MINIMALLY EMBEDDED RIEMANN SURFACES IN $\mathbb{S}^3$ AND THE CONFORMAL DEFORMATION OF THEIR METRICS

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Abstract. We prove that if $f_g : (\Sigma, g) \rightarrow (\mathbb{S}^{2+p}, \tilde{g})$ is a smooth minimal isometric embedding of a Riemannian surface (Σ, g), and $[0,1] \ni t \rightarrow g_t$ is a path of area preserving conformal deformations of g on the embedded submanifold $f_g(\Sigma)$, then there exists a path of conformal diffeomorphism $F_t : (\mathbb{S}^{2+p}, F_t^* \tilde{g}) \rightarrow (\mathbb{S}^{2+p}, \tilde{g})$ that starts at $\mathbb{1}_{\mathbb{S}^{2+p}}$, set theoretically fixes $f_g(\Sigma)$ for all t, and it is such that $F_t^* \tilde{g} \mid_{f_g(M)} = g_t$ with $f_{g_t} : (\Sigma, g_t) \rightarrow (\mathbb{S}^{2+p}, \tilde{g})$ a path of minimal embedding deformations of the initial f_g. We apply this result to the Lawson surface $(\Sigma, g) = (\xi_{k/m,m}, g_{\xi_{k/m,m}})$, m any divisor of $k > 1$, to conclude that if $a := \mu_{g_{\xi_{k/m,m}}}(\Sigma)$, and $[0,1] \ni t \rightarrow g_t$ is a path of area a metrics conformal deformations of $g_{k/m,m}$ to a metric g_a of constant scalar curvature $s_{g_a} = 4\pi\chi(\Sigma)/a$, the embedding $f_{g_{\xi_{k/m,m}}} : (\xi_{k/m,m}, g_{\xi_{k/m,m}}) \rightarrow (\mathbb{S}^3, \tilde{g})$ has associated minimal isometric conformal deformations f_{g_t} to the isometric embedding f_{g_a} of g_a, in sharp contrast with the situation of the standard sphere $(\xi_{0,1}, g_{\xi_{0,1}})$ and Clifford torus $(\xi_{1,1}, g_{\xi_{1,1}})$, which are the only orientable Riemannian surfaces of genus 0 and 1, respectively, isometrically embedded into $(\mathbb{S}^3, \tilde{g})$ as minimal surfaces. If $\sigma^2(\Sigma) := \sup_{[g] \in \mathcal{C}(\Sigma)} (4\pi\chi(\Sigma))^2 / \left(\frac{1}{4} \inf_{g \in [g]} \mathcal{W}_{f_g}(\Sigma) \right)$, $\mathcal{W}_{f_g}(\Sigma)$ the Willmore energy of the isometric embedding $f_g : (\Sigma, g) \rightarrow (\mathbb{S}^{\tilde{n}}, \tilde{g})$ and $\mathcal{C}(\Sigma)$ the space of all conformal classes of metrics on Σ, if Σ^k is orientable then $(4\pi\chi(\Sigma))^2 / \left(\frac{1}{4} \mathcal{W}_{f_g}(\Sigma) \right) \leq \sigma^2(\Sigma) = (4\pi\chi(\Sigma))^2 / \left(\frac{1}{4} \mathcal{W}_{f_{g_{\xi_{k,1}}}}(\Sigma) \right)$, and for $k \neq 1$, the equality is achieved by f_g if, and only if, $[g] = [g_{\xi_{k,1}}]$ and f_g is conformally equivalent to $f_{g_{\xi_{k,1}}}$, while for $k = 1$, $\chi(\Sigma) = 0$ so the equality is always achieved, but then $\mathcal{W}_{f_g}(\Sigma) = \mathcal{W}_{f_{\xi_{1,1}}}(\Sigma)$ if, and only if, $[g] = [g_{\xi_{1,1}}]$ and f_g is conformally equivalent to $f_{\xi_{1,1}}$.

1. Isometric embeddings into the standard sphere

If $(M^{n \geq 2}, g)$ is a closed Riemannian manifold, and

$$f_g : (M, g) \rightarrow (\tilde{M}^{\tilde{n}}, \tilde{g}) \tag{1}$$

is an isometric embedding of it into some Riemannian background $(\tilde{M}^{\tilde{n}}, \tilde{g})$, the scalar curvature s_g of g relates to extrinsic quantities of f_g by

$$s_g = \sum K^{\tilde{g}}(e_i, e_j) + \tilde{g}(H, H) - \tilde{g}(\alpha, \alpha)\,. \tag{2}$$

Here, $\{e_i\}$ is an orthonormal tangent frame of $f_g(M)$, $K^{\tilde{g}}$ is the sectional curvature of the metric $\tilde{g}$, and $\tilde{g}(H, H)$ and $\tilde{g}(\alpha, \alpha)$ are the squared $\tilde{g}$-norms of the mean curvature vector $H := H_{f_g}$, and second fundamental form $\alpha := \alpha_{f_g}$ tensors of f_g, respectively. By integration of these three extrinsic functions with respect to the intrinsic volume measure of g, we obtain extrinsic functionals $\Theta_{f_g}(M)$, $\Psi_{f_g}(M)$ and $\Pi_{f_g}(M)$, respectively, and we have that

$$\begin{aligned} \mathcal{S}_g(M) := \int_{f_g(M)} s_g d\mu_g & = \Theta_{f_g}(M) + \Psi_{f_g}(M) - \Pi_{f_g}(M) \\ & = \mathcal{W}_{f_g}(M) - \mathcal{D}_{f_g}(M)\,, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \mathcal{W}_{f_g}(M) & := \frac{n}{n-1}\Theta_{f_g}(M) + \Psi_{f_g}(M)\,, \\ \mathcal{D}_{f_g}(M) & := \frac{1}{n-1}\Theta_{f_g}(M) + \Pi_{f_g}(M)\,. \end{aligned} \tag{4}$$

The particular choice of coefficients in the linear combinations of the extrinsic functionals above is made to produce functionals $\mathcal{W}_{f_g}(M)$ and $\mathcal{D}_{f_g}(M)$ that are intrinsically defined in the space of metrics on the manifold in a fixed conformal class, a convenient fact for the geometric analysis of M if the background $(\tilde{M}, \tilde{g})$ were to carry isometric embeddings of all the metrics of M.

By the Nash isometric embedding theorem [15], any Riemannian manifold (M^n, g) can be isometrically embedded into a standard sphere $(\mathbb{S}^{\tilde{n}}, \tilde{g}) \hookrightarrow (\mathbb{R}^{\tilde{n}+1}, \| \, \|^2)$ of large but fixed dimension $\tilde{n} = \tilde{n}(n)$. We thus identify the open cone $\mathcal{M}(M)$ of smooth Riemannian metrics on M with the space of their isometric embeddings into this sphere background,

$$\mathcal{M}_{\mathbb{S}^{\tilde{n}}}(M) = \{ f_g(M) \subset \mathbb{S}^{\tilde{n}} : \; f_g : (M, g) \to (\mathbb{S}^{\tilde{n}}, \tilde{g}) \text{ isometric embedding} \}\,,$$

and the deformations of metrics in $\mathcal{M}(M)$ with the isotopic Palais' deformations of their isometric embeddings [16]. If $f_g \in \mathcal{M}_{\mathbb{S}^{\tilde{n}}}(M)$, the exterior scalar curvature on $f_g(M)$ is $\sum K^{\tilde{g}}(e_i, e_j) = n(n-1)$, and since there exist volume preserving deformations of g in the conformal class $[g]$, the isometric embedding f_g of g must be such that $\|H_{f_g}\|^2$ is a constant function on $f_g(M)$ [20, Theorem 6].

1.1. **Principal example: Lawson's minimal surfaces immersed into $\mathbb{S}^3$.** If $\Sigma = \Sigma^k$ is a a surface of topological genus k, an immersion $f : \Sigma \to \mathbb{S}^m$ into the standard sphere fixes a conformal class on Σ, who inherits an intrinsic metric representative induced by restriction of the metric of the background sphere to $f(\Sigma)$. If $\mathcal{I}_{\mathbb{S}^m}(\Sigma^k) = \{ f : \; f : \Sigma^k \to \mathbb{S}^m \text{ is an immersion} \}$ is the set of all possible immersions of Σ into $\mathbb{S}^m$, other than for $\Sigma = \mathbb{P}^2(\mathbb{R})$, $\mathcal{I}_{\mathbb{S}^3}(\Sigma^k)$ is nonempty and has distinguished elements in it given by surfaces $\xi_{m',m}$, $\tau_{m',m}$, and $\eta_{m',m}$ associated to pairs of non-negative integers m', m that determine the surface's topological genus, all minimally embedded or immersed into $\mathbb{S}^3$, discovered by Lawson back in the late 1960s [12, Corollary 1.6, Theorems 2,3,4]. The surfaces $\xi_{m',m}$ are orientable while the surfaces $\eta_{m',m}$ are not. Some but not all of the $\tau_{m',m}$s are nonorientable. The immersions of the $\xi_{m',m}$s are by embeddings.

We denote by $g_{\xi_{m',m}}$ the intrinsic Riemannian metric on the Riemann surface $\xi_{m',m} \hookrightarrow \mathbb{S}^3$. If $m'm = 0$, say $m' = 0$ and $m > 0$, $\xi_{0,m}$ is the standard two sphere in $\mathbb{S}^3$, in which case we associate it to the single pair $(m', m) = (0, 1)$; otherwise, $\xi_{m',m}$ is a Riemann surface of genus $k = m'm$ and the embedding into $\mathbb{S}^3$ is minimal. When $k > 1$ is not a prime and $1, k \neq m \mid k$, the Riemann surfaces $\xi_{k,1}$ and $\xi_{k/m,m}$, both of genus k, have intrinsic metrics in different conformal classes distinguished from each other by the symmetries of their fundamental domains. We shall single out the surfaces $\xi_{k,1}$, which for $k \geq 1$ have equilateral fundamental domain [12, Proposition 6.1], and refer to them all (including the sphere $\xi_{0,1}$) as the equilateral Lawson Riemann surface of genus k.

The sphere $\xi_{0,1} = \mathbb{S}^2$ is geodesically embedded into $\mathbb{S}^3$ with area $\mu_{g_{\xi_{0,1}}}(\xi_{0,1}) = 4\pi$. The Clifford torus $\xi_{1,1} = \mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2})$ is linearly embedded into $\mathbb{S}^3$ with area $\mu_{g_{\xi_{1,1}}}(\xi_{1,1}) = 2\pi^2$. There is no such an explicit expression for $\mu_{g_{\xi_{k,1}}}(\xi_{k,1})$ when $k > 1$, but it is known that none of the then metrics $g_{\xi_{k,1}}$ are of constant scalar curvature [12, Proposition 1.5], that $\mu_{g_{\xi_{k,1}}}(\xi_{k,1}) < 8\pi$, and that $\mu_{g_{\xi_{k,1}}}(\xi_{k,1}) \nearrow 8\pi$ as $k \nearrow \infty$ [9, 10].

If Σ is not orientable, if $k > 2$, the minimal surface $\eta_{k-1,1} = f_{\eta_{k-1,1}}(\Sigma) \hookrightarrow \mathbb{S}^3$ has topological genus k, but in general has self-intersections [12, §8]. A nonorientable surface of topological genus $k = 2$ is a Klein bottle, and Lawson realizes the most symmetric one of them as the associated bipolar surface $\Sigma = \tilde{\tau}_{3,1}$ of his manifold $\tau_{3,1}$, which is in the conformal class of the square torus $\mathbb{R}^2/\Gamma$ of lattice Γ generated by (π, π) and $(\pi, -\pi)$, and can be explicitly described by a doubly periodic immersion into $\mathbb{S}^3$ [12, §7 & §11]. The parametrization of $\Sigma = \tilde{\tau}_{3,1}$ can be given then explicitly [11, §3], and we obtain a minimal isometric embedding $f_g : (\tilde{\tau}_{3,1}, g_{\tilde{\tau}_{3,1}}) \to (\mathbb{S}^4, \tilde{g}) \hookrightarrow (\mathbb{S}^5, \tilde{g})$, where by an explicit calculation, the metric is of area $\mu_{g_{\tilde{\tau}_{3,1}}}(\tilde{\tau}_{3,1}) = 6\pi E(2\sqrt{2}/3)$, E the complete elliptic integral. The surface $\mathbb{P}^2(\mathbb{R})$, of genus 1, can be minimally embedded into $\mathbb{S}^4$ by a map from the standard sphere $\mathbb{S}^2(r)$ of radius $r = \sqrt{3/2}$ that is a 2-to-1 cover of the embedding, with the conformal structures of base and cover consistent with each other [4, 18], and area $\mu_{g_{\mathbb{P}^2}}(\mathbb{P}^2(\mathbb{R})) = 6\pi$.

2. Conformal deformations of isometric embeddings of surfaces

Given the embedding (1) of (M, g) into $(\mathbb{S}^{\tilde{n}}, \tilde{g})$, suppose that

$$[0,1] \ni t \to f_{g_t} : (M, g_t) \mapsto (\mathbb{S}^{\tilde{n}}, \tilde{g}) \tag{5}$$

is a path of conformally related isometric embedding deformations of $f_g = f_{g_t|_{t=0}}$ of conformally related Riemannian metrics

$$[0,1] \ni t \to g_t = e^{2\psi(t)} g \tag{6}$$

on M, $\psi(t)$ a path of functions, $\psi(0) = 0$. By the Palais isotopic extension theorem, there exists a smooth one parameter family of diffeomorphisms

$$F_t : \mathbb{S}^{\tilde{n}} \to \mathbb{S}^{\tilde{n}}$$

such that $F_t(f_g(x)) = f_{g_t}(x)$, and, by restriction, we obtain a diffeomorphism

$$F_t : (f_g(M), \tilde{g}) \to (f_{g_t}(M), \tilde{g}) \, .$$

The tensor $F_t^* \tilde{g}$ is just the metric $\tilde{g}$ on $\mathbb{S}^{\tilde{n}}$ acted on by the diffeomorphism F_t, and since the g_ts are conformal deformations of g, we have that

$$F_t^* \tilde{g} \mid_{f_{g_t}(M)} = e^{2u(t)(f_g(\,\cdot\,))} \tilde{g} \mid_{f_g(M)} = e^{2u(t)(f_g(\,\cdot\,))} g \, ,$$

where the conformal factor $e^{2u(t)}$ and that in (6) are related to each other by $e^{2\psi(t)(\,\cdot\,)} = e^{2u(t) \circ f_g(\,\cdot\,)}$. We extend $u(t)$ conveniently to a function on the whole of $\mathbb{S}^{\tilde{n}}$, and view the family (5) as the family of conformal isometric embeddings

$$f_{g_t} : (M, e^{2u(t) \circ f_g} g) \hookrightarrow (\mathbb{S}^{\tilde{n}}, e^{2u(t)} \tilde{g})$$

of the fixed manifold $f_g(M)$ with a varying metric on it. We have then equations arising by the transformations of the exterior quantities in (2) associated to $f_{g_t}(M)$ and $f_g(M)$, respectively.

The transformation of the exterior scalar curvature yields the equation

$$n(n-1) = e^{-2u(t)}(n(n-1)-2(n-1)(\mathrm{div}_{f_g(M)}\nabla^{\tilde{g}}u^\tau - \tilde{g}(H_{f_g}, \nabla^{\tilde{g}}u^\nu) - \|du^\tau\|^2 + \frac{n}{2}\|du\|^2_{\tilde{g}})) , \tag{7}$$

where the upper indices τ and ν stand for the tangential and normal components, respectively, while the transformations of the squared norm of the mean curvature vector and second fundamental forms on $f_{g_t}(M)$ and $f_g(M)$ yield the equations

$$\|H_{f_{g_t}}\|^2 = e^{-2u(t)}(\|H_{f_g}\|^2 - 2n\tilde{g}(H_{f_g}, \nabla^{\tilde{g}}u^\nu) + n^2\tilde{g}(\nabla^{\tilde{g}}u^\nu, \nabla^{\tilde{g}}u^\nu)) , \tag{8}$$

$$\|\alpha_{f_{g_t}}\|^2 = e^{-2u(t)}(\|\alpha_{f_g}\|^2 - 2\tilde{g}(H_{f_g}, \nabla^{\tilde{g}}u^\nu) + n\tilde{g}(\nabla^{\tilde{g}}u^\nu, \nabla^{\tilde{g}}u^\nu)) , \tag{9}$$

respectively. By (2), we then have that

$$s_{g_t} = e^{-2u(t)}\left(s_g - 2(n-1)\mathrm{div}_{f_g(M)}(\nabla^{\tilde{g}}u^\tau) - (n-1)(n-2)\tilde{g}(\nabla^{\tilde{g}}u^\tau, \nabla^{\tilde{g}}u^\tau)\right) , \tag{10}$$

and that the densities of the functionals $\mathcal{W}_{f_{g_t}}(M)$ and $\mathcal{D}_{f_{g_t}}(M)$ in (4) are

$$\begin{array}{l} e^{-2u(t)}(n^2 + 2n\Delta^g u + \|H_{f_g}\|^2 + n(n-2)\|du^\tau\|^2) , \\ e^{-2u(t)}(n + 2\Delta^g u + \|\alpha_{f_g}\|^2 + (n-2)\|du^\tau\|^2) , \end{array}$$

respectively, so though defined through their isometric embeddings, $\mathcal{W}_{f_g}(M)$ and $\mathcal{D}_{f_g}(M)$ are well-defined intrinsic functionals in the space $\mathcal{M}_{[g]}(M)$ of metrics in the conformal class of g, as we mentioned earlier.

2.1. **Minimal embeddings of surfaces.** In the two dimensional case, the functional $\mathcal{W}_{f_g}(M)$ in (4) over the sphere $\mathbb{S}^3 \hookrightarrow \mathbb{S}^{\tilde{n}}$ background is known as the Willmore energy of $f_g(M)$. We rename M^2 as Σ, and denote by $\mathcal{M}_{[g]}(\Sigma)$ and $\mathcal{C}(\Sigma)$ the spaces of metrics of Σ in the conformal class $[g]$ and set of conformal classes of metrics, respectively.

By (7), (8) and (9) we obtain the identities

$$\begin{array}{rcl} \Theta_{f_{g_t}}(\Sigma) & = & \Theta_{f_g}(\Sigma) + 2\displaystyle\int_{f_g(\Sigma)} (\langle H_{f_g}, \nabla^{\tilde{g}}u^\nu\rangle - \langle \nabla^{\tilde{g}}u^\nu, \nabla^{\tilde{g}}u^\nu\rangle) d\mu_g , \\ \mathcal{W}_{f_{g_t}}(\Sigma) = 2\Theta_{f_{g_t}}(\Sigma) + \Psi_{f_{g_t}}(\Sigma) & = & 2\Theta_{f_g}(\Sigma) + \Psi_{f_g}(\Sigma) = \mathcal{W}_{f_g}(\Sigma) , \\ \mathcal{D}_{f_{g_t}}(\Sigma) = \Theta_{f_{g_t}}(\Sigma) + \Pi_{f_{g_t}}(\Sigma) & = & \Theta_{f_g}(\Sigma) + \Pi_{f_g}(\Sigma) = \mathcal{W}_{f_g}(\Sigma) , \end{array}$$

the first one of which implies that if f_{g_t} is conformally related to a minimal f_g, then $\mathrm{area}_{g_t}(\Sigma) \leq \mathrm{area}_g(\Sigma)$, while the latter two show the invariance of each, $\mathcal{W}_{f_g}(\Sigma)$ and $\mathcal{D}_{f_g}(M)$, under conformal deformations. The total scalar curvature functional of (Σ, g) is the topological constant

$$\mathcal{S}_g(\Sigma) = 4\pi\chi(\Sigma) = \mathcal{W}_{f_g}(\Sigma) - \mathcal{D}_{f_g}(\Sigma) ,$$

written as the difference of functionals that are invariant under conformal deformations f_{g_t} of f_g. We thus have that if $F : (\mathbb{S}^{\tilde{n}}, \tilde{g}) \to (\mathbb{S}^{\tilde{n}}, \tilde{g})$ is a conformal diffeomorphism and $f_g(\Sigma) \in \mathcal{M}_{\mathbb{S}^{\tilde{n}}}(\Sigma)$, then $F(f_g(\Sigma)) \hookrightarrow (\mathbb{S}^{\tilde{n}}, \tilde{g})$ has intrinsic metric $\tilde{g}\mid_{F(f_g(\Sigma))}$, and $\mathcal{W}_{f_g}(\Sigma) = \mathcal{W}_{F\circ f_g}(\Sigma)$, so we naturally define

$$\mathcal{W}(\Sigma, [g]) = \inf_{g' \in \mathcal{M}_{[g]}(\Sigma)} \mathcal{W}_{f_{g'}}(\Sigma) = \inf_{g' \in \mathcal{M}_{[g]}(\Sigma)} \int_{f_{g'}(\Sigma)} \left(4 + \|H_{f_{g'}}\|^2\right) d\mu_{f_{g'}} , \tag{11}$$

and attempt to find its optimal isometric embedding realizer so that we can compare the values of the class invariant differentiable function

$$\mathcal{C}(\Sigma) \ni [g] \to \mathcal{W}(\Sigma, [g])$$

across conformal classes of metrics by comparing the values that the optimal realizers of each class achieve. For any surface Σ^k of diffeomorphism type fixed by the topological genus k, there exists a distinguished conformal class $[g_k]$, and metric g_k representing it of minimal isometric embedding $f_{g_k} \in \mathcal{M}_{\mathbb{S}^{\tilde{n}}}(\Sigma)$, such that

$$\mathcal{W}_{f_g}(\Sigma) = \mathcal{W}(\Sigma, [g]) \geq \mathcal{W}(\Sigma, [g_k]) = \mathcal{W}_{f_{g_k}}(\Sigma)\,, \tag{12}$$

with the equality achieved if, and only if, $[g] = [g_k]$ and f_g is conformally equivalent to f_{g_k} in the sphere background [19, Theorems 1 & 9]. Notice that if g_t is a path of conformal deformations of g so $[g_t] = [g]$, the Palais' path of diffeomorphism $F_t : \mathbb{S}^{\tilde{n}} \to \mathbb{S}^{\tilde{n}}$ for which $F_t^* \tilde{g} \mid_{f_{g_t}(\Sigma)} = g_t$ starts at $\mathbb{1}_{\mathbb{S}^{\tilde{n}}}$, and set theoretically fixes $f_g(\Sigma)$ for all t, with the metric $F_t^* \tilde{g} = e^{2u(t)} \tilde{g}$ inducing the intrinsic conformal metric deformation of g when restricted to the submanifold $f_g(\Sigma)$, and $\mathcal{W}_{f_{g_t}}(\Sigma) = \mathcal{W}(\Sigma, [g])$. Although the values of conformally invariant Riemannian functionals along the g_ts stay the same, the nontensorial quantities ∇^{g_t} vary with the t-dependent gauge over $f_g(\Sigma)$. If the starting metric g has minimal isometric embedding f_g and the conformal deformations g_t are through area preserving metrics, the minimality of the embeddings f_{g_t} remains, assertion last that comes at the expense of knowing the area of the metric on Σ with minimal isometric embedding f_g, and the finding of an appropriately related path of background sphere diffeomorphism F_t realizing the minimal conformal deformations of f_g.

Theorem 1. *Suppose that $f_g : (\Sigma^k, g) \to (\mathbb{S}^{2+q}, \tilde{g})$ is a minimal isometric embedding of a closed surface (Σ^k, g), and that $[0,1] \ni t \to g_t = e^{2u(t)} g$ is a path of area preserving conformal deformations of g on $f_g(\Sigma)$. Then there exists a path of conformal diffeomorphism $F_t : (\mathbb{S}^{2+p}, F_t^* \tilde{g}) \to (\mathbb{S}^{2+p}, \tilde{g})$ such that $F_t \mid_{t=0} = \mathbb{1}_{\mathbb{S}^{2+p}}$, $F_t(f_g(M)) = f_g(M)$, $F_t^* \tilde{g} \mid_{f_g(M)} = g_t$ and $f_{g_t} : (\Sigma, g_t) \to (\mathbb{S}^{2+p}, \tilde{g})$ is a path of conformal deformations of f_g by minimal embeddings. In particular, if s_g is not constant, there exists a path $f_{g_t} : (\Sigma, g_t) \to (\mathbb{S}^{2+q}, \tilde{g})$ of area preserving minimal conformal deformations of f_g to the isometric embedding $f_{g'}$ of a metric $g' \in \mathcal{M}_{[g]}(\Sigma)$ of scalar curvature $4\pi\chi(\Sigma)/\mu_g(\Sigma)$.*

Proof. Along any conformal deformation of f_g, by (7) we have that

$$2 = e^{-2u}(2 + 2\Delta^g u - 2\tilde{g}(\nabla^{\tilde{g}} u^\nu, \nabla^{\tilde{g}} u^\nu))\,,$$

so if the deformation is area preserving, by integration of this identity with respect to the area measure $e^{2u} d\mu_g$ of g_t, we conclude that $\|\nabla^{\tilde{g}} u^\nu\|^2 = 0$, and so $\nabla^{\tilde{g}} u^\nu$ is the zero vector along points of $f_{g_t}(\Sigma) \hookrightarrow \mathbb{S}^{2+q}$.

We let $u = u(t)$ be the function on $f_g(\Sigma)$ defining the conformal deformation $g_t = e^{2u(t)} g$ of g. By the tube lemma, there exist some $\varepsilon > 0$ and a smooth extension u_t of $u(t)$ to the whole of $\mathbb{S}^{2+q}$ that is a function of the distance in the normal directions to the submanifold $f_g(\Sigma)$, supported on the 2ε tubular neighborhood of $f_g(\Sigma)$, and constant in the normal directions in the ε tubular neighborhood inside it. The flow F_t of the time dependent vector field $\nabla^{\tilde{g}} u_t$ on $\mathbb{S}^{2+q}$ generates Palais' extension diffeomorphisms $F_t : \mathbb{S}^{2+q} \to \mathbb{S}^{2+q}$ such that such that $F_t^* \tilde{g} = e^{2u_t} \tilde{g}$ (cf. [17, Theorem p. 936]), and so $F_t^* \tilde{g} \mid_{f_{g_t}(\Sigma)} = e^{2u(t)} \tilde{g} \mid_{f_g(\Sigma)} = g_t$. Since $\nabla^{\tilde{g}} u_t^\nu \mid_{f_g(\Sigma)} = 0$, by (8), $\|H_{f_{g_t}}\|^2 = 0$. This proves the first part of the statement.

Suppose now that the scalar curvature s_g of $(f_g(\Sigma), g = \tilde{g} \mid_{f_g(M)})$ is not constant. By the classic uniformization theorem if the surface is orientable, or by a combination of that theorem with a 2-to-1 covering of orientable cover if the surface is not, there exists a path $u = u(t)$ of conformal factors e^{2u}, $u \mid_{t=0} = 0$, such that $[0, 1] \ni t \to g_t = e^{2u} g$ is a path of area preserving metrics on $f_g(\Sigma)$ satisfying the equation

$$s_{g_t} = e^{-2u}(s_g + 2\Delta^g u)$$

and $s_{g_t|_{t=1}} = \frac{4\pi\chi(\Sigma)}{\mu_g(\Sigma)}$. By the preliminary portion of the proof, we may then realize this path as the metrics induced by the associated gauge changing metric $F_t^* \tilde{g}$ of $\mathbb{S}^{2+q}$ on the minimally embedded surface $f_{g_t}(\Sigma)$ that set theoretically is $f_g(\Sigma)$. $\square$

In the case of a Riemann surface Σ^k of genus k, the distinguished conformal class in (12) is $[g_k] = [g_{\xi_{k,1}}]$, and we have that

$$\mathcal{W}_{f_g}(\Sigma) = \mathcal{W}(\Sigma, [g]) \geq \mathcal{W}_{f_{g_{\xi_{k,1}}}}(\Sigma) = \mathcal{W}(\xi_{k,1}, [g_{\xi_{k,1}}]) = 4\mu_{g_{\xi_{k,1}}}(\xi_{k,1}) \,, \tag{13}$$

with the equality achieved if, and only if, f_g is conformally equivalent to the isometric embedding $f_{g_{\xi_{k,1}}}$ of the equilateral Lawson Riemann surface [19, Theorem 1], and $\mathcal{W}(\xi_{k,1}, [g_{\xi_{k,1}}])$ increases monotonically with k to the upper bound 32π. The result in the elliptic case is due to Almgren, who proved the more general version asserting that the only minimal immersion of a genus 0 surface into $\mathbb{S}^3$ must be that of the standard sphere [1]. The case of $k = 1$, when the optimal lower bound is achieved by the Clifford torus $\xi_{1,1}$, is due to Codá & Neves [6], who proved also a more general statement when solving in the affirmative the original Willmore conjecture. These two cases can be obtained by using the lower and upper end, respectively, of the gap theorem of Simons [21, Theorem 5.3.2, Corollary 5.3.2] [4, Main Theorem] [13, Corollary 2], though the $k = 1$ case follows by a general argument that applies for any $k \geq 1$. But its alternative proof using Simons' gap theorem leads in addition to a proof of the theorem of Brendle [2] on an earlier conjecture of Lawson [14] stating that the flat torus $(\xi_{1,1}, g_{\xi_{1,1}})$ is the only Riemann surface of genus one that can be minimally embedded into $\mathbb{S}^3$ [19, Corollary 3]. For the metric g of any such embedded torus can be conformally deformed to a flat metric g' of the same area; by the gap theorem, $g' = g_{\xi_{1,1}}$ up to isometries of the background; and by Simons' identity on the rough Laplacian of the second fundamental form [21, Theorem 5.3.1], we can then show that $\mu_g = \mu_{g_{\xi_{1,1}}}$, and so $g = g' = g_{\xi_{1,1}}$.

If Σ^k is nonorientable, if $k = 1$, $\Sigma^1 = \mathbb{P}^2(\mathbb{R})$ and $[g_1]$ is the projective space only conformal class, with $\mu_{g_1}(\mathbb{P}^2(\mathbb{R})) = 6\pi$. If $k = 2$, $\Sigma^k = \tilde{\tau}_{3,1}$, $[g_2] = [g_{\tilde{\tau}_{3,1}}]$ and $\mu_{g_2}(\tilde{\tau}_{3,1}) = 6\pi E(2\sqrt{2}/3)$, as observed earlier. The remaining cases are not so explicitly known, but the areas of their metrics are known to be bounded above by 8π, and monotonically increase with k to that upper bound.

Theorem 2. *Suppose that $(\Sigma, g) = (\xi_{k/m,m}, g_{\xi_{k/m,m}})$, m any integer divisor of $k > 1$, and let us set $a = \mu_{g_{\xi_{k/m,m}}}(\Sigma)$. Then for any smooth path $[0, 1] \ni t \to g_t$ of area a conformal deformations of $g_{\xi_{k/m,m}}$ in the conformal class $[g_{\xi_{k/m,m}}]$ to a metric g_a of constant scalar curvature $4\pi\chi(\Sigma)/a$, there exists an associated path $f_{g_t} : (\Sigma, g_t) \to (\mathbb{S}^3, \tilde{g})$ of minimal isometric conformal deformations of $f_{g_{\xi_{k/m,m}}}$ to the isometric embedding f_{g_a} of g_a.*

Proof. Since $k > 1$, there are points on Σ where the metric $g_{\xi_{k/m,m}}$ is such that $s_{g_{\xi_{k/m,m}}} = 2$ [12, Proposition 1.5], and so, by the Gauss-Bonnet theorem, $s_{g_{\xi_{k/m,m}}}$ is not the constant function. By the uniformization theorem, the metric $g_{\xi_{k/m,m}}$ on $f_g(\Sigma) = \xi_{k/m,m}$ can be deformed by area preserving metrics in its conformal class to a metric g_a of constant scalar curvature, which by the Gauss-Bonnet theorem again, must be of scalar curvature $4\pi\chi(\Sigma)/a$. The result now follows by Theorem 1, as any area preserving conformal deformation of $g_{\xi_{k/m,m}}$ to g_a is realized by an associated gauge changing conformal diffeomorphism $F_t : \mathbb{S}^3 \to \mathbb{S}^3$ that fixes $f_g(\Sigma) = \xi_{k/m,m}$ and is such that $F_t^*\tilde{g} \mid_{f_g(\Sigma)}= g_t$, with corresponding family of minimal isometric conformal embedding deformations f_{g_t} of $f_{g_{\xi_{k/m,m}}}$ to the isometric embedding f_{g_a} of g_a. □

Remark 3. Besides the $\xi_{m',m}$s of Lawson, there are other families of Riemann surfaces embedded minimally into $\mathbb{S}^3$, which interestingly enough, are all of relatively large genus [8, 5, 7]. With each of them playing the role of the $\xi_{k/m,m}$ surfaces in the statement, Theorem 2 applies. Theorem 1 applies to any of the Willmore surfaces of Bryant [3] immersed into $\mathbb{S}^4$ when the immersion is an embedding, or equally well to the bipolar surface $\tilde{\tau}_{3,1}$ of topological genus 2 minimally embedded into $\mathbb{S}^4$ with area $6\pi E(2\sqrt{2}/3)$, and for which $g_{\tilde{\tau}_{3,1}}$ is not itself of constant scalar curvature.

3. The notion of σ invariant for surfaces

We let $\mathcal{M}_{a,[g]}(\Sigma)$ be the submanifold of metrics of area a in the conformal class of g, set $a = a([g]) := \frac{1}{4}\mathcal{W}(\Sigma, [g])$, and consider the functional

$$\mathcal{S}_a^2 : \mathcal{M}_{a,[g]}(\Sigma) \ni g \to \int s_g^2 d\mu_g \, .$$

This is not a conformally invariant functional, but its minimum is always achieved by a metric of constant scalar curvature $4\pi\chi(\Sigma)/a$, and for dimensional reasons, this minimum is the end point of a path of area preserving conformal deformations of g. By the Gauss-Bonnet theorem, and the extreme case of the Cauchy-Schwarz inequality, the critical value at this minimizer is

$$\mathcal{S}_a^2(\Sigma, [g]) = \frac{(4\pi\chi(\Sigma))^2}{a} \, .$$

The canonical realizer of the infimum (11) defining $\mathcal{W}(\Sigma, [g])$ is thus the f_g of an Einstein metric g in $[g]$ of minimal area [19, Theorem 6], which if $\chi(\Sigma) \leq 0$, can be uniquely associated to the fundamental domain of the conformal class $[g]$ when Σ is orientable, or to that of the conformal class of the lifted metric to its 2-to-1 orientable cover when Σ is not [19, Theorem 8]. The distinguished conformal class $[g_k] \in \mathcal{C}(\Sigma)$ in (12) is the one whose associated fundamental domain is the most symmetric of them all, and the associated metric representative g_k has the smallest area among the canonical realizers of the various elements of $\mathcal{C}(\Sigma)$, fixing the optimal scale among them.

Thus, we define

$$\sigma^2(\Sigma) = \sup_{[g]\in\mathcal{C}(\Sigma))} \mathcal{S}^2_{a([g])}(\Sigma, [g]) \, .$$

In the orientable case, the bounds (13) yield the following:

Theorem 4. *If $\Sigma = \Sigma^k$ is a Riemann surface of genus k, and $f_g : (\Sigma, g) \to (\mathbb{S}^{\tilde{n}}, \tilde{g})$, we have that*

$$\frac{(4\pi\chi(\Sigma))^2}{\frac{1}{4}\mathcal{W}_{f_g}(\Sigma)} \leq \sigma^2(\Sigma) = \frac{(4\pi\chi(\Sigma))^2}{\frac{1}{4}\mathcal{W}(\Sigma, [g_{\xi_{k,1}}])} = \frac{(4\pi\chi(\Sigma))^2}{\mu_{g_{\xi_{k,1}}}(\xi_{k,1})} ,$$

and for $k \neq 1$, the equality is achieved by f_g if, and only if, $[g] = [g_{\xi_{k,1}}]$ and f_g is conformally equivalent to $f_{g_{\xi_{k,1}}}$, while for $k = 1$, if $\mathcal{W}(\Sigma, [g]) = \mathcal{W}(\Sigma, [g_{1,1}])$ then $[g] = [g_{1,1}]$ and f_g is conformally equivalent to $f_{g_{1,1}}$.

In the general situation, irrespective of the orientation, of a surface Σ^k of topological genus k and distinguished conformal class $[g_k]$ for which (12) holds, we have that

$$\frac{(4\pi\chi(\Sigma))^2}{\frac{1}{4}\mathcal{W}_{f_g}(\Sigma)} \leq \sigma^2(\Sigma) = \frac{(4\pi\chi(\Sigma))^2}{\frac{1}{4}\mathcal{W}(\Sigma, [g_k])} ,$$

and if $\chi(\Sigma) \neq 0$, the equality is achieved by f_g if, and only if, $[g] = [g_k]$ and f_g is conformally equivalent to f_{g_k}, while if $\chi(\Sigma) = 0$ and $\mathcal{W}_{f_g}(\Sigma) = \mathcal{W}(\Sigma, [g_k])$ then $[g] = [g_{\xi_{1,1}}]$ if Σ^k is orientable or $[g] = [g_{\tilde{\tau}_{3,1}}]$ otherwise, and in either case, f_g is conformally equivalent to f_{g_k}. Thus, we always have an optimal conformal class that achieves $\sigma^2(\Sigma)$, even when $\chi(\Sigma) = 0$. We set

$$\sigma(\Sigma, [g]) = \frac{8\pi\chi(\Sigma)}{\mathcal{W}(\Sigma, [g])^{\frac{1}{2}}} ,$$

and define the sigma invariant of the surface by

$$\sigma(\Sigma) = \sup_{[g]\in\mathcal{C}(\Sigma)} \sigma(\Sigma, [g]) = \mathrm{sign}(\chi(\Sigma))\sqrt{\sigma^2(\Sigma)} = \frac{8\pi\chi(\Sigma)}{\mathcal{W}(\Sigma, [g_k])^{\frac{1}{2}}} ,$$

taking f_{g_k} as its optimal realizer. The sign of $\chi(\Sigma)$ determines the sign of any constant scalar curvature metric g in $\mathcal{M}(\Sigma)$, regardless of scale or conformal class, and we have the universal bound

$$\sigma(\Sigma) \leq \sigma(\mathbb{S}^2) = 4\pi^{\frac{1}{2}} .$$

Theorem 5. *Let Σ^k be a surface of topological genus k and distinguished conformal class $[g_k]$. Then $16\pi \leq \mathcal{W}(\Sigma, [g_k]) < 32\pi$, and*

$$\mathcal{W}(\Sigma, [g_k]) \leq \mathcal{W}(\Sigma, [g])$$

for any conformal class of metrics $[g]$ on Σ.

a) *If $\chi(\Sigma) < 0$, then $\sigma(\Sigma, [g_k]) \leq \sigma(\Sigma, [g])$.*
b) *If $\chi(\Sigma) \geq 0$, then $\sigma(\Sigma, [g]) \leq \sigma(\Sigma, [g_k])$. If $\chi(\Sigma) = 0$, the distinguished conformal classes in the orientable and nonorientable cases are $[g_{\xi_{1,1}}]$ and $[g_{\tilde{\eta}_{3,1}}]$ with Willmore energies $8\pi^2$ and $24\pi E(2\sqrt{2}/3)$, respectively, while if $\chi(\Sigma) > 0$, Σ orientable or not carries a unique conformal class of metrics, $\sigma(\mathbb{S}^2) = 4\pi^{\frac{1}{2}}$ and the Willmore energy of its class is 16π, while $\sigma(\mathbb{P}^2(\mathbb{R})) = \frac{4}{\sqrt{6}}\pi^{\frac{1}{2}}$ and the Willmore energy of its class is 24π.*

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