

On the existence of fibered three-dimensional perfect fluid equilibria without continuous Euclidean symmetry

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October 8, 2025

Abstract

Following Lortz [17], we construct a family of smooth steady states of the ideal, incompressible Euler equation in three dimensions that possess no continuous Euclidean symmetry. As in Lortz, they do possess a planar reflection symmetry and, as such, all the orbits of the velocity are closed. Different from Lortz, our construction has a discrete m -fold symmetry and is foliated by invariant cylindrical level sets of a non-degenerate Bernoulli pressure, and are flexible (come in infinite dimensional families). Such examples are distinct from but broadly fit in the category of those constructed explicitly by Woolley [22] and Salat–Kaiser [18, 15]. These narrow the scope of validity of Grad’s conjecture on non-existence of fibered equilibria without continuous symmetry. [10].

Dedicated to Prof. Peter Constantin, our mentor and friend.

1 Introduction

Stationary states of the Euler equations play an important role in understanding the fluid motion. In two dimensions, they are plentiful, and their structure can be rich [7]. In three dimensions, far less is understood. Given a domain $M \subset \mathbb{R}^3$ endowed with the standard Euclidean metric, they are defined by a vector field $u : M \rightarrow \mathbb{R}^3$ which is tangent to the boundary of M if non-empty and satisfies

$$u \cdot \nabla u + \nabla p = 0, \tag{1.1}$$

$$\operatorname{div} u = 0. \tag{1.2}$$

In the above, $p : M \rightarrow \mathbb{R}$ is the pressure, defined by solving $-\Delta p = \operatorname{div}(u \cdot \nabla u)$ on M with appropriate Neumann boundary data inherited from the equation. Provided the solution is classical, this can be rephrased in an illuminating way

$$u \times \operatorname{curl} u = \nabla H, \quad H := \frac{1}{2}|u|^2 + p, \tag{1.3}$$

where the function $H : M \rightarrow \mathbb{R}$ is called the Bernoulli pressure. As such, the steady 3D Euler equations also have the interpretation of magnetohydrostatic equilibria [10, 12, 13, 5].

Steady states in 3D can be essentially divided into two categories: those that are nearly two-dimensional in the sense that all orbits are confined to hypersurfaces, and those whose orbits explore volumes. The former are termed *fibered* solutions. Arnol’d famously classified a subclass of 3D steady states: roughly, those with non-constant Bernoulli pressure are fibered by invariant tori or cylinders [2, 3]. See also [20]. The motion on each torus is quasiperiodic (the field lines are either all closed or all dense). The reason behind this result is that non-constant Bernoulli pressure implies that the velocity and vorticity vector fields are non-vanishing, and transverse at every point. Moreover, the Euler equation (1.3) implies that H is a first integral of these fields, e.g. $u \cdot \nabla H = 0$. Hence, if ∇H is non-zero, particles are constrained to the level sets of H in a neighborhood of that point. So, on any regular level set $S := \{H = c\}$, one finds that $u(x)$ and $\operatorname{curl} u(x)$ form a basis for the tangent space for S at each point x . The only connected two-dimensional manifolds for which such fields can exist are tori or cylinders. He also conjectured that steady states having aligned velocity and vorticity (such as Beltrami flows), and therefore constant Bernoulli function, could behave much more wildly.

So far, with the notable exceptions of the work of Lortz [17] (see also [14]), Woolley [22] and Salat–Kaiser [18, 15], all known steady states on $\mathbb{R}^3$ with non-constant Bernoulli pressure have a continuous Euclidean symmetry. In particular, such fields that are axisymmetric can be build out of a scalar potential solving the Grad-Shafranov equation. Outside of Euclidean symmetry, there are very few existence results. This, together with an understanding of the difficulty for existence, led Grad to conjecture that no fibered smooth solutions, in particular those with non-constant Bernoulli pressure, exist outside of Euclidean symmetry. Grad, who was a plasma physicist, was thinking about the confinement fusion problem and was interested in this non-existence result in light of devices such as the tokamak and stellarator, aimed at producing such states for the purposes of confining a nuclear fusion reaction.

The only well understood class of steady states outside of symmetry are Beltrami flows, for which there is an easy existence theory. As mentioned above, in general their field lines may be completely chaotic [8, 9]. There is one construction of note due to Lortz, originating from an idea of Grad and Rubin [13], which perturbs a harmonic vector field (and therefore a particular Beltrami flow) with a reflection symmetry which has all of its orbits closed and is therefore fibered by tori [17]. In his construction, the Bernoulli pressure is a given small-amplitude function of the period of revolution of particles on the closed orbits. Being that the base state is harmonic, in general this period function is general not understood and in simple cases (e.g. the vector field $u_H = e_z$ on the $\mathbb{R}^2 \times \mathbb{T}$), the period function is constant. As such, it is not clear from Lortz’s construction that the pressure levels are cylinders which fiber the domain. There can be islands separated by regions of compact level sets, of constancy etc. See also [21] for a construction with non-compact pressure surfaces and [19] for an extension of Lortz’s construction to other systems. Thus, neither Lortz nor Beltrami flows show whether Arnol’d’s theorem applies to any steady state having cylindrical levels of the Bernoulli pressure but no continuous symmetry. On the other hand, the works of Woolley [22] and Salat–Kaiser [18, 15] produce *explicit* forms of cylindrical solutions which are far from having continuous symmetry (and indeed, are not perturbative of axisymmetric solutions). These solutions illustrate Arnol’d’s theorem applies outside continuous symmetry, but the issue of flexibility and isolation of the explicit objects they find is unclear.

In this work, we demonstrate the existence of such a non-symmetric 3D steady state, which are perturbations of axisymmetric solutions and come in infinite dimensional families (parametrized by the shape of the outer cylinder).

Theorem 1. *There exists a cylindrical domain $M \subset \mathbb{R}^2 \times \mathbb{T}$, and a C^∞ smooth stationary solution of Euler $u : M \rightarrow \mathbb{R}^3$, tangent to ∂M , such that ∇H is non-vanishing away from a line. The solution is periodic in the direction set by said line, m -fold symmetric about this line, but possesses no continuous Euclidean symmetry. This solution is a perturbation of a monotone, non-degenerate two-dimensional rotational flow, with the outer boundary of the cylinder a freely prescribed perturbation, up to the discrete symmetries.*

Such a solution is fibered by invariant (wobbly) cylinders which are levels of the Bernoulli function. We remark that the size of the pressure of the solution may be large; see Fig 1.

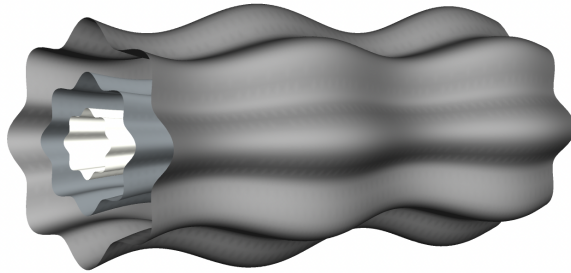


Figure 1: An 8-fold symmetric steady state, fibered by levels of the Bernoulli pressure H . All orbits of the velocity are confined to the isosurfaces of H , and wrap the “short way”.

In fact, our solution is constructed by modifying Lortz's argument to apply to perturbations of solutions with large Bernoulli pressure (as opposed to harmonic fields, with trivial Bernoulli pressure). The reason for Beltrami fields in Lortz's construction is that the pressure provides that small parameter which he uses to break the symmetry and close an iteration. To overcome this difficulty, our small parameter is instead $1/m$, where m is the multiplicity of discrete reflection symmetries enjoyed by the solution.

A couple concluding remarks should be made. First, we do not believe the discrete symmetry of the solution is essential, and that there should exist steady state nearby those constructed here which break *all* Euclidean symmetries. Such an object would certainly be interesting to construct. However, second, it is clear from the discussion of Grad and Rubin [13] and Grad [11] that Grad understood that steady states with all closed field lines could exist outside symmetry, although Lortz had the first rigorous such construction [17]. Our present contribution is to furthermore guarantee the pressure levels are cylinders, allow for large pressure (in the language of confinement fusion, large plasma β), and are flexible (come in large families of such solutions). The former two properties appear also in the works of Woolley [22] and Salat–Kaiser [18, 15]. As such, it seems reasonable to stipulate that the Grad's conjecture should include the provision that the vector field has ergodic orbits on nearly all its invariant tori. Such things, Grad clearly states, should be either non-existent or exceptionally rare (isolated) outside of (continuous) Euclidean symmetry [12]. Moreover, these hypothetical asymmetric twisting equilibria – stellarators – are believed to be relevant objects to advance plasma confinement fusion [16].

2 Steady Euler and Clebsch Variables

We aim to produce a vector field u which solves steady Euler. We will do this by perturbing a given axisymmetric solution u_* which occupies the “straight” periodic unit cylinder $D_* := \mathbb{D} \times \mathbb{T}$. Specifically, for a smooth Ω with further conditions to be specified, we will perturb

$$u_* := \Omega(r)re_\theta, \quad e_\theta := \frac{x_1}{r}e_2 - \frac{x_2}{r}e_1 \quad (2.1)$$

where $r := \sqrt{x_1^2 + x_2^2}$, which satisfies (1.1)-(1.2) with hydrodynamic pressure

$$p_*(r) = \int_0^r \rho \Omega^2(\rho) d\rho.$$

We note that Bernoulli pressure $H_* = \frac{1}{2}|u_*|^2 + p_*$ takes the form

$$H_*(r) = \frac{1}{2}(r\Omega(r))^2 + \int_0^r \rho \Omega^2(\rho) d\rho = \int_0^r [\rho \Omega'(\rho) + 2\Omega(\rho)] \rho \Omega(\rho) d\rho$$

and so if Ω is monotone and does not change sign, ∇H_* is nonzero except at $\{r = 0\}$.

The orbits of u_* are closed and have period $T_*(r) = \frac{2\pi}{\Omega(r)}$. For our construction we will want to be able to write the Bernoulli function H_* as a smooth function of T_* . For this, we will assume that Ω satisfies the following two properties:

(H1) $r \mapsto \Omega(r)$ is invertible and does not change sign for $r \in [0, \infty)$, and

(H2) $|\Omega''(0)| > 0$

The condition **(H1)** guarantees that the period T_* is a bounded and invertible function, with inverse $r_*(T) = \Omega^{-1}(\frac{2\pi}{T})$. This, together with **(H2)**, lets us think of the Bernoulli function H_* as a smooth function of the period T . Indeed, if we define $\mathcal{H}_*(T) = H_*(r_*(T))$, and, noting that $r'_*(T) = -\frac{2\pi}{T^2} \frac{1}{\Omega'(r_*(T))}$ we have the formula

$$\mathcal{H}'_*(T) = -\frac{2\pi}{T^2} \frac{H'_*(r_*(T))}{\Omega'(r_*(T))} = \frac{2\pi}{T^2} \frac{[\rho \Omega'(\rho) + 2\Omega(\rho)] \rho \Omega(\rho)}{\Omega'(\rho)} \bigg|_{\rho=r_*(T)}, \quad (2.2)$$

where we used the explicit formula for H_* . Noting that smoothness of u_* demands that $\Omega^{(2n+1)}(0) = 0$ for all $n \geq 0$, smoothness of $\mathcal{H}_*$ can then be checked by Taylor expansion of the period function near $r_* = 0$. Repeating the same argument shows higher-order derivatives of $\mathcal{H}_*$ are also continuous.

The starting point for the construction is to seek *Clebsch variables* H, τ associated to u ,

$$u \cdot \nabla H = 0, \quad (2.3)$$

$$u \cdot \nabla \tau = 1. \quad (2.4)$$

See [4] for a clear introduction. This is also the starting point of [17, 22, 18, 15]. Given such H, τ , we define

$$\omega = \nabla \tau \times \nabla H. \quad (2.5)$$

Note that, by construction, we have $\operatorname{div} \omega = 0$. Moreover, it holds

$$u \times \omega = (u \cdot \nabla \tau) \nabla H - (u \cdot \nabla H) \nabla \tau = -\nabla H. \quad (2.6)$$

Provided that u satisfies

$$\operatorname{div} u = 0, \quad (2.7)$$

$$\operatorname{curl} u = \omega, \quad (2.8)$$

the resulting u would satisfy (1.3) and thus the original system (1.1)-(1.2). In the rest of this section, we describe our strategy for solving the above system.

Equations (2.3) and (2.4) are steady transport equation in three dimensions, and are somewhat delicate because u may have a mix of both closed and ergodic field lines. Indeed Grad's original objection concerned solving (2.4) in the presence of ergodic field lines. To sidestep this difficulty, we follow Lortz and impose a reflection symmetry so that the constructed vector field will have all closed orbits. Let us now define the notion of parity:

Definition 2. Fix a plane $\Pi \subset \mathbb{R}^3$. Let R_Π be the reflection about Π . We say that a function $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ is odd if $g \circ R_\Pi = -g$ and even if $g \circ R_\Pi = g$. We say that a vector field $X : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is odd if $X \circ R_\Pi = -R_\Pi X$. We say that X is even if $X \circ R_\Pi = R_\Pi X$.

Note that X is odd (resp. even) if and only if $R_\Pi^* X = -X$ (resp. $R_\Pi^* X = X$), where $R_\Pi^* X = R_\Pi^{-1} X \circ R_\Pi = R_\Pi X \circ R_\Pi$ denotes the pullback of X by R_Π .

From hereon, without loss of generality we choose $\Pi = \{x_2 = 0\}$ to be the $x_1 - x_3$ plane. In this case, $R_\Pi := R_\Pi(x_1, x_2, x_3) = (x_1, -x_2, x_3)$. Moreover, if X is odd, it means that $X \cdot e_1$ and $X \cdot e_3$ are odd in x_2 and $X \cdot e_2$ is even in x_2 . With this definition, for any smooth function Ω the vector field u_* from (2.1) is odd.

We now define the perturbed domain we will construct our solution in. We consider a perturbed cylindrical domain of this form

$$D_\varepsilon = \{r = 1 + \varepsilon g(\theta, z)\} \quad (2.9)$$

where the function $g : \mathbb{S}^1 \times \mathbb{T}$ which is even with respect to Π and which is m -fold symmetric,

$$g(\theta + 2\pi/m, z) = g(\theta, z) \quad (2.10)$$

for some $m \in \mathbb{Z}_{>0}$. We aim to construct a solution $u = u_* + \varepsilon u'$ in D_ε with

$$u \cdot n = 0, \quad \text{on } \partial D_\varepsilon, \quad (2.11)$$

and so that u is odd with respect to Π (recall Definition 2).

Once again, the importance of the parity of u is that, together with proximity to the field u_* , it guarantees that all field lines of u are closed and this will let us find τ and H satisfying (2.3)-(2.4). To see this, we start with the following lemma.

Lemma 3. *Let X be an odd and Lipschitz vector field on $\mathbb{R}^3$. Let γ be any trajectory of X which passes through the half-planes $\{\theta = 0\}$ and $\{\theta = \pi\}$. Then γ is closed.*

Proof. Write $\gamma(t) = r(t)e_r + \theta(t)e_\theta + z(t)e_z$. Without loss of generality, we can assume that $\theta(0) = 0$ and $\theta(1) = \pi$. We claim that for all t ,

$$\theta(1+t) = 2\pi - \theta(1-t), \quad r(1+t) = r(1-t), \quad z(1+t) = z(1-t). \quad (2.12)$$

If this holds, then setting $t = 1$ shows that the curve γ is closed (with period equal to 2). Clearly (2.12) holds at $t = 0$. If we write $\theta_+(t) = \theta(1+t)$, $\theta_-(t) = 2\pi - \theta(1-t)$ as well as $r_\pm(t) = r(1 \pm t)$ and $z_\pm(t) = z(1 \pm t)$, then by the parity of X , we find

$$\frac{d}{dt}r_\pm = X_r(r_\pm, \theta_\pm, z_\pm), \quad \frac{d}{dt}\theta_\pm = X_\theta(r_\pm, \theta_\pm, z_\pm), \quad \frac{d}{dt}z_\pm = X_z(r_\pm, \theta_\pm, z_\pm), \quad (2.13)$$

and by the uniqueness theorem for ODE this gives (2.12). $\square$

In our construction to follow, we will build iteratively a u which perturbs u_* : $u = u_* + \varepsilon u'$. By the discrete symmetry assumptions on u , it must vanish along the z -axis. Using this, since all field lines of u_* pass through both $\theta = 0$ and $\theta = \pi$, so do those of u for sufficiently small ε . Thus, by the Lemma, all orbits of such u are closed.

With this knowledge, for such a u we can therefore define a function τ by

$$u \cdot \nabla \tau = 1. \quad (2.14)$$

Since u has closed field lines, such τ is necessarily multi-valued; we orient the domain so that it is well-behaved away from the plane of symmetry Π but it jumps across $\Pi \cap \{x_1 \geq 0\}$. Letting T denote the period of the trajectories of u , then $u \cdot \nabla T = 0$, and if we let $\tau|_{\Pi_\pm}$ denote the limits of τ taken from either side of Π , the period satisfies

$$\tau|_{\Pi_+} - \tau|_{\Pi_-} = T. \quad (2.15)$$

For $p \in D_\varepsilon$, $\tau(p)$ is the travel time from Π to p along the orbits of u , up to an integer multiple of $T(x)$, where $x \in \Pi$ is the unique point on Π lying on the same orbit as p .

Recalling the period-pressure relation $\mathcal{H}_*(T) = H_*(r_*(T))$, we define a Bernoulli function H for u by

$$H = \mathcal{H}_*(T), \quad (2.16)$$

noting that $u \cdot \nabla H = \mathcal{H}'_*(T)u \cdot \nabla T = 0$ by construction. We now define

$$\omega = \nabla \tau \times \nabla H = \mathcal{H}'_*(T)\nabla \tau \times \nabla T. \quad (2.17)$$

Because τ is not continuous, a priori ω may fail to be continuous across Π . However, since u is odd with respect to Π , $u|_\Pi$ is tangent to Π . Thus, because $u \cdot \nabla T = 0$, it follows that $\nabla T \times \nabla$ involves only derivatives tangent to Π , and so the jump in ω across Π is

$$\omega|_{\Pi_+} - \omega|_{\Pi_-} = \mathcal{H}'_*(T)\nabla(\tau|_{\Pi_+} - \tau|_{\Pi_-}) \times \nabla T = 0, \quad (2.18)$$

by (2.15). As a consequence, ω is continuous across Π . We will see later on that in fact ω is more regular.

To solve the system, it remains to guarantee that with ω defined as in (2.17), we have $\text{curl} u = \omega$. The above motivates the iteration scheme described in the next section.

3 The Lortz iteration

We now describe the iteration we will use to construct solutions, following Lortz [17]. Let u_N be an odd and m -fold symmetric vector field. By Lemma 3, because u_N is odd, provided ε is sufficiently small, each integral curve γ_N of u_N is closed. We seek $\tau_N = \tau_* + \tau'_N$ satisfying

$$u_N \cdot \nabla \tau_N = 1. \quad (3.1)$$

For this, we define τ'_N by solving

$$u_N \cdot \nabla \tau'_N = (u_* - u_N) \cdot \nabla \tau_*, \quad \oint_{\gamma_N} \tau'_N ds = 0. \quad (3.2)$$

The mean-zero condition will be used later on to ensure smallness of some terms in our iteration; see Lemma 5.

We note that τ_N has a jump discontinuity across Π , but it is smooth away from Π if u_N is. Given this τ_N , we let $T_N = \tau_N|_{\Pi_+} - \tau_N|_{\Pi_-}$ denote the period of γ_N , and define H_N by

$$H_N = \mathcal{H}_*(T_N), \quad (3.3)$$

and we now define

$$\omega_N = \nabla \tau_N \times \nabla H_N = \mathcal{H}'_*(T_N) \nabla \tau_N \times \nabla T_N. \quad (3.4)$$

Then, as mentioned in the previous section, even though τ_N jumps across Π , ω_N is continuous across Π , and if $\tau_N \in C^{k,\alpha}$ away from Π then $\omega \in C^{k-1,\alpha}$ away from Π as well. We also note that τ_N is odd and H is even, so ω_N is even.

To pass to the next step of the iteration, we define u_{N+1} by solving the div-curl system

$$\operatorname{div} u_{N+1} = 0, \quad \text{in } D_\varepsilon, \quad (3.5)$$

$$\operatorname{curl} u_{N+1} = \omega_N, \quad \text{in } D_\varepsilon, \quad (3.6)$$

$$u_{N+1} \cdot n = 0, \quad \text{on } \partial D_\varepsilon. \quad (3.7)$$

This does not uniquely determine u_{N+1} , because there is a one-dimensional family $\mathbf{H}$ of harmonic vector fields on D_ε which are tangent to ∂D_ε . To get a unique solution u_{N+1} , we add the requirement that

$$\mathbb{P}_\mathbf{H} u_{N+1} = 0, \quad (3.8)$$

where $\mathbb{P}_\mathbf{H}$ denotes the L^2 -orthogonal projection onto $\mathbf{H}$. We claim the resulting u_{N+1} is odd:

Lemma 4. *Define D_ε as in (2.9) and suppose that ω_N is an even and m -fold symmetric vector field on D_ε . If u_{N+1} is the unique solution of the system (3.5)-(3.7) satisfying the condition (3.8), then u_{N+1} is odd and m -fold symmetric.*

Proof. To cut down on notation, we drop the subscripts $N, N+1$. Let $v = u + R_\Pi^* u$ where recall $R_\Pi^* u$ denotes the pullback of u by the reflection R_Π . The parity assumptions on ω and on the domain ensure that v is harmonic.

Letting $u_\mathbf{H}$ be an L^2 -normalized harmonic vector field on D_ε , we can write $\mathbb{P}_\mathbf{H} u = \int_{D_\varepsilon} u(x) u_\mathbf{H}(x) dx$. We claim that under our assumptions, $u_\mathbf{H}$ is even. Indeed, both its even and odd parts are harmonic and are therefore a multiple of $u_\mathbf{H}$ (since the space of harmonics is one-dimensional). It follows that $u_\mathbf{H}$ must be either even or odd. Since $u_\mathbf{H}$ is a perturbation of the even vector field $u_{\mathbf{H}_0} = e_z$, it follows that it is even. As a result, $\mathbb{P}_\mathbf{H} R_\Pi^* u dx = \int_{D_\varepsilon} R_\Pi^* u u_\mathbf{H} = - \int_{D_\varepsilon} u R_\Pi^* u_\mathbf{H} dx = - \int_{D_\varepsilon} u u_\mathbf{H} dx = 0$, by assumption, so both u and $R_\Pi^* u$ are orthogonal to $u_\mathbf{H}$. Thus $u + R_\Pi^* u = 0$, as needed. The m -fold symmetry follows similarly. $\square$

We now show that the sequence $\{u_N\}$ converges. Because the travel time defined in (3.1) is discontinuous across Π , and because each u_N will be odd, it is convenient to work in the domain $\tilde{D}_\varepsilon = D_\varepsilon \cap \{y \geq 0\}$ instead of D_ε . We will show that the restrictions $\tilde{u}_N = u_N|_{\tilde{D}_\varepsilon}$ converge with respect to the Hölder norms

$$\|u\|_{k,\alpha} = \|u\|_{C^{k,\alpha}(\tilde{D}_\varepsilon)}, \quad (3.9)$$

for $k \geq 2, \alpha \in (0, 1)$, provided ε is taken sufficiently small. Letting $\tilde{u}$ denote the limit of the $\tilde{u}_N$, we then extend $\tilde{u}$ to a vector field u defined in all of D_ε by parity, and the resulting vector field will be in $C^{k,\alpha}$ away from Π . We will show later on that in fact it is $C^{k,\alpha}$ across Π as well.

We now prove the needed bounds. In order to close our estimates, we will use the following Poincaré-type inequality.

Lemma 5. *Let U be a domain foliated by a family of simple, $C^{k,\alpha}$ -smooth and m -fold symmetric curves; that is, for each curve γ , there is a curve γ_0 so that $\gamma = \bigcup_{j=0}^{m-1} O_{2\pi/m}^j \gamma_0$. Let X be a $C^{k,\alpha}$ vector field on U so that $X|_\gamma$ is tangent to γ for each curve γ in the foliation, and that the X period $T(\gamma) := \int_\gamma \frac{ds}{|X|} \leq C_{\text{per}}$ is bounded uniformly. Further assume that X is m -fold symmetric in the sense that $X = (O_{2\pi/m})_* X$ where $(O_{2\pi/m})_*$ denotes the pushforward. Then, if f is an m -fold symmetric function with $\int_\gamma f ds = 0$ for each curve γ in the foliation, then there is a constant $C = C(C_{\text{per}}, \|X\|_{C^{k,\alpha}}, \|\gamma\|_{C^{k,\alpha}}, k, \alpha)$ so*

$$\|f\|_{C^{k,\alpha}(U)} \leq \frac{C}{m} \|\nabla_X f\|_{C^{k,\alpha}(U)}. \quad (3.10)$$

Proof. By the m -fold symmetry, we can write $U = \bigcup_{j=0}^{m-1} (O_{2\pi/m})^j U_0$ for a domain U_0 , and it suffices to prove the bound (3.10) with U replaced by U_0 . Fix a curve γ in the foliation and let γ_0 denote its restriction to U_0 . Let $x, y \in \gamma_0$ and let Φ_t be the flowmap for X , i.e.

$$\frac{d}{dt} \Phi_t = X \circ \Phi_t, \quad \Phi_0 = \text{id}.$$

Let $T(x, y)$ be such that $\Phi_{T(x,y)}(x) = y$. Since f is mean zero on each γ and is m -fold symmetric, f is mean zero on γ_0 , so integrating the expression $f(x) - f(y) = \int_0^{T(x,y)} (\nabla_X f) \circ \Phi_t(x) dt$ in y over γ_0 , we find

$$f(x) = \frac{1}{\text{length}(\gamma_0)} \int_{\gamma_0} \int_0^{T(x,y)} (\nabla_X f) \circ \Phi_t(x) dt dy, \quad (3.11)$$

and it follows that for any x on γ_0 ,

$$|f(x)| \leq \frac{1}{\text{length}(\gamma_0)} \int_{\gamma_0} \int_0^{T(x,y)} |(\nabla_X f) \circ \Phi_t(x)| dt dy \leq T(\gamma_0) \|\nabla_X f\|_{L^\infty}$$

where

$$T(\gamma_0) := \int_{\gamma_0} \frac{ds}{|X|} \quad (3.12)$$

is the time taken to traverse the segment γ_0 , end-to-end. By the m -fold symmetry, $T_*(\gamma_0) = \frac{1}{m} T_*(\gamma)$. Since the curves γ_0 foliate the domain U_0 , this gives the bound when $k = \alpha = 0$. We now show how to bound the first derivative, higher-order derivatives being similar. We express the line integral via its parametrization by the flow of X , namely $\Phi_t(x) : [s_1(x), s_2(x)] \rightarrow \gamma_0$:

$$f(x) = \frac{1}{\int_{s_1(x)}^{s_2(x)} |X \circ \Phi_s(x)| ds} \int_{s_1(x)}^{s_2(x)} \int_0^{T(x, \Phi_s(x))} (\nabla_X f) \circ \Phi_t(x) |X \circ \Phi_s(x)| dt ds, \quad (3.13)$$

Note that derivatives of $s_i(x)$ and $\Phi_t(x)$ are bounded by derivatives of X . Thus

$$\begin{aligned} |\nabla f(x)| \leq & \left[\frac{1}{m} T(\gamma) \|s_i\|_{C^1} \|X\|_{L^\infty} \|\nabla_X f\|_{L^\infty} + \frac{\sup_{t \in [s_1, s_2]} \|\nabla_x T(\cdot, \Phi_t(\cdot))\|_{L^\infty}}{\text{length}(\gamma_0)} \int_{\gamma_0} |\nabla_X f(y)| dy \right. \\ & \left. + \frac{\sup_t \|\nabla \Phi_t(\cdot)\|_{L^\infty}}{\text{length}(\gamma_0)} \int_{\gamma_0} \int_0^{T(x,y)} |\nabla \nabla_X f \circ \Phi_t(x)| dt dy \right]. \end{aligned}$$

Note that, if f is m -fold symmetric, then so is $\nabla_X f$ and, moreover, it is mean zero on each γ_0 . As such, we can apply the estimate for $\|f\|_{L^\infty}$ to arrive at $\|\nabla_X f\|_{L^\infty} \lesssim 1/m \|\nabla \nabla_X f\|_{L^\infty}$. Likewise, the last term is bounded by the same argument. This concludes the sketch of the argument. Higher derivatives follow similarly. $\square$

We now apply this lemma to $U = \tilde{D}_\varepsilon$ with $X = u_N$. The period function of u_N satisfies all assumptions of the above theorem, because u_N is odd, close to u_* and vanishes along the axis. Since $\oint_{\gamma_N} \tau_N ds = 0$ for each field line γ_N , taking m even, it follows from the m -fold symmetry that $\int_{\gamma_N \cap \tilde{D}_\varepsilon} \tau_N ds = 0$, and so by (3.10) and the equation (3.2), τ'_N satisfies

$$\|\tau'_N\|_{k,\alpha} \leq \frac{C}{m} \|u'_N\|_{k,\alpha} \|\nabla \tau_*\|_{k,\alpha}, \quad (3.14)$$

for a constant $C > 0$, where $u'_N = u_N - u_*$ is the perturbation of the velocity field. Since $T_N = \tau_N|_{\Pi_+} - \tau_N|_{\Pi_-}$, we also have that $T'_N = T_N - T_*$ satisfies

$$\|T'_N\|_{k,\alpha} \leq \frac{C'}{m} \|u'_N\|_{k,\alpha} \|\nabla \tau_*\|_{k,\alpha}, \quad (3.15)$$

for a constant $C' > 0$, and it follows that with $\omega'_N = \omega_N - \omega_*$ satisfies

$$\|\omega'_N\|_{k,\alpha} \lesssim \|T_N\|_{k+1,\alpha} \|\tau'_N\|_{k+1,\alpha} + \|T'_N\|_{k+1,\alpha} \|\tau_N\|_{k+1,\alpha} \lesssim \frac{1}{m} \|u'_N\|_{k+1,\alpha}, \quad (3.16)$$

provided $\|u'_N\|_{k+1,\alpha} \leq 1$, say, where the implicit constant depends on k, α, u_* and $\mathcal{H}_*$, but not on m . If we now define u_{N+1} by solving (3.5)–(3.8), the perturbed velocity field $u'_{N+1} = u_{N+1} - u_*$ satisfies the system

$$\operatorname{div} u'_{N+1} = 0, \quad \text{in } D_\varepsilon, \quad (3.17)$$

$$\operatorname{curl} u'_{N+1} = \omega'_N \quad \text{in } D_\varepsilon, \quad (3.18)$$

$$u'_{N+1} \cdot n = -\varepsilon u_* \cdot \nabla g, \quad \text{on } \partial D_\varepsilon \quad (3.19)$$

and so by standard elliptic estimates (see e.g. [1]), it satisfies the bounds

$$\|u'_{N+1}\|_{k+1,\alpha} \lesssim \|\omega'_N\|_{k,\alpha} + \varepsilon \|g\|_{C^{k+1,\alpha}(\mathbb{S}^1 \times \mathbb{S}^1)} \quad (3.20)$$

$$\lesssim \frac{1}{m} \|u'_N\|_{k+1,\alpha} + \varepsilon \|g\|_{C^{k+1,\alpha}(\mathbb{S}^1 \times \mathbb{S}^1)}. \quad (3.21)$$

It follows that there is a constant $M > 0$ and m_1, ε_1 so that if $m > m_1$ and $\varepsilon < \varepsilon_1$, then the bound $\|u'_N\|_{k+1,\alpha} \leq M$ implies the same bound for $\|u_{N+1}\|_{k+1,\alpha}$. In the same way, we find

$$\|u'_{N+1} - u'_N\|_{k+1,\alpha} \lesssim \frac{1}{m} \|u'_N - u'_{N-1}\|_{k+1,\alpha}, \quad (3.22)$$

for an implicit constant independent of ε and m , if $\|u_N\|_{k+1,\alpha}, \|u'_{N+1}\|_{k+1,\alpha} \leq M$. It follows that, taking $\varepsilon, 1/m$ smaller if needed, the sequence $\{u_N\}$ converges in $C^{k+1,\alpha}(\tilde{D}_\varepsilon)$. The limit $\tilde{u}$ satisfies (1.3) in $\tilde{D}_\varepsilon$, and we extend $\tilde{u}$ to all of D_ε by $u(p) = (R_\Pi \tilde{u})(R_\Pi p)$, whenever $p \in D_\varepsilon \setminus \tilde{D}_\varepsilon$, noting that the reflection R_Π maps $D_\varepsilon \setminus \tilde{D}_\varepsilon$ to $\tilde{D}_\varepsilon$. Then u is a $C^{k+1,\alpha}$ solution of (1.3) away from Π , and by construction it has continuous vorticity across Π .

We now claim that in fact $u \in C^{k+1,\alpha}$ everywhere. Indeed, since $\omega \in C^{k,\alpha}$ up to Π and since it is continuous across Π , it follows that ω is actually Lipschitz across Π , and since u satisfies the div-curl system

$$\operatorname{div} u = 0, \quad \text{in } D_\varepsilon, \quad (3.23)$$

$$\operatorname{curl} u = \omega, \quad \text{in } D_\varepsilon \quad (3.24)$$

$$u \cdot n = 0, \quad \text{on } \partial D_\varepsilon, \quad (3.25)$$

it follows that $u \in C^{1,\alpha}(D_\varepsilon)$ (in fact, $u \in C^{1,1-}(D_\varepsilon)$). Since u satisfies (1.3) by construction, it follows that u is a strong solution of the steady Euler equations (1.1) and thus that the steady vorticity equation

$$u \cdot \nabla \omega = \omega \cdot \nabla u \quad (3.26)$$

holds. By construction, ω is $C^{k,\alpha}$ regular in the directions tangent to Π but so far we have only shown that it is continuous across Π . Recalling that u is normal to Π at Π , from (3.26)

it follows from the above that $u \cdot \nabla \omega \in C^{0,\alpha}$ across Π , and so we have shown that $\omega \in C^{1,\alpha}$ globally. This then implies $u \in C^{2,\alpha}$, and repeatedly applying this argument shows that $u \in C^{k,\alpha}$, as needed. Since the above holds for any k , this completes the proof of Thm. 1.

Acknowledgements. We are grateful to Peter Constantin for his guidance and for sharing with us his vision of fluid dynamics and PDE. We thank D. Peralta Salas for point out the work of Salat–Kaiser [18, 15]. We thank also the members of the Simons collaboration for Hidden Symmetries and Fusion Energy for discussion and insight on Grad’s conjecture. The work of TDD was supported by the NSF CAREER award #2235395, a Stony Brook University Trustee’s award as well as an Alfred P. Sloan Fellowship. The work of TME was supported by a Simons Fellowship and the NSF grants DMS-2043024 and DMS-2510472. The work of DG was supported by a startup grant from Brooklyn College, PSC-CUNY grant TRADB-55-214, and the NSF grant DMS-2406852.

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