Mean-field analysis of a neural network with stochastic STDP

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Analysing biological spiking neural network models with synaptic plasticity has proven to be challenging both theoretically and numerically. In a network with N all-to-all connected neurons, the number of synaptic connections is on the order of N^2, making these models computationally demanding. Furthermore, the intricate coupling between neuron and synapse dynamics, along with the heterogeneity generated by plasticity, hinder the use of classic theoretical tools such as mean-field or slow-fast analyses. To address these challenges, we introduce a new variable which we term a typical neuron X. Viewed as a post-synaptic neuron, X is composed of the activity state V, the time since its last spike S, and the empirical distribution ξ of the triplet V, S and W (incoming weight) associated to the pre-synaptic neurons. In particular, we study a stochastic spike-timing-dependent plasticity (STDP) model of connection in a probabilistic Wilson-Cowan spiking neural network model, which features binary neural activity. Taking the large N limit, we obtain from the empirical distribution of the typical neuron a simplified yet accurate representation of the original spiking network. This mean-field limit is a piecewise deterministic Markov process (PDMP) of McKean-Vlasov type, where the typical neuron dynamics depends on its own distribution. We term this analysis McKean-Vlasov mean-field (MKV-MF). Our approach not only reduces computational complexity but also provides insights into the dynamics of this spiking neural network with plasticity. The model obtained is mathematically exact and capable of tracking transient changes. This analysis marks the first exploration of MKV-MF dynamics in a network of spiking neurons interacting with STDP.

During the last few decades, synaptic plasticity has been widely studied and its importance in the study of neural networks is far from being completely understood. Many fascinating questions remain on the network structure formation [1], its stability [2, 3] and its function [4]. From a modelling point of view, one of the most studied plasticity rule is the so-called spike-timing-dependent plasticity (STDP). Biological neural network models with STDP pose challenges for both theoretical and numerical analyses. The numerical bottleneck arises from the N^2 scaling of synapses' number when a large number N of neurons is considered. Furthermore, the intricate coupling between the dynamics of the spiking neurons and the synapses, along with plasticity-induced heterogeneity pose many difficulties.

One approach to address these challenges is to assume that plasticity is very slow compared to neural activity, and thus to leverage slow-fast theory. In that context, given a pair of spikes, the model of synaptic weight dynamics can be deterministic or stochastic. In the deterministic case, each pair leads to a tiny (infinitesimal) increase in weights [5] that usually depends on the time between spikes according to biological experiments of STDP [6]; see [7, 8] for a general mathematical formulation of STDP rules and their scaling analysis on one synapse dynamics. Hence, we may observe a significant (macroscopic) change of the weights only after several of such spike pairs. In the stochastic case, every pair of spikes can lead to a macroscopic weight change with a probability function of the spike time differences [9]: in this context, the slow property of the plasticity is a consequence of the small value of the probability to change for each pair of spikes [10–13]. While these models show similarities when averaging across realisations, their differing definitions require the use of separate analytical approaches (deterministic versus probabilistic).

However, for both stochastic and deterministic cases, results remain incomplete when considering neural networks with STDP. Indeed, they involve intractable quantities; respectively the stationary distribution of the neural network [12, 13] and complex spike statistics in heterogeneous networks [1]. In addition, they both still involve N neurons as no mean-field (MF) has been carried out on them. To be complete here, a MF limit was derived in a spiking network assuming slow plasticity depending on the global activity (in contrary to pairwise activity dependence in STDP) and using probabilistic tools [14] or deterministic tools (PDE analysis on the Fokker-Planck equations) [15].

Finally, the fact that plasticity dynamics is slow compared to the neural dynamics is controversial. Indeed, it has been shown that synaptic plasticity can occur at very different timescales. For example, the shrinkage and

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enlargement of dendritic spines are respectively of the order of dozen of minutes and one minute [16]. In addition, receptor resources (responsible for the strength of a synaptic connection) can get in/out the cell at the order of milliseconds [17]. On the pre-synaptic side, bouton formation occurs on the order of a few minutes [18]. Finally, it was recently shown that the Drosophila mushroom body is segmented in different clusters where plasticity has different timescales regulated by dopamine; see [19] for the original paper and [20] for a review of recent advances on this topic. All of this tends towards multiple plasticity timescales within the brain and not only slow ones compared to neural dynamics [21].

Hence, the slow-fast assumption is not universally applicable in the brain [22, 23], and it is unclear how to analyse such complex systems without relying on the latter. The numerical implementation of STDP in a neural network model has already been done in many previous studies; see for example [1, 24–27]. However, rigorous mathematical analysis of such complex interaction has yet to be done. In addition, no simplified model – such as the one obtained with MF – has been proposed, hence leaving the computational bottleneck unsolved. In the setting proposed by Pechersky et al. [28], the synaptic plasticity and the neural dynamics operate at comparable timescales. The weights in their plastic spin model form a transient Markov chain that almost surely diverges to infinity, limiting the biological plausibility of this framework. In non-transient cases, one natural mathematical method to use is a MF approximation which has already been used to analyse some interacting neural networks without plasticity; see for instance [29–35].

However, when dealing with spiking networks with plastic interactions, traditional MF theory tools become ineffective due to the heterogeneity and dynamic nature of these interactions. As a result, a belief has emerged suggesting that MF theory is unsuitable for analysing networks with STDP [36, 37]. Despite this, a MF limit can still be derived by making certain compromises, such as assuming a symmetric STDP curve, and employing theories like the Ott-Antonsen ansatz [38]. Additionally, it has been suggested that MF theory does not account for spike correlations as it dilutes the impact of individual spike pairs by a factor of $\frac{1}{N}$ and mainly depend on the mean of synaptic weights [39]. Thereby, MF theory does not take into account the synaptic weights' heterogeneity [35]. This explains in part the lack of results in this area. Another reason is that MF limits have been derived using ansatz (like the famous Ott-Antonsen ansatz [29, 38]) or trying to find closed equations on the first order statistics of the variable of interest; the latter methods break down when considering heterogeneous and plastic networks.

Here, we use another approach based on deriving the limit equation for the empirical distribution of the variable of interest [40]; we call it the McKean-Vlasov mean-field (MKV-MF) limit. Hence, our analysis marks the first exploration of MKV-MF dynamics in a spiking neural network of interacting neurons with STDP; where, in response to the previous critics on MF, weights are heterogeneous and spike correlations are part of the time since last spike distributions. More specifically, we study in this article a probabilistic Wilson-Cowan neural network model in which neurons elicit spikes and interact through a stochastic STDP rule. The approach we developed is broadly applicable and may be adapted to other neural network models, for example featuring more biophysically realistic membrane potential dynamics.

The proposed framework is not limited to the study of STPD; it can also be used to derive MF models of other systems composed of interacting units such as spiking lasers [41], Ising models with plastic interactions [28] or epidemic processes on complex networks [42]. In addition, it opens the door to new mathematical questions such as establishing the uniqueness of the solution to an equation similar to the limit system we derived and confirming the convergence to a deterministic limit distribution solution of the latter [40]. Moreover, studying the limit system would provide insight in the initial model. In particular, this study opens new avenues for preventing weight divergence without relying on soft or hard bounds. In addition, our approach significantly reduces simulation costs, crucial for models involving synaptic plasticity in neural networks. A pertinent example is the recent development of adaptive deep brain stimulation (DBS). Therefore, modelling the effects of DBS on the synaptic weights of stimulated neurons using STDP could aid in identifying stimulation patterns that lead to long-term changes [38]. The potential consequences are huge as DBS is used to alleviate symptoms in many brain diseases like movement disorders, depression, obsessive-compulsive disorders (OCD), Alzheimer's disease and epilepsy [43].

The paper is organised as follows. In section I, we first present the microscopic model (initial neural network description) before presenting the first hints on the macroscopic model. In particular, we introduce the *new* variables describing the neural network from which we perform a MF analysis in section II. Hence, the latter is devoted to the derivation of the limit equation from the analysis of the empirical distribution of the *new* neural network description. Thereby, a McKean-Vlasov equation [44] is derived on a typical neuron which is composed of: the neuron state, the time from its last spike and the distribution of the triplet composed of the neuron state, the time since its last spike and its pre-synaptic weights. We finally perform numerical simulations on these last equations in section III and compare the resulting activity to the finite system one (ground truth).

I. PRESENTATION OF THE MODEL

The model is a special case of the one studied in [12, 13]. We first present it before deriving its new description based on the typical neuron definition.

A. The microscopic model

We study a network of N binary neurons, $V_t^{i,N} \in \{0,1\}$, all-to-all connected via the matrix of synaptic weights, $W_t^N \in \mathbb{Z}^{N^2}$. Neuron i is said to *spike* when its membrane potential, $V_t^{i,N}$, jumps from 0 to 1. This occurs at rate $\alpha(I_t^{i,N})$ where

$$I_t^{i,N} := \frac{1}{N}\sum_j W_t^{ij,N} V_t^{j,N} \tag{1}$$

is the incoming synaptic current into neuron i at time t. The function α is positive and bounded (usually a sigmoid function).

We denote $S_t^{i,N} \geq 0$ the time spent since the last spike of the neuron i. We are interested in the PDMP such that for all i the dynamics between jumps is

- $dS_t^{i,N} = dt$.

The jumps are

- First, the jumps of $(V_t^{i,N}, S_t^{i,N})$

$$(0, S_t^{i,N}) \xrightarrow{\alpha(I_t^{i,N})} (1,0),$$
$$(1, S_t^{i,N}) \xrightarrow{\beta>0} (0, S_t^{i,N}),$$

- **at a spiking time τ_{i_0} of neuron i_0**, we have $V_{\tau_{i_0}^-}^{i_0,N} = 0$, $V_{\tau_{i_0}}^{i_0,N} = 1$ and $S_{\tau_{i_0}}^{i_0,N} = 0$. The weights $W_{\tau_{i_0}^-}^{ji_0,N}$ and $W_{\tau_{i_0}^-}^{i_0 j,N}$ jump independently for all j:

$$W_{\tau_{i_0}^-}^{ji_0,N} \xrightarrow{p^-(S^j_{\tau_{i_0}^-}, W^{ji_0,N}_{\tau_{i_0}^-})} W_{\tau_{i_0}^-}^{ji_0,N} - 1$$
$$W_{\tau_{i_0}^-}^{i_0 j,N} \xrightarrow{p^+(S^j_{\tau_{i_0}^-}, W^{i_0 j,N}_{\tau_{i_0}^-})} W_{\tau_{i_0}^-}^{i_0 j,N} + 1,$$

where $p^\pm$ are functions (representing jump probabilities) from $\mathbb{R}^+ \times \mathbb{Z}$ to $[0,1]$. Thus, the firing rate is given by $\alpha(I_t^{i,N})$ and the neurons return to their resting potential 0 at constant rate $\beta > 0$. The weight $\frac{1}{N} W_t^{ij,N}$ represents the effect of a neuron j on the neuron i at time t. Note that we do not assume that the diagonal elements $W_t^{ii,N}$ are null. Instead, we assume that they follow dynamics similar to that of $W_t^{ij,N}$ for $i \neq j$. This means that whenever neuron i spikes, $W_t^{ii,N}$ can either stay the same, increase, or decrease. Regardless of these changes, these weights do not affect the dynamics because they appear in the firing rates of neuron i only as $W_t^{ii} V_t^i$. Since, when potentially firing, neuron i is at rest, $V_t^i = 0$, it makes the latter product null.

The neuron model (stochastic Wilson-Cowan model) has been widely studied [45, 46]. It reproduces many biological features of a network such as oscillations, bi-stability, metastability, and memory formation [47]. Thus, we obtain a model rich enough to reproduce biological phenomena, but still simple enough to be mathematically tractable and easy to be simulated with thousands of neurons.

We end the description of the microscopic model by giving assumptions on the initial conditions.

Assumption A1. *For any number of neurons N, the initial conditions $\left((V_0^{i,N}, S_0^{i,N}, (W_0^{ij,N})_{1\leq j\leq N})\right)_{1\leq i\leq N}$ are assumed to be independent and identically distributed. The distribution ρ_0 of the second component $S_0^{i,N}$ does not depend on N, is absolutely continuous with respect to the Lebesgue measure and has a bounded density. However, the third component W belongs to $\mathbb{Z}^{N^2}$ and thus its distribution depends on N.*

Under this assumption, the neural network possesses two important properties that are extensively used throughout this paper. First, the times from the last spike are almost surely distinct, see Lemma 8. Second, at any time $t \geq 0$, their distributions are also absolutely continuous with respect to the Lebesgue measure (admits a probability density), see Lemma 9.

B. Towards the macroscopic model

Definitions of the auxiliary variables

In the following, with a slight abuse of notation, we also call *neuron* the process describing the neuron. The neuron i, $X_t^{i,N}$, includes the state of the neuron $V_t^{i,N}$, its time since last spike $S_t^{i,N}$ and the empirical distribution $\xi_t^{i,N}$ of the triplets $(V_t^{j,N}, S_t^{j,N}, W_t^{ij,N})_{1\leq j\leq N}$, that is the joint distribution of the incoming weights and the neural state (V,S) associated to them.

Definition 1. *We define the following random variable on the space of probability distributions on*

$$E_m := \{0,1\} \times \mathbb{R}^+ \times \mathbb{Z} : \tag{2}$$

$$\forall i \in \{1, \cdots, N\}, \qquad \xi_t^{i,N} := \frac{1}{N}\sum_j \delta_{(V_t^{j,N}, S_t^{j,N}, W_t^{ij,N})},$$

where δ is the Dirac delta distribution.

In particular, denoting by $\mathcal{P}_N(G)$ the set of atomic probability distributions on G with N distinct atoms with uniform weights $\frac{1}{N}$, we have that for all outcome $\omega \in \Omega$ $\xi_t^{i,N}(\omega) \in \mathcal{P}_N(E_m)$.

Using the function $I : \mathcal{P}(E_m) \to \mathbb{R}$:

$$\forall \xi \in \mathcal{P}(E_m), \quad I(\xi) := \mathbb{E}^\xi(WV) = \int_{E_m} wv \; \xi(dv, ds, dw), \tag{3}$$

we note that the knowledge of $\xi_t^{i,N}$ is enough to obtain the incoming synaptic current $I_t^{i,N} = I(\xi_t^{i,N})$.

We can now re-write the model with the new definition of a neuron. The neurons are now described by the following triplets, for all $i \in \{1, \cdots, N\}$,

$$X_t^{i,N} \coloneqq (V_t^{i,N}, S_t^{i,N}, \xi_t^{i,N})$$

and the system state is then given by

$$X_t^N \coloneqq (X_t^{1,N}, \cdots, X_t^{N,N}).$$

Now that we have defined the auxiliary variables that describe the neural network, we can define the empirical distribution associated to the *new* neural network description, $(\mu_t^N)_{t\geq 0}$. Importantly, it is defined on a space that does not depend on N.

Definition 2. *The empirical distribution*

$$\mu_t^N \coloneqq \frac{1}{N}\sum_i \delta_{X_t^{i,N}} = \frac{1}{N}\sum_i \delta_{(V_t^{i,N}, S_t^{i,N}, \xi_t^{i,N})} \tag{4}$$

is a random probability distributions over the space of probability distribution on

$$E \coloneqq \{0,1\} \times \mathbb{R}^+ \times \mathcal{P}(E_m). \tag{5}$$

Hence, for all $\omega \in \Omega$, $\mu_t^N(\omega) \in \mathcal{P}_N(E) \subset \mathcal{P}(E)$.

In the following, we keep in mind the randomness of μ_t^N and $(\xi_t^{i,N})_{1\leq i\leq N}$ and we alleviate the notations by referring to the outcomes $\omega \in \Omega$ only when it helps understanding. From the definition of $(\mu_t^N)_{t\geq 0}$ and the ones of $(\xi_t^{1,N})_{t\geq 0}, \cdots, (\xi_t^{N,N})_{t\geq 0}$, we note that they have the same joint distribution on V and S.

Remark 3. *For all* $i \in [\![1, N]\!]$ *and* $t \geq 0$, *we have by Definitions 1 and 2 that for all* $(v, A) \in \{0,1\} \times \mathcal{B}(\mathbb{R}^+)$,

$$\mu_t^N(v, A, \mathcal{P}(E_m)) = \xi_t^{i,N}(v, A, \mathbb{Z}).$$

In what follows, we denote this property by: $\mu_t^N(\cdot,\cdot,\mathcal{P}(E_m)) = \xi_t^{i,N}(\cdot,\cdot,\mathbb{Z})$.

Dynamics of the system

Under Assumption A1, the initial *neurons* $X_0^{1,N}, \cdots, X_0^{N,N}$ are N independent copies. This assumption is at the heart of our ability to describe the dynamics of X_t^N.

Remark 4. *Even though the knowledge of* $(\xi_t^{i,N})_{1\leq i\leq N}$ *is sufficient to evaluate the spiking rates of the neurons* $(X_t^{i,N})_{1\leq i\leq N}$ *, they are empirical distributions and thus do not contain the labels of the pre-synaptic neurons. In particular, as soon as one neuron, say neuron* i_0, *has a jump of* $V_t^{i_0,N}$, *we have to propagate this jump (and its potential consequences, such as a reset of* $S_t^{i_0,N}$ *and changes in the synaptic weights) to every* $(\xi_t^{i,N})$. *For any* i, *we know that one atom of* $\xi_t^{i,N}$ *corresponds to the neuron* i_0 *but we have to find it. Thanks to Lemma 8, we identify this atom with the value* $S_t^{i_0,N}$ *which is unique for every neuron* $X_t^{i_0,N}$.

To do so, we denote by $\mathcal{T}_t$ the set of times $\tau \in [0,t]$ such that $V_\tau^{i,N} \neq V_{\tau^-}^{i,N}$ for some $i \in \{1, \cdots, n\}$. First, in between two consecutive jumps at times τ and $\tilde{\tau}$, $\tau < \tilde{\tau}$, the time since the last spike of all neurons increases linearly: for all $i \in \{1, \cdots, N\}$ and $t \in [\tau, \tilde{\tau}[$,

$$\begin{aligned} S_t^{i,N} &= S_t^{i,N} + t - \tau, \\ \xi_t^{i,N} &= \frac{1}{N}\sum_{(\tilde{V},\tilde{S},\tilde{W})\in \mathrm{supp}(\xi_\tau^{i,N})} \delta_{(\tilde{V},\tilde{S}+t-\tau,\tilde{W})}. \end{aligned}$$

Now considering the jump part, we denote by $\mathcal{T}_t^i \subset \mathcal{T}_t$ the spiking times of neuron i and $\tilde{\mathcal{T}}_t^i \subset \mathcal{T}_t$ the return to resting potential of neuron i ($V^i : 1 \to 0$). Among these jumping times, $\mathcal{T}_t^{ij,\pm} \subset \mathcal{T}_t^i$ stands for the spiking times of neuron i such that W^{ij} potentiates/increases ($\mathcal{T}_t^{ij,+}$) and W^{ji} depreciates/decreases ($\mathcal{T}_t^{ij,-}$). Hence, for all $i \in \{1, \cdots, N\}$, and $\tau \in \mathcal{T}_t$,

$$\begin{aligned} V_\tau^{i,N} &= 1 - V_{\tau^-}^{i,N}, \\ S_\tau^{i,N} &= S_{\tau^-}^{i,N}(1 - V_{\tau^-}^{i,N}), \\ \Delta\xi_\tau^{i,N} &= \xi_\tau^{i,N} - \xi_{\tau^-}^{i,N} \\ &= \frac{1}{N}\sum_{(\tilde{V},\tilde{S},\tilde{W})\in \mathrm{supp}(\xi_{\tau^-}^{i,N})} \Bigg\{ \\ &\quad \sum_j \mathbb{1}_{\{\tau\in\mathcal{T}_t^j, \tilde{S}=S_{\tau^-}^{j,N}\}} \left[\delta_{\left(1,0,\tilde{W}-\mathbb{1}_{\{\tau\in\mathcal{T}_t^{ji,-}\}}\right)} - \delta_{\left(0,\tilde{S},\tilde{W}\right)}\right] \\ &\quad + \sum_j \mathbb{1}_{\{\tau\in\mathcal{T}_t^{ij,+}, \tilde{S}=S_{\tau^-}^{j,N}\}} \left[\delta_{\left(\tilde{V},\tilde{S},\tilde{W}+1\right)} - \delta_{\left(\tilde{V},\tilde{S},\tilde{W}\right)}\right] \\ &\quad + \sum_j \mathbb{1}_{\{\tau\in\tilde{\mathcal{T}}_t^j, \tilde{S}=S_{\tau^-}^{j,N}\}} \left[\delta_{\left(0,\tilde{S},\tilde{W}\right)} - \delta_{\left(1,\tilde{S},\tilde{W}\right)}\right] \Bigg\}, \end{aligned} \tag{6}$$

where the characteristic function $\mathbb{1}_{\{x\in A\}}$ is equal to 1 if $x \in A$ and 0 otherwise. The first line contains the jumps related to the spike of neuron $j \neq i$; reset of S^j to 0 and potential depression of W^{ji}. The second line contains the potentiations of the weights $(W^{ij})_{j\neq i}$, this happens when the neuron i spikes. The final line is the return of the neurons to their resting potential, from $V^j = 1$ to $V^j = 0$. Note that we don't see anymore the functions $p^\pm$ which are in fact hidden in the jumps $\tau^{ij,\pm}$. We describe them rigorously when giving the dynamics of μ_t^N. The latter follows directly from the one of X_t^N as the only difference is that there is no label anymore (as for the ξ). They can be recovered again thanks to the time since the last spike $\tilde{S}$ present in each atom $\tilde{X} = (\tilde{V}, \tilde{S}, \tilde{\xi})$ of μ_t^N.

II. MAIN RESULTS

Our first important result is the Markov property of the process $(\mu_t^N)_{t\geq 0}$ defined on $\mathcal{P}_N(E)$, see Proposition 10 in the Supplementary Material.

The main result of this paper is to provide the possible deterministic limit $(\mu_t^*)_{t\geq 0}$ of $(\mu_t^N)_{t\geq 0}$ when N tends to infinity. In particular, we consider a process with distribution $(\mu_t^*)_{t\geq 0}$ as *typical neuron* and denote it by $(X_t^*)_{t\geq 0} = (V_t^*, S_t^*, \xi_t^*)_{t\geq 0}$.

Main Result 5. *Under the assumptions A1, A2, A3 and A4 detailed in the Supplementary Material, the limit objects μ_t^* and $(V_t^*, S_t^*, \xi_t^*)_{t\geq 0}$ satisfy the following McKean-Vlasov SDE:*

1. *μ_t^* is the distribution of (V_t^*, S_t^*, ξ_t^*): $\mu_t^* = \mathcal{L}(V_t^*, S_t^*, \xi_t^*)$,*
2. *$V_t^* : 0 \underset{\beta}{\overset{\alpha(I(\xi_t^*))}{\rightleftarrows}} 1$. We denote by $\mathcal{T}_t^*$ the spikes times, that is the times on $[0,t]$ when V jumps from 0 to 1,*
3. *$S_t^* = S_0^* + t - \sum_{\tau\in\mathcal{T}_t^*} S_{\tau^-}^*$,*
4. *ξ_t^* admits a density in s such that $\xi_t^*(v, ds, w) = \xi_t^*(v, s, w)ds$ (abuse of notation) and between the spikes*

$$\begin{cases} \partial_t \xi_t^*(0,s,w) = -\partial_s \xi_t^*(0,s,w) + \beta \xi_t^*(1,s,w) - \int_{\mathcal{P}(E_m)} \alpha\big(I(\xi')\big) \frac{\xi_t^*(0,s,w)}{\xi_t^*(0,s,\mathbb{Z})} \mu_t^*(0,s,d\xi') \\ \xi_t^*(0,0,w) \ = 0, \end{cases}$$

$$\begin{cases} \partial_t \xi_t^*(1,s,w) = -\partial_s \xi_t^*(1,s,w) - \beta \xi_t^*(1,s,w) \\ \xi_t^*(1,0,w) \ = \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \alpha\big(I(\xi')\big) \frac{p^-(S_t^*,w+1)\xi_t^*(0,s',w+1) + (1-p^-(S_t^*,w))\xi_t^*(0,s',w)}{\xi_t^*(0,s',\mathbb{Z})} \mu_t^*(0,s',d\xi')ds'. \end{cases}$$

5. *At a spike time $\tau \in \mathcal{T}_\infty^*$, $\xi_{\tau^-}^*$ jumps to $\nu^+\{\xi_{\tau^-}^*\}$ where for all PDF ξ on E_m,*

$$\nu^+\{\xi\}(v,s,w) \coloneqq p^+(s,w-1)\xi(v,s,w-1) + (1-p^+(s,w))\xi(v,s,w).$$

We now give some details on this limit system. The second point details the membrane potential (spiking) activity of the neuron. Note that the spiking jump depends on the distribution ξ_t^* through the function I defined in (3). The third point is the linear increase of the time since last spike between the spikes of the neuron. The fourth point gives the partial differential equations (PDE) between the jumps (spikes). The PDE on ξ_t^* is made up of three parts. First, the linear increase of s which is the term in $-\partial_s$. Second, the mass transport from $\xi_t^*(1,\cdot,\cdot)$ to $\xi_t^*(0,\cdot,\cdot)$ at rate β. Finally, the mass transport from $\xi_t^*(0,\cdot,\cdot)$ to $\xi_t^*(1,\cdot,\cdot)$ at a rate that depends on μ_t^* with the reset of s in 0 giving the boundary condition in $s=0$; see the $\xi_t^*(1,0,\cdot)$ term. This corresponds to a spike of a pre-synaptic neuron hence leading to the depression of the associated weight (outgoing weight for this pre-synaptic neuron). Then, the fifth and last point. When the *typical neuron* spikes, $V_t^* : 0 \to 1$, the weights' potentiation occurs; meaning a transfer of mass from $\xi_t^*(\cdot,\cdot,w-1)$ to $\xi_t^*(\cdot,\cdot,w)$.

Note that all the changes of $(\xi_t^*)_{t\geq 0}$ are transfers of mass on the space on which it is defined; hence showing that its total mass is conserved and thus $(\xi_t^*)_{t\geq 0}$ remains a probability measure (of mass 1) over time. In addition, this system of equations involves the distribution of its solution making $(X_t^*)_{t\geq 0}$ follows a McKean–Vlasov SDE. In particular, the process $(X_t^*)_{t\geq 0}$ solves the SDE

$$X_t^* = X_0^* + \int_0^t b(X_u^*, \mu_u^*)du + \int_0^t \int_{\mathbb{R}^+} h(X_{u^-}^*, z)\zeta^*(dz, du)$$

where ζ^* is a Point Poisson measure on $\mathbb{R}^+\times\mathbb{R}^+$ with intensity $dzdu$, the probability distribution μ_u^* is the distribution of the process X_u^*, and the functions b and h are defined according to the Main Result 5:

$$\begin{aligned} h(X_{u^-}^*, z) &= \Big(1, -S_{u^-}^*, \nu^+(\xi_{u^-}^*) - \xi_{u^-}^*\Big)\mathbb{1}_{\left\{z\leq \alpha\left(I(\xi_{u^-}^*)\right)\mathbb{1}_{\{V_{u^-}^*=0\}}\right\}} \\ &\quad + (-1,0,0)\mathbb{1}_{\left\{z\leq\beta\mathbb{1}_{\{V_{u^-}^*=1\}}\right\}} \\ b(X_u^*, \mu_u^*) &= \Big(0, 1, -\partial_s\xi_u^* + \beta\delta_0\otimes\xi_u^*(1,\cdot,\cdot) - \beta\delta_1\otimes\xi_u^*(1,\cdot,\cdot) \\ &\quad + \delta_1\otimes\nu^{-,1}(S_u^*, \xi_u^*, \mu_u^*) - \delta_0\otimes\nu^0(\xi_u^*, \mu_u^*)\Big) \end{aligned}$$

where ν^+, ν^0 and $\nu^{-,1}$ are defined respectively in (S24), (S11) and (S22). Note that the process $(X_t^*)_{t\geq 0}$ lives on the peculiar space $E = \{0,1\}\times\mathbb{R}^+\times\mathcal{P}(E_m)$ which contains the space of distributions on E_m. We give the computations leading to such a limit candidate in the Supplementary Material where we first study the system without plasticity S1 C and then with S1 D plasticity.

Remark 6. *Classically, the proof of the convergence of the particle system to the limit dynamics is done when the particles are exchangeable. By Assumption A1, the distributions* $\xi_0^{1,N}, \cdots, \xi_0^{N,N}$ *have the same distribution. This assumption added to the fact that the variables* $V_t^{i,N}$, $S_t^{i,N}$ *and* $W_t^{ij,N}$ *have for all* i, j, *the same dynamics, implies that for all* i *and* $t \geq 0$, *the* $\xi_t^{i,N}$ *are equal in distribution. We conclude this remark with the crucial following point: using Assumption A1, we deduce that the distribution of* X_t^N *is exchangeable, which means that for any permutation* σ *of* $\{1, \cdots, N\}$, $X_t^{\sigma,N} = (X_t^{\sigma(1),N}, \cdots, X_t^{\sigma(N),N})$ *has the same distribution as* X_t^N.

III. NUMERICAL COMPARISON WITH THE NEURAL NETWORK

We simulate the stochastic STDP model. In particular, we illustrate our results by comparing the simulation of the finite size neural network versus the simulation of the limit system.

We use a sigmoid for the function α and the classical STDP curves for p^+ and p^-,

$$\alpha(x) = \frac{\alpha_M - \alpha_m}{1 + e^{\sigma(\theta - x)}} + \alpha_m$$
$$\text{and} \quad p^{+/-}(s, w) = A_{+/-} e^{-\frac{s}{\tau_{+/-}}} \mathbb{1}_{[\![w_{min}, w_{max}]\!]}(w),$$

with the following parameters:

$$N = 5000,\ \alpha_m = 0.05\ ms^{-1},\ \alpha_M = \beta = 1\ ms^{-1},$$
$$\tau_+ = 1.5\ ms, \tau_- = 2\ ms,\ A_+ = 0.8,\ A_- = 0.6,$$
$$dt = 0.05\ ms,\ \sigma = 1.5\ mV^{-1},\ \theta = 0\ mV,$$
$$w_{max} = -w_{min} = 10.$$

For the limit system, we simulate N neurons $X_t^{1,*}, \cdots, X_t^{N,*}$ having the same dynamics as X_t^* (see Main Result 5 and Remark 6), except that instead of μ_t^* we use $\mu_t^{*,N} = \frac{1}{N}\sum_i \delta_{X_t^{i,*}}$. Hence, we do not need to compute μ_t^*. For instance, we use $\xi_t^{i,*}$ to compute the $I_t^{i,*} = I(\xi_t^{i,*})$. For more details on the code, see the Supplementary Material S2.

The results obtained with these simulations are compared with those obtained from a finite-size neural network: $(V_t^{i,N}, S_t^{i,N}, W_t^{ij,N})_{1 \leq i,j \leq N, t \geq 0}$. The mathematical objects illustrated in the Figure 1 are the empirical mean and distributions of different variables.

Definition 7. *Consider a finite ensemble of size* M *that we denote by* $Z = Z^1, \cdots, Z^M$, *its empirical mean is*

$$\bar{Z} = \frac{1}{M}\sum_{i=1}^{M} Z^i,$$

and its empirical distribution is

$$\hat{P}_Z^M = \frac{1}{M}\sum_{i=1}^{M} \delta_{Z^i}.$$

We compare the temporal evolution of the empirical mean of the potential (Figure 1a), the time since the last spike (Figure 1b) and synaptic weight (Figure 1c). In particular,

$$Z \in \{V_t^N, V_t^*, \xi_t^*(\cdot, \mathbb{R}^+, \mathbb{Z})\} \text{ in Figure 1a,}$$
$$Z \in \{S_t^N, S_t^*, \xi_t^*(\{0,1\}, \cdot, \mathbb{Z})\} \text{ in Figure 1b,}$$
$$Z \in \left\{\{W_t^{ij,N}\}_{i,j}, \{\xi_t^{i,N}(\{0,1\}, \mathbb{R}^+, \cdot)\}_i\right\} \text{ in Figure 1c.}$$

Moreover, we compare, at the end of the simulation $t = 500ms$, the distributions of the intensities of the incoming currents onto the neurons (Figure 1d), the distributions of the time since the last spikes for neurons in state $V = 0$ (Figure 1e) and $V = 1$ (Figure 1f). The latter are estimated in three ways: using the empirical distribution from the original system and the MF system with N typical neurons (see Definition 7), and directly from the distribution $\xi_t^{i,*}$ for a given neuron $i = 1$. The distance between the curves for long times in the Figure 1b is due to the approximation done in $\xi_t^{i,*}$ as we have to bound it in S (see Supplementary Material S2 for more details); bound which does not exist in the original model. The slight shift in the synaptic current follows from this error of approximation which leads to an accumulation in the upper bound as it can be seen in the Figure 1e.

Overall, the simulations' results show a good agreement between the initial network and the limit one. In the timeseries plots, we see that on average, the MF variables follow the ground truth closely. We also observe that the whole distribution of these variables is captured. Interestingly, we see that the synaptic currents have a wide distribution (far from the initial condition which is a Dirac distribution), showing that each neuron receives different inputs even though the latter is an expectation, see equation (3).

IV. DISCUSSION

We derived a McKean-Vlasov mean-field (MKV-MF) equation from a neural network model with stochastic STDP. To our knowledge, this is the first MF model of a network with STDP that does not rely on the symmetric STDP curve [38] or the slow-fast framework where the plasticity is slow compared to the neuronal dynamics [15]; assumptions in conflict with experimental observations [16, 17, 20, 22, 23]. When considering spiking neural networks without plasticity, MFs have been derived with, for example, the Ott-Antonsen ansatz [29] when weights are homogeneous. In the case of heterogeneous fixed weights, methods based on the estimation of the cross-correlation have been used [48]. On the other hand, MKV-MFs have already been shown to be needed in such networks with both binary neurons and synaptic weights [35] and more recently in integrate-and-fire neurons using the theory of dense graph limits (graphons) [49].

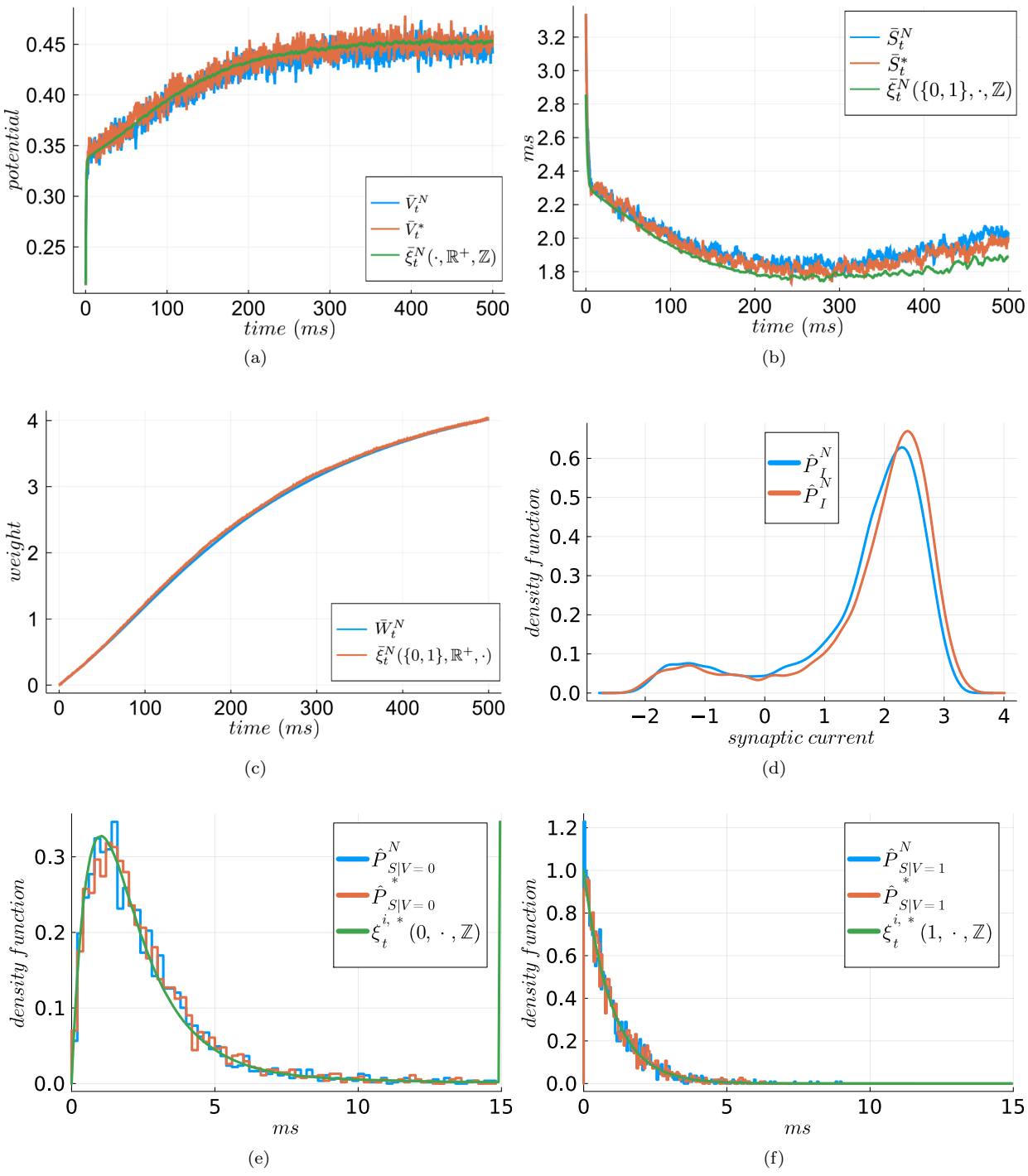

FIG. 1: **Comparisons between the MKV-MF model and the initial model.** (1a-c): Empirical mean (see Definition 7 and below) of the potentials (1a), time since last spike (1b), and the weights (1c) across time. (1d): Empirical distributions (see Definition 7) at time $t = 500ms$ of the synaptic currents onto the neurons. (1e-f): Distributions (see Definition 7 and below) at time $t = 500ms$ of the time since last spike for neurons in state $V = 0$ (1e) and $V = 1$ (1f).

This MF opens the door for the development of new mathematical tools necessary to prove convergence to the limit system and its analysis. The next steps would then be to show that the limit system has a unique solution and to prove the tightness [40] of the empirical distributions to conclude on their convergence to a unique deterministic limit. Finally, studying the limit system would provide insight into the finite-sized model and a particularly interesting study would be its long time behaviour [50]. We are currently working on a simplified version of this model for these analytical derivations.

We believe that the model cannot be additionally reduced as we tried different model descriptions without success. A crucial element is the knowledge of the last spike time of each neuron. We could have used the outgoing (instead of incoming) weights in ξ but this leads to much more complex equations; see for example the synaptic current definition (3). To insist on this, in the simulations, when starting with a Dirac distribution of synaptic currents, we end up with a wide distribution of them. This happens even though these currents are defined as expectations; see (3). It shows that the limit model has some strong heterogeneity that our MKV-MF is able to capture.

In addition, the MF is quite flexible as we have only weak assumptions on the model parameters. For example, we could easily extend the results to account for Dale's law by making sure the initial weights follow it and then preventing them to change sign using the weight dependence on $p^{\pm}$. Such flexibility is not allowed in the MF recently derived on rate-based neural network models with plasticity [51].

Finally, the simulation complexity of the limit system is here on the order of N rather than N^2 originally. Hence, we have drastically reduced the simulation costs; despite the typical neuron being infinite dimensional. Additionally, it closely follows the finite size ground truth system both on average and distribution-wise as we showed in Figure 1.

V. CONCLUSION

Our approach paves the way for new mathematical tools to analyse complex neural systems more efficiently. It effectively addresses the combination of two of the most intricate phenomena in neuroscience: spikes and synaptic plasticity. This combination has been inadequately analysed in recurrent network settings until now. Using advanced mathematical tools, we reveal the MKV-MF limit of a biological spiking neural network with STDP.

VI. ACKNOWLEDGEMENTS

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SUPPLEMENTARY MATERIAL

S1. PROOFS

A. Density and Markovianity of the empirical distribution

1. Properties of S

Under Assumption A1, the neural network possesses two important properties that are extensively used throughout this paper. First, the times from the last spike are almost surely distinct.

Lemma 8. *Grant Assumption A1. Consider the process X_t^N solution of the microscopic model described in Section I A. Then, for any $t \geq 0$ and for all $i \neq j \in [\![1, N]\!]$, we have almost surely $S_t^{i,N} \neq S_t^{j,N}$.*

Proof. By assumption, for all i, the distribution of $S_0^{i,N}$ admits a density and hence, the $(S_0^{i,N})_{0\leq i\leq N}$ are almost surely distinct. Between the jumps of $(V_t^{i,N})_{0\leq i\leq N}$ from 0 to 1, we have $d(S_t^{i,N} - S_t^{j,N}) = 0$. Finally, the probability that two different neurons spike at the exact same time is zero. □

Second, at any time $t \geq 0$, the distribution of $S_t^{i,N}$ is absolutely continuous with respect to the Lebesgue measure λ and has a density ρ_t almost surely finite.

Lemma 9. *We denote $\|\rho_0\|_\infty$ the upper bound of ρ_0 and we assume that Assumption A1 holds. Then, for any $t \geq 0$, the distribution ρ_t of $S_t^{i,N}$ also admits a density. In particular, for any $A \in \mathcal{B}(\mathbb{R}^+)$,*

$$\rho_t(A) \leq (\|\alpha\|_\infty + \|\rho_0\|_\infty)\lambda(A).$$

Proof. We have for all $A \in \mathcal{B}(\mathbb{R}^+)$, $t > 0$,

$$\mathbb{P}(S_t^{i,N} \in A) = \mathbb{P}(S_t^{i,N} \in A \cap [0,t[) + \mathbb{P}(S_t^{i,N} \in A \cap [t, +\infty[).$$

First, denoting $A - t = \{x \in \mathbb{R}^+, x + t \in A\}$, one has

$$\left\{S_t^{i,N} \in A \cap [t, +\infty[\right\} \subset \left\{S_0^{i,N} \in A - t\right\}.$$

Thus,

$$\begin{aligned}\mathbb{P}\left(S_t^{i,N} \in A \cap [t, +\infty[\right) &\leq \mathbb{P}\left(S_0^{i,N} \in A - t\right) \\ &\leq \|\rho_0\|_\infty \lambda(A - t) = \|\rho_0\|_\infty \lambda(A).\end{aligned}$$

Second, the event $\left\{S_t^{i,N} \in A \cap [0, t[\right\}$ means that the last spike of the neuron i occurred at a time in $t - A$. The probability of this event is less than the probability that there is at least one spike of the neuron i in $t - A$. So, one has,

$$\begin{aligned}\mathbb{P}(S_t^{i,N} \in &A \cap [0, t[) \\ &\leq \mathbb{E}\left[1 - \exp\left(-\int_{t-A} \mathbb{1}_{\{V_u^{i,N}=0\}} \alpha(I_u^{i,N}) du\right)\right] \\ &\leq \mathbb{E}\left[\int_{t-A} \alpha(I_u^{i,N}) du\right] \\ &\leq \|\alpha\|_\infty \lambda(t - A) = \|\alpha\|_\infty \lambda(A).\end{aligned}$$

□

2. $(\mu_t^N)_t$ is Markov

Proposition 10. *The random process $(\mu_t^N)_{t\geq 0}$ is a Markov process on $\mathcal{P}_N(E)$.*

Proof. We propose a constructive proof by showing how to construct such a process. If we know the process at time t_0, we need to find its next jumping time that we denote by τ. This time is obtained by drawing a random variable following an exponential distribution with parameter

$$\lambda_{tot} = \sum_{(V,S,\xi)\in \mathrm{supp}(\mu_{t_0}^N)} \beta V + \alpha\big(I(\xi)\big)(1 - V).$$

Until this jumping time τ, we have

$$\forall t_0 \leq t < \tau, \quad \mu_t^N = \frac{1}{N} \sum_{(V,S,\xi)\in \mathrm{supp}(\mu_{t_0}^N)} \delta_{\left(V,\ S+t-t_0,\ \frac{1}{N} \sum\limits_{(\tilde{V},\tilde{S},\tilde{W})\in \mathrm{supp}(\xi)} \delta_{(\tilde{V},\ \tilde{S}+t-t_0,\ \tilde{W})}\right)}.$$

The jump at time τ is first associated to a neuron $\hat{X} = (\hat{V}, \hat{S}, \hat{\xi}) \in \mathrm{supp}(\mu_{t_0}^N)$ for which $\hat{V}$ jumps to $1 - \hat{V}$. Any such neuron has probability $\frac{\beta\hat{V} + \alpha\left(I(\hat{\xi})\right)(1-\hat{V})}{\lambda_{tot}}$ to be chosen. Hence, in addition to the jump of $\hat{V}$, $\hat{S}$ jumps to 0 if $\hat{V} = 0$. It stays in $\hat{S} + \tau - t_0$ if $\hat{V} = 1$. Furthermore, all the $\xi \in \mathrm{supp}(\mu_{\tau-}^N)$ jump. To do the latter changes, we remind that each neuron can be identify by $\hat{S} + \tau - t_0$ in all $\xi \in \mathrm{supp}(\mu_{\tau-}^N)$ (see Remark 4). We then have to control the potentiations

and the depressions: for any $(V,S,\xi)\in \mathrm{supp}(\mu^N_{\tau^-})$, we introduce independent random variables $\tilde U^+(S)$ and $\tilde U^-(S)$, uniformly distributed on $[0,1]$. We can now describe the jumps of ξ. First, if $\hat V=1$, we have for any ξ

$$\Delta\xi=\frac{1}{N}\sum_{(\tilde V,\tilde S,\tilde W)\in\mathrm{supp}(\xi)}\left\{\mathbb{1}_{\{\tilde S=\hat S+\tau-t_0\}}\big[\delta_{\big(0,\tilde S,\tilde W\big)}-\delta_{\big(1,\tilde S,\tilde W\big)}\big]\right\}.$$

Then, if $\hat V=0$, for the *spiking* neuron $\hat X$, that is for $(V,S,\xi)\in\mathrm{supp}(\mu^N_{\tau^-})$ with $S=\hat S+\tau-t_0$, we have

$$\begin{aligned}\Delta\xi=\frac{1}{N}\sum_{(\tilde V,\tilde S,\tilde W)\in\mathrm{supp}(\xi)}\Bigg\{&\mathbb{1}_{\{\tilde S=\hat S+\tau-t_0\}}\big[\delta_{\big(1,0,\tilde W+\mathbb{1}_{\{\tilde U^+(\tilde S)\le p^+(\tilde S,\tilde W)\}}-\mathbb{1}_{\{\tilde U^-(\tilde S)\le p^-(\tilde S,\tilde W)\}}\big)}-\delta_{\big(0,\tilde S,\tilde W\big)}\big]\\&+\mathbb{1}_{\{\tilde S\neq\hat S+\tau-t_0\}}\big[\delta_{\big(\tilde V,\tilde S,\tilde W+\mathbb{1}_{\{\tilde U^+(\tilde S)\le p^+(\tilde S,\tilde W)\}}\big)}-\delta_{\big(\tilde V,\tilde S,\tilde W\big)}\big]\Bigg\},\end{aligned}$$

and for all $(V,S,\xi)\in\mathrm{supp}(\mu^N_{\tau^-})$ with $S\neq\hat S+\tau-t_0$,

$$\Delta\xi=\frac{1}{N}\sum_{(\tilde V,\tilde S,\tilde W)\in\mathrm{supp}(\xi)}\left\{\mathbb{1}_{\{\tilde S=\hat S+\tau-t_0\}}\big[\delta_{\big(1,0,\tilde W-\mathbb{1}_{\{\tilde U^-(\tilde S)\le p^-(S,\tilde W)\}}\big)}-\delta_{\big(0,\tilde S,\tilde W\big)}\big]\right\}.$$

This ends the construction of the dynamics of the process $(\mu^N_{t_0})_t$. □

B. Preliminaries for the derivation of the main result

The purpose of this section is to set up the necessary elements to derive the conjecture on the evolution equations of the typical neuron $(X^*_t)_{t\ge0}$ from those of the finite-size neural network $(X^N_t)_{t\ge0}$. To do so, we first detail the necessary definitions and notations, before giving the assumptions needed.

Definitions and notations

We are interested in finding the possible deterministic limit processes $(\mu^*_t)_{t\ge0}$ of $(\mu^N_t)_{t\ge0}$ when N tends to infinity, for the weak topology. To do so, for $T>0$, we consider the empirical distributions on $D_E[0,T]$, the space of càdlàg functions from $[0,T]$ to E.

Notation 11. *Let Z be a complete separable metric space. We denote by $D_Z[0,T]$ the space of càdlàg functions (right continuous with left limits) from $[0,T]$ to Z.*

We define

$$X^{i,N}=(X^{i,N}_t)_{0\le t\le T}\text{ and }X^*=(X^*_t)_{0\le t\le T},$$

with the distribution of X^*_t being μ^*_t, which we denote by $\mathcal{L}(X^*_t)=\mu^*_t$. Thus, we can then denote by

$$\mu^N=\frac{1}{N}\sum_{i=1}^N\delta_{X^{i,N}}\quad\text{and}\quad\mu^*=\mathcal{L}(X^*).$$

Thereby, we consider the convergence in distribution of $(\mu^N)\in\mathcal{P}(D_E[0,T])$ where we recall that

$$E=\{0,1\}\times\mathbb{R}^+\times\mathcal{P}\big(\{0,1\}\times\mathbb{R}^+\times\mathbb{Z}\big).$$

The space $D_E[0,T]$ is equipped with the J_1 Skorohod topology. We also consider a distance d on $D_E[0,T]$ with the two properties: first d induces the J_1 Skorohod topology; second, the space $(D_E[0,T],d)$ is a separable and complete space (i.e a Polish space). The existence of such a distance is proved in Billingsley [52, Sec 12]. We also use the property that if a space G is Polish, then $\mathcal{P}(G)$ equipped with the associated weak convergence, is also Polish [53, Thm 17.23].

Assuming that a limit $\mu^*\in\mathcal{P}(D_E[0,T])$ is deterministic (i.e. $\mathcal{L}(\mu^N)\to_{N_\infty}\delta_{\mu^*}$), the aim of this article is to find the system satisfied by any limit point which satisfies

$$\begin{aligned}&\forall\Psi\in C^{1,1}_b(E),\\&\mathbb{E}\left[\int_0^T\langle\mu^N_u,\Psi\rangle du\right]\to_{N_\infty}\mathbb{E}\left[\int_0^T\langle\mu^*_u,\Psi\rangle du\right]\\&\qquad\qquad=\int_0^T\langle\mu^*_u,\Psi\rangle du,\end{aligned}\tag{S1}$$

where the derivation is in the Fréchet sense (see the $C^{1,1}_b$ Notation 13 below). Indeed, knowing $\langle\mu^*_t,\Psi\rangle$ for all $\Psi\in C^{1,1}_b(E)$ is sufficient to determine the distribution μ^*_t. Hence, we restrict ourselves to **the study of** $\mathbb{E}\langle\mu^{\mathbf{N}}_{\mathbf{t}},\mathbf{\Psi}\rangle$ for Ψ in $C^{1,1}_b(E)$. Thereby, we are looking for the limit equation of a *typical neuron* that we denote $X^*_t=(V^*_t,S^*_t,\xi^*_t)$ with distribution μ^*_t.

We recall the notion of Fréchet differentiation. We use the definition given in [54] at the beginning of page 6.

Definition 12. *Let F and G be two Banach spaces embedded with the norms $\|\cdot\|_F$ and $\|\cdot\|_G$. Let D be an open set of F. We say that $\varphi : D \to G$ is Fréchet differentiable in $y \in D$ if there exists a linear bounded operator $L(y) : h \in F \mapsto L(y)h \in G$ such that for all $h \in F$ satisfying $y + h \in D$, we have*

$$\|\varphi(y+h) - \varphi(y) - L(y)h\|_G \leq \rho(\|h\|_F, y),$$

where $\rho(\|h\|_F, y) \in o(\|h\|_F)$.

We call $L(y)$ the Fréchet derivative of φ in y and $L(y)h$ the Fréchet differential of φ in y in the direction h.

We say that φ is Fréchet differentiable on D if for all $y \in D$,

Then, we can define a space suitable for our following computations.

Notation 13. *We denote by $C_b^{1,1}(E)$ the space of functions from E to $\mathbb{R}$ that are bounded, continuously differentiable with respect to their second variable, Fréchet differentiable (see Definition 12) with respect to their third variable and finally, both these derivatives are bounded.*

The Fréchet derivative with respect to the third variable is defined on the larger space of the signed distribution on E_m, $\mathcal{M}(E_m)$, equipped with the total variation norm, $\|\cdot\|_{TV}$.

Definition 14. *Let $(X, \mathcal{X})$ be a measurable space equipped with a signed distribution η. One associates to η the (unique) pair of positive distributions η^+ and η^- defined for any $A \in \mathcal{X}$ by*

$$\eta^+(A) = \sup\{\eta(B), B \in \mathcal{X}, B \subset A\}$$
$$\eta^-(A) = \sup\{-\eta(B), B \in \mathcal{X}, B \subset A\}.$$

The Jordan decomposition of the signed distribution η writes

$$\eta(A) = \eta^+(A) - \eta^-(A).$$

The variation $|\eta|$ of the signed distribution η is defined, for all $A \in \mathcal{X}$, by

$$|\eta|(A) \coloneqq \eta^+(A) + \eta^-(A).$$

We define the total variation of η as

$$\|\eta\|_{TV} \coloneqq |\eta|(X) = \eta^+(X) + \eta^-(X).$$

The space of signed distributions embedded with the norm $\|\cdot\|_{TV}$ is a Banach space.

In particular, $\left(\mathcal{M}(E_m), \|\cdot\|_{TV}\right)$ is a Banach space, see [55, Rk 1.7], enabling us to define the Fréchet derivative of $\Psi \in C_b^{1,1}(E)$. For all $\Psi \in \mathcal{C}_b^{1,1}(E)$, for all $(v, s, \xi) \in E$ and $h \in \mathcal{M}(E_m)$, we denote by $\partial_\xi \Psi(v, s, \xi) \cdot h$ the Fréchet derivative of Ψ at (v, s, ξ) in the direction h (see Definition 12).

Now, we define two operators η^N and ζ^N to describe the jumps of the empirical measure when one neuron V jumps from 0 to 1 or from 1 to 0. They both associate to a pair $(s, \xi) \in \mathbb{R}^+ \times \mathcal{P}_N(E_m)$ the following signed distributions on E_m:

$$\eta^N(s,\xi) \coloneqq \frac{1}{N} \sum_{(1,s,\tilde{w}) \in \mathrm{supp}(\xi)} \left[\delta_{(0,s,\tilde{w})} - \delta_{(1,s,\tilde{w})}\right]$$
$$\zeta^N(s,\xi) \coloneqq \frac{1}{N} \sum_{(0,s,\tilde{w}) \in \mathrm{supp}(\xi)} \left[\delta_{(1,0,\tilde{w})} - \delta_{(0,s,\tilde{w})}\right].$$

Notation 15. *The symbol $\overset{\mathbb{E}}{=}$ refers to the equality between the expectations of two random variables:*

$$\mathbb{E}[X] = \mathbb{E}[Y] \quad \Leftrightarrow \quad X \overset{\mathbb{E}}{=} Y.$$

Remark 16. *Let τ be any spiking time of the neuron i, i.e. $V^{i,N}_{\tau^-} = 0$ and $V^{i,N}_{\tau} = 1$. Then, according to Lemma 8, for every j, there is a unique $(\tilde{v}, \tilde{s}, \tilde{w})$ in the support of $\xi^{j,N}_{\tau^-}$ such that $\tilde{s} = S^{i,N}_{\tau^-}$. So, $\zeta^N(S^{i,N}_{\tau^-}, \xi^{j,N}_{\tau^-})$ is a signed distribution with two atoms.*

$$\zeta^N(S^{i,N}_{\tau^-}, \xi^{j,N}_{\tau^-}) = \frac{1}{N}\delta_{(1,0,\tilde{w})} - \frac{1}{N}\delta_{(0,S^{i,N}_{\tau^-},\tilde{w})}.$$

Similarly, at a time t when $V^{i,N}_{\tau^-} = 1$ and $V^{i,N}_{\tau} = 0$, there is a unique $(\tilde{v}, \tilde{s}, \tilde{w})$ in the support of $\xi^{j,N}_{\tau^-}$ such that $\tilde{s} = S^{i,N}_{\tau^-}$ and $\eta^N(S^{i,N}_{\tau^-}, \xi^{j,N}_{\tau^-})$ is a signed distribution with two atoms

$$\eta^N(S^{i,N}_{\tau^-}, \xi^{j,N}_{\tau^-}) = \frac{1}{N}\delta_{(0,S^{i,N}_{\tau^-},\tilde{w})} - \frac{1}{N}\delta_{(1,S^{i,N}_{\tau^-},\tilde{w})}.$$

We define here η^N and ζ^N only on $\mathcal{P}_N(E_m)$, the set of empirical distributions of order N over E_m. With the changes occurring when passing to the large N limit, these distributions will no longer be required. New ones are defined on the full spaces $\mathcal{P}(E_m)$ later on, see Propositions 18 and 19.

Third, we define the derivative ε' of any probability distribution ε on $\mathbb{R}^+$ using the space of continuous functions with bounded derivative C_b^1 by,

$$\forall \phi \in C_b^1(\mathbb{R}^+), \quad \langle \varepsilon', \phi \rangle \coloneqq -\langle \varepsilon, \phi' \rangle. \tag{S2}$$

Finally, let us define for all $\xi \in \mathcal{P}(E_m)$ and $t \geq 0$, the probability distribution $\xi \oplus t$ such that

$$\forall (v, A, w) \in \mathcal{B}(\{0,1\}, [t, +\infty[, \mathbb{Z}),$$
$$\xi \oplus t(v, A, w) = \xi(v, A - t, w).$$

Notation 17. *For all $x, y \in \{0,1\}$, we denote by $\delta_x \otimes \xi$ the distribution such that*

$$\forall (v, A, w) \in \{0,1\} \times \mathcal{B}(\mathbb{R}^+) \times \mathbb{Z},$$
$$\{\delta_x \otimes \xi^y\}(v, A, w) \coloneqq \delta_x(v)\xi(y, A, w).$$

Finally, when μ^*_t and ξ^*_t admit densities in s, we use the abuse of notation

$$\mu^*_t(v, ds, d\xi) = \mu^*_t(v, s, d\xi)ds$$
$$\xi^*_t(v, ds, w) = \xi^*_t(v, s, w)ds.$$

Assumptions

Assumption A2. *Assume that for all $T > 0$, the empirical distributions of the system $(\mu^N) \in \mathcal{P}(D_E[0,T])$ converges in distribution to a (deterministic) probability distribution, $\mu^* \in \mathcal{P}(D_E[0,T])$, when N tends to infinity. In particular, the limit* (S1) *holds.*

Then, we need the following assumptions for our computations to hold:

Assumption A3. *The functions $\xi \mapsto \alpha(I(\xi))$ and for all $w \in \mathbb{Z}$, $s \mapsto p^{\pm}(s,w)$, are bounded continuous functions.*

Our calculations concerning the dynamics of $\mathbb{E}\langle \mu_t^N, \Psi\rangle$ enable us to conjecture the limit distribution dynamics. In order to make this conjecture a theorem, we still need to prove the convergence as N tends to infinity of some terms of this dynamics. It will be explained in further details just after Proposition 19 for one term and just before Conjecture 20 for another one. Our Main Result 5 concerns the dynamics of the typical neuron $(X_t^*)_{t\geq 0}$. In particular, it exposes the PDE that we expect to be satisfied by the density of ξ_t^* in s and requires a compatibility assumption: the boundary condition of the mean field must be satisfied at time $t = 0$.

Assumption A4. *Assume that for all $t \geq 0$, ξ_t^* admits a density of class C^1 in s and in particular at time $t = 0$, for all $w \in \mathbb{Z}$, the following densities satisfy the boundary conditions for μ_0^* almost all ξ_0^*,*

$$\xi_0^*(0,0,w) = 0$$
$$\xi_0^*(1,0,w) = \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \alpha\big(I(\xi')\big)\frac{p^-(s',w+1)\xi_0^*(0,s',w+1)}{\xi_0^*(0,s',\mathbb{Z})}\mu_0^*(0,s',d\xi')ds' + \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \alpha\big(I(\xi')\big)\frac{(1-p^-(s',w))\xi_0^*(0,s',w)}{\xi_0^*(0,s',\mathbb{Z})}\mu_0^*(0,s',d\xi')ds'.$$

C. The case without plasticity

The purpose of this section is to conjecture the evolution equations of the typical neuron $(X_t^*)_{t\geq 0}$ when $p^+ \equiv p^- \equiv 0$ (without plasticity). To do so, we first derive the infinitesimal generator of $(\mu_t^N)_{t\geq 0}$ and second, derive the dynamics of $\mathbb{E}\langle \mu_t^N, \Psi\rangle$ where Ψ is a test function with properties described below. Then, we study in detail the most complex terms of this dynamics. Finally, we conjecture the dynamics of any limit point $(X_t^*)_{t\geq 0}$.

1. *Generator of $(\mu_t^N)_{t\geq 0}$*

We now look for the generator of the Markov process $(\mu_t^N)_{t\geq 0}$. We describe the generator only on the set of cylindrical functions $\Phi \in C_b(\mathcal{P}(E))$ that is there exists $\Psi \in C_b^{1,1}(E)$ such that

$$\forall \mu \in \mathcal{P}(E), \qquad \Phi(\mu) = \langle \mu, \Psi\rangle.$$

We consider the first jump of $(\mu_t^N)_{t\geq 0}$ that we denote by $\tau > 0$. We split the generator into the drift term and the jump term using the equality $1 = \mathbb{1}_{t<\tau} + \mathbb{1}_{t\geq\tau}$, hence splitting the generator (obtained when t tends to 0) of $(\mu_t^N)_{t\geq 0}$ into a drift term (D) and a jump term (J). Let $\mu_0 \in \mathcal{P}_N(E)$ such that $\mu_0 = \frac{1}{N}\sum_i \delta_{(v^i,s^i,\xi^i)}$ with $(v^1,s^1,\xi^1),\dots,(v^N,s^N,\xi^N)$ are in E. Let $\Psi \in C_b^{1,1}(E)$, then

$$\langle \mu_t^N - \mu_0, \Psi\rangle = \langle \mathbb{1}_{t<\tau}\mu_t^N - \mu_0, \Psi\rangle + \langle \mathbb{1}_{t\geq\tau}\mu_t^N, \Psi\rangle = \frac{1}{N}\sum_i \big(\Psi\big(v^i, s^i + t, \xi^i \oplus t\big)\mathbb{1}_{t<\tau} - \Psi\big(v^i,s^i,\xi^i\big)\big) + \langle \mathbb{1}_{t\geq\tau}\mu_t^N, \Psi\rangle$$
$$= \underbrace{\frac{1}{N}\sum_i \big(\Psi\big(v^i, s^i + t, \xi^i \oplus t\big) - \Psi\big(v^i,s^i,\xi^i\big)\big)}_{\text{(D)}} + \underbrace{\langle \mathbb{1}_{t\geq\tau}\mu_t^N, \Psi\rangle - \frac{1}{N}\sum_i \Psi\big(v^i, s^i + t, \xi^i \oplus t\big)\mathbb{1}_{t\geq\tau}}_{\text{(J)}}.$$

Thereby, we obtain that the drift term satisfies, with $-\partial_s\xi$ the derivative defined as in (S2),

$$\begin{aligned}\lim_{t\to 0}\frac{\mathbb{E}\textcircled{D}}{t} &= \lim_{t\to 0}\frac{1}{N}\sum_i \frac{\mathbb{E}\left[\Psi\big(v^i, s^i+t, \xi^i\oplus t\big)\right]-\Psi\big(v^i,s^i,\xi^i\big)}{t}\\ &= \frac{1}{N}\sum_i \partial_s\Psi\big(v^i,s^i,\xi^i\big)+\partial_\xi\Psi\big(v^i,s^i,\xi^i\big)\cdot(-\partial_s\xi^i)\\ &= \int_E \Big(\partial_s\Psi\left(v,s,\xi\right)+\partial_\xi\Psi\left(v,s,\xi\right)\cdot(-\partial_s\xi)\Big)\mu_0(dv,ds,d\xi).\end{aligned}$$

For the jump term $\textcircled{J}$, when we take the limit as t tends to 0 in $\frac{\mathbb{E}\textcircled{J}}{t}$, the only term left is the one due to the first jump at time τ. Indeed, all the other terms are of order t or more. We thus obtain that

$$\begin{aligned}\lim_{t\to 0}\frac{\mathbb{E}\textcircled{J}}{t} = \frac{1}{N}\sum_i \mathbb{1}_{\{v^i=1\}}\beta\Bigg\{&\Big[\Psi\Big(0,s^i,\xi^i+\eta^N(s^i,\xi^i)\Big)-\Psi\Big(1,s^i,\xi^i\Big)\Big]\\ &+\sum_{j\neq i}\Big[\Psi\Big(v^j,s^j,\xi^j+\eta^N(s^i,\xi^j)\Big)-\Psi\Big(v^j,s^j,\xi^j\Big)\Big]\Bigg\}\\ +\frac{1}{N}\sum_i \mathbb{1}_{\{v^i=0\}}\alpha(I(\xi^i))\Bigg\{&\Big[\Psi\Big(1,0,\xi^i+\zeta^N(s^i,\xi^i)\Big)-\Psi\Big(0,s^i,\xi^i\Big)\Big]\\ &+\sum_{j\neq i}\Big[\Psi\Big(v^j,s^j,\xi^j+\zeta^N(s^i,\xi^j)\Big)-\Psi\Big(v^j,s^j,\xi^j\Big)\Big]\Bigg\}.\end{aligned}$$

In the previous equation, by adding and removing the missing terms in the sums inside the braces, we get

$$\begin{aligned}\lim_{t\to 0}\frac{\mathbb{E}\textcircled{J}}{t} = \frac{1}{N}\sum_i \mathbb{1}_{\{v^i=1\}}\beta\Bigg\{&\Big[\Psi\Big(0,s^i,\xi^i+\eta^N(s^i,\xi^i)\Big)-\Psi\Big(1,s^i,\xi^i+\eta^N(s^i,\xi^i)\Big)\Big]\\ &+\sum_{j}\Big[\Psi\Big(v^j,s^j,\xi^j+\eta^N(s^i,\xi^j)\Big)-\Psi\Big(v^j,s^j,\xi^j\Big)\Big]\Bigg\}\\ +\frac{1}{N}\sum_i \mathbb{1}_{\{v^i=0\}}\alpha(I(\xi^i))\Bigg\{&\Big[\Psi\Big(1,0,\xi^i+\zeta^N(s^i,\xi^i)\Big)-\Psi\Big(0,s^i,\xi^i+\zeta^N(s^i,\xi^i)\Big)\Big]\\ &+\sum_{j}\Big[\Psi\Big(v^j,s^j,\xi^j+\zeta^N(s^i,\xi^j)\Big)-\Psi\Big(v^j,s^j,\xi^j\Big)\Big]\Bigg\}\\ =\int_{\mathbb{R}^+\times\mathcal{P}(E_m)}&\beta\Big[\Psi\Big(0,s,\xi+\eta^N(s,\xi)\Big)-\Psi\Big(1,s,\xi+\eta^N(s,\xi)\Big)\Big]\mu_0(1,ds,d\xi)\\ +\frac{1}{N}\sum_i \mathbb{1}_{\{v^i=1\}}&\beta\sum_j\Big[\Psi\Big(v^j,s^j,\xi^j+\eta^N(s^i,\xi^j)\Big)-\Psi\Big(v^j,s^j,\xi^j\Big)\Big]\\ +\int_{\mathbb{R}^+\times\mathcal{P}(E_m)}&\alpha\big(I(\xi)\big)\Big[\Psi\Big(1,0,\xi+\zeta^N(s,\xi)\Big)-\Psi\Big(0,s,\xi+\zeta^N(s,\xi)\Big)\Big]\mu_0(0,ds,d\xi)\\ +\frac{1}{N}\sum_i \mathbb{1}_{\{v^i=0\}}&\alpha(I(\xi^i))\sum_j\Big[\Psi\Big(v^j,s^j,\xi^j+\zeta^N(s^i,\xi^j)\Big)-\Psi\Big(v^j,s^j,\xi^j\Big)\Big].\end{aligned}$$

We can now deduce the dynamics of $\mathbb{E}\langle\mu_t^N,\Psi\rangle$ from $\textcircled{D}$, $\textcircled{J}$ and the Markov property of the process $(\mu_t^N)_{t\geq 0}$:

for all $\Psi \in C_b^{1,1}(E)$,

$$\begin{aligned}\langle \mu_t^N, \Psi\rangle \overset{\mathbb{E}}{=} \langle \mu_0^N, \Psi\rangle &+ \int_0^t \int_E \Big(\partial_s \Psi(v,s,\xi) + \partial_\xi \Psi(v,s,\xi)\cdot(-\partial_s \xi)\Big)\mu_u^N(dv,ds,d\xi)du \\ &+ \int_0^t \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \beta\Big[\Psi\Big(0,s,\xi+\eta^N(s,\xi)\Big) - \Psi\Big(1,s,\xi+\eta^N(s,\xi)\Big)\Big]\mu_{u^-}^N(1,ds,d\xi)du + \textcircled{1} \\ &+ \int_0^t \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \alpha\big(I(\xi)\big)\Big[\Psi\Big(1,0,\xi+\zeta^N(s,\xi)\Big) - \Psi\Big(0,s,\xi+\zeta^N(s,\xi)\Big)\Big]\mu_{u^-}^N(0,ds,d\xi)du + \textcircled{2}\end{aligned} \tag{S3}$$

where $\textcircled{1}$ and $\textcircled{2}$ are the most complex elements:

$$\textcircled{1} \coloneqq \int_0^t \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=1\}} \frac{\beta}{N} \sum_j \Big[\Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N} + \eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\Big) - \Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N}\Big)\Big]du, \tag{S4}$$

$$\textcircled{2} \coloneqq \int_0^t \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_j \Big[\Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N} + \zeta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\Big) - \Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N}\Big)\Big]du. \tag{S5}$$

2. Drift term due to the returns to resting potential

When a neuron returns to its resting potential state 0, the total variation of the jumps of the empirical distributions $(\xi_t^{1,N}), \cdots, (\xi_t^{N,N})$ are of order $\frac{1}{N}$. Hence, as N tends to infinity, the limit of the term $\textcircled{1}$ depends on the Fréchet derivative of Ψ. The direction of this derivative describes the way some of the mass of the distribution ξ^* is transported. This transport can be described informally as follows: between time $t = 0$ and time $t = \epsilon$ small, for all s, w, a proportion $\beta\epsilon$ of the mass of ξ_0^* that was in $(1, s, w)$ is transported to $(0, s, w)$.

Proposition 18. *Under Assumptions A1 and A2, for all $\Psi \in C_b^{1,1}(E)$,*

$$\lim_{N\to\infty} \textcircled{1} \overset{\mathbb{E}}{=} \int_0^t \int_E \partial_\xi \Psi(v,s,\xi)\cdot(\beta\delta_0 \otimes \xi^1 - \beta\delta_1 \otimes \xi^1)\mu_{u^-}^*(dv,ds,d\xi)du. \tag{S6}$$

Proof. By Lemma 8, for all j, the support of $\xi_{u^-}^{j,N}$ almost surely does not possess two points with the same second coordinate. Hence, we have the following upper bound, for all $s \in \mathbb{R}^+$, almost surely,

$$\|\eta^N(s, \xi_{u^-}^{j,N})\|_{TV} \leq \frac{2}{N}. \tag{S7}$$

We write the term of $\textcircled{1}$ which is between square brackets using the Fréchet derivative:

$$\begin{aligned}&\Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N} + \eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\Big) - \Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N}\Big) \\ &\qquad = \partial_\xi \Psi(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N})\cdot\Big(\eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\Big) + \mathcal{O}_{N\infty}\Big(\|\eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\|_{TV}\Big).\end{aligned}$$

Hence, we have:

$$\textcircled{1} \overset{\mathbb{E}}{=} \int_0^t \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=1\}} \frac{\beta}{N} \sum_j \Big[\partial_\xi \Psi(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N})\cdot\Big(\eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\Big) + \mathcal{O}_{N\infty}\Big(\|\eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\|_{TV}\Big)\Big]du.$$

Thus, with the upper bound (S7), we obtain that

$$\textcircled{1} \overset{\mathbb{E}}{=} \int_0^t \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=1\}} \frac{\beta}{N} \sum_j \Big[\partial_\xi \Psi(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N})\cdot\Big(\eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\Big)\Big]du + \mathcal{O}_{N\infty}(1).$$

Now, by linearity of $\partial_\xi \Psi$ and inverting the sums, we get

$$\textcircled{1} \overset{\mathbb{E}}{=} \int_0^t \frac{1}{N} \sum_j \Big[\partial_\xi \Psi(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N})\cdot\Big(\sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=1\}}\beta\eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\Big)\Big]du + \mathcal{O}_{N\infty}(1).$$

From Remark 3, for all j and $u \geq 0$, $\xi^{j,N}_{u^-}$ and $\mu^N_{u^-}$ have the same support in V and S, and from Lemma 8, these supports are N almost surely distinct points. Hence, almost surely, for any function $f : \mathbb{R}^+ \times \mathbb{Z} \to \mathbb{R}^+$,

$$\forall j, \quad \sum_i \sum_{(1,S^{i,N}_{u^-},\tilde{w}) \in \mathrm{supp}(\xi^{j,N}_{u^-})} f(S^{i,N}_{u^-}, \tilde{w}) = \sum_{(1,\tilde{s},\tilde{w}) \in \mathrm{supp}(\xi^{j,N}_{u^-})} f(\tilde{s}, \tilde{w}).$$

We deduce that almost surely,

$$\begin{aligned}
\sum_i \mathbb{1}_{\{V^{i,N}_{u^-}=1\}} \beta \eta^N(S^{i,N}_{u^-}, \xi^{j,N}_{u^-}) &= \sum_i \frac{\beta}{N} \sum_{(1,S^{i,N}_{u^-},\tilde{w}) \in \mathrm{supp}(\xi^{j,N}_{u^-})} \left[\delta_{\left(0,S^{i,N}_{u^-},\tilde{w}\right)} - \delta_{\left(1,S^{i,N}_{u^-},\tilde{w}\right)}\right] \\
&= \frac{\beta}{N} \sum_{(1,\tilde{s},\tilde{w}) \in \mathrm{supp}(\xi^{j,N}_{u^-})} \left[\delta_{\left(0,\tilde{s},\tilde{w}\right)} - \delta_{\left(1,\tilde{s},\tilde{w}\right)}\right] \\
&= \beta \delta_0 \otimes \xi^{j,N}_{u^-}(1,\cdot,\cdot) - \beta \delta_1 \otimes \xi^{j,N}_{u^-}(1,\cdot,\cdot),
\end{aligned}$$

where we used Notation 17. We deduce that

$$\textcircled{1} \overset{\mathbb{E}}{=} \int_0^t \int_E \partial_\xi \Psi(v,s,\xi) \cdot (\beta \delta_0 \otimes \xi^1 - \beta \delta_1 \otimes \xi^1)\ \mu^N_{u^-}(dv,ds,d\xi)du + \mathcal{O}_{N\infty}(1).$$

The application $\xi \mapsto \xi(1,\cdot,\cdot)$ is bounded continuous. Thus, as $\Psi \in C^{1,1}_b(E)$, the application

$$(s,v,\xi) \mapsto \partial_\xi \Psi(v,s,\xi) \cdot (\beta \delta_0 \otimes \xi^1 - \beta \delta_1 \otimes \xi^1)$$

is bounded continuous. We conclude that under Assumption A2,

$$\lim_{N\to\infty} \textcircled{1} \overset{\mathbb{E}}{=} \int_0^t \int_E \partial_\xi \Psi(v,s,\xi) \cdot (\beta \delta_0 \otimes \xi^1 - \beta \delta_1 \otimes \xi^1) \mu^*_{u^-}(dv,ds,d\xi)du.$$

□

3. Drift term due to action potentials

As N tends to infinity, the limit of the term $\textcircled{2}$ also gives a drift term for ξ^*. This term is more difficult to deal with because the spiking rates $(\mathbb{1}_{\{V^{i,N}_t=0\}}\alpha(I^{i,N}_t))_{1\leq i\leq N}$ depend on the actual state of the neural network X^N_t. Informally, the limit of the term $\textcircled{2}$ represents the following mass transport on the probability distributions: between time $t=0$ and time $t=\epsilon$ small, for all $s,\ w$, a proportion $\epsilon \int_{\mathcal{P}(E_m)} \frac{\alpha(I(\tilde{\xi}))}{\mu^*_0(0,s,\mathcal{P}(E_m))} \mu^*_0(0,s,d\tilde{\xi})$ of the mass of ξ^*_0 that was at $(0,s,w)$ is transported to $(1,[0,\epsilon],w)$.

Proposition 19. *We assume that Assumptions A1 and A3 hold. For all Ψ Fréchet differentiable with respect to its third variable:*

$$\textcircled{2} \overset{\mathbb{E}}{=} \int_0^t \int_E \partial_\xi \Psi(v,s,\xi) \cdot \Big(\delta_1 \otimes \nu^1(\xi,\mu^N_{u^-}) - \delta_0 \otimes \nu^0(\xi,\mu^N_{u^-})\Big) \mu^N_{u^-}(dv,ds,d\xi)du + \mathcal{O}_{N\infty}(1).$$

where ν^0 and ν^1 are defined in the proof: equations (S11) *and* (S12).

Proof. By Lemma 8, for any j, the support of $\xi^{j,N}_{u^-}$ almost surely does not contain two atoms with the same S. Furthermore, we have assumed that α is bounded. Thus, we have the following upper bounds, for all $i,\ j \in [\![1,N]\!]$, almost surely,

$$\|\alpha(I^{i,N}_{u^-})\zeta^N(S^{i,N}_{u^-}, \xi^{j,N}_{u^-})\|_{TV} \leq \frac{2\|\alpha\|_\infty}{N}. \tag{S8}$$

Using the Fréchet derivative as previously, from (S5), we obtain:

$$\textcircled{2} \overset{\mathbb{E}}{=} \int_0^t \frac{1}{N} \sum_j \Big[\partial_\xi \Psi(V^{j,N}_{u^-}, S^{j,N}_{u^-}, \xi^{j,N}_{u^-}) \cdot \Big(\sum_i \mathbb{1}_{\{V^{i,N}_{u^-}=0\}} \alpha(I^{i,N}_{u^-}) \zeta^N(S^{i,N}_{u^-}, \xi^{j,N}_{u^-})\Big)\Big] du + \mathcal{O}_{N\infty}(1).$$

We now look at the term between round brackets, it gives the direction of the Fréchet derivative. It is a distribution on E_m. For all j, we denote

$$\begin{aligned}
\textcircled{2.a} &\coloneqq \sum_i \mathbb{1}_{\{V^{i,N}_{u^-}=0\}} \alpha(I^{i,N}_{u^-}) \zeta^N(S^{i,N}_{u^-}, \xi^{j,N}_{u^-}) \\
&= \sum_i \mathbb{1}_{\{V^{i,N}_{u^-}=0\}} \alpha(I^{i,N}_{u^-}) \frac{1}{N} \sum_{(0,S^{i,N}_{u^-},\tilde{w})\in \mathrm{supp}(\xi^{j,N}_{u^-})} \Big[\delta_{\left(1,0,\tilde{w}\right)} - \delta_{\left(0,S^{i,N}_{u^-},\tilde{w}\right)}\Big] \\
&= \frac{1}{N} \sum_i \mathbb{1}_{\{V^{i,N}_{u^-}=0\}} \alpha(I^{i,N}_{u^-}) \sum_{\tilde{w}\in\mathbb{Z}} \mathbb{1}_{\{(0,S^{i,N}_{u^-},\tilde{w})\in \mathrm{supp}(\xi^{j,N}_{u^-})\}} \Big[\delta_{\left(1,0,\tilde{w}\right)} - \delta_{\left(0,S^{i,N}_{u^-},\tilde{w}\right)}\Big].
\end{aligned}$$

But

$$\mathbb{1}_{\{(0,S^{i,N}_{u^-},\tilde{w})\in \mathrm{supp}(\xi^{j,N}_{u^-})\}} = \frac{\frac{1}{N}\sum_{(v,s,w)\in\mathrm{supp}(\xi^{j,N}_{u^-})} \mathbb{1}_{\{v=0,s=S^{i,N}_{u^-},w=\tilde{w}\}}}{\frac{1}{N}\sum_{(v,s)\in\mathrm{supp}(\xi^{j,N}_{u^-}(\cdot,\cdot,\mathbb{Z}))} \mathbb{1}_{\{v=0,s=S^{i,N}_{u^-}\}}} = \frac{\xi^{j,N}_{u^-}(0,S^{i,N}_{u^-},\tilde{w})}{\xi^{j,N}_{u^-}(0,S^{i,N}_{u^-},\mathbb{Z})}, \tag{S9}$$

hence inciting us to define for any $w\in\mathbb{Z}$, the Radon-Nikodym derivative γ_ξ

$$\forall (s,w)\in\mathbb{R}^+\times\mathbb{Z}, \qquad \gamma_\xi(s,w) = \frac{d\xi(0,\cdot,w)}{d\xi(0,\cdot,\mathbb{Z})}(s). \tag{S10}$$

Note that for all $A\in\mathcal{B}(\mathbb{R}^+)$, we have

$$\xi(0,A,w) = \int_A \gamma_\xi(s,w)\xi(0,ds,\mathbb{Z}).$$

As $\xi(0,A,w)\le\xi(0,A,\mathbb{Z})$, we deduce that $\gamma_\xi\le 1$. In particular, using (S9), we obtain that

$$\mathbb{1}_{\{(0,S^{i,N}_{u^-},\tilde{w})\in \mathrm{supp}(\xi^{j,N}_{u^-})\}} = \gamma_{\xi^{j,N}_{u^-}}(S^{i,N}_{u^-},\tilde{w}).$$

Thereby, using the distribution $\mu^N_{u^-}(0,\cdot,\cdot) = \frac{1}{N}\sum_i \mathbb{1}_{\{V^{i,N}_{u^-}=0\}}\delta_{(S^{i,N}_{u^-},\xi^{i,N}_{u^-})}$, we obtain that

$$\begin{aligned}
\textcircled{2.a} &= \sum_{\tilde{w}\in\mathbb{Z}} \frac{1}{N}\sum_i \mathbb{1}_{\{V^{i,N}_{u^-}=0\}} \alpha(I^{i,N}_{u^-}) \gamma_{\xi^{j,N}_{u^-}}(S^{i,N}_{u^-},\tilde{w}) \Big(\delta_{(1,0,\tilde{w})} - \delta_{(0,S^{i,N}_{u^-},\tilde{w})}\Big) \\
&= \sum_{\tilde{w}\in\mathbb{Z}} \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \alpha(I(\tilde{\xi})) \gamma_{\xi^{j,N}_{u^-}}(s,\tilde{w}) \Big(\delta_{(1,0,\tilde{w})} - \delta_{(0,s,\tilde{w})}\Big) \mu^N_{u^-}(0,ds,d\tilde{\xi}).
\end{aligned}$$

This last equation incites us to define the two following distributions. For $\xi\in\mathcal{P}(E_m)$ and $\mu\in\mathcal{P}(E)$, we define $\nu^0(\xi,\mu)$ and $\nu^1(\xi,\mu)$ the distributions on $\mathbb{R}^+\times\mathbb{Z}$ such that for all $(A,w)\in\mathcal{B}(\mathbb{R}^+)\times\mathbb{Z}$,

$$\nu^0(\xi,\mu)(A,w) \coloneqq \int_{\mathcal{P}(E_m)} \int_A \alpha\big(I(\tilde{\xi})\big)\gamma_\xi(s,w)\mu(0,ds,d\tilde{\xi}), \tag{S11}$$

$$\nu^1(\xi,\mu)(A,w) \coloneqq \mathbb{1}_{\{0\in A\}}\nu^0(\xi,\mu)(\mathbb{R}^+,w). \tag{S12}$$

Hence, we deduce that

$$\textcircled{2.a} = \delta_1\otimes\nu^1(\xi^{j,N}_{u^-},\mu^N_{u^-}) - \delta_0\otimes\nu^0(\xi^{j,N}_{u^-},\mu^N_{u^-})$$

and then

$$\begin{aligned}
\textcircled{2} &\overset{\mathbb{E}}{=} \int_0^t \frac{1}{N}\sum_j \partial_\xi\Psi(V^{j,N}_{u^-},S^{j,N}_{u^-},\xi^{j,N}_{u^-})\cdot \textcircled{2.a} du + \mathcal{O}_{N\infty}(1) \\
&\overset{\mathbb{E}}{=} \int_0^t \int_E \partial_\xi\Psi(v,s,\xi)\cdot\Big(\delta_1\otimes\nu^1(\xi,\mu^N_{u^-}) - \delta_0\otimes\nu^0(\xi,\mu^N_{u^-})\Big)\mu^N_{u^-}(dv,ds,d\xi)du + \mathcal{O}_{N\infty}(1).
\end{aligned}$$

□

From this result and under Assumption A2, we expect $\mathbb{E}(2)$ to converge – as N tends to infinity – to

$$\int_0^t \int_E \partial_\xi \Psi(v,s,\xi) \cdot (\delta_1 \otimes \nu^1(\xi, \mu^*_{u^-}) - \delta_0 \otimes \nu^0(\xi, \mu^*_{u^-}))\ \mu^*_{u^-}(dv,ds,d\xi)du. \tag{S13}$$

This convergence does not hold a priori because the functions

$$(v,s,\xi) \mapsto \partial_\xi \Psi(v,s,\xi) \cdot [\delta_1 \otimes \nu^1(\xi, \mu^N_{u^-}) - \delta_0 \otimes \nu^0(\xi, \mu^N_{u^-})] \tag{S14}$$

are not a priori continuous. We expect to show this convergence using a sequence (indexed by N) of continuous functions both getting closer and closer to (S14) and converging to

$$(v,s,\xi) \mapsto \partial_\xi \Psi(v,s,\xi) \cdot [\delta_1 \otimes \nu^1(\xi, \mu^*_{u^-}) - \delta_0 \otimes \nu^0(\xi, \mu^*_{u^-})].$$

Equation on μ^*_t

We denote by $\mu^*_t = \mathcal{L}(X^*_t) = \mathcal{L}(V^*_t, S^*_t, \xi^*_t)$. Thus, under Assumption A2 and from Propositions 18- 19, taking the limit as N tends to infinity in equation (S3) we can formulate the following conjecture.

Conjecture 20. *Assume that Assumptions A1, A2 and A3 hold. Then, the dynamics of μ^*_t is given by:*

- $\mathcal{L}(X^*_0) = \lim_{N\to\infty} \mathcal{L}(X^{1,N}_0) = \mu^*_0$ *and in particular S^*_0 is distributed by ρ_0 which is absolutely continuous with respect to the Lebesgue measure,*
- *for all $t \geq 0$, for all $\Psi \in C^{1,1}_b(E)$ (continuously differentiable with respect to its second variable and Fréchet differentiable with respect to its third variable), and using Notation 17,*

$$\begin{aligned}
\langle \mu^*_t, \Psi\rangle = \langle \mu^*_0, \Psi\rangle &+ \int_0^t \int_E \Big(\partial_s \Psi(v,s,\xi) + \partial_\xi \Psi(v,s,\xi)\cdot(-\partial_s \xi)\Big)\mu^*_u(dv,ds,d\xi)du \\
&+ \int_0^t \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \beta\Big[\Psi\Big(0,s,\xi\Big) - \Psi\Big(1,s,\xi\Big)\Big]\mu^*_{u^-}(1,ds,d\xi)du \\
&+ \int_0^t \int_E \partial_\xi \Psi(v,s,\xi)\cdot(\beta\delta_0 \otimes \xi^1 - \beta\delta_1 \otimes \xi^1)\mu^*_{u^-}(dv,ds,d\xi)du \\
&+ \int_0^t \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \alpha\big(I(\xi)\big)\Big[\Psi\Big(1,0,\xi\Big) - \Psi\Big(0,s,\xi\Big)\Big]\mu^*_{u^-}(0,ds,d\xi)du \\
&+ \int_0^t \int_E \partial_\xi \Psi(v,s,\xi)\cdot(\delta_1 \otimes \nu^1(\xi,\mu^*_{u^-}) - \delta_0 \otimes \nu^0(\xi,\mu^*_{u^-}))\ \mu^*_{u^-}(dv,ds,d\xi)du.
\end{aligned} \tag{S15}$$

We deduce from this conjecture that the joint distribution of the two first components of ξ^*_t and μ^*_t would then be equal for all t.

Consequence 21. *Grant Conjecture 20. Then, for all $t \geq 0$,*

$$\xi^*_t(\cdot,\cdot,\mathbb{Z}) = \mu^*_t(\cdot,\cdot,\mathcal{P}(E_m)).$$

Proof. For all $\xi^*_0 \in \mathrm{supp}(\mu^*_0)$, we have $\mu^*_0(\cdot,\cdot,\mathcal{P}(E_m)) = \xi^*_0(\cdot,\cdot,\mathbb{Z})$. Then, we show that $(\mu^*_t(\cdot,\cdot,\mathcal{P}(E_m)))_{t\geq 0}$ and $(\xi^*_t(\cdot,\cdot,\mathbb{Z}))_{t\geq 0}$ have the same dynamics, which enables us to conclude the proof. First, in equation (S15), taking a function Ψ which depends only on v and s, we obtain the dynamics of $(\mu^*_t(\cdot,\cdot,\mathcal{P}(E_m)))_{t\geq 0}$:

$$\begin{aligned}
\langle \mu^*_t, \Psi\rangle = \langle \mu^*_0, \Psi\rangle &+ \int_0^t \int_E \partial_s \Psi(v,s)\mu^*_u(dv,ds,d\xi)du \\
&+ \int_0^t \int \beta\left[\Psi(0,s) - \Psi(1,s)\right]\mu^*_{u^-}(1,ds,d\xi)du \\
&+ \int_0^t \int \alpha\big(I(\xi)\big)\left[\Psi(1,0) - \Psi(0,s)\right]\mu^*_{u^-}(0,ds,d\xi)du.
\end{aligned} \tag{S16}$$

Second, taking Ψ depending only on ξ in equation (S15), we obtain that

$$\langle \mu_t^*, \Psi\rangle - \langle \mu_0^*, \Psi\rangle = \int_0^t \int_E \partial_\xi \Psi(\xi) \cdot \Big[\beta\delta_0 \otimes \xi^1 - \beta\delta_1 \otimes \xi^1 + \delta_1 \otimes \nu^1(\xi, \mu_{u^-}^*) - \delta_0 \otimes \nu^0(\xi, \mu_{u^-}^*) - \partial_s \xi\Big]\, \mu_{u^-}^*(dv, ds, d\xi)du.$$

Working on the right hand side term, we obtain

$$\langle \mu_t^*, \Psi\rangle - \langle \mu_0^*, \Psi\rangle = \mathbb{E}\left[\Psi(\xi_t^*)\right] - \mathbb{E}\left[\Psi(\xi_0^*)\right] = \int_0^t \mathbb{E}\left[\partial_\xi \Psi(\xi_u^*) \cdot \partial \xi_u^*\right] du$$

and then deduce that $\phi \in C_b^1(E_m)$ with bounded derivative,

$$\begin{aligned}\langle \xi_t^*, \phi\rangle = \langle \xi_0^*, \phi\rangle + \int_0^t \langle -\partial_s \xi_u^*, \phi\rangle du + \int_0^t \beta\langle \delta_0 \otimes {\xi_{u^-}^*}^1 - \delta_1 \otimes {\xi_{u^-}^*}^1, \phi\rangle du \\ + \int_0^t \langle \delta_1 \otimes \nu^1(\xi_{u^-}^*, \mu_{u^-}^*) - \delta_0 \otimes \nu^0(\xi_{u^-}^*, \mu_{u^-}^*), \phi\rangle du.\end{aligned} \tag{S17}$$

Evaluating this last equation for a function ϕ depending only on v and s gives us the dynamics of $\xi_t^*(\cdot, \cdot, \mathbb{Z})$. We obtain exactly the same equation as (S16):

- using (S2), we have for all $u \in [0, t]$, $\langle -\partial_s \xi_u^*, \phi\rangle = \langle \xi_u^*, \partial_s \phi\rangle$,
- as for all s in the support of $\xi \in \mathcal{P}(E_m)$, $\sum_{w\in\mathbb{Z}} \gamma_\xi(s, w) = 1$, we obtain that (see (S11) and (S12) for the definitions of ν^0 and ν^1) for all $A \in \mathcal{B}(\mathbb{R}^+)$:

$$\begin{aligned}\nu^0(\xi, \mu)(A, \mathbb{Z}) &= \int_{\mathcal{P}(E_m)} \int_A \alpha\big(I(\tilde{\xi})\big)\mu(0, ds, d\tilde{\xi}),\\ \nu^1(\xi, \mu)(A, \mathbb{Z}) &= \mathbb{1}_{\{0\in A\}} \int_{\mathcal{P}(E_m)} \int_{\mathbb{R}^+} \alpha\big(I(\tilde{\xi})\big)\mu(0, ds, d\tilde{\xi}).\end{aligned}$$

□

The typical neuron dynamics

Consequence 22. *Grant Conjecture 20 and Assumption A4 with $p^\pm \equiv 0$. Then, we have*

- $dS_t^* = dt$.
- *ξ_t^* admits a density in s that satisfies the following equations,*

$$\begin{cases} \partial_t \xi_t^*(0, s, w) &= -\partial_s \xi_t^*(0, s, w) + \beta\xi_t^*(1, s, w) - \int_{\mathcal{P}(E_m)} \alpha\big(I(\xi')\big) \frac{\xi_t^*(0,s,w)}{\xi_t^*(0,s,\mathbb{Z})} \mu_t^*(0, s, d\xi') \\ \xi_t^*(0, 0, w) &= 0, \end{cases}$$

$$\begin{cases} \partial_t \xi_t^*(1, s, w) &= -\partial_s \xi_t^*(1, s, w) - \beta\xi_t^*(1, s, w) \\ \xi_t^*(1, 0, w) &= \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \alpha\big(I(\xi')\big) \frac{\xi_t^*(0,s',w)}{\xi_t^*(0,s',\mathbb{Z})} \mu_t^*(0, s', d\xi')ds'. \end{cases}$$

- *At rate $\beta\mathbb{1}_{\{V_{t^-}^*=1\}}$, $(V_{t^-}^*, S_{t^-}^*, \xi_{t^-}^*)$ jumps to $(0, S_{t^-}^*, \xi_{t^-}^*)$.*
- *At rate $\alpha\big(I(\xi_{t^-}^*)\big)\mathbb{1}_{\{V_{t^-}^*=0\}}$, $(V_{t^-}^*, S_{t^-}^*, \xi_{t^-}^*)$ jumps to $(1, 0, \xi_{t^-}^*)$.*

Proof. Both the first two and last two points are clear with Conjecture 20. We detail the third point. First, from the regularity assumption on the density in s of μ_t^*, Consequence 21 tells us that ξ_t^* admits a density C^1 in s and satisfies (S17). Using (S10), (S11), (S12), we thus obtain by integration by parts the following equations on the density functions $s \mapsto \xi_t^*(v,s,w)$:

$$\begin{aligned}\partial_t \xi_t^*(0,s,w) =& -\partial_s \xi_t^*(0,s,w) - \delta_0(s)\xi_t^*(0,0,w) + \beta \xi_t^*(1,s,w)\\ &- \int_{\mathcal{P}(E_m)} \alpha\big(I(\xi')\big)\frac{\xi_t^*(0,s,w)}{\xi_t^*(0,s,\mathbb{Z})}\mu_t^*(0,s,d\xi'),\end{aligned}$$

$$\begin{aligned}\partial_t \xi_t^*(1,s,w) =& -\partial_s \xi_t^*(1,s,w) - \delta_0(s)\xi_t^*(1,0,w) - \beta \xi_t^*(1,s,w)\\ &+ \delta_0(s)\int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \alpha\big(I(\xi')\big)\frac{\xi_t^*(0,s',w)}{\xi_t^*(0,s',\mathbb{Z})}\mu_t^*(0,ds',d\xi').\end{aligned}$$

By integrating the previous equations on $s \in [0,\epsilon]$ and making ϵ tends to 0 we obtain the equations with conditions at boundaries $s = 0$ of the corollary. □

D. The case with plasticity

We now deal with the synaptic weight jumps. As soon as a neuron spikes, several different synaptic weight jumps are possible. We denote by J_i^N the possible increments of the weights when the neuron i spikes

$$J_i^N = \{A = (a^{kl})_{1\le k,l\le N} : a^{ii} \in \{-1,0,1\} \text{ and } \forall k,l \neq i, a^{il} \in \{0,1\}, a^{ki} \in \{0,-1\}, a^{kl} = 0\}.$$

For all $\Delta \in J_i^N$, we denote by $P_{t,i}^{\Delta}$ the probability that the weight matrix be incremented by Δ conditionally to X_t^N. In order to ease its understanding, we express it with the weight matrix W_t^N rather than using the empirical distributions $(\xi_t^{i,N})_{1\le i\le N}$:

$$\begin{aligned}P_{t,i}^{\Delta} = \Bigg[&\frac{\Delta^{ii}(1+\Delta^{ii})}{2}p^+(S_{t^-}^{i,N}, W_{t^-}^{ii,N})\big(1-p^-(S_{t^-}^{i,N}, W_{t^-}^{ii,N})\big)\\ &+ \big(1-|\Delta^{ii}|\big)\Big(p^+(S_{t^-}^{i,N}, W_{t^-}^{ii,N})p^-(S_{t^-}^{i,N}, W_{t^-}^{ii,N})\\ &\qquad + \big(1-p^+(S_{t^-}^{i,N}, W_{t^-}^{ii,N})\big)\big(1-p^-(S_{t^-}^{i,N}, W_{t^-}^{ii,N})\big)\Big)\\ &+ \frac{\Delta^{ii}(\Delta^{ii}-1)}{2}p^-(S_{t^-}^{i,N}, W_{t^-}^{ii,N})\big(1-p^+(S_{t^-}^{i,N}, W_{t^-}^{ii,N})\big)\Bigg]\\ &\prod_{k,l\neq i}\Big(\Delta^{il}p^+(S_{t^-}^{l,N}, W_{t^-}^{il,N}) + (1-\Delta^{il})\big(1-p^+(S_{t^-}^{l,N}, W_{t^-}^{il,N})\big)\Big)\\ &\Big(-\Delta^{ki}p^-(S_{t^-}^{k,N}, W_{t^-}^{ki,N}) + (1+\Delta^{ki})\big(1-p^-(S_{t^-}^{k,N}, W_{t^-}^{ki,N})\big)\Big).\end{aligned}$$

After the spike of the neuron i at time t and assuming the weights are incremented of $\Delta \in J_i^N$ at this time, $\xi_{t^-}^{i,N}$ jumps to $\xi_{t,\Delta}^{i,N}$ and for all $j \neq i$, $\xi_{t^-}^{j,N}$ jumps to $\xi_{t,\Delta,i}^{j,N}$ such that

$$\begin{aligned}\xi_{t,\Delta}^{i,N} =& \xi_{t^-}^{i,N} + \frac{1}{N}\sum_{(0,S_{t^-}^{i,N},\tilde{W})\in\mathrm{supp}(\xi_{t^-}^{i,N})}\big(\delta_{(1,0,\tilde{W}+\Delta^{ii})} - \delta_{(0,S_{t^-}^{i,N},\tilde{W})}\big)\\ &+ \sum_{l\neq i}\frac{1}{N}\sum_{(\tilde{V},S_{t^-}^{l,N},\tilde{W})\in\mathrm{supp}(\xi_{t^-}^{i,N})}\big(\delta_{(\tilde{V},S_{t^-}^{l,N},\tilde{W}+\Delta^{il})} - \delta_{(\tilde{V},S_{t^-}^{l,N},\tilde{W})}\big)\end{aligned} \tag{S18}$$

$$\xi_{t,\Delta,i}^{j,N} = \xi_{t^-}^{j,N} + \frac{1}{N}\sum_{(0,S_{t^-}^{i,N},\tilde{W})\in\mathrm{supp}(\xi_{t^-}^{j,N})}\big(\delta_{(1,0,\tilde{W}+\Delta^{ji})} - \delta_{(0,S_{t^-}^{i,N},\tilde{W})}\big). \tag{S19}$$

For all $\Psi \in C_b^{1,1}(E)$, we now have

$$\begin{aligned}
\langle \mu_t^N, \Psi\rangle \overset{\mathbb{E}}{=} \langle \mu_0^N, \Psi\rangle &+ \int_0^t \int_E \Big(\partial_s \Psi(v,s,\xi) + \partial_\xi \Psi(v,s,\xi)\cdot(-\partial_s \xi)\Big)\mu_u^N(dv,ds,d\xi)du \\
&+ \int_0^t \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=1\}} \frac{\beta}{N}\Bigg\{\Big[\Psi\Big(0, S_{u^-}^{i,N}, \xi_{u^-}^{i,N} + \eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{i,N})\Big) - \Psi\Big(1, S_{u^-}^{i,N}, \xi_{u^-}^{i,N}\Big)\Big] \\
&\qquad + \sum_{j\neq i}\Big[\Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N} + \eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\Big) - \Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N}\Big)\Big]\Bigg\}du \\
&+ \int_0^t \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_{\Delta\in J_i^N} P_{u,i}^\Delta \Bigg\{\Big[\Psi\Big(1,0,\xi_{u,\Delta}^{i,N}\Big) - \Psi\Big(0, S_{u^-}^{i,N}, \xi_{u^-}^{i,N}\Big)\Big] \\
&\qquad + \sum_{j\neq i}\Big[\Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u,\Delta,i}^{j,N}\Big) - \Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N}\Big)\Big]\Bigg\}du.
\end{aligned}$$

By adding and removing

$$\mathbb{E}\Bigg[\int_0^t \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_{\Delta\in J_i^N} P_{u,i}^\Delta \Big[\Psi\Big(0, S_{u^-}^{i,N}, \xi_{u,\Delta,i}^{i,N}\Big) - \Psi\Big(0, S_{u^-}^{i,N}, \xi_{u^-}^{i,N}\Big)\Big]du\Bigg],$$

we can rewrite the last term to obtain a complete sum over j:

$$\begin{aligned}
\langle \mu_t^N, \Psi\rangle \overset{\mathbb{E}}{=} \langle \mu_0^N, \Psi\rangle &+ \int_0^t \int_E \Big(\partial_s \Psi(v,s,\xi) + \partial_\xi \Psi(v,s,\xi)\cdot(-\partial_s \xi)\Big)\mu_u^N(dv,ds,d\xi)du \\
&+ \int_0^t \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=1\}} \frac{\beta}{N}\Bigg\{\Big[\Psi\Big(0, S_{u^-}^{i,N}, \xi_{u^-}^{i,N} + \eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{i,N})\Big) - \Psi\Big(1, S_{u^-}^{i,N}, \xi_{u^-}^{i,N}\Big)\Big] \\
&\qquad + \sum_{j\neq i}\Big[\Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N} + \eta^N(S_{u^-}^{i,N}, \xi_{u^-}^{j,N})\Big) - \Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N}\Big)\Big]\Bigg\}du \\
&+ \int_0^t \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_{\Delta\in J_i^N} P_{u,i}^\Delta \Bigg\{\Big[\Psi\Big(1,0,\xi_{u,\Delta}^{i,N}\Big) - \Psi\Big(0, S_{u^-}^{i,N}, \xi_{u,\Delta,i}^{i,N}\Big)\Big] \\
&\qquad + \sum_{j}\Big[\Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u,\Delta,i}^{j,N}\Big) - \Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N}\Big)\Big]\Bigg\}du.
\end{aligned} \tag{S20}$$

We then denote by

$$\textcircled{3} = \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_{\Delta\in J_i^N} P_{u,i}^\Delta \Big[\Psi\Big(1,0,\xi_{u,\Delta}^{i,N}\Big) - \Psi\Big(0, S_{u^-}^{i,N}, \xi_{u,\Delta,i}^{i,N}\Big)\Big] \tag{S21}$$

and

$$\textcircled{4} = \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_{\Delta\in J_i^N} P_{u,i}^\Delta \sum_j \Big[\Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u,\Delta,i}^{j,N}\Big) - \Psi\Big(V_{u^-}^{j,N}, S_{u^-}^{j,N}, \xi_{u^-}^{j,N}\Big)\Big].$$

The depression term

We first deal with the term $\textcircled{4}$ which describes the depression of the weights.

Proposition 23. *Then, for all $u \geq 0$,*

$$\textcircled{4} \overset{\mathbb{E}}{=} \int \partial_\xi \Psi(v,s,\xi)\cdot\left[\delta_1 \otimes \nu^{-,1}(s,\xi,\mu_u^N) - \delta_0 \otimes \nu^0(\xi,\mu_u^N)\right] \mu_u^N(dv,ds,d\xi) + \mathcal{O}(1)\,.$$

where $\nu^{-,1}$ is defined in the proof, see (S22).

Proof. First, by Lemma 8, we have almost surely for all i, j,

$$\|\frac{1}{N}\sum_{(0,S^{i,N}_{u^-},\tilde{W})\in\mathrm{supp}(\xi^{j,N}_{u^-})}\big(\delta_{(1,0,\tilde{W}+\Delta^{ji})}-\delta_{(0,S^{i,N}_{u^-},\tilde{W})}\big)\|_{TV}\leq\frac{2}{N}.$$

Then, we use Fréchet derivative to obtain that

$$\Psi\Big(V^{j,N}_{u^-},S^{j,N}_{u^-},\xi^{j,N}_{u,\Delta,i}\Big)-\Psi\Big(V^{j,N}_{u^-},S^{j,N}_{u^-},\xi^{j,N}_{u^-}\Big)=\\ \partial_\xi\Psi\Big(V^{j,N}_{u^-},S^{j,N}_{u^-},\xi^{j,N}_{u^-}\Big)\cdot\Big(\frac{1}{N}\sum_{(0,S^{i,N}_{u^-},\tilde{W})\in\mathrm{supp}(\xi^{j,N}_{u^-})}\big(\delta_{(1,0,\tilde{W}+\Delta^{ji})}-\delta_{(0,S^{i,N}_{u^-},\tilde{W})}\big)\Big)+\mathcal{O}\Big(\frac{1}{N}\Big).$$

We deduce by linearity that

$$④=\frac{1}{N}\sum_j\partial_\xi\Psi\Big(V^{j,N}_{u^-},S^{j,N}_{u^-},\xi^{j,N}_{u^-}\Big)\cdot\\ \underbrace{\Big(\sum_i\mathbb{1}_{\{V^{i,N}_{u^-}=0\}}\frac{\alpha(I^{i,N}_{u^-})}{N}\sum_{(0,S^{i,N}_{u^-},\tilde{W})\in\mathrm{supp}(\xi^{j,N}_{u^-})}\sum_{\Delta\in J^N_i}P^\Delta_{u,i}\big(\delta_{(1,0,\tilde{W}+\Delta^{ji})}-\delta_{(0,S^{i,N}_{u^-},\tilde{W})}\big)\Big)}_{④a}+\mathcal{O}(1)\,.$$

But for any function f on $\{1,0,-1\}$ we have for all $i\neq j$,

$$\sum_{\Delta\in J^N_i}P^\Delta_{u,i}f(\Delta^{ji})=\sum_{\Delta\in J^N_i,\Delta^{ji}=-1}P^\Delta_{u,i}f(-1)+\sum_{\Delta\in J^N_i,\Delta^{ji}=0}P^\Delta_{u,i}f(0)\\ =p^-(S^{j,N}_{u^-},W^{ji}_{u^-})f(-1)+(1-p^-(S^{j,N}_{u^-},W^{ji}_{u^-}))f(0),$$

and for $i=j$,

$$\sum_{\Delta\in J^N_j}P^\Delta_{u,j}f(\Delta^{jj})=p^-(S^{j,N}_{u^-},W^{jj}_{u^-})(1-p^+(S^{j,N}_{u^-},W^{jj}_{u^-}))f(-1)+p^+(S^{j,N}_{u^-},W^{jj}_{u^-})(1-p^-(S^{j,N}_{u^-},W^{jj}_{u^-}))f(1)\\ +\Big((1-p^-(S^{j,N}_{u^-},W^{jj}_{u^-}))(1-p^+(S^{j,N}_{u^-},W^{jj}_{u^-}))+p^+(S^{j,N}_{u^-},W^{jj}_{u^-})p^-(S^{j,N}_{u^-},W^{jj}_{u^-})\Big)f(0).$$

Denoting by

$$C^{j,N}=\mathbb{1}_{\{V^{j,N}_{u^-}=0\}}\frac{\alpha(I^{j,N}_{u^-})}{N}\sum_{(0,S^{j,N}_{u^-},\tilde{W})\in\mathrm{supp}(\xi^{j,N}_{u^-})}\Big[-p^-(S^{j,N}_{u^-},\tilde{W})p^+(S^{j,N}_{u^-},\tilde{W})\big(\delta_{(1,0,\tilde{W}-1)}-\delta_{(0,S^{j,N}_{u^-},\tilde{W})}\big)\\ +p^+(S^{j,N}_{u^-},\tilde{W})\big(2p^-(S^{j,N}_{u^-},\tilde{W})-1\big)\big(\delta_{(1,0,\tilde{W})}-\delta_{(0,S^{j,N}_{u^-},\tilde{W})}\big)\\ +p^+(S^{j,N}_{u^-},\tilde{W})\big(1-p^-(S^{j,N}_{u^-},\tilde{W})\big)\big(\delta_{(1,0,\tilde{W}+1)}-\delta_{(0,S^{j,N}_{u^-},\tilde{W})}\big)\Big],$$

a term of order $\frac{1}{N}$, we obtain that

$$\begin{aligned}
\text{(4a)} - C^{j,N} &= \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_{(0,S_{u^-}^{i,N},\tilde{W})\in \mathrm{supp}(\xi_{u^-}^{j,N})} \Big[p^-(S_{u^-}^{j,N},\tilde{W})\big(\delta_{(1,0,\tilde{W}-1)} - \delta_{(0,S_{u^-}^{i,N},\tilde{W})}\big) \\
&\qquad + \big(1-p^-(S_{u^-}^{j,N},\tilde{W})\big)\big(\delta_{(1,0,\tilde{W})} - \delta_{(0,S_{u^-}^{i,N},\tilde{W})}\big)\Big] \\
&= \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_{\tilde{W}\in\mathbb{Z}} \mathbb{1}_{\{(0,S_{u^-}^{i,N},\tilde{W})\in \mathrm{supp}(\xi_{u^-}^{j,N})\}} \Big[p^-(S_{u^-}^{j,N},\tilde{W})\big(\delta_{(1,0,\tilde{W}-1)} - \delta_{(0,S_{u^-}^{i,N},\tilde{W})}\big) \\
&\qquad + \big(1-p^-(S_{u^-}^{j,N},\tilde{W})\big)\big(\delta_{(1,0,\tilde{W})} - \delta_{(0,S_{u^-}^{i,N},\tilde{W})}\big)\Big] \\
&= \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_{\tilde{W}\in\mathbb{Z}} \gamma_{\xi_{u^-}^{j,N}}(S_{u^-}^{i,N},\tilde{W}) \Big[p^-(S_{u^-}^{j,N},\tilde{W})\big(\delta_{(1,0,\tilde{W}-1)} - \delta_{(0,S_{u^-}^{i,N},\tilde{W})}\big) \\
&\qquad + \big(1-p^-(S_{u^-}^{j,N},\tilde{W})\big)\big(\delta_{(1,0,\tilde{W})} - \delta_{(0,S_{u^-}^{i,N},\tilde{W})}\big)\Big],
\end{aligned}$$

where we used the definition of γ_ξ given by (S10). Hence, we deduce that

$$\begin{aligned}
\text{(4a)} = \sum_{\tilde{W}\in\mathbb{Z}} \int_{\mathcal{P}(E_m)} \int_{\mathbb{R}^+} \alpha(I(\tilde{\xi}))\gamma_{\xi_{u^-}^{j,N}}(\tilde{s},\tilde{W}) \Big[p^-(S_{u^-}^{j,N},\tilde{W})\big(\delta_{(1,0,\tilde{W}-1)} - \delta_{(0,\tilde{s},\tilde{W})}\big) \\
+ \big(1-p^-(S_{u^-}^{j,N},\tilde{W})\big)\big(\delta_{(1,0,\tilde{W})} - \delta_{(0,\tilde{s},\tilde{W})}\big)\Big]\mu_{u^-}^N(0,d\tilde{s},d\tilde{\xi}) + \mathcal{O}\Big(\frac{1}{N}\Big).
\end{aligned}$$

This incites us to define the following distribution. For all triple $(s,\xi,\mu) \in \mathbb{R}^+ \times \mathcal{P}(E_m) \times \mathcal{P}(E)$, we denote by $\nu^{-,1}(s,\xi,\mu)$ the distribution on $\mathbb{R}^+ \times \mathbb{Z}$ such that for all $(A,w) \in \mathcal{B}(\mathbb{R}^+) \times \mathbb{Z}$,

$$\begin{aligned}
\nu^{-,1}(s,\xi,\mu)(A,w) \coloneqq \mathbb{1}_{\{0\in A\}} \int_{\mathcal{P}(E_m)} \alpha\big(I(\tilde{\xi})\big) p^-(s,w+1)\ \gamma_\xi(\tilde{s},w+1)\mu(0,d\tilde{s},d\tilde{\xi}) \\
+ \mathbb{1}_{\{0\in A\}} \int_{\mathcal{P}(E_m)} \alpha\big(I(\tilde{\xi})\big)(1-p^-(s,w))\ \gamma_\xi(\tilde{s},w)\mu(0,d\tilde{s},d\tilde{\xi}).
\end{aligned} \tag{S22}$$

Thereby, using the function ν^0 defined in (S11), we get

$$\text{(4a)} = \delta_1 \otimes \nu^{-,1}(S_{u^-}^{j,N},\xi_{u^-}^{j,N},\mu_{u^-}^N) - \delta_0 \otimes \nu^0(\xi_{u^-}^{j,N},\mu_{u^-}^N) + o(1)\,.$$

We finally obtain that

$$\text{(4)} \overset{\mathbb{E}}{=} \int \partial_\xi \Psi(v,s,\xi)\cdot[\delta_1 \otimes \nu^{-,1}(s,\xi,\mu_u^N) - \delta_0 \otimes \nu^0(\xi,\mu_u^N)]\ \mu_u^N(dv,ds,d\xi) + o(1)\,.$$

It ends the proof. □

Under Assumption A2, it is reasonable to expect that, as N tends to infinity, $\mathbb{E}$(4) converges to

$$\int \partial_\xi \Psi(v,s,\xi)\cdot[\delta_1 \otimes \nu^{-,1}(s,\xi,\mu_u^*) - \delta_0 \otimes \nu^0(\xi,\mu_u^*)]\ \mu_u^*(dv,ds,d\xi).$$

As noted in the remark given just after Proposition 19, this convergence does not hold a priori because the functions

$$(v,s,\xi) \mapsto \partial_\xi \Psi(v,s,\xi)\cdot[\delta_1 \otimes \nu^{-,1}(s,\xi,\mu_u^N) - \delta_0 \otimes \nu^0(\xi,\mu_u^N)] \tag{S23}$$

are not a priori continuous. We expect to show this convergence using a sequence (indexed by N) of continuous functions both getting closer and closer to (S23) and converging to

$$(v,s,\xi) \mapsto \partial_\xi \Psi(v,s,\xi)\cdot[\delta_1 \otimes \nu^{-,1}(s,\xi,\mu_u^*) - \delta_0 \otimes \nu^0(\xi,\mu_u^*)].$$

The potentiation term

Now, for the term ③ describing potentiation, see (S21), we need to assume that Ψ is linear in its third variable ξ.

Proposition 24. *Under Assumptions A1, A2 and A3 we find that for all $u \geq 0$, for all $\Psi \in C_b^{1,1}(E)$* ***linear in its third variable*** *ξ,*

$$\lim_{N\to\infty} ③ \overset{\mathbb{E}}{=} \int_E \alpha\big(I(\xi)\big)\Big[\Psi\Big(1,0,\nu^+(\xi)\Big) - \Psi\Big(0,s,\xi\Big)\Big]\mu_u^*(0,ds,d\xi),$$

where ν^+ is defined in the proof, see (S24).

Proof. By linearity of Ψ in its third variable, we obtain that

$$③ = \sum_i \mathbb{1}_{\{V_{u^-}^{i,N}=0\}} \frac{\alpha(I_{u^-}^{i,N})}{N} \sum_{\Delta\in J_i^N} P_{u,i}^{\Delta}\Big[\Psi\Big(1,0,\sum_{\Delta\in J_i^N} P_{u,i}^{\Delta}\xi_{u,\Delta}^{i,N}\Big) - \Psi\Big(0,S_{u^-}^{i,N},\sum_{\Delta\in J_i^N} P_{u,i}^{\Delta}\xi_{u,\Delta,i}^{i,N}\Big)\Big].$$

and then by definition of $\xi_{u,\Delta}^{i,N}$, see (S18), we have

$$\begin{aligned}\sum_{\Delta\in J_i^N} P_{u,i}^{\Delta}\xi_{u,\Delta}^{i,N} = \xi_{u^-}^{i,N} &+ \frac{1}{N}\sum_{(0,S_{u^-}^{i,N},\tilde{W})\in\mathrm{supp}(\xi_{u^-}^{i,N})}\sum_{\Delta\in J_i^N} P_{u,i}^{\Delta}\big(\delta_{(1,0,\tilde{W}+\Delta^{ii})} - \delta_{(0,S_{u^-}^{i,N},\tilde{W})}\big)\\ &+ \sum_{l\neq i}\frac{1}{N}\sum_{(\tilde{V},S_{u^-}^{l,N},\tilde{W})\in\mathrm{supp}(\xi_{u^-}^{i,N})}\sum_{\Delta\in J_i^N} P_{u,i}^{\Delta}\big(\delta_{(\tilde{V},S_{u^-}^{l,N},\tilde{W}+\Delta^{il})} - \delta_{(\tilde{V},S_{u^-}^{l,N},\tilde{W})}\big).\end{aligned}$$

As for the term ④, noting the second term of the right hand side of the last equation is of order $\frac{1}{N}$, we can find a term $\tilde{C}_i^N$ of $\frac{1}{N}$ such that

$$\begin{aligned}\sum_{\Delta\in J_i^N} P_{u,i}^{\Delta}\xi_{u,\Delta}^{i,N} + \tilde{C}^{i,N} &= \xi_{u^-}^{i,N} + \sum_{l\neq i}\frac{1}{N}\sum_{(\tilde{V},S_{u^-}^{l,N},\tilde{W})\in\mathrm{supp}(\xi_{u^-}^{i,N})} p^+(S_{u^-}^{l,N},\tilde{W})\big(\delta_{(\tilde{V},S_{u^-}^{l,N},\tilde{W}+1)} - \delta_{(\tilde{V},S_{u^-}^{l,N},\tilde{W})}\big)\\ &= \frac{1}{N}\sum_{(\tilde{V},\tilde{S},\tilde{W})\in\mathrm{supp}(\xi_{u^-}^{i,N})} p^+(\tilde{S},\tilde{W})\delta_{(\tilde{V},\tilde{S},\tilde{W}+1)} + \big(1-p^+(\tilde{S},\tilde{W})\big)\delta_{(\tilde{V},\tilde{S},\tilde{W})}\\ &= \nu^+(\xi_{u^-}^{i,N})\end{aligned}$$

where

$$\nu^+(\xi)(v,A,w) = \int_A p^+(s,w-1)\xi(v,ds,\{w-1\}) + \int_A (1-p^+(s,w))\xi(v,ds,w). \tag{S24}$$

But p^+ takes values in $[0,1]$ so using the triangular inequality, $\xi \mapsto \nu^+(\xi)$ is continuous. We can then conclude using Assumption A2. □

1. *Justifying Main Result 5*

We are now in position to detail why Main Result 5 can be conjectured. Under Assumptions A1, A2 and A3, by making N tends to infinity in equation (S20) and using the results of Propositions 23 and 24, we should get the following dynamics of μ_t^*:

- $\mathcal{L}(X_0^*) = \lim_{N\to\infty}\mathcal{L}(X_0^{1,N}) = \mu_0^*$ and in particular S_0^* is distributed by the ρ_0 distribution which is absolutely continuous with respect to the Lebesgue measure,

- using Notation 17 we have, for all $t \geq 0$, for all $\Psi \in C_b^{1,1}(E)$,

$$\begin{aligned}\langle \mu_t^*, \Psi\rangle = \langle \mu_0^*, \Psi\rangle &+ \int_0^t \int_E \Big(\partial_s \Psi(v,s,\xi) + \partial_\xi \Psi(v,s,\xi)\cdot(-\partial_s \xi)\Big)\mu_u^*(dv,ds,d\xi)du \\ &+ \int_0^t \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \beta\Big[\Psi\Big(0,s,\xi\Big) - \Psi\Big(1,s,\xi\Big)\Big]\mu_u^*(1,ds,d\xi)du \\ &+ \int_0^t \int_E \partial_\xi \Psi(v,s,\xi)\cdot(\beta\delta_0\otimes\xi^1 - \beta\delta_1\otimes\xi^1)\mu_u^*(dv,ds,d\xi)du \\ &+ \int_0^t \int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \alpha\big(I(\xi)\big)\Big[\Psi\Big(1,0,\nu^+(\xi)\Big) - \Psi\Big(0,s,\xi\Big)\Big]\mu_u^*(0,ds,d\xi)du \\ &+ \int_0^t \int_E \partial_\xi \Psi(v,s,\xi)\cdot[\delta_1\otimes\nu^{-,1}(s,\xi,\mu_u^*) - \delta_0\otimes\nu^0(\xi,\mu_u^*)]\ \mu_u^*(dv,ds,d\xi)du,\end{aligned} \tag{S25}$$

  where ν^+, $\nu^{-,1}$ and ν^0 are respectively defined in (S24), (S22) and (S11).

With the same arguments as the one used in Consequence 21, one has that for all $t \geq 0$,

$$\xi_t^*(\cdot,\cdot,\mathbb{Z}) = \mu_t^*(\cdot,\cdot,\mathcal{P}(E_m)).$$

Then, from Main Result 5, μ_t^* is assumed to admit a density C^1 in s. Therefore, the last equation tells us that ξ_t^* admits a density C^1 in s and by integration by parts in equation (S25), we have:

$$\partial_t \xi_t^* = -\partial_s \xi_t^* + \beta\left(\delta_0\otimes\xi_t^{*1} - \delta_1\otimes\xi_t^{*1}\right) + \delta_1\otimes\nu^{-,1}(S_t^*,\xi_t^*,\mu_t^*) - \delta_0\otimes\nu^0(\xi_t^*,\mu_t^*).$$

From this equation and (S10), (S11), (S12), we thus obtain the following equations on the density functions $s \mapsto \xi_t^*(v,s,w)$:

$$\partial_t \xi_t^*(0,s,w) = -\partial_s \xi_t^*(0,s,w) - \delta_0(s)\xi_t^*(0,0,w) + \beta\xi_t^*(1,s,w) - \int_{\mathcal{P}(E_m)} \alpha\big(I(\xi')\big)\frac{\xi_t^*(0,s,w)}{\xi_t^*(0,s,\mathbb{Z})}\mu_t^*(0,s,d\xi'),$$

$$\begin{aligned}\partial_t \xi_t^*(1,s,w) = &-\partial_s \xi_t^*(1,s,w) - \delta_0(s)\xi_t^*(1,0,w) - \beta\xi_t^*(1,s,w) \\ &+ \delta_0(s)\int_{\mathbb{R}^+\times\mathcal{P}(E_m)} \frac{\alpha\big(I(\xi')\big)}{\xi_t^*(0,s',\mathbb{Z})}\Big[p^-(S_t^*,w)\xi_t^*(0,s',\{w+1\}) + (1-p^-(S_t^*,w))\xi_t^*(0,s',w)\Big]\mu_t^*(0,ds',d\xi').\end{aligned}$$

By integrating the previous equations on $s \in [0,\epsilon]$ and making ϵ tends to 0, we obtain the equations with conditions at boundaries $s = 0$ of the Main Result 5.

S2. CODE OF THE MEAN-FIELD EQUATIONS

We explain in this section our simulations both for the microscopic model, defined through $(V_t^{i,N}, S_t^{i,N}, (W_t^{ij,N})_j)_{i,t\geq 0}$, and the mean field system, $(V_t^{i,*}, S_t^{i,*}, \xi_t^{i,*})_{i,t\geq 0}$. Indeed, we cannot simulate directly the dynamics of X_t^* because we do not have access to its law μ_t^*. Hence, instead we simulate N *typical* neurons $X_t^{1,*}, \cdots, X_t^{N,*}$ having the same dynamics as X_t^*, except that instead of μ_t^* we use an approximation of it, $\mu_t^{*,N} = \frac{1}{N}\sum_k \delta_{X_t^{k,*}}$. We start this section with an overview giving the main parameters and describing the way ξ is approximated in the mean field system. Then, we give the initial conditions and finally, we detail how the dynamics are simulated in both cases.

A. Overview

1. Parameters

First, they both have the same parameters that we list here:

- N: number of neurons,
- α: rate function with the total synaptic current as input (jump of V from 0 to 1),
- β: rate of the return to the resting potential (jump of V from 1 to 0),
- $p^{\pm}$: STDP functions with time delay between spikes as input and probability of synaptic weight jump as output,
- dt: time step of the simulation,
- t_f: final time of the simulation,
- p_V: initial probability of V being in state 1,
- $\rho_{0/1}$: respectively the distribution of the initial S depending on the corresponding V being 0 or 1,
- p_W: initial distribution of the weights.

2. *Approximating ξ*

The neuron i satisfies the following equations

- $X_0^{i,*}$ is determined from the initial state of the microscopic model.
- $dS_t^{i,*} = dt$.
- $\xi_t^{i,*}$ admits a density in s and satisfies the following equation:

$$
\begin{aligned}
\partial_t \xi_t^{i,*}(0,s,w) &= -\partial_s \xi_t^{i,*}(0,s,w) + \beta \xi_t^{i,*}(1,s,w) - \xi_t^{i,*}(0,s,w) \underbrace{\frac{\frac{1}{N}\sum_k \alpha(I(\xi_t^{k,*}))\mathbb{1}_{\{V_t^{k,*}=0, S_t^{k,*}=s\}}}{\frac{1}{N}\sum_l \mathbb{1}_{\{V_t^{l,*}=0, S_t^{l,*}=s\}}}}_{a_t^0(s)} \\
\partial_t \xi_t^{i,*}(1,s,w) &= -\partial_s \xi_t^{i,*}(1,s,w) - \beta \xi_t^{i,*}(1,s,w) \\
\xi_t^{i,*}(0,0,w) &= 0, \\
\xi_t^{i,*}(1,0,w) &= \frac{\frac{dt}{N}\sum_k \alpha(I(\xi_t^{k,*}))\left[p^-(S_t^{i,*})\xi_t^{i,*}(0,S_t^{k,*},w+1) + (1-p^-(S_t^{i,*}))\xi_t^{i,*}(0,S_t^{k,*},w)\right]}{\frac{1}{N}\sum_l \mathbb{1}_{\{V_t^{l,*}=0, S_t^{l,*}=S_t^{k,*}\}}}.
\end{aligned}
\tag{S26}
$$

  Note that we used $\mu_t^{*,N}$ instead of μ_t^* present in the limit system of the Main Result 5.

- At rate $\alpha(I(\xi_{t^-}^{i,*}))\mathbb{1}_{\{V_{t^-}^{i,*}=0\}} + \beta \mathbb{1}_{\{V_{t^-}^{i,*}=1\}}$,
  - $X_{t^-}^{i,*} = (1, S_{t^-}^{i,*}, \xi_{t^-}^{i,*}) \to X_t^{i,*} = (0, S_{t^-}^{i,*}, \xi_{t^-}^{i,*})$,
  - $X_{t^-}^{i,*} = (0, S_{t^-}^{i,*}, \xi_{t^-}^{i,*}) \to X_t^{i,*} = (1, 0, \xi_t^{i,*})$ with

$$
\xi_t^{i,*}(v,A,w) = \int_A p^+(s,w-1)\xi_{t^-}^{i,*}(v,ds,\{w-1\}) + \int_A (1-p^+(s,w))\xi_{t^-}^{i,*}(v,ds,w).
$$

We discretise the equations given in (S26) with an explicit Euler scheme to estimate the derivatives. This means that for a function $u : (t,s) \mapsto u(t,s)$,

$$
\partial_t u(t,s) \to \frac{u(t+dt,s) - u(t,s)}{dt} \quad \text{and} \quad -\partial_s u(t,s) \to \frac{u(t,s-dt) - u(t,s)}{dt}.
$$

This transformation from continuous to discrete space is done in the subsection S2 C 2.

We simulated with a fixed dt time step instead of following exactly the jumping times one by one. Then, as ξ variables are distributions (in s and w), we use 2D histograms to represent them. For the continuous variable s, we used a grid of size dt starting from 0 and ending in $m_s = M_s dt$ ($m_s = 15$ms in the simulations of Section S2) with

$M_s \in \mathbb{N}$. It is also necessary to choose bounds for synaptic weights, we denote them by $w_{min},\ w_{max} \in \mathbb{Z}$. Therefore, ξ are probability density functions evaluated on a 2D-grid,

$$\xi : (m \times dt, w) \in [\![0, M_s]\!]dt \times [\![w_{min}, w_{max}]\!] \mapsto \xi(m \times dt, w) \in \mathbb{R}^+.$$

Finally, we estimate the function $s \mapsto a_t^0(s)$ which is the average firing rate of neurons with time since last spike equals to s. We used the `Polynomials.fit` function from the `Polynomials` package in Julia to approximate a_t^0 with a polynomial of order 5.

B. Initial conditions

The initial conditions are the same for V and S in both models. We draw them in an independent and identically distributed (iid) way. For S we have to be more careful as the densities ρ_0 and ρ_1 have to satisfy the boundary conditions given by the two last equations in (S26):

$$\rho_0(0) = 0 \quad \text{and} \quad \rho_1(0) = dt \sum_{\tilde{m}=0}^{M_s} \xi_0^{i_0,*}(0, mdt, \mathbb{Z}) \frac{\frac{1}{N}\sum_k \alpha(I(\xi_t^{k,*}))\mathbb{1}_{\{V_t^{k,*}=0, S_t^{k,*}=s\}}}{\frac{1}{N}\sum_l \mathbb{1}_{\{V_t^{l,*}=0, S_t^{l,*}=s\}}}, \tag{S27}$$

where i_0 is a given neuron; we chose $i_0 = 1$. Regarding the synaptic weights, the two systems differ. For the microscopic system, we draw the synaptic weights $(W_0^{ij,N})_{i,j}$ from the distribution p_W. For the mean field system, we truncate ρ_0 and ρ_1 at m_s and normalise them.

Here are the steps we followed.

1. We draw $(V_0^{i,N})_i = (V_0^{i,*})_i$ as N iid random variables with binomial distribution of parameter p_V.
2. For i such that $V_0^{i,*} = 0$, we draw $(S_0^{i,N})_i = (S_0^{i,*})_i$ iid with distributions ρ_0.
3. We draw the synaptic weights $(W_0^{ij,N})_{i,j}$ iid with distribution p_W.
4. We define for all $(m, w) \in [\![0, M_s]\!] \times [\![w_{min}, w_{max}]\!]$,

$$\xi_0^{i,*}(0, mdt, w) = \frac{(1 - p_V)\rho_0(mdt)p_W(w)}{dt \sum_{\tilde{m}=0}^{M_s} \rho_0(\tilde{m}dt)},$$

   where ρ_0 is distributed as a LogNormal distribution with parameters $\mu = 0.8$ and $\sigma = 1$.

5. We first compute from (S27) and then we define for all $(m, w) \in [\![0, M_s]\!] \times [\![w_{min}, w_{max}]\!]$,

$$\xi_0^{i,*}(1, mdt, w) = \frac{p_V \rho_1(mdt)p_W(w)}{dt \sum_{\tilde{m}=0}^{M_s} \rho_1(\tilde{m}dt)},$$

   where ρ_1 is an exponential distribution with parameter $\rho_1(0)$.

We illustrate in Fig. S1 these initial densities that we used in our simulations.

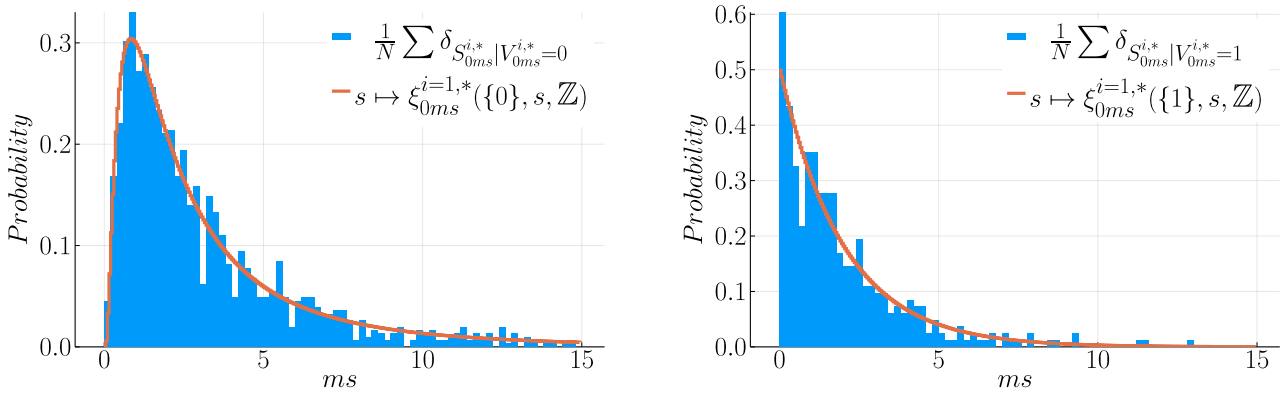
C. Dynamics: from t to t+dt

1. *The microscopic system*

We compute the synaptic currents $I_t^{i,N} = \frac{1}{N}\sum_j W_t^{ij,N} V_t^{j,N}$. We draw the jumping times $\tau_t^{i,N}$ of the neurons from exponential distributions with parameter

$$\alpha(I_t^{i,N})\delta_{\{V_t^{i,N}=0\}} + \beta\delta_{\{V_t^{i,N}=1\}}.$$

If $\tau_t^{i,N} < dt$ and



(a) (b)

FIG. S1: Distributions of the time from last spikes for neurons in state $V = 0$ in (S1a) and for neurons in state $V = 1$ in (S1b), at the initial time $t = 0\,ms$. Parameters' values of Section III are used.

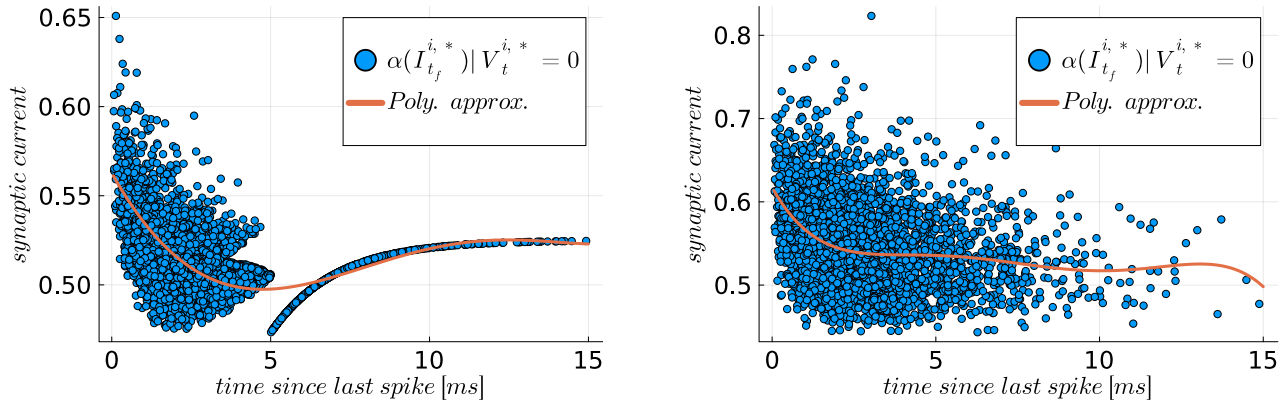
- $V_t^{i,N} = 1$, then we set $V_{t+dt}^{i,N} = 0$ and add dt to the corresponding S,
- $V_t^{i,N} = 0$, then we set $V_{t+dt}^{i,N} = 1$, $S_{t+dt}^{i,N} = dt - \tau_t^{i,N}$ and we potentiate (+1) the incoming weights $(W_t^{ij,N})_j$ with probability $p^+(S_t^{j,N})$ and depress (-1) the outgoing weights $(W_t^{ji,N})_j$ with probability $p^-(S_t^{j,N})$.

If $\tau_t^{i,N} > dt$, we just add dt to the corresponding S.

2. *The mean field system*

We describe how to pass from t to $t+dt$ in the mean field system, in particular we have to describe how to discretise equation (S26).

1. We compute the $I_t^{i,*}$ (initially all the same and deterministic).
2. We draw the jumping times $\tau_t^{i,*}$ from the exponential distributions of parameters
$$\alpha(I_t^{i,*})\mathbb{1}_{\{V_t^{i,*}=0\}} + \beta\mathbb{1}_{\{V_t^{i,*}=1\}}.$$
3. We estimate the function $a_t^0 = (a_t^0(0), \cdots, a_t^0(M_s dt))$. To do so, we can approximate this term looking back at the original term which is
$$a_t^0(mdt) = \frac{\mu_t^*(0, mdt, \mathcal{P}(E_m))}{\xi_t^*(0, mdt, \mathbb{Z})} \underbrace{\int_{\mathcal{P}(E_m)} \alpha\big(I(\xi')\big)\mu_t^*(0, mdt, d\xi')}_{\text{(c)}}. \tag{S28}$$
Regarding the quotient term in S28, it should be equal to one in theory (same marginal law in S of μ and ξ) so we keep it equal to one. Hence, we are left with the term (c) which is nothing but the expectation of the current knowing that the membrane potential is $V = 0$ and its time since last spike $s = mdt$, $\mathbb{E}\left[\alpha\big(I(\xi_t^*)\big)|V_t^* = 0, S_t^* = mdt\right]$. We illustrate the approximation of (c) in Fig. S2 where we can see the randomness of it – even though being an expectation – as soon as the initial condition is erased: transition at $5\,ms$ in Fig. S2a and in Fig. S2b nothing deterministic is left when $t_f = 30\,ms > m_s$.



(a) (b)

FIG. S2: Scatter plot of the synaptic currents from all neurons in state V=0 (blue dots) and the approximation of its expectation (term (c) in (S28)) at final time $t_f = 5ms$ in S2a and $t_f = 30ms$ in S2b. Parameters' values of Section III are used.

4. We compute the terms due to the drift of the ξ. For all $m \in [\![0, M_s]\!]$:

$$\xi^{i,*}_{t+dt}(0, mdt, w) = \xi^{i,*}_t(0, (m-1)dt, w)\left(1 - a^0_t\big((m-1)dt\big)dt\right) + \xi^{i,*}_t(1, (m-1)dt, w)\beta dt$$
$$\xi^{i,*}_{t+dt}(0, 0, w) = 0,$$

$$\xi^{i,*}_{t+dt}(1, mdt, w) = \xi^{i,*}_t(1, (m-1)dt, w)\,(1 - \beta dt)$$
$$\xi^{i,*}_{t+dt}(1, 0, w) = dt \sum_m a^0_t(mdt)\left[p^-(S^{i,*}_t)\xi^{i,*}_t(0, mdt, w+1) + \big(1 - p^-(S^{i,*}_t)\big)\xi^{i,*}_t(0, mdt, w)\right]$$

5. If $\tau^{i,*}_t < dt$ and $V^{i,*}_t = 0$,

$$\xi^{i,*}_{t+dt}(v, mdt, w) = p^+(mdt)\xi^{i,*}_t(v, mdt, w-1) + (1 - p^+(mdt))\xi^{i,*}_t(v, mdt, w).$$

6. We update

$$S^{i,*}_{t+dt} = (S^{i,*}_t + dt)\mathbb{1}_{\{\tau^{i,*}_t \geq dt\}} + (dt - \tau^{i,*}_t)\mathbb{1}_{\{\tau^{i,*}_t < dt\}},$$

7. For i such that $\tau^{i,*}_t < dt$, we make the membrane potentials $V^{i,*}$ jump accordingly.

8. We compute the elements that we keep in memory at each time step : expectations of the potentials, of the time from last the spike, of the pre-synaptic weights, distribution of the synaptic current input.

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