Spectral extremal problems for the (p, Q)-spectral radius of hypergraphs

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Abstract

Let Q be an s-vertex r-uniform hypergraph, and let H be an n-vertex r-uniform hypergraph. Denote by $\mathcal{N}(Q, H)$ the number of isomorphic copies of Q in H. For a hereditary family $\mathcal{P}$ of r-uniform hypergraphs, define

$$\pi(Q, \mathcal{P}) := \lim_{n\to\infty} \binom{n}{s}^{-1} \max\{\mathcal{N}(Q, H) : H \in \mathcal{P} \text{ and } |V(H)| = n\}.$$

For $p \geq 1$, the (p, Q)-spectral radius of H is defined as

$$\lambda^{(p)}(Q, H) := \max_{\|\mathbf{x}\|_p=1} s! \sum_{\{i_1,\ldots,i_s\}\in\binom{[n]}{s}} \mathcal{N}(Q, H[\{i_1, \ldots, i_s\}])x_{i_1} \cdots x_{i_s}.$$

In this paper, we present a systematically investigation of the parameter $\lambda^{(p)}(Q, H)$. First, we prove that the limit

$$\lambda^{(p)}(Q, \mathcal{P}) := \lim_{n\to\infty} n^{s/p-s} \max\{\lambda^{(p)}(Q, H) : H \in \mathcal{P} \text{ and } |V(H)| = n\}$$

exists, and for $p > 1$, it satisfies

$$\pi(Q, \mathcal{P}) = \lambda^{(p)}(Q, \mathcal{P}).$$

Second, we study spectral generalized Turán problems. Specifically, we establish a spectral stability result and apply it to derive a spectral version of the Erdős Pentagon Problem: for $p \geq 1$ and sufficiently large n, the balanced blow-up of C_5 maximizes $\lambda^{(p)}(C_5, H)$ among all n-vertex triangle-free graphs H, thereby improving a result of Liu [12]. Furthermore, we show that for $p \geq 1$ and sufficiently large n, the l-partite Turán graph $T_l(n)$ attains the maximum $\lambda^{(p)}(K_s, H)$ among all n-vertex F-free graphs H, where F is an edge-critical graph with $\chi(F) = l + 1$. This provides a spectral analogue of a theorem due to Ma and Qiu [14].

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1 Introduction

A *hypergraph* $H = (V(H), E(H))$ consists of a vertex set $V(H) = \{v_1, v_2, \cdots, v_n\}$ and an edge set $E(H) = \{e_1, e_2, \cdots, e_m\}$, where $e_i \subseteq V$ for $i \in [m] := \{1, 2, \ldots, m\}$. The *order* and *size* of H are defined as $\nu(H) := |V(H)|$ and $e(H) := |E(H)|$, respectively. If $|e_i| = r$ for each $i \in [m]$ and $r \geq 2$, then H is called an *r-uniform* hypergraph (or r-graph). A simple graph is exactly a 2-uniform hypergraph. Given $I \subseteq V(H)$, the subgraph of H with I as vertex set and $\{e \in E(H) : e \subseteq I\}$ as edge set is denoted by $H[I]$ (called *induced* by I). For any vertex $v \in V(H)$, we write $H - v$ for the subgraph of H induced by $V(H)\backslash\{v\}$. For $l \geq r \geq 2$, an r-graph is called *l-partite* if its vertex set can be divided into l parts such that each edge has at most one vertex from each part. An edge-maximal l-partite r-graph is called *complete l-partite*. Let $T_l^r(n)$ be the complete l-partite r-graph on n vertices without two part sizes differing by more than one; when $r = 2$, the graph $T_l^2(n)$ is Turán graph $T_l(n)$.

Given an s-vertex r-graph Q and an r-graph H, let $\mathcal{N}(Q, H)$ denote the number of isomorphic copies of Q in H. For example, for the complete r-graph K_s^r on s vertices, we have $\mathcal{N}(Q, K_s^r) = \frac{s!}{|Aut(Q)|}$, where $Aut(Q)$ is the automorphism group of r-graph Q. For a family $\mathcal{F}$ of r-graphs, we say a hypergraph G is *$\mathcal{F}$-free* if G does not contain any member of $\mathcal{F}$ as a subgraph. The generalized Turán number $ex(n, Q, \mathcal{F})$ is the largest $\mathcal{N}(Q, H)$ among all the n-vertex $\mathcal{F}$-free r-graphs H. The function $ex(n, Q, \mathcal{F})$ is a well-studied parameter; a comprehensive survey can be found in [6]. Let $E(Q, H)$ denote the collection of all s-subsets I of $V(H)$ such that $\mathcal{N}(Q, H[I]) > 0$, and define $E_{Q,H}(v) = \{I \in E(Q, H) : v \in I\}$. The *$Q$-degree* of v, denoted $d_{Q,H}(v)$, is given by

$$d_{Q,H}(v) = \sum_{I \in E_{Q,H}(v)} \mathcal{N}(Q, H[I]).$$

The minimum Q-degree of H is denoted by $\delta_Q(H)$.

Let $p \geq 1$, Q be an s-vertex r-graph and H be an n-vertex r-graph, where $r \leq s \leq n$. The *Q-Lagrangian polynomial* $P_{Q,H}(\mathbf{x})$ of H is defined as

$$\begin{aligned} P_{Q,H}(\mathbf{x}) &= s! \sum_{\{i_1,\ldots,i_s\} \in \binom{[n]}{s}} \mathcal{N}(Q, H[\{i_1, \ldots, i_s\}]) x_{i_1} \cdots x_{i_s} \\ &= s! \sum_{\{i_1,\ldots,i_s\} \in E(Q,H)} \mathcal{N}(Q, H[\{i_1, \ldots, i_s\}]) x_{i_1} \cdots x_{i_s}, \end{aligned}$$

and the *(p, Q)-spectral radius* $\lambda^{(p)}(Q, H)$ of H is defined as

$$\lambda^{(p)}(Q, H) = \max_{\|\mathbf{x}\|_p = 1} P_{Q,H}(\mathbf{x}),$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\|\mathbf{x}\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$. It is noteworthy that the definition of (p, Q)-spectral radius was recently introduced by Liu [12], and our definition here differs from Liu's by a constant factor $|Aut(Q)|$. When $Q = K_s^r$, we abbreviate $\lambda^{(p)}(Q, H)$ as $\lambda_s^{(p)}(H)$, termed the *s-clique p-spectral radius* [11] of H. If, further $Q = K_r^r$, we simple write $\lambda^{(p)}(H)$, recovering the p-spectral radius of H introduced by Keevash, Lenz, and Mubayi [8]. If $\mathbf{x} \in \mathbb{R}^n$ is a vector such that $\|\mathbf{x}\|_p = 1$ and $\lambda^{(p)}(Q, H) = P_{Q,H}(\mathbf{x})$, then $\mathbf{x}$ is called a *Q-eigenvector* of H corresponding to $\lambda^{(p)}(Q, H)$. Clearly, there always exists a nonnegative Q-eigenvector corresponding to $\lambda^{(p)}(Q, H)$, called a *principal Q-eigenvector* of H. Moreover, if a principal Q-eigenvector $\mathbf{x}$ is strictly positive (i.e., $x_v > 0$ for all $v \in V(H)$), then we call it a *Perron-Frobenius Q-eigenvector* of H.

A property of r-graphs is a family of r-graphs closed under isomorphisms. For a property $\mathcal{P}$, denoted by $\mathcal{P}_n$ the collection of r-graphs in $\mathcal{P}$ of order n. A property is called *hereditary* if it is closed under taking induced subgraphs. Given a family $\mathcal{F}$ of r-graphs, the class of all $\mathcal{F}$-free r-graphs forms a hereditary property, denoted by $\overline{\mathcal{F}}$. Throughout our discussion, we assume that for any hereditary property $\mathcal{P}$ of r-graphs, the disjoint union of H and an isolated vertex belongs to $\mathcal{P}$. Given two r-graphs Q and H, a map ϕ: $V(Q) \to V(H)$ is a *homomorphism* from Q to H if $\phi(e) \in E(H)$ for all $e \in E(Q)$. We say Q is *H-colorable* if there is a homomorphism from Q to H.

A fundamental problem in extremal combinatorics can be formulated as follows: Given an s-vertex r-graph Q and a hereditary property $\mathcal{P}$ of r-graphs, determine the extremal function

$$ex(Q, \mathcal{P}_n) := \max_{H \in \mathcal{P}_n} \mathcal{N}(Q, H).$$

By Katona-Nemetz-Simonovits averaging argument [9], the ratio $ex(Q, \mathcal{P}_n)/\binom{n}{s}$ is decreasing in n, and so the limit

$$\pi(Q, \mathcal{P}) := \lim_{n \to \infty} \frac{ex(Q, \mathcal{P}_n)}{\binom{n}{s}}$$

always exists, called the *Q-density* of $\mathcal{P}$. If $\mathcal{P} = \overline{\mathcal{F}}$ for a family $\mathcal{F}$ of r-graphs, then $ex(K_r^r, \mathcal{P}_n)$ and $\pi(K_r^r, \mathcal{P})$ are the *Turán number* and *Turán density* of $\mathcal{F}$, respectively. To maintain consistency in notation, we will use $ex(Q, \overline{\mathcal{F}}_n)$ instead of $ex(n, Q, \mathcal{F})$ in the remaining part.

Similarly, we can study the spectral analogue of the aforementioned problem. For an s-vertex r-graph Q and a hereditary property $\mathcal{P}$ of r-graphs, we define

$$\lambda^{(p)}(Q, \mathcal{P}_n) := \max_{H \in \mathcal{P}_n} \lambda^{(p)}(Q, H),$$

and the *(p, Q)-spectral density* of $\mathcal{P}$ is defined as

$$\lambda^{(p)}(Q, \mathcal{P}) := \lim_{n \to \infty} \frac{\lambda^{(p)}(Q, \mathcal{P}_n)}{n^{s - s/p}}.$$

In [17], Nikiforov conducted a systematic study of the p-spectral radius of hypergraphs using analytical methods, and proved that $\pi(K_r^r, \mathcal{P}) = \lambda^{(p)}(K_r^r, \mathcal{P})$ holds for any $p > 1$ and any hereditary property $\mathcal{P}$ of r-graphs. Liu and Bu [11] introduced the s-clique spectral radius of a graph G (equivalent to $\lambda_s^{(s)}(G)$), and extended the spectral Mantel's theorem via the clique tensor. Yu and Peng [21] gave a spectral version of the generalized Erdős-Gallai theorem via the clique tensor. In [12], Liu established a general theorem that extends the result of Keevash-Lenz-Mubayi and obtained a spectral Erdős pentagon theorem.

In this paper, we investigate spectral extremal problems concerning the (p, Q)-spectral radius of hypergraphs. For any hereditary property $\mathcal{P}$ of r-graphs, we prove that the (p, Q)-*spectral density* of $\mathcal{P}$ exists for all $p \geq 1$. Moreover, we show that the Q-density of $\mathcal{P}$ coincides with its (p, Q)-spectral density when $p > 1$. Furthermore, we study spectral generalized Turán problems. In particular, we establish a spectral stability result: if the maximum (p, Q)-spectral radius among all $\mathcal{F}$-free r-graphs satisfies a specific growth condition, then the extremal hypergraphs must possess a large minimum Q-degree. As an application, we derive a spectral analogue of the Erdős Pentagon Problem: for any $p \geq 1$ and all sufficiently large n, the balanced blowup of C_5 attains the maximal (p, C_5)-spectral radius over all n-vertex triangle-free graphs. This extends the result of Liu [12]. Additionally, we demonstrate that for $p \geq 1$ and n sufficiently large, the l-partite Turán graph $T_l(n)$ achieves the maximum s-clique p-spectral radius among all n-vertex F-free graphs, where F is an edge-critical graph with $\chi(F) = l + 1$. This establishes a spectral counterpart to the result of Ma and Qiu [14] and extends a theorem of Yu and Peng [21].

2 Preliminaries

In this section, we present some properties of the parameter $\lambda^{(p)}(Q, H)$. Hereafter, when given an s-vertex r-graph Q and an n-vertex r-graph H, it is always assumed that $n \geq s \geq r \geq 2$, provided no ambiguity arises.

Proposition 2.1. *Let Q be an s-vertex r-graph and H be an n-vertex r-graph. If $p \geq 1$, then $\lambda^{(p)}(Q, H)$ is an increasing and continuous function in p. Moreover,*

$$\lim_{p \to \infty} \lambda^{(p)}(Q, H) = s! \mathcal{N}(Q, H).$$

Proof. Since $\lambda^{(p)}(Q, H)$ always has a nonnegative Q-eigenvector, we obtain the following equivalent definition of $\lambda^{(p)}(Q, H)$:

$$\lambda^{(p)}(Q, H) = \max_{|x_1|+\cdots+|x_n|=1} s! \sum_{\{i_1,\ldots,i_s\} \in E(Q,H)} \mathcal{N}(Q, H[\{i_1, \ldots, i_s\}]) |x_{i_1}|^{1/p} \cdots |x_{i_s}|^{1/p}, \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Note that $0 \leq |x_{i_1}| \cdots |x_{i_s}| \leq 1$. We now claim that for any $b \geq a \geq 1$,

$$0 \leq |x_{i_1}|^{1/b} \cdots |x_{i_s}|^{1/b} - |x_{i_1}|^{1/a} \cdots |x_{i_s}|^{1/a} \leq b - a.$$

Observe that the left inequality holds trivially, and when $|x_{i_1}| \cdots |x_{i_s}| = 0$ or 1, the right inequality also holds. For $0 < |x_{i_1}| \cdots |x_{i_s}| < 1$, applying the Mean Value Theorem we know that there exists $\xi \in (a, b)$ such that

$$\begin{aligned}
|x_{i_1}|^{1/b} \cdots |x_{i_s}|^{1/b} - |x_{i_1}|^{1/a} \cdots |x_{i_s}|^{1/a} &= (b - a)\xi^{-2}(|x_{i_1}| \cdots |x_{i_s}|)^{\xi^{-1}} \ln(|x_{i_1}| \cdots |x_{i_s}|)^{-1} \\
&\leq (b - a)(|x_{i_1}| \cdots |x_{i_s}|)^{\xi^{-1}-\xi^{-2}} \\
&\leq b - a.
\end{aligned}$$

So the claim is confirmed.

Let $\mathbf{y} = (y_1, \ldots, y_n)$ be a nonnegative vector such that equality (1) holds for $\lambda^{(a)}(Q, H)$. Then,

$$\lambda^{(b)}(Q, H) - \lambda^{(a)}(Q, H) \geq s! \sum_{\{i_1,\ldots,i_s\} \in E(Q,H)} \mathcal{N}(Q, H[\{i_1, \ldots, i_s\}])(y_{i_1}^{1/b} \cdots y_{i_s}^{1/b} - y_{i_1}^{1/a} \cdots y_{i_s}^{1/a}) \geq 0.$$

This implies that $\lambda^{(p)}(Q, H)$ is increasing in p.

Now, let $\mathbf{z} = (z_1, \ldots, z_n)$ be a nonnegative vector such that equality (1) holds for $\lambda^{(b)}(Q, H)$. Then,

$$\begin{aligned}
0 \leq \lambda^{(b)}(Q, H) - \lambda^{(a)}(Q, H) &\leq s! \sum_{\{i_1,\ldots,i_s\} \in E(Q,H)} \mathcal{N}(Q, H[\{i_1, \ldots, i_s\}])(z_{i_1}^{1/b} \cdots z_{i_s}^{1/b} - z_{i_1}^{1/a} \cdots z_{i_s}^{1/a}) \\
&\leq (b - a)s! \mathcal{N}(Q, H).
\end{aligned}$$

Therefore, $\lambda^{(p)}(Q, H)$ satisfies the Lipschitz condition and is thus continuous.

By the definition of $\lambda^{(p)}(Q,H)$, it is evident that $\lambda^{(p)}(Q,H) \le s!\mathcal{N}(Q,H)$. On the other hand, taking the n-vector $\mathbf{x} = (n^{-1/p}, \dots, n^{-1/p})$ yields

$$\lambda^{(p)}(Q,H) \ge P_{Q,H}(\mathbf{x}) = s!\mathcal{N}(Q,H)/n^{s/p}.$$

Thus, we obtain

$$s!\mathcal{N}(Q,H)/n^{s/p} \le \lambda^{(p)}(Q,H) \le s!\mathcal{N}(Q,H),$$

which implies $\lim\limits_{p\to\infty} \lambda^{(p)}(Q,H) = s!\mathcal{N}(Q,H)$. This completes the proof. □

For a vertex subset $U \subseteq V(H)$ of an n-vertex r-graph H, we write $x_U = \Pi_{v\in U} x_v$. For $p > 1$, the principal Q-eigenvector $\mathbf{x} = (x_1, \dots, x_n)$ of H satisfies the following system of eigenequations derived from Lagrange's method:

$$\lambda^{(p)}(Q,H) x_i^{p-1} = (s-1)! \sum_{I \in E_{Q,H}(v)} \mathcal{N}(Q, H[I]) x_{I\setminus\{v\}}, \quad i = 1, 2, \dots, n. \tag{2}$$

Lemma 2.2. *Let $p \ge 1$, and let Q be an s-vertex r-graph and H be an n-vertex r-graph. Then the function*

$$f_{Q,H}(p) = \left(\frac{\lambda^{(p)}(Q,H)}{s!\mathcal{N}(Q,H)}\right)^p$$

is decreasing in p.

Proof. Set $\beta \ge \alpha \ge 1$ and $\mathcal{N} := \mathcal{N}(Q,H)$. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a principal Q-eigenvector corresponding to $\lambda^{(\beta)}(Q,H)$. Using Power-Mean inequality, we obtain

$$\frac{\lambda^{(\beta)}(Q,H)}{s!\mathcal{N}} = \frac{1}{\mathcal{N}} \sum_{I \in E(Q,H)} \mathcal{N}(Q,H[I]) x_I \le \left(\frac{1}{\mathcal{N}} \sum_{I \in E(Q,H)} \mathcal{N}(Q,H[I]) (x_I)^{\beta/\alpha}\right)^{\alpha/\beta}.$$

Note that

$$\left(x_1^{\beta/\alpha}\right)^\alpha + \cdots + \left(x_n^{\beta/\alpha}\right)^\alpha = x_1^\beta + \cdots + x_n^\beta = 1.$$

Thus, we have

$$\frac{1}{\mathcal{N}} \sum_{I \in E(Q,H)} \mathcal{N}(Q,H[I]) (x_I)^{\beta/\alpha} \le \frac{1}{s!\mathcal{N}} \lambda^{(\alpha)}(Q,H),$$

and so

$$\left(\frac{\lambda^{(\beta)}(Q,H)}{s!\mathcal{N}}\right)^\beta \le \left(\frac{\lambda^{(\alpha)}(Q,H)}{s!\mathcal{N}}\right)^\alpha,$$

completing the proof. □

We conclude this section with the following obvious result.

Proposition 2.3. *Let $p \ge 1$, and let Q be an s-vertex r-graph and H be an n-vertex r-graph. If G is a subgraph of H, then $\lambda^{(p)}(Q,G) \le \lambda^{(p)}(Q,H)$.*

3 Extremal (p, Q)-spectral radius of hereditary families

In this section, we show that for any hereditary property $\mathcal{P}$ of r-graphs, the Q-density of $\mathcal{P}$ is equal to its (p, Q)-spectral density when $p > 1$, namely $\pi(Q, \mathcal{P}) = \lambda^{(p)}(Q, \mathcal{P})$ for $p > 1$. We then investigate the (p, Q)-spectral radius of hereditary families which satisfy $\pi(Q, \mathcal{P}) = \lambda^{(1)}(Q, \mathcal{P})$.

Fact 3.1 ([23]). *If $p > 1$ and $s \geq 2$, then the function*

$$f(x) = \frac{1 - sx}{(1 - x)^{s/p}}$$

is decreasing for $0 \leq x < 1$.

For a vector $\mathbf{x} \in \mathbb{R}^n$, we use the notation $\mathbf{x}_{\min}$ to represent the smallest element in the vector $\mathbf{x}$.

Theorem 3.2. *Let $p \geq 1$, and let Q be an r-graph on s vertices. If $\mathcal{P}$ is a hereditary property of r-graphs, then the limit*

$$\lambda^{(p)}(Q, \mathcal{P}) = \lim_{n \to \infty} \lambda^{(p)}(Q, \mathcal{P}_n) n^{s/p - s}$$

exists. If $p = 1$, then $\lambda^{(1)}(Q, \mathcal{P}_n)$ is increasing, and so

$$\lambda^{(1)}(Q, \mathcal{P}_n) \leq \lambda^{(1)}(Q, \mathcal{P}).$$

If $p > 1$, then $\lambda^{(p)}(Q, \mathcal{P})$ satisfies

$$\lambda^{(p)}(Q, \mathcal{P}) \leq \frac{\lambda^{(p)}(Q, \mathcal{P}_n) n^{s/p}}{(n)_s},$$

where $(n)_s = n(n-1) \cdots (n - s + 1)$.

Proof. Let $H \in \mathcal{P}_n$ be an r-graph satisfying $\lambda^{(p)}(Q, H) = \lambda^{(p)}(Q, \mathcal{P}_n)$, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a principal Q-eigenvector corresponding to $\lambda^{(p)}(Q, H)$. By previous assumption on hereditary properties, we have

$$\lambda^{(p)}(Q, \mathcal{P}_n) \leq \lambda^{(p)}(Q, H + u) \leq \lambda^{(p)}(Q, \mathcal{P}_{n+1}),$$

where $u \notin V(H)$ and $H + u \in \mathcal{P}_{n+1}$ is an r-graph with vertex set $V(H + u) = V(H) \cup \{u\}$ and edge set $E(H + u) = E(H)$. Thus, $\lambda^{(p)}(Q, \mathcal{P}_n)$ is increasing in n.

Recall that $\mathcal{N}(Q, K_s^r) = \frac{s!}{|Aut(Q)|} \leq s!$. For $p = 1$, by Maclaurin's inequality, we have

$$\lambda^{(1)}(Q, \mathcal{P}_n) \leq s! \sum_{\{i_1, \ldots, i_s\} \in \binom{[n]}{s}} s! x_{i_1} \cdots x_{i_s} \leq s!(x_1 + \cdots + x_n)^s = s!.$$

Thus, the sequence $\left\{\lambda^{(1)}(Q, \mathcal{P}_n)\right\}_{n=1}^{\infty}$ converges to a limit λ, and we conclude

$$\lambda = \lim_{p \to \infty} \lambda^{(1)}(Q, \mathcal{P}_n) n^{s-s} = \lambda^{(1)}(Q, \mathcal{P}).$$

For $p > 1$, let $k \in V(H)$ be a vertex with $x_k = \mathbf{x}_{\min}$, and let $\mathbf{x}'$ be the $(n-1)$-vector obtained from $\mathbf{x}$ by removing the component x_k. By (2), we have

$$P_{Q, H-k}(\mathbf{x}') = \lambda^{(p)}(Q, H) - s! x_k \sum_{I \in E_{Q,H}(k)} \mathcal{N}(Q, H[I]) x_{I \setminus \{k\}} = \lambda^{(p)}(Q, \mathcal{P}_n) - s \lambda^{(p)}(Q, \mathcal{P}_n) x_k^p.$$

Since $\mathcal{P}$ is hereditary, $H-k\in\mathcal{P}_{n-1}$. Therefore,

$$\lambda^{(p)}(Q,\mathcal{P}_n)(1-sx_k^p)=P_{Q,H-k}(\mathbf{x}')\le\lambda^{(p)}(Q,H-k)(\|\mathbf{x}'\|_p^s)\le\lambda^{(p)}(Q,\mathcal{P}_{n-1})(1-x_k^p)^{s/p},$$

or equivalently,

$$\frac{\lambda^{(p)}(Q,\mathcal{P}_{n-1})}{\lambda^{(p)}(Q,\mathcal{P}_n)}\ge\frac{1-sx_k^p}{(1-x_k^p)^{s/p}}. \tag{3}$$

Noting that $(\mathbf{x}_{\min})^p\le 1/n$, by (3) and Fact 3.1, we have

$$\frac{\lambda^{(p)}(Q,\mathcal{P}_{n-1})}{\lambda^{(p)}(Q,\mathcal{P}_n)}\ge\frac{1-s(\mathbf{x}_{\min})^p}{(1-(\mathbf{x}_{\min})^p)^{s/p}}\ge\frac{1-s/n}{(1-1/n)^{s/p}}.$$

This implies that

$$\frac{\lambda^{(p)}(Q,\mathcal{P}_{n-1})(n-1)^{s/p}}{(n-1)_s}\ge\frac{\lambda^{(p)}(Q,\mathcal{P}_n)n^{s/p}}{(n)_s}.$$

Therefore, the sequence $\left\{\frac{\lambda^{(p)}(Q,\mathcal{P}_n)n^{s/p}}{(n)_s}\right\}_{n=1}^{\infty}$ is decreasing and hence convergent. This completes the proof. $\square$

3.1 The equivalence of $\lambda^{(p)}(Q,\mathcal{P})$ and $\pi(Q,\mathcal{P})$

Given a hereditary property $\mathcal{P}$ of r-graphs and an r-graph Q on s vertices. For $H\in\mathcal{P}_n$ with $\mathcal{N}(Q,H)=ex(Q,\mathcal{P}_n)$, the n-vector $\mathbf{x}=(n^{-1/p},\dots,n^{-1/p})$ yields

$$\lambda^{(p)}(Q,H)\ge P_{Q,H}(\mathbf{x})=s!\mathcal{N}(Q,H)/n^{s/p}=s!ex(Q,\mathcal{P}_n)/n^{s/p}. \tag{4}$$

Thus

$$\lambda^{(p)}(Q,\mathcal{P}_n)\ge\lambda^{(p)}(Q,H)\ge s!ex(Q,\mathcal{P}_n)/n^{s/p},$$

which implies

$$\frac{\lambda^{(p)}(Q,\mathcal{P}_n)n^{s/p}}{(n)_s}\ge\frac{ex(Q,\mathcal{P}_n)}{\binom{n}{s}}.$$

Taking $n\to\infty$ and applying Theorem 3.2, we obtain for $p\ge 1$,

$$\lambda^{(p)}(Q,\mathcal{P})\ge\pi(Q,\mathcal{P}). \tag{5}$$

We now state one of our main results: we show that for $p>1$, equality in inequality (5) always holds. This significantly extends the result of Nikiforov [17, Theorem 12].

Theorem 3.3. *If Q is an r-graph and $\mathcal{P}$ is a hereditary property of r-graphs, then for every $p>1$,*

$$\lambda^{(p)}(Q,\mathcal{P})=\pi(Q,\mathcal{P}).$$

In the following, we present several lemmas necessary for the proof of Theorem 3.3.

Lemma 3.4. *Let $p>1$, and let Q be an r-graph on s vertices and $\mathcal{P}$ be a hereditary property of r-graphs with $\lambda^{(p)}(Q,\mathcal{P})>0$. If $\lambda_n^{(p)}:=\lambda^{(p)}(Q,\mathcal{P}_n)$, then there exist infinitely many n such that*

$$\frac{\lambda_{n-1}^{(p)}(n-1)^{s/p}}{(n-1)_s}-\frac{\lambda_n^{(p)}n^{s/p}}{(n)_s}<\frac{1}{n\log n}\cdot\frac{\lambda_n^{(p)}n^{s/p}}{(n)_s}.$$

Proof. Assume for a contradiction that there exists n_0 such that for all $n \geq n_0$,

$$\frac{\lambda^{(p)}_{n-1}(n-1)^{s/p}}{(n-1)_s} - \frac{\lambda^{(p)}_n n^{s/p}}{(n)_s} \geq \frac{1}{n\log n} \cdot \frac{\lambda^{(p)}_n n^{s/p}}{(n)_s}.$$

Summing the inequalities for all $n_0, n_0+1, \ldots, k$, we get

$$\begin{aligned}
\frac{\lambda^{(p)}_{n_0-1}(n_0-1)^{s/p}}{(n_0-1)_s} - \frac{\lambda^{(p)}_k k^{s/p}}{(k)_s} &= \sum_{n=n_0}^{k} \left(\frac{\lambda^{(p)}_{n-1}(n-1)^{s/p}}{(n-1)_s} - \frac{\lambda^{(p)}_n n^{s/p}}{(n)_s} \right) \\
&\geq \sum_{n=n_0}^{k} \frac{1}{n\log n} \cdot \frac{\lambda^{(p)}_n n^{s/p}}{(n)_s} \\
&\geq \lambda^{(p)}(Q,\mathcal{P}) \sum_{n=n_0}^{k} \frac{1}{n\log n},
\end{aligned}$$

where the last inequality follows from Theorem 3.2. Note that the left-hand side is bounded and the right-hand side diverges. Taking k sufficiently large, we come to a contradiction. □

Lemma 3.5. *Let $p > 1$, and let Q be an r-graph on s vertices and $\mathcal{P}$ be a hereditary property of r-graphs with $\lambda^{(p)}(Q,\mathcal{P}) > 0$. Suppose that $H_n \in \mathcal{P}_n$ is an r-graph satisfying $\lambda^{(p)}(Q,H_n) = \lambda^{(p)}(Q,\mathcal{P}_n)$ and $\mathbf{x} = (x_1,\ldots,x_n)$ is a principal Q-eigenvector corresponding to $\lambda^{(p)}(Q,H_n)$. Then there exist infinitely many n such that*

$$(\mathbf{x}_{\min})^p \geq \frac{1}{n}\Big(1 - \frac{p}{(p-1)s\log n}\Big).$$

Proof. Assume, for contradiction, that there exists n_0 such that for all $n > n_0$,

$$(\mathbf{x}_{\min})^p < \frac{1}{n}\Big(1 - \frac{p}{(p-1)s\log n}\Big). \tag{6}$$

Set $\lambda^{(p)}_n := \lambda^{(p)}(Q,\mathcal{P}_n)$. By Lemma 3.4, we can select sufficiently large $n > n_0$ such that

$$\frac{\lambda^{(p)}_{n-1}(n-1)^{s/p}}{(n-1)_s} - \frac{\lambda^{(p)}_n n^{s/p}}{(n)_s} < \frac{1}{n\log n} \cdot \frac{\lambda^{(p)}_n n^{s/p}}{(n)_s},$$

and so

$$\frac{\lambda^{(p)}_{n-1}}{\lambda^{(p)}_n} < \frac{n^{s/p-1}(n-s)}{(n-1)^{s/p}}\Big(1 + \frac{1}{n\log n}\Big). \tag{7}$$

Let $k \in V(H_n)$ be a vertex with $x_k = \mathbf{x}_{\min}$. Then, by (3) and (7), we obtain

$$\frac{1 - sx_k^p}{(1-x_k^p)^{s/p}} \leq \frac{\lambda^{(p)}_{n-1}}{\lambda^{(p)}_n} \leq \frac{n^{s/p-1}(n-s)}{(n-1)^{s/p}}\Big(1 + \frac{1}{n\log n}\Big).$$

Applying Fact 3.1 and (6), we derive

$$\frac{1 - \frac{s}{n}(1 - \frac{p}{(p-1)s\log n})}{\Big(1 - \frac{1}{n}(1 - \frac{p}{(p-1)s\log n})\Big)^{s/p}} \leq \frac{1 - sx_k^p}{(1-x_k^p)^{s/p}} \leq \frac{n^{s/p-1}(n-s)}{(n-1)^{s/p}}\Big(1 + \frac{1}{n\log n}\Big),$$

and hence

$$\frac{\left(n-s+\frac{p}{(p-1)\log n}\right)n^{s/p-1}}{\left(n-1+\frac{p}{(p-1)s\log n}\right)^{s/p}} \le \frac{n^{s/p-1}(n-s)}{(n-1)^{s/p}}\left(1+\frac{1}{n\log n}\right).$$

This can be simplified to

$$1+\frac{p}{(p-1)(n-s)\log n} \le \left(1+\frac{p}{(p-1)s(n-1)\log n}\right)^{s/p}\left(1+\frac{1}{n\log n}\right). \tag{8}$$

For sufficiently large n, we have

$$\begin{aligned}\left(1+\frac{p}{(p-1)s(n-1)\log n}\right)^{s/p} &= 1+\frac{1}{(p-1)(n-1)\log n}+O\left(\frac{1}{(n\log n)^2}\right)\\ &\le 1+\frac{1}{(p-1)(n-1)\log n}+\frac{1}{(p-1)(n-1)(n-2)\log n}\\ &= 1+\frac{1}{(p-1)(n-2)\log n}.\end{aligned}$$

Substituting this bound into (8), we obtain

$$1+\frac{p}{(p-1)(n-s)\log n} \le \left(1+\frac{1}{(p-1)(n-2)\log n}\right)\left(1+\frac{1}{n\log n}\right).$$

By some cancellations and rearranging, we get

$$\frac{p}{n-s} \le \frac{1}{n-2}+\frac{p-1}{n}+\frac{1}{n(n-2)\log n}.$$

Noting that $\frac{p}{n-s} \ge \frac{p}{n-2}$, we have

$$2(p-1) \le \frac{1}{\log n},$$

which leads to a contradiction for sufficiently large n. □

Lemma 3.6. *Let $p>1$, and let Q be an s-vertex r-graph and H be an n-vertex r-graph with $\lambda^{(p)}(Q,H) =: \lambda$ and minimum Q-degree δ. Let $\mathbf{x}$ be a principal Q-eigenvector corresponding to λ. Then*

$$\left(\frac{\lambda(\mathbf{x}_{\min})^{p-1}}{(s-1)!}\right)^p \le \frac{s!\binom{n}{s-1}\delta^{p-1}}{n^{s-1}} - \left(s!\binom{n}{s-1}\delta^{p-1}-\delta^p\right)(\mathbf{x}_{\min})^{p(s-1)}.$$

Proof. Set $V := V(H)$, and let $k \in V$ be a vertex achieving the minimum Q-degree δ. Considering the eigenequation for $\lambda^{(p)}(Q,H)$ at vertex k:

$$\lambda(\mathbf{x}_{\min})^{p-1} \le \lambda x_k^{p-1} = (s-1)!\sum_{I\in E_{Q,H}(k)} \mathcal{N}(Q,H[I])x_{I\setminus\{k\}}.$$

By Hölder's inequality, we have

$$\left(\frac{\lambda(\mathbf{x}_{\min})^{p-1}}{(s-1)!}\right)^p \le \delta^{p-1}\sum_{I\in E_{Q,H}(k)} \mathcal{N}(Q,H[I])(x_{I\setminus\{k\}})^p. \tag{9}$$

Define $T_1 = \{I_1 \in \binom{V}{s-1} : I_1 \cup \{k\} \in E_{Q,H}(k)\}$ and $T_2 = \{I_2 \in \binom{V}{s-1} : I_2 \cup \{k\} \notin E_{Q,H}(k)\}$. Then

$$\begin{aligned}
\sum_{I \in E_{Q,H}(k)} \mathcal{N}(Q, H[I])(x_{I\setminus\{k\}})^p &= \sum_{I \in \binom{V}{s-1}} s! x_I^p - \sum_{I_1 \in T_1} (s! - \mathcal{N}(Q, H[I_1 \cup \{k\}])) x_{I_1}^p - \sum_{I_2 \in T_2} s! x_{I_2}^p \\
&\leq \sum_{I \in \binom{V}{s-1}} s! x_I^p - \sum_{I_1 \in T_1} (s! - \mathcal{N}(Q, H[I_1 \cup \{k\}]))(\mathbf{x}_{\min})^{p(s-1)} \\
&\quad - \sum_{I_2 \in T_2} s! (\mathbf{x}_{\min})^{p(s-1)} \\
&= \sum_{I \in \binom{V}{s-1}} s! x_I^p - \big(s!\tbinom{n}{s-1} - \delta\big)(\mathbf{x}_{\min})^{p(s-1)}.
\end{aligned} \tag{10}$$

By Maclaurin's inequality, we have

$$\sum_{I \in \binom{V}{s-1}} x_I^p \leq \binom{n}{s-1}\left(n^{-1}\sum_{i \in V} x_i^p\right)^{s-1} = \frac{\binom{n}{s-1}}{n^{s-1}}. \tag{11}$$

Then the conclusion follows by combining the inequalities (9), (10), and (11). $\square$

We now prove Theorem 3.3, which is based on an idea from [17].

Proof of Theorem 3.3. Observe that if $\lambda^{(p)}(Q, \mathcal{P}) = 0$, then it follows from inequality (5) that $\pi(Q, \mathcal{P}) = 0$.

Next, assume $\lambda^{(p)}(Q, \mathcal{P}) > 0$. Suppose that $H_n \in \mathcal{P}_n$ is an r-graph with $\lambda^{(p)}(Q, H_n) = \lambda^{(p)}(Q, \mathcal{P}_n) =: \lambda$ and minimum Q-degree δ. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a principal Q-eigenvector corresponding to $\lambda^{(p)}(Q, H_n)$. By Lemma 3.5, there exists an increasing infinite sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers such that for each $n \in \{n_1, n_2, \ldots\}$,

$$(\mathbf{x}_{\min})^p \geq \frac{1}{n}\Big(1 - \frac{p}{(p-1)s\log n}\Big).$$

From Theorem 3.2 and Lemma 3.6, we derive

$$\begin{aligned}
\big(1 - o(1)\big)\Big(\frac{\lambda^{(p)}(Q, \mathcal{P})}{(s-1)!}\Big)^p \cdot \frac{((n)_s)^p}{n^{s+p-1}} &\leq \Big(\frac{\lambda(\mathbf{x}_{\min})^{p-1}}{(s-1)!}\Big)^p \\
&\leq \frac{s!\binom{n}{s-1}\delta^{p-1}}{n^{s-1}} - \big(s!\tbinom{n}{s-1}\delta^{p-1} - \delta^p\big)(\mathbf{x}_{\min})^{p(s-1)} \\
&\leq \frac{s!\binom{n}{s-1}\delta^{p-1}}{n^{s-1}} - \frac{s!\binom{n}{s-1}\delta^{p-1}}{n^{s-1}}\big(1 - o(1)\big) + \frac{\delta^p}{n^{s-1}} \\
&\leq o(n^{(s-1)(p-1)}) + \frac{\delta^p}{n^{s-1}}.
\end{aligned}$$

Since $\delta \leq \frac{s\mathcal{N}(Q,H_n)}{n} \leq \frac{sex(Q,\mathcal{P}_n)}{n}$, it follows that

$$(1 - o(1))(\lambda^{(p)}(Q, \mathcal{P}))^p \leq o(1) + \Big(\frac{ex(Q, \mathcal{P}_n)}{\binom{n}{s}}\Big)^p,$$

where the term $o(1)$ tends to 0 as $n \to \infty$. Consequently,

$$(\lambda^{(p)}(Q,\mathcal{P}))^p = \lim_{i\to\infty}(1-o(1))(\lambda^{(p)}(Q,\mathcal{P}))^p \leq \lim_{i\to\infty} o(1) + \Big(\frac{ex(Q,\mathcal{P}_{n_i})}{\binom{n_i}{s}}\Big)^p = (\pi(Q,\mathcal{P}))^p.$$

Combining this inequality with (5) completes the proof of Theorem 3.3. □

The celebrated Erdős-Stone-Simonovits theorem states that

$$\mathrm{ex}(n,F) = \left(1 - \frac{1}{\chi(F)-1} + o(1)\right)\frac{n^2}{2},$$

where $\chi(F)$ is the chromatic number of F. In [1], Alon and Shikhelman extended the Erdős-Stone-Simonovits theorem to count copies of K_s.

Lemma 3.7 ([1])**.** *Let F be a graph with $\chi(F) = k$. Then*

$$ex(K_s, \overline{F}_n) = \binom{k-1}{s}\left(\frac{n}{k-1}\right)^s + o(n^s).$$

Applying Theorem 3.3 and Lemma 3.7 yields the following result directly.

Corollary 3.8. *Let $p > 1$ and F be a graph with $\chi(F) = k$. Then*

$$\lambda^{(p)}(K_s, \overline{F}_n) = \frac{(k-1)_s}{(k-1)^s} n^{s-s/p} + o(n^{s-s/p}).$$

Remark 3.9. The specail case when $p = s = 2$ in Corollary 3.8 corresponds to the spectral Erdős-Stone-Simonovits theorem by Nikiforov [16]. Thus, Corollary 3.8 can be viewed as a generalization of adjacency spectral version of the Erdős-Stone-Simonovits theorem.

Gerbner and Palmer [5] provided a further extension to count arbitrary graphs Q using the regularity lemma.

Lemma 3.10 ([5])**.** *Let Q be an s-vertex graph, and let F be a graph with $\chi(F) = k$. Then*

$$ex(Q, \overline{F}_n) = ex(H, (\overline{K_k})_n) + o(n^s).$$

Below, we present the spectral version of Lemma 3.10.

Corollary 3.11. *Let $p > 1$, and let Q be an s-vertex graph and F be a graph with $\chi(F) = k$. Then*

$$\lambda^{(p)}(Q, \overline{F}_n) = \lambda^{(p)}(Q, (\overline{K_k})_n) + o(n^{s-s/p}).$$

Proof. By Theorem 3.3, we have

$$\frac{\lambda^{(p)}(Q,\overline{F}_n)}{n^{s-s/p}} = \big(1+o(1)\big)\frac{ex(Q,\overline{F}_n)}{\binom{n}{s}},$$

and thus,

$$\lambda^{(p)}(Q, \overline{F}_n) = s!ex(Q, \overline{F}_n)n^{-s/p} + o(n^{s-s/p}).$$

Similarly,

$$\lambda^{(p)}(Q, (\overline{K_k})_n) = s!ex(Q, (\overline{K_k})_n)n^{-s/p} + o(n^{s-s/p}).$$

By Lemma 3.10, we obtain

$$\begin{aligned}\lambda^{(p)}(Q, \overline{F}_n) - \lambda^{(p)}(Q, (\overline{K_k})_n) &= s!(ex(Q, \overline{F}_n) - ex(Q, (\overline{K_k})_n))n^{-s/p} + o(n^{s-s/p}) \\ &= o(n^{s-s/p}),\end{aligned}$$

which completes the proof. □

3.2 Q-flat properties of r-graphs

Consider an r-graph H on n vertices and a sequence of positive integers $k_1, \ldots, k_n$. The *blow-up* of H with respect to $k_1, \ldots, k_n$, denoted by $H(k_1, \ldots, k_n)$, is the r-graph obtained by replacing each vertex $i \in V(H)$ with a vertex class V_i (also called a block) of size k_i, and if $\{i_1, \ldots, i_r\} \in E(H)$, then $\{i_{1,j_1}, \ldots, i_{r,j_r}\} \in E(H(k_1, \ldots, k_n))$ for every $i_{1,j_1} \in V_{i_1}, \ldots, i_{r,j_r} \in V_{i_r}$. A property $\mathcal{M}$ of r-graphs is *multiplicative* if $H \in \mathcal{M}$ implies that any blow-up of H is also in $\mathcal{M}$ (i.e., $\mathcal{M}$ is closed under the blow-up operation).

For an s-vertex r-graph Q and a hereditary property $\mathcal{P}$ of r-graphs, we say that $\mathcal{P}$ is *Q-flat* if $\lambda^{(1)}(Q, \mathcal{P}) = \pi(Q, \mathcal{P})$. We establish the following sufficient condition for Q-flat properties.

Lemma 3.12. *Let Q be an r-graph and $\mathcal{P}$ be a hereditary and multiplicative property of r-graphs. Then $\mathcal{P}$ is Q-flat; that is, $\lambda^{(p)}(Q, \mathcal{P}) = \pi(Q, \mathcal{P})$ for every $p \geq 1$.*

Proof. Inequality (5) shows that $\lambda^{(1)}(Q, \mathcal{P}) \geq \pi(Q, \mathcal{P})$. To complete the proof, we shall prove the reverse inequality: $\lambda^{(1)}(Q, \mathcal{P}) \leq \pi(Q, \mathcal{P})$.

Consider $H \in \mathcal{P}_n$ with $\lambda^{(1)}(Q, H) = \lambda^{(1)}(Q, \mathcal{P}_n)$, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a principal Q-eigenvector corresponding to $\lambda^{(1)}(Q, H)$ (with $\|\mathbf{x}\|_1 = 1$). We claim that

$$\lambda^{(1)}(Q, H) = P_{Q,H}(\mathbf{x}) \leq \pi(Q, \mathcal{P}). \tag{12}$$

Since $P_{Q,H}(\mathbf{x})$ is continuous in each variable, it suffices to prove inequality (12) for positive rational numbers $x_1, \ldots, x_n$. Thus, we can assume that

$$x_1 = k_1/k, \ldots, x_n = k_n/k,$$

where $k, k_1, \ldots, k_n$ are positive integers and $k = k_1 + \cdots + k_n$. Consequently, inequality (12) is equivalent to the statement that for any positive integers $k_1, \ldots, k_n$, the following inequality holds:

$$\frac{P_{Q,H}((k_1, \ldots, k_n))}{k^s} \leq \pi(Q, \mathcal{P}). \tag{13}$$

Let $H(k_1, \ldots, k_n)$ denote the blow-up of H with blocks $V_1, \ldots, V_n$. Then for any $i_1, \ldots, i_s \in V(H)$, the subgraph $H(k_1, \ldots, k_n)[V_{i_1} \cup \ldots \cup V_{i_s}]$ contains at least $k_{i_1} \times \cdots \times k_{i_s}$ copies of $H[\{i_1, \ldots, i_s\}]$. It follows that

$$\mathcal{N}(Q, H(k_1, \ldots, k_n)[V_{i_1} \cup \ldots \cup V_{i_s}]) \geq k_1 \times \cdots \times k_s \mathcal{N}(Q, H[\{i_1, \ldots, i_s\}]).$$

This implies that

$$\begin{aligned}
P_{Q,H}((k_1, \ldots, k_n)) &= s! \sum_{\{i_1, \ldots, i_s\} \in E(Q,H)} \mathcal{N}(Q, H[\{i_1, \ldots, i_s\}]) k_{i_1} \cdots k_{i_s} \\
&\leq s! \sum_{\{i_1, \ldots, i_s\} \in E(Q,H)} \mathcal{N}(Q, H(k_1, \ldots, k_n)[V_{i_1} \cup \ldots \cup V_{i_s}]) \\
&\leq P_{Q,H(k_1,\ldots,k_n)}((1, \ldots, 1)) = s! \mathcal{N}(Q, H(k_1, \ldots, k_n)).
\end{aligned}$$

Since $H(k_1, \ldots, k_n) \in \mathcal{P}$ (as $\mathcal{P}$ is multiplicative) and $\nu(H(k_1, \ldots, k_n)) = k$, we obtain

$$\frac{s! \mathcal{N}(Q, H(k_1, \ldots, k_n))}{k^s} \leq \frac{ex(Q, \mathcal{P}_k)}{\binom{k}{s}} \leq \pi(Q, \mathcal{P}) + o(1),$$

where the term $o(1)$ tends to 0 as $k \to \infty$.

Similarly, for every positive integer t, we have

$$\begin{aligned}\frac{P_{Q,H}((k_1,\ldots,k_n))}{k^s} &= \frac{P_{Q,H}((tk_1,\ldots,tk_n))}{(tk)^s} \\ &\le \frac{s!\mathcal{N}(Q,H(tk_1,\ldots,tk_n))}{(tk)^s} \\ &\le \pi(Q,\mathcal{P})+o(1).\end{aligned}$$

Taking $t \to \infty$, we establish inequality (13), hence inequality (12) holds. Therefore,

$$\lambda^{(1)}(Q,\mathcal{P}) = \lim_{n\to\infty} \lambda^{(1)}(Q,\mathcal{P}_n) \le \pi(Q,\mathcal{P}),$$

completing the proof. □

Remark 3.13. For an r-graph H, we say that H is 2-*covering* if for any $\{u,v\} \in V(H)$, there exists an edge $e \in E(H)$ such that $\{u,v\} \subseteq e$. If r-graph F is 2-covering, then for any F-free r-graph G, any blow-up of G is F-free. Consequently, for each r-graph Q, the family $\overline{F}$ of r-graphs is Q-flat.

Notably, the complete graph K_{l+1} is 2-covering, which implies that $\overline{K_{l+1}}$ is K_2-flat. A classical theorem of Turán [2, p. 294] establishes that for any K_{l+1}-free graph G on n vertices, the number of edges satisfies $e(G) \le (1-\frac{1}{l})\frac{n^2}{2}$. Wilf [20] later provided a spectral extension of Turán's theorem, demonstrating that if G is an n-vertex K_{l+1}-free graph, then its largest eigenvalue $\lambda(G)$ satisfies $\lambda(G) \le (1-\frac{1}{l})n$. In 2002, Nikiforov [15] further extended this result by proving that for any K_{l+1}-free graph G with m edges, $\lambda(G) \le (1-\frac{1}{l})^{1/2}(2m)^{1/2}$. In the following, we generalize these bounds to families of r-graphs with Q-flat properties.

Theorem 3.14. *If Q is an s-vertex r-graph and $\mathcal{P}$ is a Q-flat property of r-graphs, then for any $H \in \mathcal{P}_n$,*

$$\mathcal{N}(Q,H) \le \pi(Q,\mathcal{P})n^s/s!,$$

and for every $p \ge 1$,

$$\lambda^{(p)}(Q,H) \le \pi(Q,\mathcal{P})n^{s-s/p}.$$

Proof. Let $\mathbf{x} = (x_1,\ldots,x_n)$ be a Q-principal eigenvector corresponding to $\lambda^{(p)}(Q,H)$ with $\|\mathbf{x}\|_p = 1$. Then

$$\frac{\lambda^{(p)}(Q,H)}{n^{s-s/p}} = \frac{P_{Q,H}(\mathbf{x})}{n^{s-s/p}} = P_{Q,H}((x_1/n^{1-1/p},\ldots,x_n/n^{1-1/p})).$$

By Power-Mean inequality, for any $p \ge 1$,

$$\frac{x_1+\cdots+x_n}{n^{1-1/p}} \le (x_1^p+\cdots+x_n^p)^{1/p} = 1.$$

From Theorem 3.2 it follows that

$$\frac{\lambda^{(p)}(Q,H)}{n^{s-s/p}} \le \lambda^{(1)}(Q,H) \le \lambda^{(1)}(Q,\mathcal{P}_n) \le \lambda^{(1)}(Q,\mathcal{P}) = \pi(Q,\mathcal{P}).$$

Since $\lambda^{(p)}(Q,H) \ge s!\mathcal{N}(Q,H)/n^{s/p}$, we conclude

$$\mathcal{N}(Q,H) \le \pi(Q,\mathcal{P})n^s/s!,$$

completing the proof. □

Lemma 3.15. *If Q is an s-vertex r-graph and $\mathcal{P}$ is a Q-flat property of r-graphs, then for any $p \geq 1$ and $H \in \mathcal{P}$,*

$$\lambda^{(p)}(Q, H) \leq \pi(Q, \mathcal{P})^{1/p}(s!\mathcal{N}(Q, H))^{1-1/p}.$$

Proof. Lemma 2.2 implies that

$$\left(\lambda^{(p)}(Q, H)\right)^p \leq \lambda^{(1)}(Q, H)(s!\mathcal{N}(Q, H))^{p-1}.$$

Moreover, since $\mathcal{P}$ is Q-flat, Theorem 3.2 yields

$$\lambda^{(1)}(Q, H) \leq \lambda^{(1)}(Q, \mathcal{P}) = \pi(Q, \mathcal{P}).$$

Combining these inequalities, we obtain

$$\lambda^{(p)}(Q, H) \leq \pi(Q, \mathcal{P})^{1/p}(s!\mathcal{N}(Q, H))^{1-1/p},$$

completing the proof. □

4 Spectral generalized Turán problems

In this section, we study spectral generalized Turán problems for a family of $\mathcal{F}$-free r-graphs. We assume that all members of $\mathcal{F}$ contain no isolated vertices.

The following spectral stability theorem indicates that if the maximum (p, Q)-spectral radius over all $\mathcal{F}$-free r-graphs satisfies a specific growth condition, then the extremal hypergraphs must have a large minimum Q-degree.

Theorem 4.1. *Let $p > 1$, $s \geq r \geq 2$, and $0 < \varepsilon < 1$. Let Q be an s-vertex r-graph, and let $\mathcal{F}$ be a family of r-graphs with $\pi(Q, \overline{\mathcal{F}}) > 0$. Let $\mathcal{G}_n$ be the collection of all n-vertex $\mathcal{F}$-free r-graphs with minimum Q-degree at least $(1-\varepsilon)\pi(Q, \overline{\mathcal{F}})\binom{n}{s-1}$ and define $\lambda^{(p)}(Q, \mathcal{G}_n) = \max\{\lambda^{(p)}(Q, G) : G \in \mathcal{G}_n\}$. Suppose that there exists a sufficiently large $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have*

$$\lambda^{(p)}(Q, \overline{\mathcal{F}}_n) \geq \lambda^{(p)}(Q, \overline{\mathcal{F}}_{n-1}) + \pi(Q, \overline{\mathcal{F}})(s - s/p)(1 - \sigma)n^{s-s/p-1}, \tag{14}$$

where $\sigma = \varepsilon\pi(Q, \overline{\mathcal{F}})/(5s!(s-1))$. Then for any $\mathcal{F}$-free r-graph H on $n \geq n_0$ vertices, we have

$$\lambda^{(p)}(Q, H) \leq \lambda^{(p)}(Q, \mathcal{G}_n).$$

In addition, if the equality holds, then $H \in \mathcal{G}_n$.

We need the following Lemma for the proof of Theorem 4.1.

Lemma 4.2. *Let $p > 1$, $s \geq r \geq 2$, and $0 < \varepsilon < 1$. Let Q be an s-vertex r-graph, and let $\mathcal{P}$ be a hereditary property of r-graphs with $\pi(Q, \mathcal{P}) > 0$. Let $H_n \in \mathcal{P}_n$ satisfy $\lambda^{(p)}(Q, H_n) = \lambda^{(p)}(Q, \mathcal{P}_n)$. Suppose $0 \leq \varepsilon' < \varepsilon\pi(Q, \mathcal{P})/(s!(s-1))$, and let $\mathbf{x}$ be a principal Q-eigenvector corresponding to $\lambda^{(p)}(Q, H_n)$. If n is sufficiently large and $(\mathbf{x}_{\min})^p \geq \frac{1-\varepsilon'}{n}$, then*

$$\delta_Q(H_n) \geq (1 - \varepsilon)\pi(Q, \mathcal{P})\binom{n}{s-1}.$$

Proof. Set $\delta := \delta_Q(H_n)$ and $\lambda := \lambda^{(p)}(Q, H_n)$. Suppose for contradiction that $\delta < (1-\varepsilon)\pi(Q,\mathcal{P})\binom{n}{s-1}$. By Theorem 3.2 and Lemma 3.6, we obtain

$$\begin{aligned}
(1-\varepsilon')^{p-1}\Big(\frac{\lambda^{(p)}(Q,\mathcal{P})}{(s-1)!}\Big)^p \cdot \frac{((n)_s)^p}{n^{s+p-1}} &\le \Big(\frac{\lambda(\mathbf{x}_{\min})^{p-1}}{(s-1)!}\Big)^p \\
&\le \frac{s!\binom{n}{s-1}\delta^{p-1}}{n^{s-1}} - (s!\tbinom{n}{s-1}\delta^{p-1} - \delta^p)(\mathbf{x}_{\min})^{p(s-1)} \\
&\le \frac{s!\binom{n}{s-1}\delta^{p-1}}{n^{s-1}} - \frac{1}{n^{s-1}}(s!\tbinom{n}{s-1}\delta^{p-1} - \delta^p)(1-(s-1)\varepsilon') \\
&\le \frac{s!(s-1)\varepsilon'\binom{n}{s-1}\delta^{p-1}}{n^{s-1}} + \frac{\delta^p}{n^{s-1}} \\
&\le \frac{\binom{n}{s-1}\delta^{p-1}}{n^{s-1}}(s!(s-1)\varepsilon' + (1-\varepsilon)\pi(Q,\mathcal{P})) \\
&\le \frac{\pi(Q,\mathcal{P})\binom{n}{s-1}\delta^{p-1}}{n^{s-1}},
\end{aligned}$$

where the last inequality follows from $\varepsilon' < \varepsilon\pi(Q,\mathcal{P})/(s!(s-1))$. By Theorem 3.3 and the assumption $\delta < (1-\varepsilon)\pi(Q,\mathcal{P})\binom{n}{s-1}$, we further get

$$(1-\varepsilon')^{p-1}\big(\pi(Q,\mathcal{P})\tbinom{n-1}{s-1}\big)^p \le (1-\varepsilon)^{p-1}\big(\pi(Q,\mathcal{P})\tbinom{n}{s-1}\big)^p,$$

which implies that

$$1 < \Big(\frac{1-\varepsilon'}{1-\varepsilon}\Big)^{\frac{p-1}{p}} \le \frac{n}{n-s+1}.$$

This is a contradiction for sufficiently large n, completing the proof. $\square$

Proof of Theorem 4.1. Let H_n be an $\mathcal{F}$-free r-graph on n vertices that satisfies $\lambda^{(p)}(Q,H_n) = \lambda^{(p)}(Q,\overline{\mathcal{F}}_n)$, and let $\mathbf{x} = (x_1,\ldots,x_n)$ be a principal Q-eigenvector corresponding to $\lambda^{(p)}(Q,H_n)$. In view of Lemma 4.2, it suffices to show that for $n \ge n_0$,

$$(\mathbf{x}_{\min})^p \ge \frac{1-\varepsilon'}{n},$$

where $\varepsilon' = \varepsilon\pi(Q,\overline{\mathcal{F}})/(2s!(s-1))$. Suppose for contradiction that for some n,

$$(\mathbf{x}_{\min})^p < \frac{1-\varepsilon'}{n}.$$

Applying (3), Fact 3.1, and Bernoulli's inequality, we obtain

$$\begin{aligned}
\frac{\lambda^{(p)}(Q,H_{n-1})}{\lambda^{(p)}(Q,H_n)} &\ge \frac{1-s(\mathbf{x}_{\min})^p}{(1-(\mathbf{x}_{\min})^p)^{s/p}} \\
&\ge \Big(1-\frac{s(1-\varepsilon')}{n}\Big)\Big(1-\frac{1-\varepsilon'}{n}\Big)^{-s/p} \\
&\ge \Big(1-\frac{s(1-\varepsilon')}{n}\Big)\Big(1+\frac{s(1-\varepsilon')}{pn}\Big) \\
&= 1-\frac{(s-s/p)(1-\varepsilon')}{n} - \frac{s^2(1-\varepsilon')^2}{pn^2}.
\end{aligned}$$

From Lemma 3.3, it follows that $\lambda^{(p)}(Q,H_n)=(\pi(Q,\overline{\mathcal{F}})+o(1))n^{s-s/p}$, and hence

$$\lambda^{(p)}(Q,H_{n-1})\geq\lambda^{(p)}(Q,H_n)-\pi(Q,\overline{\mathcal{F}})(s-s/p)(1-\varepsilon'/2)n^{s-s/p-1}. \tag{15}$$

On the other hand, by (14), we have

$$\lambda^{(p)}(Q,H_n)\geq\lambda^{(p)}(Q,H_{n-1})+\pi(Q,\overline{\mathcal{F}})(s-s/p)(1-\sigma)n^{s-s/p-1}. \tag{16}$$

Combining (15) and (16) yields

$$\pi(Q,\overline{\mathcal{F}})(s-s/p)(1-\sigma)n^{s-s/p-1}\leq\pi(Q,\overline{\mathcal{F}})(s-s/p)(1-\varepsilon'/2)n^{s-s/p-1},$$

which contradicts $\sigma=\varepsilon\pi(Q,\overline{\mathcal{F}})/(5s!(s-1))$. This completes the proof of Theorem 4.1. □

4.1 Spectral Erdős pentagon problem

In 1984, Erdős conjectured that for every $n\geq 5$, the balanced blow-up of C_5 contains the maximum number of copies of C_5 among all n-vertex triangle-free graphs. This conjecture was first resolved independently by Grzesik [4] and Hatami et al. [7] for sufficiently large n. Later, Lidický and Pfender [13] completed the proof by extending the result to all n.

Lemma 4.3 ([13])**.** *For all n, the maximum number of copies of C_5 in K_3-free graphs on n vertices is*

$$\prod_{i=0}^{4}\left\lfloor\frac{n+i}{5}\right\rfloor.$$

Moreover, for $n\geq 9$, the only K_3-free graph on n vertices maximizing the number of copies of C_5 is the balanced blow-up of C_5.

Lemma 4.4 ([3])**.** *There exist $\varepsilon>0$ and N_0 such that the following holds for all $n\geq N_0$: If G is an n-vertex K_3-free graph with $\delta_{C_5}(G)\geq(1/5^4-\varepsilon)n^4$, then G is C_5-colorable.*

Remark 4.5. We remark that the minimum Q-degree in [3] differs from ours by a constant factor of $|Aut(Q)|$. Additionally, observe that $|Aut(C_5)|=10$.

Lemma 4.6 ([22])**.** *Let $l\geq r\geq 2$. Then $e(T_l^r(n))=\frac{(l)_r}{r!l^r}n^r+O(n^{r-2})$.*

Lemma 4.7 ([10])**.** *Let $l\geq r\geq 2$, and let G be an l-partite r-graph of order n. For every $p>1$,*

$$\lambda^{(p)}(G)\leq\lambda^{(p)}(T_l^r(n)),$$

with equality if and only if $G=T_l^r(n)$.

For any r-graph Q on s vertices and any r-graph H, we define $D(Q,H)$ as the s-graph derived from H with vertex set $V(D(Q,H))=V(H)$ and edge set

$$E(D(Q,H))=\{\{v_1,\ldots,v_s\}:H[v_1,\ldots,v_s]\supseteq Q\}.$$

Note that if $\mathcal{N}(Q,H[v_1,\ldots,v_s])=1$ for any $\{v_1,\ldots,v_s\}\in E(D(Q,H))$, then

$$\mathcal{N}(Q,H)=e(D(Q,H))\ \text{ and }\ \lambda^{(p)}(Q,H)=\lambda^{(p)}(D(Q,H)). \tag{17}$$

Recently, Liu [12, Theorem 1.5] established a general theorem that extends the result of Keevash-Lenz-Mubayi and applied it to obtain a spectral Erdős pentagon theorem. We extend Liu's result via a different approach. For any $p\geq 1$, an r-graph Q and a family $\mathcal{G}_n$ of r-graphs on n vertices, let $\lambda^{(p)}(Q,\mathcal{G}_n)$ (resp. $\lambda^{(p)}(\mathcal{G}_n)$) denote the maximum (p,Q)-spectral radius (resp. p-spectral radius) among all r-graphs in $\mathcal{G}_n$.

Theorem 4.8. *Let $p \geq 1$, and let $\mathcal{L}_n$ be the balanced blowup of C_5 on n vertices. Then, for all sufficiently large n and any n-vertex K_3-free graph G, we have $\lambda^{(p)}(C_5, G) \leq \lambda^{(p)}(C_5, \mathcal{L}_n)$. The equality holds if and only if $G = \mathcal{L}_n$ for $p > 1$, and if $G \supseteq C_5$ for $p = 1$.*

Proof. For any C_5-colorable graph H with a homomorphism ϕ from $V(H)$ to $V(C_5)$, denote $V(C_5)$ as $\{1, 2, 3, 4, 5\}$. For each $i \in [5]$, define $V_i = \{v \in V(H) : \phi(v) = i\}$ (some V_i may be empty). This defines a natural partition of $V(H)$.

We claim that for any five vertices $v_1, \ldots, v_5$ in $V(H)$, if $H[v_1, \ldots, v_5]$ contains a copy of C_5 (in fact, $H[v_1, \ldots, v_5] \cong C_5$), then these five vertices must belong to five distinct parts in the partition.

Suppose $v_1v_2v_3v_4v_5v_1$ forms a copy of C_5 in $H[v_1, \ldots, v_5]$, with $\{v_iv_{i+1}\} \in E(H)$ for $i \in [5]$ (indices modulo 5). Assume for contradiction that the claim fails. By symmetry, we may assume that v_1 and v_3 belong to the same part, i.e., $\phi(v_1) = \phi(v_3)$. Since H is C_5-colorable and $\{v_5v_1\} \in E(H)$, it follows that

$$\{\{\phi(v_5)\phi(v_3)\}, \{\phi(v_3)\phi(v_4)\}, \{\phi(v_4)\phi(v_5)\}\} \subseteq E(C_5).$$

which contradicts the fact that C_5 is K_3-free. Therefore, the claim holds.

The above claim implies that $D(C_5, H)$ is a 5-partite 5-graph. It follows that $D(C_5, \mathcal{L}_n)$ is isomorphic to $T_5^5(n)$, and hence $\mathcal{N}(C_5, \mathcal{L}_n) = e(T_5^5(n))$. By Lemmas 4.3 and 4.6, we have

$$ex(C_5, (\overline{K_3})_n) = \mathcal{N}(C_5, \mathcal{L}_n) = e(T_5^5(n)) = \frac{n^5}{5^5} + O(n^3), \tag{18}$$

and hence $\pi(C_5, \overline{K_3}) = 5!/5^5$.

Let $Col(C_5)_n$ be the set of all C_5-colorable graphs on n vertices, and let

$$\mathcal{R}_n := \{D(C_5, H) : H \in Col(C_5)_n\}.$$

Then, by (17),

$$\lambda^{(p)}(C_5, Col(C_5)_n) = \lambda^{(p)}(\mathcal{R}_n). \tag{19}$$

Lemma 4.4 shows that there exist $\varepsilon > 0$ and N_0 such that for every n-vertex K_3-free graph G with $\delta_{C_5}(G) \geq (1/5^4 - \varepsilon)n^4$ is contained in $Col(C_5)$.

By (4) and (18), we have

$$\lambda^{(p)}(C_5, (\overline{K_3})_n) \geq 5! ex(C_5, (\overline{K_3})_n)/n^{5/p} \geq \pi(C_5, \overline{K_3})n^{5-5/p} + O(n^{3-5/p}). \tag{20}$$

Note that K_3 is a 2-covering graph. Lemma 3.12 and Theorem 3.14 imply that

$$\begin{aligned} \lambda^{(p)}(C_5, (\overline{K_3})_{n-1}) &\leq \pi(C_5, \overline{K_3})(n-1)^{5-5/p} \\ &= \pi(C_5, \overline{K_3})n^{5-5/p} - \pi(C_5, \overline{K_3})(5 - 5/p)n^{4-5/p} + O(n^{3-5/p}). \end{aligned} \tag{21}$$

Combining (20) and (21) yields

$$\lambda^{(p)}(C_5, (\overline{K_3})_n) - \lambda^{(p)}(C_5, (\overline{K_3})_{n-1}) \geq \pi(C_5, \overline{K_3})(5 - 5/p)n^{4-5/p} + o(n^{4-5/p}).$$

Thus, by Theorem 4.1 and equality (19), for $p > 1$ and enough large n, we have

$$\lambda^{(p)}(C_5, (\overline{K_3})_n) \leq \lambda^{(p)}(C_5, Col(C_5)_n) = \lambda^{(p)}(\mathcal{R}_n) \leq \lambda^{(p)}(T_5^5(n)) = \lambda^{(p)}(C_5, \mathcal{L}_n),$$

where the third inequality follows from Lemma 4.7.

For $p = 1$, by Theorem 3.14, we have $\lambda^{(1)}(C_5, (\overline{K_3})_n) \leq \pi(C_5, \overline{K_3}) = 5!/5^5$. Moreover, observe that

$$\lambda^{(1)}(C_5, G) \geq \lambda^{(1)}(C_5, C_5) = \lambda^{(1)}(K_5^5) = 5!/5^5,$$

which implies $\lambda^{(1)}(C_5, G) = \lambda^{(1)}(C_5, (\overline{K_3})_n)$, completing the proof. □

4.2 Spectral generalized Turán problems for edge-critical graphs

For a graph H and $e \in E(H)$, let $H - e$ denote the graph with vertex set $V(H)$ and edge set $E(H)\backslash\{e\}$. A graph H is called *edge-critical* if there exists an edge e of H such that $\chi(H-e) = \chi(H) - 1$. The *s-expansion* $H^{(s)}$ of H is the s-graph obtained from H by enlarging each edge of H with $s-2$ new vertices disjoint from $V(H)$ such that distinct edges of H are enlarged by distinct vertices.

Simonovits [19] extended Turán's theorem to any edge-critical graph F and established the critical edge theorem. Later, Ma and Qiu [14] generalized Simonovit's result as follows:

Theorem 4.9 ([14]). *Let $l \geq s \geq 2$, and let F be an edge-critical graph with $\chi(F) = l+1$. Then for sufficiently large n, the unique n-vertex F-free graph with the maximum number of copies of K_s is the Turán graph $T_l(n)$.*

Very recently, Zheng, Li and Su [23, Theorem 4.8] determined the maximum p-spectral radius among all n-vertex $F^{(r)}$-free r-graphs, where F is an edge-critical graph.

Lemma 4.10 ([23]). *Let $p \geq 1$, $l \geq s \geq 2$, and let F be an edge-critical graph with $\chi(F) = l+1$. Then there exists n_0, such that for any $F^{(s)}$-free s-graph G on $n > n_0$ vertices, $\lambda^{(p)}(G) \leq \lambda^{(p)}(T_l^s(n))$. The equality holds if and only if $G = T_l^s(n)$ for $p > 1$, and if $G \supseteq K_l^s$ for $p = 1$.*

Let H be an r-graph. The 2-*shadow* of H, denoted by $\partial_2 H$, is the graph with vertex set $V(\partial_2 H) = V(H)$ and edge set $E(\partial_2 H) = \{\{v_1, v_2\} : \{v_1, v_2\} \subseteq e \in E(H)\}$.

We present a spectral analogue of Theorem 4.9.

Theorem 4.11. *Let $p \geq 1$, $l \geq s \geq 2$, and let F be an edge-critical graph with $\chi(F) = l+1$. Then there exists n_0, such that for any F-free graph G on $n > n_0$ vertices, $\lambda_s^{(p)}(G) \leq \lambda_s^{(p)}(T_l(n))$. The equality holds if and only if $G = T_l(n)$ for $p > 1$, and if $G \supseteq K_l$ for $p = 1$.*

Proof. We define the following sets for a given integer n:

$$\mathcal{A}_n := \{D(K_s, G) : G \text{ is an } F\text{-free graph on } n \text{ vertices}\},$$

$$\mathcal{B}_n := \{H : H \text{ is an } s\text{-graph on } n \text{ vertices and } \partial_2 H \text{ is } F\text{-free}\},$$

$$\mathcal{C}_n := \{H : H \text{ is an } s\text{-graph on } n \text{ vertices and } H \text{ is } F^{(s)}\text{-free}\}.$$

Observe that for any F-free graph G, we have $\partial_2 D(K_s, G) \subseteq G$, which implies $\mathcal{A}_n \subseteq \mathcal{B}_n \subseteq \mathcal{C}_n$. By Lemma 4.10, for $p \geq 1$, it follows that

$$\lambda^{(p)}(\mathcal{A}_n) \leq \lambda^{(p)}(\mathcal{C}_n) = \lambda^{(p)}(T_l^s(n)).$$

Note that $D(K_s, T_l(n)) = T_l^s(n)$. From equality (17), we obtain

$$\lambda^{(p)}(K_s, \overline{F}_n) = \lambda^{(p)}(\mathcal{A}_n) = \lambda_s^{(p)}(T_l(n)).$$

The result follows from Lemma 4.10. $\square$

Remark 4.12. Letting $p \to \infty$ in Theorem 4.11 and applying Proposition 2.1, we immediately obtain Theorem 4.9. Moreover, Theorem 4.11 can be viewed as a generalization of the result of Yu and Peng [21, Theorem 1.7].

5 Concluding remarks

In this paper, we systematically investigate the (p, Q)-spectral radius of hypergraphs and derive several results concerning spectral generalized Turán problems. Specifically, Theorem 4.1 establishes a spectral stability result. We conjecture that the conclusion of Theorem 4.1 holds even without condition (14), leading to the following:

Conjecture 5.1. *Let $p > 1$, $s \geq r \geq 2$, Q be an s-vertex r-graph and $\mathcal{F}$ be a family of r-graphs with $\pi(Q, \overline{\mathcal{F}}) > 0$. Let $\mathcal{G}_n$ be the collection of all n-vertex $\mathcal{F}$-free r-graphs with minimum Q-degree at least $(1-\varepsilon)\pi(Q, \overline{\mathcal{F}})\binom{n}{s-1}$ and define $\lambda^{(p)}(Q, \mathcal{G}_n) = \max\{\lambda^{(p)}(Q, G) : G \in \mathcal{G}_n\}$. Then for any $\mathcal{F}$-free r-graph H on $n \geq n_0$ vertices, we have*

$$\lambda^{(p)}(Q, H) \leq \lambda^{(p)}(Q, \mathcal{G}_n).$$

In addition, if the equality holds, then $H \in \mathcal{G}_n$.

To address this conjecture, we propose two potential approaches.

Problem 5.2. *Let Q be an r-graph on s vertices, and $\mathcal{P}$ be a hereditary property of r-graphs with $\pi(Q, \mathcal{P}) > 0$. Suppose that $H_n \in \mathcal{P}_n$ is an r-graph satisfying $\lambda^{(p)}(Q, H_n) = \lambda^{(p)}(Q, \mathcal{P}_n)$ for $p > 1$ and $\mathbf{x} = (x_1, \ldots, x_n)$ is a principal Q-eigenvector for $\lambda^{(p)}(Q, H_n)$. Does there exist a constant n_0 such that for all $n \geq n_0$,*

$$(\mathbf{x}_{\min})^p \geq \frac{1}{n}\Big(1 - \frac{p}{(p-1)s\log n}\Big)?$$

An affirmative answer to Problem 5.2 would, via Lemma 4.2, imply conjecture 5.1.

Problem 5.3. *Let Q be an s-vertex r-graph, and $\mathcal{F}$ be a family of r-graphs with $\pi(Q, \overline{\mathcal{F}}) > 0$. For $p > 1$, does there exist a sequence $\{a_n\}$ such that*

$$\lambda^{(p)}(Q, \overline{\mathcal{F}}_n) = \pi(Q, \overline{\mathcal{F}})n^{s-s/p} + a_n n^{s-s/p-1},$$

and the limit $\lim\limits_{n\to\infty} a_n$ exists? If answered affirmatively, then

$$\lambda^{(p)}(Q, \overline{\mathcal{F}}_n) = \lambda^{(p)}(Q, \overline{\mathcal{F}}_{n-1}) + \pi(Q, \overline{\mathcal{F}})(s - s/p)n^{s-s/p-1} + o(n^{s-s/p-1}),$$

and conjecture 5.1 would follow from Theorem 4.1.

Moreover, Problem 5.3 is of independent interest. Another natural question is to ask: to what structural parameters of the graph is $\{a_n\}$ related and for which hereditary families does the limit of $\{a_n\}$ exist?

For any r-graph Q on s vertices and any r-graph H, recall the definition of s-graph $D(Q, H)$: its edge set is defined as

$$E(D(Q, H)) = \{\{v_1, \ldots, v_s\} : H[v_1, \ldots, v_s] \supseteq Q\}.$$

Assigning a weight $\mathcal{N}(Q, H[v_1, \ldots, v_s])$ to each edge $\{v_1, \ldots, v_s\}$, we define its p-spectral radius as:

$$\lambda^{(p)}(D(Q, H)) = \max_{\|\mathbf{x}\|_p = 1} s! \sum_{\{i_1,\ldots,i_s\} \in E(D(Q,H))} \mathcal{N}(Q, H[\{i_1, \ldots, i_s\}])x_{i_1} \cdots x_{i_s}.$$

Then, it follows that

$$\lambda^{(p)}(Q, H) = \lambda^{(p)}(D(Q, H)).$$

This establishes a connection between the (p, Q)-spectral radius of the r-graph H and the p-spectral radius of the weighted s-graph $D(Q, H)$. For relevant conclusions regarding the p-spectral radius of weighted hypergraphs, one may refer to the results of Nikiforov [18], such as the Perron-Frobenius theory for the weighted hypergraphs discussed in Section 5 of [18].

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