}) = d_{\boldsymbol{w}}.$$

This proves the claim.

It follows that for all $j \in J$, we have $w_j = \prod_{k \in J \setminus \{j\}} q_k$. Since $\pi_j(\boldsymbol{w}) = 1$ for all $j \in J$, we find that $\gcd(w_{i_0}, q_j) = 1$ for all $j \in J$. We find $w_{i_0} = \gcd(w_{i_0}, d_{\boldsymbol{w}}) = 1$, where the second equality comes from the fact that gcd and lcm are distributive over each other. By Proposition 3.9, $\boldsymbol{w} \cdot \boldsymbol{u} < (n-1)d_{\boldsymbol{w}}$, a contradiction. $\square$

Corollary 4.6. *Every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ such that $d_{\boldsymbol{w}}$ divides $\boldsymbol{w} \cdot \boldsymbol{u}$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u} \leq (n-2)d_{\boldsymbol{w}}$.*

Proof. By Corollary 3.2, in order to prove the corollary, it is enough to show that there is no $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} = (n-1)d_{\boldsymbol{w}}$. Moreover, according to Lemma 3.13(b) and Lemma 3.14 we may assume, without loss of generality, that $\gcd(\boldsymbol{w}) = \gcd(\pi_i(\boldsymbol{w})) = 1$ for all $i \in \{1, \dots, n\}$. The corollary now follows from Theorem 4.3. $\square$

5 Small weights

In this section we treat the main computational part of the paper, describing an efficient algorithm for enumerating all the bubbles with positive coordinates of a given weight vector. We apply this algorithm to list all the weight vectors with maximum less than 42 which admit a bubble.

By **list** we mean a finite sequence.

Definition 5.1. Let $W \subseteq \mathbb{Z}^n_{\geq 0}$ be a list of vectors.

- We say W is **arranged** if W is lexicographically ordered and no two consecutive elements are comparable for the product order.
- We say W is **marked** if there exists $\boldsymbol{c} \in \mathbb{Z}^n_{\geq 0}$ such that W contains $c_i \boldsymbol{e}_i$ for every $i \in \{1, \dots, n\}$.
- We call $\widetilde{W}$ an **arrangement** of W if $\widetilde{W}$ is an *arranged list* containing all the minimal elements of W.

Proposition 5.2. *Algorithm 1 in Appendix A returns the correct output.*

Proof. Let $n \in \mathbb{Z}_{\geq 1}$ be a positive integer, let $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$ be a positive vector, let $\texttt{mins}, \texttt{maxs} \in \mathbb{Z}^n$ be vectors, let $\texttt{min\_product} \in \mathbb{Z}$ be an integer and let $\texttt{thresholds} \subseteq \mathbb{Z}^n_{\geq 0}$ be an arranged marked list of nonnegative vectors. We let $\texttt{a}$ and $\texttt{b}$ be defined as in lines 1 and 2, and we denote by V the output of

$$\textsc{Recursive}(\boldsymbol{w}, \texttt{mins}, \texttt{maxs}, \texttt{min\_product}, \texttt{thresholds}, \texttt{false}).$$

Suppose that there exists a vector $\boldsymbol{u} \in \mathbb{Z}^n$ such that $\texttt{mins} \preceq \boldsymbol{u} \preceq \texttt{maxs}$ and $\texttt{min\_product} \leq \boldsymbol{w} \cdot \boldsymbol{u}$ hold, and such that there are no vectors $\boldsymbol{v} \in \texttt{thresholds}$ with $\boldsymbol{v} \preceq \boldsymbol{u}$. Since $\texttt{thresholds}$ is marked and arranged, we know that its first element $\boldsymbol{v}$ is of the form $\boldsymbol{v} = c_n \boldsymbol{e}_n$ for some $c_n \geq 0$. We thus deduce, by assumption, that $u_n \leq \texttt{b}$. Moreover, we know that

$$w_n u_n \geq \texttt{min\_product} - \pi_n(\boldsymbol{w}) \cdot \pi_n(\boldsymbol{u})$$

and together with $\pi_n(\boldsymbol{u}) \preceq \pi_n(\texttt{maxs})$ we obtain that $u_n \geq \texttt{a}$. In particular $\texttt{a} \leq \texttt{b}$, which justifies line 4.

(1) If $n = 1$ (line 5), then `thresholds` consists of a single vector $\boldsymbol{v}$ which is a nonnegative multiple of $\boldsymbol{e}_1$. Moreover $\boldsymbol{u}$ consists of one coordinate u_1 which must be contained, by assumption, between $\max\Big(\texttt{mins}_1, \lceil \frac{\texttt{min\_product}}{w_1} \rceil\Big)$ and $\min(\texttt{maxs}_1, v_1 - 1)$. In this situation, the former coincides with `a` and the latter with `b`. Thus $\boldsymbol{u}$ is in V and by construction, all the elements in V satisfy the same assumptions as $\boldsymbol{u}$. In particular, $\texttt{a}\boldsymbol{e}_1 \in V$ is the smallest element for the lexicographic order to satisfy the wanted assumptions.

(2) In the case where $n \geq 2$ is a positive integer, we observe as before that

$$\pi_n(\boldsymbol{w}) \cdot \pi_n(\boldsymbol{u}) \geq \texttt{min\_product} - w_n u_n.$$

If one denotes $\boldsymbol{u}' := \pi_n(\boldsymbol{u})$, then there are no vectors

$$\boldsymbol{v}' \in \texttt{thrs} := \{\pi_n(\boldsymbol{v}) \mid \boldsymbol{v} \in \texttt{thresholds},\ v_n \leq u_n\}$$

with $\boldsymbol{v}' \preceq \boldsymbol{u}'$. Note that the arrangement $\widetilde{\texttt{thrs}}$ of `thrs` is again marked. Moreover, by the assumptions on $\boldsymbol{u}$, we already know that the inequalities $\pi_n(\texttt{mins}) \preceq \boldsymbol{u}' \preceq \pi_n(\texttt{maxs})$ hold. Hence, by induction, $\boldsymbol{u}'$ lies in the output V' of

$$\textsc{Recursive}(\pi_n(\boldsymbol{w}),\ \pi_n(\texttt{mins}),\ \pi_n(\texttt{maxs}),\ \texttt{min\_product} - u_n w_n,\ \widetilde{\texttt{thrs}},\ \texttt{false})$$

and $\boldsymbol{u} = \kappa_n(\boldsymbol{u}') + u_n \boldsymbol{e}_n \in V$. Moreover, once again, all the elements in V satisfy the wanted assumptions.

Note that by induction, if V is nonempty, the output of

$$\textsc{Recursive}(\boldsymbol{w},\ \texttt{mins},\ \texttt{maxs},\ \texttt{min\_product},\ \texttt{thresholds},\ \texttt{true})$$

consists of the smallest element of V, for the reverse lexicographic order. □

Proposition 5.3. *Algorithm 2 in Appendix A returns the correct output.*

Proof. Let n be a positive integer, let $\boldsymbol{w} := (w_1, \ldots, w_n) \in \mathbb{Z}^n_{\geq 1}$ be a positive vector and let $\texttt{low} \in \{0, 1\}$ be an integer.

Step 1 (line 1). According to Corollary 3.2, if $n < 3$ holds then there are no $\boldsymbol{w}$-bubbles and the procedure terminates. In what follows, we will therefore assume that $n \geq 3$ holds.

Step 2 (line 4). According to Lemma 3.13(a), if $g > 1$ holds then we may replace $\boldsymbol{w}$ with $(w_1/g, \ldots, w_n/g)$ since both vectors have the same set of bubbles. In what follows, we will therefore assume that $g = 1$.

Step 3 (line 7). According to Lemma 3.4, if

$$\boldsymbol{w} \cdot \boldsymbol{\alpha} < 2d_{\boldsymbol{w}}$$

holds, then there are no $\boldsymbol{w}$-bubbles and the procedure terminates.

Step 4 (line 10). By Lemmas 3.1 and 3.4, if $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble, we have that $u_i \leq \varepsilon_i$ holds for every $i \in \{1, \ldots, n\}$. Moreover, following an argument similar to the beginning of the proof of Lemma 3.8, if such a $\boldsymbol{w}$-bubble exists then there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}'$ such that $u'_i \leq \zeta_i - 1$ holds for every $i \in \{1, \ldots, n\}$. By definition, every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies the inequality $\boldsymbol{w} \cdot \boldsymbol{u} \geq 2d_{\boldsymbol{w}}$. Hence if $\boldsymbol{w} \cdot \boldsymbol{\beta} < 2d_{\boldsymbol{w}}$ holds, then there are no $\boldsymbol{w}$-bubbles and the procedure terminates. Note that for all $i \in \{1, \ldots, n\}$, we add $\varepsilon_i + 1$ to the definition of ζ_i since w_i might divide none of the w_j's for all $j > i$.

Step 5 (line 16). Note that by Lemma 3.20, the inequality $\delta_i \leq \alpha_i$ holds for every $i \in \{1, \ldots, n\}$. The vector $\boldsymbol{\delta}$ is harder to compute than $\boldsymbol{\alpha}$ but it gives upper bounds that are often stronger. Again, by Lemmas 3.1 and 3.17, we have that $u_i \leq \delta_i$ holds for every $i \in \{1, \ldots, n\}$ and for every $\boldsymbol{w}$-bubble $\boldsymbol{u}$. Similarly as before, if there exists a $\boldsymbol{w}$-bubble with positive coordinates (line 12), then there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ such that $u_i \leq \gamma_i$ holds for every $i \in \{1, \ldots n\}$. Hence, if $\boldsymbol{w} \cdot \boldsymbol{\gamma} < 2d_{\boldsymbol{w}}$ holds then there are no $\boldsymbol{w}$-bubbles with positive coordinates and the procedure terminates if `low` is set to 1.

Step 6 (line 23). If there exists a $\boldsymbol{w}$-bubble (resp. with positive coordinates), then there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ (resp. with positive coordinates) such that $u_i \leq \gamma_i$ holds for every $i \in \{1, \ldots, n\}$ (where $\boldsymbol{\gamma} = \boldsymbol{\beta}$ when `low` $= 0$). By definition of such a $\boldsymbol{w}$-bubble $\boldsymbol{u}$, the vector $\boldsymbol{u}$ is not greater than any element $\boldsymbol{v} \in V_{\boldsymbol{w}}(d_{\boldsymbol{w}})$: since $\boldsymbol{u} \preceq \boldsymbol{\gamma}$ holds, we just need to try to compare $\boldsymbol{u}$ with the elements $\boldsymbol{v} \in V_{\boldsymbol{w}}(d_{\boldsymbol{w}})$ satisfying $\boldsymbol{v} \preceq \boldsymbol{\gamma}$. Hence, by Proposition 5.2, if such a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ exists, the output of

$$\textsc{Recursive}(\boldsymbol{w},\, \texttt{mins},\, \boldsymbol{\gamma},\, 2d_{\boldsymbol{w}},\, \widetilde{\texttt{W}_{\boldsymbol{\gamma}}},\, \boldsymbol{\beta},\, \texttt{true})$$

is nonempty. If it is empty, then there are no $\boldsymbol{w}$-bubbles satisfying the wanted assumptions, and the procedure terminates.

Step 7 (line 27). We have previously shown that for every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying the wanted assumptions, we must observe that $u_i \leq \delta_i$ holds (where $\boldsymbol{\delta} = \boldsymbol{\varepsilon}$ when `low` $= 0$). Hence, by Proposition 5.2, the output of

$$\textsc{Recursive}(\boldsymbol{w},\, \texttt{mins},\, \boldsymbol{\gamma},\, 2d_{\boldsymbol{w}},\, \widetilde{\texttt{W}_{\boldsymbol{\delta}}},\, \boldsymbol{\beta},\, \texttt{false})$$

consists exactly of the $\boldsymbol{w}$-bubbles with coordinates greater than or equal to `low`. □

Remark 5.4. We make the following remarks on computational complexity.

(a) To prove Proposition 5.5, we need to check over two trillion weight vectors, so the speed of the algorithm is crucial. Computations for Proposition 5.5 have shown that condition in line 7 of Algorithm 2 is often more than ten times faster compared to line 10. In the computations for Table 1, we have observed that Algorithm 2 terminates at line 8 for most of the strictly increasing vectors in $\mathbb{Z}^n_{\geq 1}$ with bounded coordinates.

(b) Given a finite list of vectors $W \subseteq \mathbb{Z}^n_{\geq 0}$, computing the set W' of minimal elements for the product order can be expensive. The set W' or the process of computing it has various names in the literature:

- Pareto front or Pareto frontier or Pareto set,
- skyline operator or skyline query,
- maxima of a point set,
- multi-objective optimisation.

Instead of computing W', we compute an approximation to it, namely an arrangement.

(c) The upper bounds defined by `maxs` in Algorithm 1 are often not sharp after a few iterations. For instance, if after k iterations we have chosen particular values for $u_{n-k+1}, \ldots, u_n$ such that $\widetilde{\texttt{thrs}}$ (defined in line 14) contains an element of the form $c\boldsymbol{e}_{n-k}$ with $c \leq \texttt{maxs}_{n-k}$, then we would obtain a strictly sharper upper bound $u_{n-k} \leq c - 1$. This drastically reduces the amount of cases to check.

(d) Since the list `thresholds` in Algorithm 1 is lexicographically ordered and consists of nonnegative vectors, its first element is of the form $(0,\ldots,0,v_n)$. At line 2, we use this to give a bound $u_n \le v_n - 1$. Computationally, there is almost no performance loss if instead of taking the first element of `thresholds`, we loop through the entire list `thresholds` to find all elements of the form $(0,\ldots,0,v_n)$ and then take the minimum over all v_n.

(e) The average computation time for computing Table 1 for a single weight vector $\boldsymbol{w}$ on a single CPU core was 3.305 µs. One possibility to further improve the computation time is to reduce the amount of weight vectors we look through. For example, if the three biggest entries are $(w_{n-2}, w_{n-1}, w_n) = (39, 40, 41)$, then by Corollary 3.5, there are no $\boldsymbol{w}$-bubbles since $(\max(\boldsymbol{w}) - 2)\cdot\sum_i w_i \le 33579 < \operatorname{lcm}(\boldsymbol{w})$.

(f) There were 82 weight vectors that took longer than half an hour to compute on a single CPU core. Another possibility to further improve the computation time is to develop stronger inequalities on the coordinates of bubbles. The five weight vectors that took the longest were

- (2, 5, 9, 21, 28, 30, 35, 40) — 47 h 2 min,
- (1, 16, 24, 26, 30, 39, 40) — 36 h 28 min,
- (2, 5, 7, 12, 14, 16, 21, 28, 35, 40) — 20 h 51 min,
- (2, 7, 9, 21, 28, 30, 35, 40) — 19 h 53 min,
- (2, 21, 28, 30, 32, 35, 40) — 13 h 29 min.

Proposition 5.5. *Table 1 in appendix B contains all the $\boldsymbol{w}$-bubbles with positive coordinates where $\boldsymbol{w} \in \mathbb{Z}^n_{\ge 1}$ is such that $\max(\boldsymbol{w}) < 42$ and $w_1 < \ldots < w_n$ hold. Moreover, every such $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{w}\cdot\boldsymbol{u} < 3d_{\boldsymbol{w}}$.*

Proof. Since there are no strictly increasing sequences of positive integers of length greater than the last element, we have $n \le 41$. For every $1 \le n \le 41$, we consider each vector $(w_1,\ldots,w_n) \in \mathbb{Z}^n_{\ge 1}$ separately, where $1 \le w_1 < \ldots < w_n \le 41$. For a fixed n, there are exactly $\binom{41}{n}$ such vectors, so there are exactly $2^{41}-1$ vectors to consider in total. According to Proposition 5.3, for each vector $\boldsymbol{w}$ as above, the output of the function BUBBLES($\boldsymbol{w}$, 1) of Algorithm 2 in Appendix A consists exactly of the $\boldsymbol{w}$-bubbles with positive coordinates. Table 1 in appendix B is now obtained by a computer calculation. □

We use Notation 5.6 to describe all the small weights with bubbles in Theorem 5.7.

Notation 5.6. For every positive integer n and vector $\boldsymbol{w} \in \mathbb{Z}^n_{\ge 1}$, define the **expansion of $\boldsymbol{w}$** to be the set of all vectors $(\boldsymbol{w},\boldsymbol{v}) := (w_1,\ldots,w_n,v_1,\ldots,v_k) \in \mathbb{Z}^{n+k}_{\ge 1}$ and their permutations where k is a nonnegative integer and for all $i \in \{1,\ldots,k\}$, v_i divides $\operatorname{lcm}(\boldsymbol{w})$. For every subset S of the power set of $\mathbb{Z}^n_{\ge 1}$, define the **expansion of S** to be the union of the expansions of the elements of S.

Theorem 5.7. *Let n be a positive integer and let $\boldsymbol{w} \in \mathbb{Z}^n_{\ge 1}$ be such that $\max(\boldsymbol{w}) < 42$. Then, all of the following hold:*

(a) *there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ such that $\boldsymbol{w}\cdot\boldsymbol{u}$ is divisible by $d_{\boldsymbol{w}}$ if and only if $\boldsymbol{w}$ is in the expansion of the set*

$$\left\{\begin{matrix}(1,6,10,15),\ (1,6,22,33),\ (1,12,20,30),\\ (1,12,21,28),\ (1,12,22,33),\ (1,15,21,35),\\ (1,21,28,36),\ (1,21,30,35),\ (2,12,15,20),\\ (2,12,20,30),\ (2,21,30,35),\ (4,15,24,40),\\ (4,24,30,40),\ (1,3,24,30,40)\end{matrix}\right\},$$

(b) *there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}\notin 2V^+_{\boldsymbol{w}}(d_{\boldsymbol{w}})$ if and only if $\boldsymbol{w}$ is in the expansion of the set*

$$\left\{\begin{matrix}(6,14,21),\ (6,26,39),\ (10,14,35),\ (12,15,20),\\ (12,26,39),\ (14,20,35),\ (15,20,24),\ (15,24,40),\\ (20,28,35),\ (24,30,40),\ (28,35,40),\\ (1,6,10,15),\ (1,6,22,33),\ (1,12,20,30),\\ (1,12,21,28),\ (1,12,22,33),\ (1,15,21,35),\\ (1,21,28,36),\ (1,21,30,35),\ (2,12,20,30),\\ (2,21,30,35),\ (3,12,20,30)\end{matrix}\right\},$$

(c) *every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies $\boldsymbol{w}\cdot\boldsymbol{u}<3d_{\boldsymbol{w}}$.*

Proof. For every subset $J=\{j_1,\dots,j_r\}$ of $\{1,\dots,n\}$, let π_J denote the composition $\pi_{j_1}\circ\ldots\circ\pi_{j_r}\colon\mathbb{Z}^n_{\geq 0}\to\mathbb{Z}^{n-|I|}_{\geq 0}$ and let κ_J denote the composition $\kappa_{j_1}\circ\ldots\circ\kappa_{j_r}\colon\mathbb{Z}^{n-|I|}_{\geq 0}\to\mathbb{Z}^n_{\geq 0}$. It suffices to prove the case where $w_1<\ldots<w_n$ hold. Note that a strictly increasing vector $\boldsymbol{a}\in\mathbb{Z}^n_{\geq 1}$ is in the expansion of a strictly increasing vector $\boldsymbol{b}\in\mathbb{Z}^m_{\geq 1}$, where $m\in\{1,\dots,n\}$, if and only if there exists a proper subset I of $\{1,\dots,n\}$ such that $\pi_I(\boldsymbol{a})=\boldsymbol{b}$ and $d_{\boldsymbol{a}}=d_{\boldsymbol{b}}$.

We prove (c). Let $\boldsymbol{u}$ be a $\boldsymbol{w}$-bubble. By induction, using Lemma 3.15, we find a proper subset I of $\{1,\dots,n\}$ and a $\pi_I(\boldsymbol{w})$-bubble $\boldsymbol{u}'$ with strictly positive coordinates satisfying $\pi_I(\boldsymbol{w})\cdot\boldsymbol{u}'\geq\boldsymbol{w}\cdot\boldsymbol{u}+d_{\pi_I(\boldsymbol{w})}-d_{\boldsymbol{w}}$. If $\boldsymbol{w}\cdot\boldsymbol{u}\geq 3d_{\boldsymbol{w}}$ holds, then $\pi_I(\boldsymbol{w})\cdot\boldsymbol{u}'\geq 3d_{\pi_I(\boldsymbol{w})}$ holds too, which is impossible by Proposition 5.5. Therefore, the inequality $\boldsymbol{w}\cdot\boldsymbol{u}<3d_{\boldsymbol{w}}$ is satisfied.

We show that if there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$, then there exists a proper subset I of $\{1,\dots,n\}$ and a $\pi_I(\boldsymbol{w})$-bubble $\boldsymbol{u}'\preceq\pi_I(\boldsymbol{u})$ with positive coordinates such that $d_{\boldsymbol{w}}=d_{\pi_I(\boldsymbol{w})}$. This is clear if $\boldsymbol{u}$ has positive coordinates. Now, consider the case where u_i is zero for some $i\in\{1,\dots,n\}$. By induction, using Lemma 3.15, there is a nonempty subset I of $\{1,\dots,n\}$ and a $\pi_I(\boldsymbol{w})$-bubble $\boldsymbol{u}'\preceq\pi_I(\boldsymbol{u})$ with positive coordinates satisfying $\pi_I(\boldsymbol{w})\cdot\boldsymbol{u}'\geq\boldsymbol{w}\cdot\boldsymbol{u}+d_{\pi_I(\boldsymbol{w})}-d_{\boldsymbol{w}}$. If $d_{\pi_I(\boldsymbol{w})}\neq d_{\boldsymbol{w}}$, then $\pi_I(\boldsymbol{w})\cdot\boldsymbol{u}'\geq 3d_{\pi_I(\boldsymbol{w})}$ holds, which contradicts Proposition 5.5. Therefore, $d_{\pi_I(\boldsymbol{w})}=d_{\boldsymbol{w}}$.

We prove (a). By the above, if there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ such that $\boldsymbol{w}\cdot\boldsymbol{u}$ is divisible by $d_{\boldsymbol{w}}$, then there exist a set I, a vector $\boldsymbol{w}'$ in Table 1 and a $\boldsymbol{w}'$-bubble $\boldsymbol{u}'\preceq\pi_I(\boldsymbol{u})$ with positive coordinates such that $\pi_I(\boldsymbol{w})=\boldsymbol{w}'$, $\boldsymbol{w}'\cdot\boldsymbol{u}'=2d_{\boldsymbol{w}'}$ and $d_{\boldsymbol{w}}=d_{\boldsymbol{w}'}$. Every such $\boldsymbol{w}'$ is in the expansion of the set given in (a). Conversely, if there exists a set I and a vector $\boldsymbol{w}'$ in Table 1 such that $\pi_I(\boldsymbol{w})=\boldsymbol{w}'$, then, as can be checked from Table 1, there exists a $\boldsymbol{w}'$-bubble $\boldsymbol{u}'$ satisfying $\boldsymbol{w}'\cdot\boldsymbol{u}'=2d_{\boldsymbol{w}'}$. The vector $\boldsymbol{u}:=\kappa_I(\boldsymbol{u}')$ is a $\boldsymbol{w}$-bubble satisfying $\boldsymbol{w}\cdot\boldsymbol{u}=2d_{\boldsymbol{w}}$. This proves (a).

Last, we prove (b). By the above, if there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ that is not the sum of any two vectors $\boldsymbol{a},\boldsymbol{b}\in\mathbb{Z}^n_{\geq 0}$ satisfying $\boldsymbol{w}\cdot\boldsymbol{a}\geq d_{\boldsymbol{w}}$ and $\boldsymbol{w}\cdot\boldsymbol{b}\geq d_{\boldsymbol{w}}$ respectively, then there exist a set I, a vector $\boldsymbol{w}'$ in Table 1 and a $\boldsymbol{w}'$-bubble $\boldsymbol{u}'\preceq\pi_I(\boldsymbol{u})$ with positive coordinates such that $\boldsymbol{w}' = \pi_I(\boldsymbol{w})$ and $d_{\boldsymbol{w}} = d_{\boldsymbol{w}'}$. If $\boldsymbol{u}'$ is the sum of two vectors $\boldsymbol{a}', \boldsymbol{b}' \in \mathbb{Z}_{\geq 0}^{n-|I|}$ such that $\boldsymbol{w}' \cdot \boldsymbol{a}' \geq d_{\boldsymbol{w}'}$ and $\boldsymbol{w}' \cdot \boldsymbol{b}' \geq d_{\boldsymbol{w}'}$ simultaneously hold, then defining $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_{\geq 0}^n$ by $\boldsymbol{a} = \kappa_I(\boldsymbol{a}')$ and $\boldsymbol{b} = \boldsymbol{u} - \boldsymbol{a}$, we have that both inequalities $\boldsymbol{w} \cdot \boldsymbol{a} \geq d_{\boldsymbol{w}}$ and $\boldsymbol{w} \cdot \boldsymbol{b} \geq d_{\boldsymbol{w}}$ hold, which contradicts the assumptions on $\boldsymbol{u}$. Therefore, $\boldsymbol{w}$ is in the expansion of the set of vectors $\boldsymbol{w}'$ in Table 1 such that there exists a $\boldsymbol{w}'$-bubble $\boldsymbol{u}'$ with positive coordinates that is not the sum of any two vectors $\boldsymbol{a}', \boldsymbol{b}' \in \mathbb{Z}_{\geq 0}^{n-|I|}$ satisfying $\boldsymbol{w}' \cdot \boldsymbol{a}' \geq d_{\boldsymbol{w}'}$ and $\boldsymbol{w}' \cdot \boldsymbol{b}' \geq d_{\boldsymbol{w}'}$. Every such $\boldsymbol{w}'$ is the expansion of the set given in (b). Conversely, if there exists a proper subset I of $\{1, \ldots, n\}$ such that $\pi_I(\boldsymbol{w}) = \boldsymbol{w}'$ for some $\boldsymbol{w}'$ in the set in (b), then, as can be checked from Table 1, there exists a $\boldsymbol{w}'$-bubble $\boldsymbol{u}'$ such that $\boldsymbol{u}'$ is not the sum of any two vectors $\boldsymbol{a}', \boldsymbol{b}' \in \mathbb{Z}_{\geq 0}^{n-|I|}$ satisfying $\boldsymbol{w}' \cdot \boldsymbol{a}' \geq d_{\boldsymbol{w}'}$ and $\boldsymbol{w}' \cdot \boldsymbol{b}' \geq d_{\boldsymbol{w}'}$. Then, $\boldsymbol{u} := \kappa_I(\boldsymbol{u}')$ is a $\boldsymbol{w}$-bubble that is not the sum of any two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_{\geq 0}^n$ satisfying $\boldsymbol{w} \cdot \boldsymbol{a} \geq d_{\boldsymbol{w}}$ and $\boldsymbol{w} \cdot \boldsymbol{b} \geq d_{\boldsymbol{w}}$. This proves (b). □

Remark 5.8. Since the definition of *expansion* is very simple, it is very easy to check whether a vector is in the expansion of another vector, but it also means that there is some redundancy in the sets in Theorem 5.7. For every vector $(w_1, \ldots, w_n)$ in the two sets and every positive integer c such that the inequality $cw_n < 42$ holds, the vector $(cw_1, \ldots, cw_n)$ is also in the corresponding set. If $(w_1, \ldots, w_n)$ is in the set in (a), then for every positive integer a, every $i \in \{1, \ldots, n\}$ and every positive integer b coprime to w_i, a permutation of the vector $(abw_1, \ldots, abw_{i-1}, aw_i, abw_{i+1}, \ldots, abw_n)$ is also in the set, provided the maximum of its elements is less than 42.

Theorem 5.7 implies that certain graded rings are not generated in degree 1. Proposition 5.10 implies further that $\mathcal{O}(d_{\boldsymbol{w}})$ is not very ample on the Proj of those graded rings. For this, we use the following lemma:

Lemma 5.9. *Let $n \geq 3$ and $\boldsymbol{w} := (w_1, \ldots, w_n) \in \mathbb{Z}_{\geq 1}^n$ be positive integers. Let $\boldsymbol{u} \in \mathbb{Z}_{\geq 0}^n$ be a $\boldsymbol{w}$-bubble satisfying both of the following conditions:*

(1) $\boldsymbol{w} \cdot \boldsymbol{u} < 3d_{\boldsymbol{w}}$,

(2) *there exists $i \in \{1, \ldots, n\}$ such that* $u_i = \dfrac{d_{\boldsymbol{w}}}{w_i} - 1$.

Then $(\boldsymbol{w}, \boldsymbol{u}, 1)$ satisfies Condition 2.2 if and only if $\boldsymbol{u} \notin 2V_{\boldsymbol{w}}^+(d_{\boldsymbol{w}})$.

Proof. Already note that if $\boldsymbol{u} \in 2V_{\boldsymbol{w}}^+(d_{\boldsymbol{w}})$ then $(\boldsymbol{w}, \boldsymbol{u}, 1)$ does not satisfy Condition 2.2.

From now on, let us assume that $\boldsymbol{u} \notin 2V_{\boldsymbol{w}}^+(d_{\boldsymbol{w}})$, let $i \in \{1, \ldots, n\}$ be as in part (2) of the statement, and let $m \geq 2$. Seeking a contradiction, we suppose that there exist $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^m \in V_{\boldsymbol{w}}^+(d_{\boldsymbol{w}})$ such that the following holds:

$$\boldsymbol{u} + \frac{(m-2)d_{\boldsymbol{w}}}{w_i}\boldsymbol{e}_i = \sum_{j=1}^m \boldsymbol{v}^j.$$

Since the scalar product $\boldsymbol{w} \cdot \sum_{j=1}^m \boldsymbol{v}^j$ is equal to $\boldsymbol{w} \cdot \boldsymbol{u} + (m-2)d_{\boldsymbol{w}}$, for every $j \in \{1, \ldots, m\}$, the following inequalities hold:

$$d_{\boldsymbol{w}} \leq \boldsymbol{w} \cdot \boldsymbol{v}^j \leq \boldsymbol{w} \cdot \boldsymbol{u} - d_{\boldsymbol{w}}. \tag{5.1}$$

Now, by assumption, we observe that for all $j \in \{1, \ldots, m\}$ and for all $l \in \{1, \ldots, n\} \setminus \{i\}$, $v_l^j \leq u_l$ and moreover:

$$(m-1)\frac{d_{\boldsymbol{w}}}{w_i} - 1 = \left(\boldsymbol{u} + \frac{(m-2)d_{\boldsymbol{w}}}{w_i}\boldsymbol{e}_i\right)_i = \sum_{j=1}^m v_i^j.$$

In particular, there exists $j \in \{1, \dots, m\}$ such that $v_i^j \leq \dfrac{d_{\boldsymbol{w}}}{w_i} - 1 = u_i$, which implies that $\boldsymbol{v}^j \preceq \boldsymbol{u}$. Let us denote $\boldsymbol{u}' := \boldsymbol{u} - \boldsymbol{v}^j$: it is a nonnegative vector, and equation (5.1) gives the following inequality:

$$\boldsymbol{w} \cdot \boldsymbol{u}' = \boldsymbol{w} \cdot \boldsymbol{u} - \boldsymbol{w} \cdot \boldsymbol{v}^j \geq d_{\boldsymbol{w}}.$$

Hence, $\boldsymbol{u} = \boldsymbol{v}^j + \boldsymbol{u}' \in 2V_{\boldsymbol{w}}^+(d_{\boldsymbol{w}})$, which is a contradiction. Hence $(\boldsymbol{w}, \boldsymbol{u}, 1)$ satisfies Condition 2.2. □

Proposition 5.10. *Both of the following hold:*

(a) *if $\boldsymbol{w}$ is in the expansion of the set defined in Theorem 5.7(a), then there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} = 2d_{\boldsymbol{w}}$ such that the triple $(\boldsymbol{w}, \boldsymbol{u}, 1)$ satisfies Condition 2.2,*

(b) *if $\boldsymbol{w}$ is in the expansion of the set defined in Theorem 5.7(b), then there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{u} \in \{2d_{\boldsymbol{w}}, 2d_{\boldsymbol{w}} + 1, 2d_{\boldsymbol{w}} + 2, 2d_{\boldsymbol{w}} + 3\}$ such that the triple $(\boldsymbol{w}, \boldsymbol{u}, 1)$ satisfies Condition 2.2.*

Proof. First, we prove the statement for vectors $\boldsymbol{w}'$ in the two sets of Theorem 5.7.

(a) Let $\boldsymbol{w}' := (w_1', \dots, w_n')$ be one of the vectors in the set defined in Theorem 5.7(a). According to Table 1 in appendix B, there exists a $\boldsymbol{w}'$-bubble $\boldsymbol{u}'$ such that $\boldsymbol{w}' \cdot \boldsymbol{u}' = 2d_{\boldsymbol{w}'}$ (coloured in blue). Since $\boldsymbol{u}'$ is a $\boldsymbol{w}'$-bubble, it is not contained in $2V_{\boldsymbol{w}'}^+(d_{\boldsymbol{w}'})$. Moreover, one can easily check that there exists $i \in \{1, \dots, n\}$ such that $u_i' = \dfrac{d_{\boldsymbol{w}'}}{w_i'} - 1$. It therefore follows from Lemma 5.9 that $(\boldsymbol{w}', \boldsymbol{u}', 1)$ satisfies Condition 2.2.

(b) Let $\boldsymbol{w}' := (w_1', \dots, w_n')$ be one of the vectors in the set defined in Theorem 5.7(b). This time, according to Table 1 in appendix B, there exists a $\boldsymbol{w}'$-bubble $\boldsymbol{u}'$ such that $\boldsymbol{u}' \notin 2V_{\boldsymbol{w}'}^+(d_{\boldsymbol{w}'})$ and $\boldsymbol{w}' \cdot \boldsymbol{u}' \in \{2d_{\boldsymbol{w}'}, 2d_{\boldsymbol{w}'} + 1, 2d_{\boldsymbol{w}'} + 2, 2d_{\boldsymbol{w}'} + 3\}$ (coloured in orange). As before, by straightforward calculations, there exists $i \in \{1, \dots, n\}$ such that $u_i' = \dfrac{d_{\boldsymbol{w}'}}{w_i'} - 1$. In this situation, Lemma 5.9 tells us that $(\boldsymbol{w}', \boldsymbol{u}', 1)$ satisfies Condition 2.2.

Now, we prove the statement for the vectors $\boldsymbol{w}$ in the expansion of one of the sets in Theorem 5.7. Let $(\boldsymbol{w}', \boldsymbol{u}', i)$ be as in one of the previous items and let $\boldsymbol{w}$ be in the expansion of $\boldsymbol{w}'$. Up to reordering the coordinates, we may write $\boldsymbol{w} := (w_1', \dots, w_n', v_1, \dots, v_k) \in \mathbb{Z}_{\geq 1}^k$. Let $\boldsymbol{u} := \kappa_{\{n+1,\dots,n+k\}}(\boldsymbol{u}')$ (following the notation defined at the beginning of the proof of Theorem 5.7). The vector $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble satisfying $\boldsymbol{u} \notin 2V_{\boldsymbol{w}}^+(d_{\boldsymbol{w}})$, $\boldsymbol{w} \cdot \boldsymbol{u} = \boldsymbol{w}' \cdot \boldsymbol{u}'$ and $u_i = u_i' = \dfrac{d_{\boldsymbol{w}'}}{w_i'} - 1 = \dfrac{d_{\boldsymbol{w}}}{w_i} - 1$. Therefore, we can apply again Lemma 5.9 to conclude that the triple $(\boldsymbol{w}, \boldsymbol{u}, 1)$ satisfies Condition 2.2. □

Remark 5.11. By Lemma 5.9, for every vector $\boldsymbol{w} \in \mathbb{Z}_{\geq 1}^n$ satisfying $\max \boldsymbol{w} < 42$ and every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ with positive coordinates, the following conditions are equivalent:

(i) $(\boldsymbol{w}, \boldsymbol{u}, 1)$ satisfies Condition 2.2,

(ii) the vector $\boldsymbol{u}$ in the row corresponding to $\boldsymbol{w}$ is coloured in Table 1.

This is because by Theorem 5.7, Table 1 contains all possible vectors $\boldsymbol{u}$ in the row corresponding to $\boldsymbol{w}$, and it is straightforward to check that every coloured $\boldsymbol{u}$ satisfies both of the following:

- $\boldsymbol{w} \cdot \boldsymbol{u} < 3d_{\boldsymbol{w}}$, and
- there exists $i \in \{1, \dots, n\}$ such that $u_i = \dfrac{d_{\boldsymbol{w}}}{w_i} - 1$.

To conclude this section, we give a small application of Theorem 5.7.

Proposition 5.12. *Every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies*

$$\boldsymbol{w}\cdot\boldsymbol{u} < \operatorname{lcm}(1,2,\dots,\max(\boldsymbol{w})).$$

Proof. By Lemma 3.13(b), it suffices to prove the case where the weights are pairwise distinct. If $\max(\boldsymbol{w}) < 7$ holds, then the result follows from Theorem 5.7. Otherwise, by Corollary 3.5, it suffices to note that for all integers $k \geq 7$, we have the following inequality:

$$\operatorname{lcm}(1,2,\dots,k) > (k-2)(1+2+\ldots+k). \qquad \square$$

6 Nullity of the set of weights with a bubble

In this section, we show that the asymptotic density of weight vectors which admit bubbles is zero. In what follows, we denote by $\{1,\dots,M\}^n$ the set of all vectors $(w_1,\dots,w_n)$ of positive integers such that $w_i \leq M$ holds for every $i \in \{1,\dots,n\}$.

Lemma 6.1. *For all integers $n \geq 2$ and for all real numbers $0 < \varepsilon < 1$, there exists a positive integer M_0 such that for all $M \geq M_0$ the following holds:*

$$\left|\left\{\boldsymbol{w} \in \{1,\dots,M\}^n \;\middle|\; \begin{array}{l} \exists\, 1 \leq i < j \leq n\colon \\ \gcd(w_i,w_j) \geq M^\varepsilon \end{array}\right\}\right| < M^{n-\frac{\varepsilon}{2}}.$$

Proof. For every positive integer M, the average greatest common divisor of two integers in $\{1,\dots,M\}$ is equal to

$$\frac{\sum_{k=1}^{M} |\{(w_1,w_2) \in \{1,\dots,M\}^2 \mid \gcd(w_1,w_2) = k\}|}{M^2},$$

which can be written as

$$\frac{\sum_{k=1}^{M} k \cdot \left|\left\{(w_1,w_2) \in \left\{1,\dots,\left\lfloor \frac{M}{k} \right\rfloor\right\}^2 \mid \gcd(w_1,w_2) = 1\right\}\right|}{M^2}.$$

This is bounded above by

$$\frac{\sum_{k=1}^{M} k \cdot \left(\frac{M}{k}\right)^2}{M^2} = \sum_{k=1}^{M} \frac{1}{k}.$$

Therefore,

$$\frac{\sum_{(w_1,w_2)\in\{1,\dots,M\}^2} \gcd(w_1,w_2)}{M^2} < 1 + \ln(M). \tag{6.1}$$

Seeking a contradiction, assume there exists an arbitrarily large positive integer M such that the following holds:

$$\left|\left\{\boldsymbol{w} \in \{1,\dots,M\}^n \;\middle|\; \begin{array}{l} \exists\, 1 \leq i < j \leq n\colon \\ \gcd(w_i,w_j) \geq M^\varepsilon \end{array}\right\}\right| \geq M^{n-\frac{\varepsilon}{2}}.$$

It follows that

$$\left|\{(w_1,w_2)\in\{1,\dots,M\}^2 \mid \gcd(w_1,w_2)\geq M^{\varepsilon}\}\right| \geq \frac{2}{n(n-1)}M^{2-\frac{\varepsilon}{2}}.$$

After replacing M with a suitable greater integer, we have the inequality:

$$\frac{2}{n(n-1)}M^{2-\frac{\varepsilon}{2}} > M^{2-\frac{2\varepsilon}{3}}.$$

Using this, we give a lower bound for the average greatest common divisor of two integers in $\{1,\dots,M\}^2$, namely:

$$M^{\frac{\varepsilon}{3}} < \frac{\sum\limits_{\{(w_1,w_2)\in\{1,\dots,M\}^2\}} \gcd(w_1,w_2)}{M^2}.$$

Replacing M by a suitable greater integer, we find a contradiction with inequality (6.1). $\square$

Lemma 6.2. *For all positive integers n and M, and for all real numbers $0\leq\varepsilon\leq 1$, the following holds:*

$$\left|\left\{\boldsymbol{w}\in\{1,\dots,M\}^n \;\middle|\; \begin{array}{c}\exists i\in\{1,\dots n\}:\\ w_i\leq M^{1-\varepsilon}\end{array}\right\}\right| \leq nM^{n-\varepsilon}.$$

Proof. Obvious. $\square$

Theorem 6.3. *For all positive integers n and M, we define*

$$p(n,M) := \frac{|\{\boldsymbol{w}\in\{1,2,\dots,M\}^n \mid \textit{there are no } \boldsymbol{w}\textit{-bubbles}\}|}{M^n}.$$

Then, the following holds:

$$\lim_{M\to+\infty} p(n,M) = 1.$$

Proof. First suppose that $n\in\{1,2\}$. By Proposition 3.9, we know that for every $\boldsymbol{w}\in\mathbb{Z}^n_{>0}$, there are no $\boldsymbol{w}$-bubbles. In particular, for every positive integer M, we have $p(n,M)=1$.

Now let $n\geq 3$ be an integer and let $0<\varepsilon<\frac{1}{7}$ be a real number. Let M_0 be as in Lemma 6.1. For all $M\geq M_0$, define

$$C(n,\epsilon,M) := \left\{\boldsymbol{w}\in\{1,\dots,M\}^n \;\middle|\; \begin{array}{c}\forall 1\leq i<j\leq n:\\ \gcd(w_i,w_j)<M^{\varepsilon},\\ w_i,w_j>M^{1-\varepsilon}\end{array}\right\}.$$

It follows from Lemmas 6.1 and 6.2 that the following inequality holds:

$$|C(n,\epsilon,M)| > M^n - M^{n-\frac{\varepsilon}{2}} - nM^{n-\varepsilon}.$$

Now let $\boldsymbol{w}\in C(n,\varepsilon,M)$. One computes:

$$\begin{aligned} d_{\boldsymbol{w}} &= \operatorname{lcm}(w_1,\dots,w_n)\\ &\geq \operatorname{lcm}(w_1,w_2,w_3)\\ &= \frac{w_1w_2w_3\gcd(w_1,w_2,w_3)}{\gcd(w_1,w_2)\gcd(w_1,w_3)\gcd(w_2,w_3)} \end{aligned}$$

$$> M^{3-6\varepsilon}$$
$$> M^{2+\varepsilon}.$$

Since the inequality $(\max(\boldsymbol{w}) - 2)\sum_{1\le i\le n} w_i < nM^2$ holds, there exists $M_1 > 0$ such that for all $M \ge \max(M_0, M_1)$ and for all $\boldsymbol{w} \in C(n, \varepsilon, M)$, the inequality

$$2d_{\boldsymbol{w}} > (\max(\boldsymbol{w}) - 2) \sum_{1\le i\le n} w_i$$

holds. Now Corollary 3.5 tells us that there are no $\boldsymbol{w}$-bubbles. We conclude by observing that for all $M \ge \max(M_0, M_1)$, the inequalities

$$1 - \frac{1}{M^{\frac{\varepsilon}{2}}} - \frac{n}{M^{\varepsilon}} \le \frac{|C(n, \epsilon, M)|}{M^n} \le p(n, M) \le 1$$

hold, and thus $\lim_{M\to+\infty} p(n, M)$ exists and it is equal to 1. $\square$

7 Infinitude of the set of weights with a bubble

In this section, we construct infinitely many examples of weight vectors which admit a bubble. These examples are at the heart of the proof of Theorems 7.5 and 7.6.

Lemma 7.1. *Let $n \ge 2$ and let $\boldsymbol{p} := (p_1, \dots, p_n)$ be pairwise distinct prime numbers. Then, there exists a unique $\boldsymbol{u} \in \mathbb{Z}^n_{\ge 1}$ with $u_i \in \{1, \dots, p_i - 1\}$ for all $1 \le i \le n$, such that*

$$\sum_{i=1}^{n} \frac{u_i}{p_i} - \prod_{i=1}^{n} \frac{1}{p_i}$$

is an integer, which we denote by $k_{\boldsymbol{p}}$. Moreover, for every $1 \le i \le n$, the following holds:

$$\left\lceil \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \frac{u_j}{p_j} \right\rceil = k_{\boldsymbol{p}}.$$

Proof. Already note that for every tuples $(u_1, \dots, u_n)$ where $u_i \in \{1, \dots, p_i - 1\}$ for all $1 \le i \le n$, we have the following inequalities:

$$0 < \sum_{i=1}^{n} \frac{u_i}{p_i} - \prod_{1}^{n} \frac{1}{p_i} < n.$$

Suppose that we are given a tuple $(u_1, \dots, u_n)$, with $u_i \in \{1, \dots, p_i - 1\}$ for all $1 \le i \le n$, such that

$$k := \sum_{i=1}^{n} \frac{u_i}{p_i} - \prod_{i=1}^{n} \frac{1}{p_i}$$

is an integer. Let $d := \prod_{i=1}^{n} p_i$, and for all $1 \le i \le n$, let us denote $w_i := \prod_{j\in\{1,\dots,n\}\setminus\{i\}} p_j$. The following equality holds:

$$1 + kd = \sum_{i=1}^{n} w_i u_i.$$

In particular, for all $1 \le i \le n$, we get that

$$u_i \equiv w_i^{-1} \mod p_i,$$

and since $1 \leq u_i \leq p_i - 1$ hold, we see that u_i is uniquely determined by $p_1, \ldots, p_n$. Conversely if we choose, for all $1 \leq i \leq n$, the integer $u_i \in \{1, \ldots p_i - 1\}$ to be such that

$$u_i w_i \equiv 1 \mod p_i,$$

then, for all $j \in \{1, \ldots, n\}$, it follows:

$$-1 + \sum_{i=1}^{n} w_i u_i \equiv 0 \mod p_j.$$

Hence by the Chinese Remainder Theorem, we have that

$$\sum_{i=1}^{n} \frac{u_i}{p_i} - \prod_{i=1}^{n} \frac{1}{p_i} = \frac{\sum_{i=1}^{n}(u_i w_i) - 1}{d}$$

is an integer. This proves the existence and uniqueness of the tuple $(u_1, \ldots, u_n)$ as in the statement.

Now let $i \in \{1, \ldots n\}$: since $u_i \in \{1, \ldots, p_i - 1\}$, we observe that

$$0 < \frac{u_1}{p_1} - \prod_{j=1}^{n} \frac{1}{p_j} < 1$$

hold where the first inequality holds because $n \geq 2$. Therefore, we have that

$$k_{\boldsymbol{p}} - 1 < \sum_{j \in \{1,\ldots,n\} \setminus \{i\}} \frac{u_j}{p_j} < k_{\boldsymbol{p}}$$

hold too, giving us the wanted result. □

Lemma 7.2. *Let $n \geq 3$ be an integer and let $p_2, \ldots, p_n$ be pairwise distinct prime numbers. Let $u_2, \ldots, u_n$ be any integers with $u_i \in \{1, \ldots, p_i - 1\}$ for all $i \in \{2, \ldots, n\}$. Then, there exists a unique integer k in $\{1, \ldots n-1\}$ such that the following inequalities hold:*

$$k - 1 < \sum_{i=2}^{n} \frac{u_i}{p_i} < k.$$

Moreover, there exist infinitely many prime numbers p_1 such that the equality

$$\sum_{i=1}^{n} \frac{u_i}{p_i} - \prod_{i=1}^{n} \frac{1}{p_i} = k$$

holds for some $u_1 \in \{1, \ldots, p_1 - 1\}$.

Proof. The uniqueness of k is clear. To prove existence of k, note that if $\sum_{i=2}^{n} \frac{u_i}{p_i}$ were an integer, then p_i would divide u_i for all $i \in \{2, \ldots, n\}$, which is not possible by the hypotheses.

Now we prove the existence of infinitely many prime numbers p_1 satisfying the given equality. Let $q := \prod_{i=2}^{n} p_i$. By the Chinese remainder theorem and the Dirichlet theorem on arithmetic progressions, there are infinitely many prime number solutions to the following system of congruence equations in the variable $x \in \mathbb{Z}$:

$$\begin{cases} x \equiv (u_2 q/p_2)^{-1} \mod p_2 \\ x \equiv (u_3 q/p_3)^{-1} \mod p_3 \\ \vdots \\ x \equiv (u_n q/p_n)^{-1} \mod p_n. \end{cases}$$

Let p_1 be one such prime solution and define $u_1 \in \{1, \dots, p_1 - 1\}$ by:

$$u_1 q \equiv 1 \mod p_1.$$

Note that u_1 exists, and it is unique, by remarking that $\gcd(p_1, q) = 1$. For every $j \in \{1, \dots, n\}$, we have the congruence

$$\sum_{i \in \{1,\dots,n\}} \frac{p_1 q}{p_i} u_i \equiv 1 \mod p_j.$$

Therefore,

$$\sum_{i=1}^{n} \frac{u_i}{p_i} - \prod_{i=1}^{n} \frac{1}{p_i}$$

is an integer. By Lemma 7.1, we have

$$\sum_{i=1}^{n} \frac{u_i}{p_i} - \prod_{i=1}^{n} \frac{1}{p_i} = k_{\boldsymbol{p}} = \left\lceil \sum_{i=2}^{n} \frac{u_i}{p_i} \right\rceil = k. \qquad \square$$

Corollary 7.3. *Let $n \geq 2$ be an integer and let $k \in \{1, \dots, n-1\}$. Then, there are infinitely many tuples $\boldsymbol{p} := (p_1, \dots, p_n)$ of pairwise distinct prime numbers such that $k_{\boldsymbol{p}}$ (defined as in Lemma 7.1) is equal to k.*

Proof. For all prime numbers $p_2, p_3, \dots, p_k$ greater than k and all sufficiently large prime numbers $p_{k+1}, p_{k+2}, \dots, p_n$, defining $u_i = p_i - 1$ for $i \in \{1, 2, \dots, k\}$ and $u_i = 1$ for $i \in \{k+1, k+2, \dots, n\}$, we find

$$k - 1 < \sum_{i \in \{2,\dots,n\}} \frac{u_i}{p_i} < k.$$

The corollary now follows from Lemma 7.2. $\square$

We expect that examples satisfying $k_{\boldsymbol{p}} = 1$ and $k_{\boldsymbol{p}} = n - 1$ are the rarest. We make this statement precise in Remark 7.4.

Remark 7.4. Let n and k be integers satisfying $1 \leq k \leq n - 1$. For every positive integer M, let $S(n, M)$ denote the set of all vectors $(m, p_2, \dots, p_n) \in \mathbb{Z}^n_{\geq 1}$ where m is less than $\operatorname{lcm}(p_2, \dots, p_n)$ and the integers $p_2, \dots, p_n$ are pairwise distinct prime numbers such that for all $i \in \{2, \dots, n\}$, $p_i \nmid m$ and $p_i \leq M$ hold. For every positive integer M, let $S(n, k, M)$ denote the set of all vectors $(m, p_2, \dots, p_n) \in S(n, M)$ such that there exists a prime number p_1 satisfying all of the following:

(1) The numbers $\boldsymbol{p} := (p_1, \dots, p_n)$ are pairwise distinct,
(2) $k_{\boldsymbol{p}} = k$, and
(3) $p_1 \equiv m \mod \operatorname{lcm}(p_2, \dots, p_n)$.

By Lemma 7.2, Chinese remainder theorem, Dirichlet's theorem on arithmetic progressions and [Pet15, Proposition 7.1], we expect the following asymptotic behaviour:

$$\lim_{M \to \infty} \frac{|S(n, k, M)|}{|S(n, M)|} = A(n - 1, k - 1),$$

where $A(n - 1, k - 1)$ denotes the *Eulerian number*, meaning the number of permutations of $1, 2, \dots, n - 1$ with $k - 1$ ascents.

At the same time, for our purposes, the cases with $k_{\boldsymbol{p}} \in \{1, n-1\}$ are the most interesting. In Table 2 and Table 3 in appendix B, we give the smallest examples of vectors $\boldsymbol{p}$ satisfying respectively $k_{\boldsymbol{p}} = 1$ and $k_{\boldsymbol{p}} = n-1$ for every $n < 15$.

Theorem 7.5. *Let $n \geq 4$ be an integer. Then for every $k \in \{2, \dots, n-2\}$ and for all pairwise distinct prime numbers $\boldsymbol{p} := (p_1, \dots, p_{k+1}) \in \mathbb{Z}_{\geq 1}^{k+1}$ such that $k_{\boldsymbol{p}} = 1$, the vector $\boldsymbol{w} \in \mathbb{Z}_{\geq 1}^{n}$ defined by*

$$w_i := \begin{cases} \prod_{j \in \{1,\dots,k+1\} \setminus \{i\}} p_j & \text{if} \quad i \leq k+1 \\ 1 & \text{if} \quad i = k+2 \\ \prod_{j=1}^{k+1} p_j & \text{if} \quad i \geq k+3 \end{cases}$$

satisfies all of the following:

(a) *$\boldsymbol{w}$ is well-formed and it satisfies Condition 2.12,*

(b) *there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ with $\boldsymbol{w} \cdot \boldsymbol{u} = k d_{\boldsymbol{w}}$ such that $(\boldsymbol{w}, \boldsymbol{u}, k-1)$ satisfies Condition 2.2, and*

(c) *every $\boldsymbol{w}$-bubble $\boldsymbol{u}'$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u}' < (k+1) d_{\boldsymbol{w}}$.*

Proof. It is sufficient to prove the case $k = n-2$: a similar proof applies for any $k < n-2$ by setting, for the $\boldsymbol{w}$-bubble $\boldsymbol{u}$, $u_i = 0$ for all $i \geq k+3$.

In what follows, let $\boldsymbol{p} := (p_1, \dots, p_{n-1}) \in \mathbb{Z}_{\geq 1}^{n-1}$ be pairwise distinct prime numbers such that $k_{\boldsymbol{p}} = 1$. Let moreover $\boldsymbol{w} \in \mathbb{Z}_{\geq 1}^{n}$ be defined as in the statement.

(a) This follows directly from the definition of $\boldsymbol{w}$ and Corollary 2.15.

(b) Let $\boldsymbol{\alpha} \in \mathbb{Z}_{\geq 1}^{n-1}$ be such that $\alpha_i \in \{1, \dots, p_i - 1\}$ for all $i \in \{1, \dots, n-1\}$ and

$$\sum_{i=1}^{n-1} \frac{\alpha_i}{p_i} - \prod_{i=1}^{n-1} \frac{1}{p_i} = 1$$

holds. According to Lemma 7.1, $\boldsymbol{\alpha}$ exists and it is uniquely determined by $\boldsymbol{p}$. Now, for all $i \in \{1, \dots, n\}$, we define

$$u_i := \begin{cases} p_i - \alpha_i & \text{if } i \leq n-1 \\ 1 & \text{otherwise} \end{cases}.$$

By construction, it follows that the equality

$$\boldsymbol{w} \cdot \boldsymbol{u} = (n-2) d_{\boldsymbol{w}}, \tag{7.1}$$

holds. We now prove that $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble. Seeking a contradiction, we assume that there exists $\boldsymbol{v} \in \mathbb{Z}_{\geq 0}^{n}$ such that both $\boldsymbol{w} \cdot \boldsymbol{v} = d_{\boldsymbol{w}}$ and $\boldsymbol{v} \preceq \boldsymbol{u}$ hold.

(1) if v_n were zero, similar argument as the one used at the beginning of the proof of Lemma 7.2, equation (7.1) actually tells us that for all $i \in \{1, \dots, n-1\}$, p_i divides v_i and one of the v_i's must be nonzero. Hence, there exists $i \in \{1, \dots, n-1\}$ such that $v_i \geq p_i$ holds, contradicting that $\boldsymbol{v} \preceq \boldsymbol{u}$;

(2) if v_n were equal to 1, we define $\boldsymbol{v}' := \pi_n(\boldsymbol{v}) \in \mathbb{Z}_{\geq 0}^{n-1}$ and observe the following equality:

$$\pi_n(\boldsymbol{w}) \cdot \boldsymbol{v}' = d_{\boldsymbol{w}} - 1.$$

In particular, for all $i \in \{1, \dots, n-1\}$, we have that $v_i < p_i$ holds. Define, for all $i \in \{1, \dots, n-1\}$, $\tilde{v}_i := p_i - v'_i$ and remark that

$$\pi_n(\boldsymbol{w}) \cdot \tilde{\boldsymbol{v}} = (n-2) d_{\boldsymbol{w}} + 1.$$

But this contradicts the uniqueness of $k_{\boldsymbol{p}}$ from Lemma 7.1 and the assumption that $k_{\boldsymbol{p}} = 1 < n-2$. Hence v_n cannot be 1.

(3) if $v_n > 1$ held, then $\boldsymbol{v} \not\preceq \boldsymbol{u}$ would hold: a contradiction.

By (1)–(3) above, we indeed have that $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble, as claimed.

We now prove that $(\boldsymbol{w}, \boldsymbol{u}, n-3)$ satisfies Condition 2.2. Fix any $i \in \{1, \ldots, n-1\}$. We proceed again by proof by contradiction. Assume that there exists $m \geq 2$ such that there exist $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^m \in V_{\boldsymbol{w}}^+((n-3)d_{\boldsymbol{w}})$ satisfying the following equation:

$$\boldsymbol{u}' := \boldsymbol{u} + ((n-3)m - (n-2))p_i \boldsymbol{e}_i = \sum_{j=1}^{m} \boldsymbol{v}^j.$$

Note that since $\boldsymbol{w} \cdot \boldsymbol{u}' = (n-3)md_{\boldsymbol{w}}$ we have that $\boldsymbol{v}^j \in V_{\boldsymbol{w}}((n-3)d_{\boldsymbol{w}})$ for all $j \in \{1, \ldots, m\}$. Now, since $u'_n = u_n = 1$, there exists a unique $j_0 \in \{1, \ldots, m\}$ such that $v_n^{j_0} = 1$ and $v_n^j = 0$ for all $j \in \{1, \ldots, m\} \setminus \{j_0\}$. Without loss of generality, we assume $j_0 = m$. Then, for all $l \in \{1, \ldots, n-1\} \setminus \{i\}$ and for all $j \in \{1, \ldots, m-1\}$, we have that $v_l^j \leq u_l < p_l$ hold, and v_l^j is divisible by p_l (following the same ideas as in the beginning of the proof of Lemma 7.2). This implies the following:

$$\boldsymbol{v}^1 = \ldots = \boldsymbol{v}^{m-1} = (n-3)p_i \boldsymbol{e}_i.$$

Moreover, the previous implies that

$$u'_i = u_i + ((n-3)m - (n-2))p_i = \sum_{j=1}^{m} v_i^j = (m-1)(n-3)p_i + v_i^m$$

and in particular, we obtain that

$$p_i + v_i^m = u_i < p_i$$

holds. However, this is absurd since v_i^m must be nonnegative. Hence $(\boldsymbol{w}, \boldsymbol{u}, n-3)$ satisfies Condition 2.2.

(c) To conclude, we observe that since $w_n = 1$,

$$I_1 := \{1\},\ I_2 := \{2\},\ \ldots,\ I_{n-2} := \{n-2\},\ I_{n-1} := \{n-1, n\}$$

form a partition of $\{1, \ldots, n\}$ and each I_l is the vertex set of a directed path in the directed graph $G_{\boldsymbol{w}}$ associated to $\boldsymbol{w}$ (see Notation 3.7 for the definition of $G_{\boldsymbol{w}}$). Therefore, according to Proposition 3.9, every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ satisfies that $\boldsymbol{w} \cdot \boldsymbol{u} < (n-1)d_{\boldsymbol{w}}$.

□

Theorem 7.6. *Let $n \geq 3$ be an integer. Then for any $k \in \{2, \ldots, n-1\}$ and for any pairwise distinct prime numbers $\boldsymbol{p} := (p_1, \ldots, p_{k+1}) \in \mathbb{Z}_{\geq 1}^{k+1}$ such that $k_{\boldsymbol{p}} = k$, the vector $\boldsymbol{w} \in \mathbb{Z}_{\geq 1}^n$ defined by*

$$w_i := \begin{cases} \prod_{j \in \{1, \ldots, k+1\} \setminus \{i\}} p_j & \text{if} \quad i \leq k+1 \\ \prod_{j=1}^{k+1} p_j & \text{if} \quad i > k+1 \end{cases},$$

satisfies all of the following:

(a) *there exists a $\boldsymbol{w}$-bubble $\boldsymbol{u}$ with $\boldsymbol{w} \cdot \boldsymbol{u} = kd_{\boldsymbol{w}} + 1$ such that $(\boldsymbol{w}, \boldsymbol{u}, k-1)$ satisfies Condition 2.2, and*

(b) *every $\boldsymbol{w}$-bubble $\boldsymbol{u}'$ satisfies $\boldsymbol{w} \cdot \boldsymbol{u}' < (k+1)d_{\boldsymbol{w}}$.*

Proof. Again, it is sufficient to prove the case $k = n - 1$. In what follows, let $\boldsymbol{p} := (p_1, \ldots, p_n) \in \mathbb{Z}^n_{\geq 1}$ be pairwise distinct prime numbers such that $k_{\boldsymbol{p}} = k$. Let moreover $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$ be defined as in the statement.

(a) This time, we let $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 1}$ be such that $\alpha_i \in \{1, \ldots, p_i - 1\}$ for all $i \in \{1, \ldots, n\}$ and

$$\sum_{i=1}^{n} \frac{\alpha_i}{p_i} - \prod_{i=1}^{n} \frac{1}{p_i} = k$$

holds. Moreover, we define $\boldsymbol{u} := \boldsymbol{\alpha}$. By construction, it follows that the equality

$$\boldsymbol{w} \cdot \boldsymbol{u} = k d_{\boldsymbol{w}} + 1,$$

holds, and the vector $\boldsymbol{u}$ is a $\boldsymbol{w}$-bubble according to item (1) of the proof of Theorem 7.5(b).

We now prove that $(\boldsymbol{w}, \boldsymbol{u}, n-2)$ satisfies Condition 2.2. Fix any $i \in \{1, \ldots, k+1\}$. We proceed once again by proof by contradiction. Assume that there exists $m \geq 2$ such that there exist $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^m \in V^+_{\boldsymbol{w}}((n-2)d_{\boldsymbol{w}})$ satisfying the following equation:

$$\boldsymbol{u}' := \boldsymbol{u} + ((n-2)m - (n-1))p_i \boldsymbol{e}_i = \sum_{j=1}^{m} \boldsymbol{v}^j.$$

Note that since $\boldsymbol{w} \cdot \boldsymbol{u}' = (n-2)m d_{\boldsymbol{w}} + 1$, there exists $j_0 \in \{1, \ldots, m\}$ such that $\boldsymbol{w} \cdot \boldsymbol{v}^{j_0} = (n-2)d_{\boldsymbol{w}} + 1$ and $\boldsymbol{v}^j \in V_{\boldsymbol{w}}((n-2)d_{\boldsymbol{w}})$ for all $j \in \{1, \ldots, m\} \setminus \{j_0\}$. Without loss of generality, we may assume that $j_0 = m$. For all $l \in \{1, \ldots, n\} \setminus \{i\}$ and for all $j \in \{1, \ldots, m-1\}$, we have that $v^j_l \leq u_l < p_l$ hold, and v^j_l is divisible by p_l (following again the same ideas as in the beginning of the proof of Lemma 7.2). This implies the following:

$$\boldsymbol{v}^1 = \ldots = \boldsymbol{v}^{m-1} = (n-2)p_i \boldsymbol{e}_i.$$

Moreover, the previous implies that

$$u'_i = u_i + ((n-2)m - (n-1))p_i = \sum_{j=1}^{m} v^j_i = (m-1)(n-2)p_i + v^m_i$$

and in particular, we obtain that

$$p_i + v^m_i = u_i < p_i$$

holds. However, this is absurd since v^m_i must be nonnegative. Hence $(\boldsymbol{w}, \boldsymbol{u}, n-2)$ satisfies Condition 2.2.

(b) Similar as Theorem 7.5(c).

□

A Algorithms

Algorithm 1: Recursive

Input:

- A vector $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$,
- a vector $\texttt{mins} \in \mathbb{Z}^n$,
- a vector $\texttt{maxs} \in \mathbb{Z}^n$,
- an integer `min_product`,
- a finite arranged marked list of vectors $\texttt{thresholds} \subseteq \mathbb{Z}^n_{\geq 0}$ and
- a boolean $\texttt{return\_one} \in \{\texttt{true}, \texttt{false}\}$.

Output: If $\texttt{return\_one} = \texttt{false}$, then returns the list of all vectors $\boldsymbol{u} \in \mathbb{Z}^n$ satisfying the inequalities $\texttt{mins} \preceq u \preceq \texttt{maxs}$ and $\texttt{min\_product} \leq \boldsymbol{w} \cdot \boldsymbol{u}$ such that there are no vectors $\boldsymbol{v} \in \texttt{thresholds}$ with $\boldsymbol{v} \preceq \boldsymbol{u}$. If $\texttt{return\_one} = \texttt{true}$, then returns exactly one such vector if it exists, otherwise returns the empty list.

$\texttt{a} \leftarrow \max\left(\texttt{mins}_n, \left\lceil \dfrac{\texttt{min\_product} - \pi_n(\boldsymbol{w}) \cdot \pi_n(\texttt{maxs})}{w_n} \right\rceil\right)$.
$\texttt{b} \leftarrow \min(\texttt{maxs}_n, v_n - 1)$ where $\boldsymbol{v}$ is the first element of `thresholds`.
if $\texttt{a} > \texttt{b}$ **then**
    return [].
if $n = 1$ **then**
    if $\texttt{return\_one} = \texttt{true}$ **then**
        return $[\texttt{a}\boldsymbol{e}_1]$.
    else
        return $[\texttt{a}\boldsymbol{e}_1,\ (\texttt{a}+1)\boldsymbol{e}_1,\ \ldots,\ \texttt{b}\boldsymbol{e}_1]$.
$L \leftarrow []$.
for $u_n \in \{\texttt{a}, \ldots, \texttt{b}\}$ **do**
    $\texttt{low} \leftarrow \texttt{min\_product} - u_n w_n$.
    $\texttt{thrs} \leftarrow \{\pi_n(\boldsymbol{v}) \mid \boldsymbol{v} \in \texttt{thresholds},\ v_n \leq u_n\}$.
    Let $\widetilde{\texttt{thrs}}$ be an arrangement of `thrs`.
    $V' \leftarrow \textsc{Recursive}(\pi_n(\boldsymbol{w}),\ \pi_n(\texttt{mins}),\ \pi_n(\texttt{maxs}),\ \texttt{low},\ \widetilde{\texttt{thrs}},\ \texttt{return\_one})$
    for $\boldsymbol{u}' \in V'$ **do**
        append $\kappa_n(\boldsymbol{u}') + u_n\boldsymbol{e}_n$ to L.
        if $\texttt{return\_one} = \texttt{true}$ **then**
            return L.
return L.

Algorithm 2: Bubbles

Input: A nondecreasing vector $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$ and an integer $\texttt{low} \in \{0, 1\}$.

Output: The complete list of all $\boldsymbol{w}$-bubbles $\boldsymbol{u}$ such that $\texttt{low} \leq u_i$ for every $i \in \{1, \ldots, n\}$.

if $n < 3$ **then**
  return [].

$g \leftarrow \gcd(w_1, \ldots, w_n)$.

if $g > 1$ **then**
  $\boldsymbol{w} \leftarrow \left(\frac{w_1}{g}, \ldots, \frac{w_n}{g}\right)$.

For every $i \in \{1, \ldots, n\}$, define $\alpha_i := \begin{cases} w_n - 2 & \text{if } i < n, \\ w_{n-1} - 2 & \text{otherwise.} \end{cases}$

if $\boldsymbol{w} \cdot \boldsymbol{\alpha} < 2d_{\boldsymbol{w}}$ **then**
  return [].

For every $i \in \{1, \ldots, n\}$, define:

$$\varepsilon_i := \min\left(\alpha_i, \frac{d_{\boldsymbol{w}}}{w_i} - 1\right),$$
$$\zeta_i := \min\left(\{\varepsilon_i + 1\} \cup \left\{\frac{w_j}{w_i} \;\middle|\; j \in \{i+1, \ldots, n\},\, w_i \mid w_j\right\}\right),$$
$$\beta_i := \min(\varepsilon_i, \zeta_i - 1).$$

if $\boldsymbol{w} \cdot \boldsymbol{\beta} < 2d_{\boldsymbol{w}}$ **then**
  return [].

if $\texttt{low} = 1$ **then**
  For every $i \in \{1, \ldots, n\}$, use Notation 3.16 to define:

$$\delta_i := \min\left(\varepsilon_i, \left\lfloor \frac{\max(\pi_i(\boldsymbol{w})) + c_{\boldsymbol{w},i}}{w_i} \right\rfloor - 2\right),$$
$$\gamma_i := \min(\delta_i, \zeta_i).$$

  if $\exists i \in \{1, \ldots, n\}$: $\gamma_i \leq 0$ **then**
    return [].
  if $\boldsymbol{w} \cdot \boldsymbol{\gamma} < 2d_{\boldsymbol{w}}$ **then**
    return [].

else
  $(\boldsymbol{\gamma}, \boldsymbol{\delta}) \leftarrow (\boldsymbol{\beta}, \boldsymbol{\varepsilon})$.

$\texttt{W}_{\boldsymbol{\gamma}} \leftarrow \{(\gamma_i + 1)\boldsymbol{e}_i \mid i \in \{1, \ldots, n\}\} \cup \{\boldsymbol{v} \in V_{\boldsymbol{w}}(d_{\boldsymbol{w}}) \mid \boldsymbol{v} \preceq \boldsymbol{\gamma}\}$.

Let $\widetilde{\texttt{W}_{\boldsymbol{\gamma}}}$ be an arrangement of $\texttt{W}_{\boldsymbol{\gamma}}$.

$\texttt{mins} \leftarrow \texttt{low} \cdot \sum_{i=1}^{n} \boldsymbol{e}_i$.

if $\textsc{Recursive}(\boldsymbol{w}, \texttt{mins}, \boldsymbol{\gamma}, 2d_{\boldsymbol{w}}, \widetilde{\texttt{W}_{\boldsymbol{\gamma}}}, \texttt{true}) = \emptyset$ **then**
  return [].

$\texttt{W}_{\boldsymbol{\delta}} \leftarrow \{(\delta_i + 1)\boldsymbol{e}_i \mid i \in \{1, \ldots, n\}\} \cup \{\boldsymbol{v} \in V_{\boldsymbol{w}}(d_{\boldsymbol{w}}) \mid \boldsymbol{v} \preceq \boldsymbol{\delta}\}$.

Let $\widetilde{\texttt{W}_{\boldsymbol{\delta}}}$ be an arrangement of $\texttt{W}_{\boldsymbol{\delta}}$.

return $\textsc{Recursive}(\boldsymbol{w}, \texttt{mins}, \boldsymbol{\delta}, 2d_{\boldsymbol{w}}, \widetilde{\texttt{W}_{\boldsymbol{\delta}}}, \texttt{false})$.

B Tables

Table 1 contains all the $\boldsymbol{w}$-bubbles with positive coordinates where $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 1}$ is such that $\max(\boldsymbol{w}) < 42$ and $w_1 < \ldots < w_n$ hold (see Proposition 5.5). For every such $\boldsymbol{w}$, the $\boldsymbol{w}$-bubbles $\boldsymbol{u}$ in Table 1 are sorted by the value of $\boldsymbol{w} \cdot \boldsymbol{u}$ with the first bubble $\boldsymbol{v}$ having the least $\boldsymbol{w} \cdot \boldsymbol{v}$. For every $\boldsymbol{w}$-bubble $\boldsymbol{u}$ in Table 1,

(1) if $d_{\boldsymbol{w}} \mid \boldsymbol{w} \cdot \boldsymbol{u}$, then $\boldsymbol{u}$ is coloured blue,

(2) if $\boldsymbol{w} \cdot \boldsymbol{u} \geq 2d_{\boldsymbol{w}} + 1$ and $\boldsymbol{u} \notin 2V^+_{\boldsymbol{w}}(d_{\boldsymbol{w}})$, then $\boldsymbol{u}$ is coloured orange.

All the $\boldsymbol{w}$-bubbles $\boldsymbol{u}$ in Table 1 satisfy the inequality $\boldsymbol{w} \cdot \boldsymbol{u} < 3d_{\boldsymbol{w}}$. If $\boldsymbol{u}$ is coloured, then $\boldsymbol{w} \cdot \boldsymbol{u} \in \{2d_{\boldsymbol{w}},\, 2d_{\boldsymbol{w}} + 1,\, 2d_{\boldsymbol{w}} + 2,\, 2d_{\boldsymbol{w}} + 3\}$.

Table 1: Bubbles with positive coordinates for weights less than 42

$\boldsymbol{w}$	Bubbles
$(6, 14, 21)$	$\{(6, 2, 1)\}$
$(6, 22, 33)$	$\{(10, 2, 1)\}$
$(6, 26, 39)$	$\{(11, 2, 1),\ (12, 2, 1)\}$
$(10, 14, 35)$	$\{(5, 4, 1),\ (6, 4, 1)\}$
$(12, 15, 20)$	$\{(3, 3, 2),\ (4, 3, 2)\}$
$(12, 21, 28)$	$\{(6, 2, 2),\ (5, 3, 2),\ (6, 3, 2)\}$
$(12, 26, 39)$	$\{(12, 2, 3),\ (12, 5, 1)\}$
$(14, 20, 35)$	$\{(4, 6, 3),\ (9, 6, 1)\}$
$(15, 20, 24)$	$\{(3, 5, 4),\ (7, 2, 4)\}$
$(15, 21, 35)$	$\{(4, 4, 2),\ (6, 3, 2),\ (5, 4, 2),\ (6, 4, 2)\}$
$(15, 24, 40)$	$\{(7, 4, 1),\ (6, 3, 2),\ (5, 4, 2),\ (7, 3, 2),\ (6, 4, 2),\ (7, 4, 2)\}$
$(20, 28, 35)$	$\{(6, 2, 3),\ (5, 4, 2),\ (5, 3, 3),\ (4, 4, 3),\ (6, 4, 2),\ (6, 3, 3),\ (5, 4, 3),\ (6, 4, 3)\}$
$(21, 24, 28)$	$\{(3, 6, 5),\ (7, 6, 2)\}$
$(21, 30, 35)$	$\{(4, 6, 5),\ (9, 6, 2)\}$
$(24, 30, 40)$	$\{(3, 3, 2),\ (4, 3, 2)\}$
$(28, 35, 40)$	$\{(4, 6, 6),\ (9, 2, 6),\ (3, 7, 6),\ (8, 3, 6),\ (4, 7, 6),\ (9, 3, 6)\}$
$(1, 6, 10, 15)$	$\{(1, 4, 2, 1)\}$
$(1, 6, 22, 33)$	$\{(1, 9, 2, 1),\ (1, 10, 2, 1)\}$
$(1, 12, 20, 30)$	$\{(2, 4, 2, 1),\ (3, 4, 2, 1)\}$
$(1, 12, 21, 28)$	$\{(1, 4, 3, 2),\ (1, 6, 2, 2),\ (1, 5, 3, 2),\ (1, 6, 3, 2)\}$
$(1, 12, 22, 33)$	$\{(1, 10, 2, 3),\ (1, 10, 5, 1)\}$
$(1, 15, 21, 35)$	$\{(1, 6, 4, 1),\ (2, 5, 3, 2),\ (2, 6, 4, 1),\ (1, 4, 4, 2),\ (2, 4, 4, 2),\ (1, 6, 3, 2),\ (2, 6, 3, 2),\ (1, 5, 4, 2),\ (2, 5, 4, 2),\ (1, 6, 4, 2),\ (2, 6, 4, 2)\}$
$(1, 15, 24, 40)$	$\{(1, 7, 4, 1)\}$
$(1, 20, 28, 35)$	$\{(1, 6, 2, 3)\}$
$(1, 21, 28, 36)$	$\{(1, 3, 8, 6),\ (1, 7, 5, 6),\ (1, 11, 2, 6)\}$
$(1, 21, 30, 35)$	$\{(2, 3, 6, 5)\}$
$(1, 24, 30, 40)$	$\{(1, 3, 3, 2),\ (1, 4, 3, 2)\}$
$(2, 12, 15, 20)$	$\{(1, 4, 2, 2),\ (1, 3, 3, 2),\ (1, 4, 3, 2)\}$

$\boldsymbol{w}$	Bubbles
$(2,12,20,30)$	$\{(1,4,2,1)\}$
$(2,21,30,35)$	$\{(1,3,6,5),\ (1,8,6,2),\ (1,4,6,5),\ (1,9,6,2)\}$
$(3,12,15,20)$	$\{(1,4,2,2)\}$
$(3,12,20,30)$	$\{(1,4,2,1)\}$
$(3,12,21,28)$	$\{(1,4,3,2)\}$
$(3,24,30,40)$	$\{(2,4,2,2),\ (1,3,3,2),\ (3,4,2,2),\ (1,4,3,2)\}$
$(4,15,24,40)$	$\left\{\begin{matrix}(1,4,4,2),\ (2,7,2,2),\ (1,7,4,1),\ (1,6,3,2),\\ (1,5,4,2),\ (1,7,3,2),\ (1,6,4,2),\ (1,7,4,2)\end{matrix}\right\}$
$(4,20,28,35)$	$\{(1,3,4,3)\}$
$(4,24,30,40)$	$\{(1,4,2,2),\ (1,3,3,2),\ (1,4,3,2)\}$
$(5,12,20,30)$	$\{(1,4,2,1)\}$
$(5,15,21,35)$	$\{(1,6,4,1)\}$
$(5,15,24,40)$	$\{(1,4,4,2)\}$
$(5,20,28,35)$	$\{(1,3,4,3),\ (1,4,4,3)\}$
$(5,24,30,40)$	$\{(1,4,2,2),\ (3,4,3,1),\ (1,3,3,2),\ (1,4,3,2)\}$
$(5,28,35,40)$	$\{(1,4,7,5),\ (1,9,3,5)\}$
$(6,15,24,40)$	$\{(1,4,4,2),\ (1,5,4,2)\}$
$(6,21,30,35)$	$\{(1,3,6,5)\}$
$(6,24,30,40)$	$\{(1,4,2,2)\}$
$(7,12,21,28)$	$\{(1,6,3,1)\}$
$(7,20,28,35)$	$\{(1,6,3,2),\ (2,6,4,1),\ (1,5,4,2),\ (1,6,4,2)\}$
$(7,28,35,40)$	$\{(1,4,6,6),\ (1,9,2,6)\}$
$(8,15,24,40)$	$\{(1,7,2,2)\}$
$(8,28,35,40)$	$\{(1,3,7,6)\}$
$(10,15,24,40)$	$\{(1,7,4,1)\}$
$(12,15,20,30)$	$\{(3,1,2,1),\ (4,1,2,1)\}$
$(14,20,28,35)$	$\left\{\begin{matrix}(1,6,4,1),\ (2,6,1,3),\ (3,6,3,1),\\ (5,6,2,1),\ (7,6,1,1)\end{matrix}\right\}$
$(15,20,24,30)$	$\{(1,2,4,3),\ (1,5,4,1),\ (3,2,4,2),\ (5,2,4,1)\}$
$(15,20,24,40)$	$\{(3,1,4,2),\ (3,3,4,1)\}$
$(15,24,30,40)$	$\left\{\begin{matrix}(1,4,3,1),\ (3,4,2,1),\ (5,4,1,1),\ (2,3,2,2),\\ (4,3,1,2),\ (1,4,2,2),\ (3,4,1,2),\ (1,3,3,2),\\ (3,3,2,2),\ (5,3,1,2),\ (2,4,2,2),\ (4,4,1,2),\\ (1,4,3,2),\ (3,4,2,2),\ (5,4,1,2)\end{matrix}\right\}$
$(1,2,12,20,30)$	$\{(1,1,4,2,1)\}$
$(1,3,24,30,40)$	$\left\{\begin{matrix}(1,1,4,2,2),\ (1,2,4,2,2),\\ (1,1,3,3,2),\ (1,1,4,3,2)\end{matrix}\right\}$
$(1,4,12,21,28)$	$\{(1,1,6,3,1)\}$
$(1,4,24,30,40)$	$\{(1,1,4,2,2),\ (1,1,3,3,2),\ (1,1,4,3,2)\}$
$(1,5,24,30,40)$	$\{(1,1,4,2,2)\}$
$(1,6,24,30,40)$	$\{(1,1,4,2,2)\}$
$(1,7,15,21,35)$	$\{(1,1,6,2,2)\}$
$(1,15,24,30,40)$	$\{(1,1,4,3,1),\ (1,3,4,2,1),\ (1,5,4,1,1)\}$
$(2,3,12,20,30)$	$\{(1,1,4,2,1)\}$
$(2,5,12,15,20)$	$\{(1,1,4,3,1)\}$
$(2,5,12,20,30)$	$\{(1,1,4,2,1)\}$

$\boldsymbol{w}$	Bubbles
$(2,12,15,20,30)$	$\{(1,3,1,2,1),\ (1,4,1,2,1)\}$
$(2,14,21,30,35)$	$\{(1,1,4,6,4),\ (1,1,9,6,1)\}$
$(3,4,15,24,40)$	$\{(1,1,7,2,2)\}$
$(3,4,24,30,40)$	$\{(1,1,4,2,2),\ (1,1,3,3,2),\ (1,1,4,3,2)\}$
$(3,6,24,30,40)$	$\{(1,1,4,2,2)\}$
$(4,5,24,30,40)$	$\{(1,2,4,3,1),\ (1,1,4,2,2),\ (1,3,4,3,1),\ (1,1,3,3,2),\ (1,1,4,3,2)\}$
$(4,10,15,24,40)$	$\{(1,1,6,4,1),\ (1,1,7,4,1)\}$
$(4,10,24,30,40)$	$\{(1,1,4,3,1)\}$
$(4,15,24,30,40)$	$\{(1,2,4,1,2),\ (2,1,2,3,2),\ (2,3,2,2,2),\ (2,5,2,1,2),\ (1,1,4,3,1),\ (1,3,4,2,1),\ (1,5,4,1,1),\ (1,2,3,2,2),\ (1,4,3,1,2),\ (1,1,4,2,2),\ (1,3,4,1,2),\ (1,1,3,3,2),\ (1,3,3,2,2),\ (1,5,3,1,2),\ (1,2,4,2,2),\ (1,4,4,1,2),\ (1,1,4,3,2),\ (1,3,4,2,2),\ (1,5,4,1,2)\}$
$(5,6,24,30,40)$	$\{(1,1,4,2,2)\}$
$(5,10,24,30,40)$	$\{(1,1,4,3,1)\}$
$(5,15,24,30,40)$	$\{(1,2,4,1,2)\}$
$(6,15,24,30,40)$	$\{(1,2,4,1,2),\ (1,1,4,2,2),\ (1,3,4,1,2)\}$
$(8,15,24,30,40)$	$\{(1,1,2,3,2),\ (1,3,2,2,2),\ (1,5,2,1,2)\}$
$(10,15,24,30,40)$	$\{(1,1,4,3,1),\ (1,3,4,2,1),\ (1,5,4,1,1)\}$
$(15,20,24,30,40)$	$\{(1,1,4,1,2),\ (1,3,4,1,1)\}$
$(1,3,10,24,30,40)$	$\{(1,1,1,4,3,1)\}$
$(1,4,10,24,30,40)$	$\{(1,1,1,4,3,1)\}$
$(2,5,12,15,20,30)$	$\{(1,1,4,1,1,1)\}$
$(3,4,10,24,30,40)$	$\{(1,1,1,4,3,1)\}$
$(3,4,15,24,30,40)$	$\{(1,1,1,2,3,2),\ (1,1,3,2,2,2),\ (1,1,5,2,1,2)\}$
$(4,5,10,24,30,40)$	$\{(1,1,1,4,3,1)\}$
$(4,10,15,24,30,40)$	$\{(1,1,2,4,2,1),\ (1,1,4,4,1,1),\ (1,1,1,4,3,1),\ (1,1,3,4,2,1),\ (1,1,5,4,1,1)\}$

For every $m < 15$, Table 2 contains the lexicographically smallest vector $\boldsymbol{p} := (p_1, \ldots, p_m)$ of Corollary 7.3 satisfying $p_1 > \ldots > p_m$ and $k_{\boldsymbol{p}} = 1$.

Table 2: Smallest examples of Corollary 7.3 with $k_{\boldsymbol{p}} = 1$

m	$(p_1, \ldots, p_m)$
2	$(3,2)$
3	$(5,3,2)$
4	$(11,7,5,3)$
5	$(23,19,17,11,7)$
6	$(37,31,19,17,11,3)$

m	$(p_1, \ldots, p_m)$
7	$(59, 43, 41, 23, 19, 7, 5)$
8	$(43, 41, 37, 19, 17, 13, 11, 7)$
9	$(71, 59, 53, 47, 41, 19, 17, 13, 5)$
10	$(107, 97, 89, 83, 79, 37, 29, 23, 7, 5)$
11	$(103, 101, 79, 71, 61, 59, 41, 29, 23, 19, 13)$
12	$(151, 149, 139, 127, 59, 47, 43, 37, 29, 23, 13, 5)$
13	$(167, 163, 151, 149, 131, 89, 83, 73, 43, 37, 23, 13, 11)$
14	$(191, 181, 163, 149, 139, 137, 127, 107, 97, 83, 79, 59, 37, 13)$

For every $n < 15$, Table 3 contains the lexicographically smallest vector $\boldsymbol{p}$ of Corollary 7.3 satisfying $p_1 > \ldots > p_n$ and $k_{\boldsymbol{p}} = n - 1$.

Table 3: Smallest examples of Corollary 7.3 with $k_{\boldsymbol{p}} = n - 1$

n	$(p_1, \ldots, p_n)$
2	$(3, 2)$
3	$(7, 3, 2)$
4	$(17, 11, 5, 2)$
5	$(29, 17, 13, 11, 3)$
6	$(31, 29, 23, 17, 13, 3)$
7	$(47, 41, 37, 31, 19, 17, 3)$
8	$(47, 43, 23, 19, 17, 13, 7, 5)$
9	$(83, 73, 61, 41, 31, 19, 13, 11, 7)$
10	$(103, 97, 67, 43, 41, 31, 19, 17, 13, 7)$
11	$(109, 107, 83, 79, 53, 43, 37, 29, 19, 17, 3)$
12	$(137, 113, 101, 97, 83, 73, 47, 43, 41, 23, 19, 7)$
13	$(181, 167, 151, 149, 137, 127, 113, 103, 73, 43, 29, 23, 19)$
14	$(193, 167, 163, 157, 149, 139, 113, 97, 83, 71, 53, 41, 37, 23)$

Acknowledgments

We would like to thank Simon Brandhorst and Qing Liu for useful comments.

References

[BR86] Mauro Beltrametti and Lorenzo Robbiano, *Introduction to the theory of weighted projective spaces*, Exposition. Math. **4** (1986), no. 2, 111–162.

[CCC11] Jheng-Jie Chen, Jungkai A. Chen, and Meng Chen, *On quasismooth weighted complete intersections*, J. Algebraic Geom. **20** (2011), no. 2, 239–262.

[Del75] Charles Delorme, *Espaces projectifs anisotropes*, Bull. Soc. Math. Fr. **103** (1975), 203–223 (French).

[GW20] Ulrich Görtz and Torsten Wedhorn, *Algebraic Geometry I: Schemes. With Examples and Exercises*, second ed., Springer Studium Mathematik—Master, Springer Spektrum, Wiesbaden, 2020.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[HL85] Helmut A. Hamm and Dũng Tráng Lê, *Lefschetz theorems on quasi-projective varieties*, Bull. Soc. Math. Fr. **113** (1985), 123–142 (English).

[HL20] ———, *The Lefschetz theorem for hyperplane sections*, Handbook of geometry and topology of singularities I, Cham: Springer, 2020, pp. 491–540 (English).

[IF00] A. R. Iano-Fletcher, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 101–173.

[Kaw03] Masayuki Kawakita, *General elephants of three-fold divisorial contractions*, J. Amer. Math. Soc. **16** (2003), no. 2, 331–362, see https://www.kurims.kyoto-u.ac.jp/~masayuki/Website/Documents/erratum-03.pdf for erratum.

[KM92] János Kollár and Shigefumi Mori, *Classification of three-dimensional flips*, J. Amer. Math. Soc. **5** (1992), no. 3, 533–703.

[Oka23] Takuzo Okada, *Birationally solid Fano 3-fold hypersurfaces*, Preprint, 2023, arXiv:2305.06183v3.

[Pet15] T. Kyle Petersen, *Eulerian numbers*, Birkhäuser Adv. Texts, Basler Lehrbüch., New York, NY: Birkhäuser/Springer, 2015.

[PS] Victor Przyjalkowski and Constantin Shramov, *Weighted complete intersections*, to be published.

[Rei80] Miles Reid, *Canonical* 3*-folds*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, pp. 273–310.

[Sta25] The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu, 2025.

[Yam18] Yuki Yamamoto, *Divisorial contractions to cDV points with discrepancy greater than 1*, Kyoto J. Math. **58** (2018), no. 3, 529–567.

Stevell Muller `muller@math.uni-hannover.de`
Mathematik und Informatik, Universität des Saarlandes, 66123 Saarbrücken, Germany.
Current address: Institut für Algebraische Geometrie, Leibniz Universität Hannover, 30167 Hannover, Germany.

Erik Paemurru `algebraic.geometry@runbox.com`
Mathematik und Informatik, Universität des Saarlandes, 66123 Saarbrücken, Germany.
Current address: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. bl. 8, 1113, Sofia, Bulgaria.