

VARIATION OF ALGEBRAICALLY INTEGRABLE ADJOINT FOLIATED STRUCTURES

PAOLO CASCINI, JIHAO LIU, FANJUN MENG, ROBERTO SVALDI, AND LINGYAO XIE

ABSTRACT. Given a canonical algebraically integrable foliation on a klt projective variety, we study the variation of the ample models of the associated adjoint foliated structures with respect to the parameter. When the foliation is of general type, we show the finiteness of ample models if the parameter is sufficiently close to 1. When the ambient variety is of general type, we show the finiteness of ample models for all parameters. A key ingredient in our proof is the equivalence between the existence of minimal models and the termination of MMP with scaling for algebraically integrable adjoint foliated structures.

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1. INTRODUCTION

We work over the field of complex numbers $\mathbb{C}$.

The existence of good minimal models for klt algebraically integrable adjoint foliated structures $(X, \mathcal{F}, t)$ of general type was established in [Cas+25]. Equivalently, for $t \in [0, 1)$ these are the good minimal models of the $\mathbb{R}$ -divisors $tK_{\mathcal{F}} + (1 - t)K_X$. Several people (e.g. D. Bejleri and K. DeVleming) asked us whether there is a wall-crossing phenomenon for such adjoint foliated structures as t varies in $[0, 1)$, in analogy with the wall crossing for moduli of stable pairs developed in [ABIP23, MZ23].

Since a moduli theory for adjoint foliated structures of general type has not yet been developed and several necessary tools are still unavailable, our first step toward this question is to prove the following finiteness of ample models as t varies and to describe their interrelations. This will eventually yield a wall-and-chamber decomposition in the

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anticipated wall-crossing phenomenon. A complete wall-crossing theory (e.g. construction of reduction morphisms and wall-crossing morphisms) for adjoint foliated structures of general type requires substantial further developments and lies beyond the scope of this work. In the setting of foliations, the question becomes significantly subtler as $t \rightarrow 1$, since $K_{\mathcal{F}}$ may fail to admit a good minimal model owing to the failure of abundance for algebraically integrable foliations. Indeed, the question is closely tied to the existence of minimal models for $K_{\mathcal{F}}$, a fundamental problem in the minimal model program for foliations.

For clarity of exposition, in the following, we assume that X has klt singularities, that $\mathcal{F}$ has canonical singularities, and that either X or $\mathcal{F}$ is of general type, or more generally, $\lambda K_{\mathcal{F}} + (1 - \lambda)K_X$ is big for some $\lambda \in [0, 1]$. We provide a more general statement in Theorem 5.4.

Theorem 1.1. *Let X be a klt projective variety and $\mathcal{F}$ a canonical algebraically integrable foliation on X . For $t \in [0, 1]$ set $\mathfrak{A}_t := (X, \mathcal{F}, t)$. Assume that $\mathfrak{A}_\lambda$ is of general type for some $\lambda \in [0, 1]$. Then there exist a rational number $\epsilon < \lambda$ and finitely many rational numbers*

$$\max\{\epsilon, 0\} := t_1 < t_2 < \cdots < t_n := 1$$

with the following properties. Let $\Gamma := \{\{t_i\}, (t_i, t_{i+1}) \mid 1 \leq i \leq n-1\}$. For any $\mathcal{P} \in \Gamma$ there exists a rational map $\psi_{\mathcal{P}}: X \dashrightarrow Z_{\mathcal{P}}$ such that:

- (1) *For any $\mathcal{P} \in \Gamma$ and $t \in \mathcal{P}$, the map $\psi_{\mathcal{P}}$ is the ample model of $\mathfrak{A}_t$.*
- (2) *For any $\mathcal{P}, \mathcal{Q} \in \Gamma$ with $\mathcal{Q} \subset \partial\mathcal{P}$, there exists a unique contraction $\mu_{\mathcal{P}, \mathcal{Q}}: Z_{\mathcal{P}} \rightarrow Z_{\mathcal{Q}}$ such that $\mu_{\mathcal{P}, \mathcal{Q}} \circ \psi_{\mathcal{P}} = \psi_{\mathcal{Q}}$.*

Compared with the case of pairs, a key difference is that $K_{\mathcal{F}}$ may not admit a good minimal model (cf. [McQ08, Theorem 3 IV.5.11], [ACSS21, Subsection 5.4]). Consequently, we work over the half-open interval $[\epsilon, 1)$, whereas classical results for pairs typically hold over a closed interval.

A central ingredient in the proof of Theorem 1.1 is an equivalence between the existence of minimal models and the termination of the MMP with scaling for algebraically integrable adjoint foliated structures.

Theorem 1.2. *Let $(X, \mathcal{F}, t)$ be a projective lc algebraically integrable adjoint foliated structure with X klt. If $(X, \mathcal{F}, t)$ admits a minimal model, then one can run a $(tK_{\mathcal{F}} + (1 - t)K_X)$ -MMP with scaling of an ample divisor on X , and any such MMP terminates with a minimal model of $(X, \mathcal{F}, t)$.*

We also expect Theorem 1.2 to be useful for further developments in the minimal model program for algebraically integrable adjoint foliated structures. More general versions of Theorem 1.2 are given in Theorems 4.1 and 4.6.

It is interesting to ask that, in the setting of Theorem 1.1, whether we can choose ϵ and $\{t_i\}$ to be independent of X or $\mathcal{F}$. More precisely, we ask the following questions:

Question 1.3. Notation and conditions as in 1.1.

- (1) Assume that $\dim X = d$ is fixed.
 - (a) Can we choose $\epsilon = \epsilon(d, \lambda)$ so that it only depends on d and λ ?
 - (b) Can we find an explicit or the optimal value of $\epsilon(d, 1)$?
- (2) Assume that X is fixed and $\lambda = 0$, i.e. X is of general type. We have $t_1 = 0$.
 - (a) Can we choose t_2 so that it only depends on X ?

- (b) Can we choose n so that it only depends on X ? What about the set $\{t_i\}_{i=1}^n$?

Structure of the paper. In Section 2 we recall some preliminary results and prove some perturbation results that will be used later. In Section 3 we define different types of models, particularly models in the sense of Birkar-Shokurov, of adjoint foliated structures, and study their basic behaviors. In Section 4 we show the equivalence between existence of minimal models and termination of MMP with scaling, and prove Theorem 1.2. In Section 5 we prove a finiteness of ample models result which implies Theorem 1.1.

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2. PRELIMINARIES

We will adopt the standard notation and definitions on MMP in [KM98, BCHM10] and use them freely. For adjoint foliated structures, generalized foliated quadruples, foliated triples, foliations, and generalized pairs, we adopt the notation and definitions in [Cas+24, Cas+25] (for adjoint foliated structures) which generally align with [LLM23, CHLX23] (for generalized foliated quadruples), [CS20, ACSS21, CS21] (for foliations and foliated triples), and [BZ16, HL23] (for generalized pairs and $\mathbf{b}$ -divisors), possibly with minor differences.

2.1. Notation.

Definition 2.1. A *contraction* is a projective morphism of varieties $f: X \rightarrow Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Notation 2.2. Let $h: X \dashrightarrow X'$ be a birational map between normal varieties. We denote by $\text{Exc}(h)$ the reduced divisor supported on the codimension one part of the exceptional locus of h .

Definition 2.3. Let $X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a variety. Let D be an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X and $\phi: X \dashrightarrow X'$ a birational map/ U . We say that ϕ is D -negative (resp. D -trivial) if the following conditions hold:

- (1) ϕ does not extract any divisor.
- (2) $D' := \phi_*D$ is $\mathbb{R}$ -Cartier.
- (3) There exists a resolution of indeterminacy $p: W \rightarrow X$ and $q: W \rightarrow X'$, such that

$$p^*D = q^*D' + F$$

where $F \geq 0$ and $\text{Supp } p_*F = \text{Exc}(\phi)$ (resp. $F = 0$).

Definition 2.4 (NQC). Let $X \rightarrow U$ be a projective morphism between normal quasi-projective varieties. Let D be a nef $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X and $\mathbf{M}$ a nef $\mathbf{b}$ -divisor on X . We say that D is *NQC/U* if $D = \sum d_i D_i$, where each $d_i \geq 0$ and each D_i is a nef/ U Cartier divisor. We say that $\mathbf{M}$ is *NQC/U* if $\mathbf{M} = \sum \mu_i \mathbf{M}_i$, where each $\mu_i \geq 0$ and each $\mathbf{M}_i$ is a nef/ U $\mathbf{b}$ -Cartier $\mathbf{b}$ -divisor.

Definition 2.5. Let $\pi: X \rightarrow U$ be a projective morphism from a normal quasi-projective variety to a quasi-projective variety, D a pseudo-effective/ U $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on X , and P a prime divisor on X . We define $\sigma_P(X/U, D)$ as in [LX25, Definition 3.1] by considering $\sigma_P(X/U, D)$ as a number in $[0, +\infty) \cup \{+\infty\}$. We define $N_\sigma(X/U, D) = \sum_Q \sigma_Q(X/U, D)Q$ where the sum runs through all prime divisors on X and consider it as a formal sum of divisors with coefficients in $[0, +\infty) \cup \{+\infty\}$. We say that D is *movable/U* if $N_\sigma(X/U, D) = 0$.

2.2. Adjoint foliated structures.

Definition 2.6 (Foliations, cf. [ACSS21, CS21]). Let X be a normal variety. A *foliation* on X is a coherent subsheaf $T_{\mathcal{F}} \subset T_X$ such that

- (1) $T_{\mathcal{F}}$ is saturated in T_X , i.e. $T_X/T_{\mathcal{F}}$ is torsion free, and
- (2) $T_{\mathcal{F}}$ is closed under the Lie bracket.

The *canonical divisor* of $\mathcal{F}$ is a divisor $K_{\mathcal{F}}$ such that $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(T_{\mathcal{F}})$. If $T_{\mathcal{F}} = 0$, then we say that $\mathcal{F}$ is a *foliation by points*.

Given any dominant map $h: Y \dashrightarrow X$ and a foliation $\mathcal{F}$ on X , we denote by $h^{-1}\mathcal{F}$ the *pullback* of $\mathcal{F}$ on Y as constructed in [Dru21, 3.2]. Given any birational map $g: X \dashrightarrow X'$, we denote by $g_*\mathcal{F} := (g^{-1})^{-1}\mathcal{F}$ the *pushforward* of $\mathcal{F}$ on X' . We say that $\mathcal{F}$ is an *algebraically integrable foliation* if there exists a dominant map $f: X \dashrightarrow Z$ such that $\mathcal{F} = f^{-1}\mathcal{F}_Z$, where $\mathcal{F}_Z$ is the foliation by points on Z , and we say that $\mathcal{F}$ is *induced by f*.

A subvariety $S \subset X$ is called *$\mathcal{F}$ -invariant* if for any open subset $U \subset X$ and any section $\partial \in H^0(U, \mathcal{F})$, we have $\partial(\mathcal{I}_{S \cap U}) \subset \mathcal{I}_{S \cap U}$, where $\mathcal{I}_{S \cap U}$ denotes the ideal sheaf of $S \cap U$ in U . For any prime divisor P on X , we define $\epsilon_{\mathcal{F}}(P) := 1$ if P is not $\mathcal{F}$ -invariant and $\epsilon_{\mathcal{F}}(P) := 0$ if P is $\mathcal{F}$ -invariant. For any prime divisor E over X , we define $\epsilon_{\mathcal{F}}(E) := \epsilon_{\mathcal{F}_Y}(E)$ where $h: Y \dashrightarrow X$ is a birational map such that E is on Y and $\mathcal{F}_Y := h^{-1}\mathcal{F}$.

Suppose that the foliation structure $\mathcal{F}$ on X is clear in the context. Then, given an $\mathbb{R}$ -divisor $D = \sum a_i D_i$ where each D_i is a prime divisor, we denote by $D^{\text{nin}} := \sum \epsilon_{\mathcal{F}}(D_i) a_i D_i$ and $D^{\text{inv}} := D - D^{\text{nin}}$.

Definition 2.7. An *adjoint foliated structure* $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ is the datum of a normal quasi-projective variety X and a projective morphism $X \rightarrow U$, a foliation $\mathcal{F}$ on X , an $\mathbb{R}$ -divisor $B \geq 0$ on X , a $\mathbf{b}$ -divisor $\mathbf{M}$ nef/ U , and a real number $t \in [0, 1]$ such that $K_{\mathfrak{A}} := tK_{\mathcal{F}} + (1-t)K_X + B + \mathbf{M}_X$ is $\mathbb{R}$ -Cartier. We may simply say that “ $\mathfrak{A}/U$ is an adjoint foliated structure” without mentioning $X, \mathcal{F}, B, \mathbf{M}, t$ at all. X, t are called the *ambient variety* and *parameter* of $\mathfrak{A}$ respectively.

We say that $\mathfrak{A}/U$ is *of general type* if $K_{\mathfrak{A}}$ is big/ U . We say that $\mathfrak{A}$ is *algebraically integrable* if $\mathcal{F}$ is algebraically integrable. We say that $\mathfrak{A}/U$ is *KNQC* if $K_{\mathfrak{A}}$ is NQC/ U .

For any $\mathbb{R}$ -divisor D on X and nef/ U $\mathbf{b}$ -divisor $\mathbf{N}$ on X such that $D + \mathbf{N}_X$ is $\mathbb{R}$ -Cartier, we denote by $(\mathfrak{A}, D, \mathbf{N}) := (X, \mathcal{F}, B + D, \mathbf{M} + \mathbf{N}, t)$. If $D = 0$ then we may drop D , and if $\mathbf{N} = \mathbf{0}$ then we may drop $\mathbf{N}$.

When $t = 0$ or $\mathcal{F} = T_X$, we call $(X, B, \mathbf{M})/U$ a *generalized pair*, and in addition, if $\mathbf{M} = \mathbf{0}$, then we call $(X, B)/U$ a *pair*. If $B = 0$, or if $\mathbf{M} = \mathbf{0}$, or if U is not important,

then we may drop $B, \mathbf{M}, U$ respectively. If $U = \{pt\}$ then we also drop U and say that $(X, \mathcal{F}, B, \mathbf{M}, t)$ is *projective*.

For any birational map $\phi : X \dashrightarrow X'$, we define $\phi_*\mathfrak{A} := (X', \phi_*\mathcal{F}, \phi_*B, \mathbf{M}, t)$ and say that $\phi_*\mathfrak{A}$ is the *image* of $\mathfrak{A}$ on X' . For any projective birational morphism $h : X' \rightarrow X$, we define

$$h^*\mathfrak{A} := (X', \mathcal{F}', B', \mathbf{M}, t)$$

where $\mathcal{F}' := h^{-1}\mathcal{F}$ and B' is the unique $\mathbb{R}$ -divisor such that $K_{h^*\mathfrak{A}} = h^*K_{\mathfrak{A}}$. For any prime divisor E on X' , we denote by

$$a(E, \mathfrak{A}) := -\text{mult}_E B'$$

the *discrepancy* of E with respect to $\mathfrak{A}$. The *total minimal log discrepancy* of $\mathfrak{A}$ is

$$\text{tml}(\mathfrak{A}) := \inf\{a(E, \mathfrak{A}) + t\epsilon_{\mathcal{F}}(E) + (1-t) \mid E \text{ is over } X\}.$$

For any non-negative real number ϵ , we say that $\mathfrak{A}$ is ϵ -lc (resp. ϵ -klt) if $\text{tml}(\mathfrak{A}) \geq \epsilon$ (resp. $> \epsilon$). We say that $\mathfrak{A}$ is lc (resp. klt) if $\mathfrak{A}$ is 0-lc (resp. 0-klt). An *lc place* of $\mathfrak{A}$ is a prime divisor E over X such that $a(E, \mathfrak{A}) = -t\epsilon_{\mathcal{F}}(E) - (1-t)$. An *lc center* of $\mathfrak{A}$ is the image of an lc place of $\mathfrak{A}$ on X .

For any adjoint foliated structures $\mathfrak{A}_i/U = (X, \mathcal{F}, B_i, \mathbf{M}_i, t_i)/U$ and real numbers $a_i \in [0, 1]$ such that $\sum a_i = 1$, we denote by

$$\sum a_i \mathfrak{A}_i := \left(X, \mathcal{F}, \sum a_i B_i, \sum a_i \mathbf{M}_i, \sum a_i t_i \right).$$

Definition 2.8 (Qdlt, [Cas+24, Definition 3.1]). We say that an adjoint foliated structure $\mathfrak{A}/U := (X, \mathcal{F}, B^{\text{inv}} + (1-t)B^{\text{inv}}, \mathbf{M}, t)/U$ is *qdl* if $t < 1$, $\mathfrak{A}$ is lc, $(X, B, \mathbf{M})$ is qdlt (cf. [CHLX23, Definition 7.1.1]), and any lc place of $\mathfrak{A}$ is an lc place of $(X, B, \mathbf{M})$.

Definition-Lemma 2.9. Let $\mathfrak{A}/U$ be an lc algebraically integrable adjoint foliated structure with ambient variety X and parameter $t < 1$. A $\mathbb{Q}$ -factorial *qdl modification* of $\mathfrak{A}$ is a birational morphism $h : X' \rightarrow X$ such that $h^*\mathfrak{A}$ is $\mathbb{Q}$ -factorial qdlt and h only extracts lc places of $\mathfrak{A}$, which always exists by [Cas+24, Theorem 1.9].

Definition 2.10 (Potentially klt). Let X be a normal quasi-projective variety. We say that X is *potentially klt* if (X, Δ) is klt for some $\mathbb{R}$ -divisor $\Delta \geq 0$.

Definition 2.11 ([Cas+25, Definition 3.11]). Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ be an algebraically integrable adjoint foliated structure. We say that $\mathfrak{A}$ is *foliated log smooth* if there exists a contraction $f : X \rightarrow Z$ satisfying the following.

- (1) X has at most quotient singularities.
- (2) $\mathcal{F}$ is induced by f .
- (3) (X, Σ_X) is toroidal for some reduced divisor Σ_X such that $\text{Supp } B \subset \Sigma_X$. In particular, $(X, \text{Supp } B)$ is toroidal, and X is $\mathbb{Q}$ -factorial klt.
- (4) There exists a log smooth pair (Z, Σ_Z) such that

$$f : (X, \Sigma_X) \rightarrow (Z, \Sigma_Z)$$

is an equidimensional toroidal contraction.

- (5) $\mathbf{M}$ descends to X .

We say that $f : (X, \Sigma_X) \rightarrow (Z, \Sigma_Z)$ is *associated with* $(X, \mathcal{F}, B, \mathbf{M}, t)$.

Definition 2.12 (Foliated log resolutions). Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ be an algebraically integrable adjoint foliated structure. A *foliated log resolution* of $\mathfrak{A}$ is a birational morphism $h : X' \rightarrow X$ such that

$$(X', \mathcal{F}' := h^{-1}\mathcal{F}, B' := h_*^{-1}B + \text{Exc}(h), \mathbf{M}, t)$$

is foliated log smooth. By [CHLX23, Lemma 6.2.4], a foliated log resolution for $\mathfrak{A}$ always exists.

Definition 2.13 (Property (*) foliations, [ACSS21, Definition 3.8], [CHLX23, Definition 7.2.2]). Let $(X, \mathcal{F}, B, \mathbf{M})/U$ be a generalized foliated quadruple, $G \geq 0$ be a reduced divisor on X , and $f : X \rightarrow Z$ a contraction. We say that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies *Property (*)* if the following conditions hold.

- (1) $\mathcal{F}$ is induced by f and G is an $\mathcal{F}$ -invariant divisor.
- (2) $f(G)$ is of pure codimension 1, $(Z, f(G))$ is log smooth, and $G = f^{-1}(f(G))$.
- (3) For any closed point $z \in Z$ and any reduced divisor $\Sigma \geq f(G)$ on Z such that (Z, Σ) is log smooth near z , $(X, B + G + f^*(\Sigma - f(G)), \mathbf{M})$ is lc over a neighborhood of z .

We say that f , Z , and G are *associated* with $(X, \mathcal{F}, B, \mathbf{M})$.

Definition 2.14 (ACSS, cf. [CHLX23, Definitions 5.4.2, 7.2.2, 7.2.3]). Let $(X, \mathcal{F}, B, \mathbf{M})$ be an lc generalized foliated quadruple, $G \geq 0$ a reduced divisor on X , and $f : X \rightarrow Z$ a contraction. We say that $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is *ACSS* if the following conditions hold:

- (1) $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ satisfies Property (*).
- (2) f is equidimensional.
- (3) There exists an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor $D \geq 0$ on X and a nef/ X $\mathbf{b}$ -divisor $\mathbf{N}$ on X , such that $\text{Supp}\{B\} \subset \text{Supp } D$, $\mathbf{N} - \alpha\mathbf{M}$ is nef/ X for some $\alpha > 1$, and for any reduced divisor $\Sigma \geq f(G)$ such that (Z, Σ) is log smooth,

$$(X, B + D + G + f^*(\Sigma - f(G)), \mathbf{N})$$

is qdlt (cf. [CHLX23, Definition 7.1.1], [dFKX17, Definition 35]).

- (4) For any lc center of $(X, \mathcal{F}, B, \mathbf{M})$ with generic point η , over a neighborhood of η ,
 - (a) η is the generic point of an lc center of $(X, \mathcal{F}, \lfloor B \rfloor)$, and
 - (b) $f : (X, B + G) \rightarrow (Z, f(G))$ is a toroidal morphism.

If $(X, \mathcal{F}, B, \mathbf{M}; G)/Z$ is ACSS, then we say that $(X, \mathcal{F}, B, \mathbf{M})/Z$ and $(X, \mathcal{F}, B, \mathbf{M})$ are ACSS.

Definition 2.15 (ACSS modifications). Let $\mathfrak{A} = (X, \mathcal{F}, B, \mathbf{M})$ and $\mathfrak{A}' = (X', \mathcal{F}', B', \mathbf{M})$ be two lc algebraically integrable generalized foliated quadruples and let $h : X' \rightarrow X$ be a birational morphism. We say that $h : X' \rightarrow X$ is a $\mathbb{Q}$ -factorial ACSS modification of $\mathfrak{A}$ if there exists a contraction $f : X' \rightarrow Z$ and a reduced divisor G on X' , such that

- (1) $\mathfrak{A}' = h^*\mathfrak{A}$,
- (2) $(X', \mathcal{F}', B', \mathbf{M}; G)/Z$ is $\mathbb{Q}$ -factorial ACSS, and
- (3) $a(E, \mathfrak{A}) = -\epsilon_{\mathcal{F}}(E)$ for any h -exceptional prime divisor E .

$\mathbb{Q}$ -factorial ACSS modifications always exist by [CHLX23, Theorem 8.2.2].

3. MODELS OF ADJOINT FOLIATED STRUCTURES

This section aligns with [Bir12, Section 2], [HL23, Section 3], and [LMX24, Section 4] and the proofs are similar.

Definition 3.1 (Log birational models). Let $\mathfrak{A}/U$ be an adjoint foliated structure with ambient variety X , $\phi : X \dashrightarrow X'$ a birational map/ U , and $E := \text{Exc}(\phi^{-1})$ the reduced ϕ^{-1} -exceptional divisor. Assume that $a(D, \mathfrak{A}) \leq -t\epsilon_{\mathcal{F}}(D) - (1-t)$ for any component D of E . We let

$$\mathfrak{A}' := \left(\phi_* \mathfrak{A}, - \sum_D a(D, \mathfrak{A}) D \right)$$

where the sum runs through all irreducible components of E . If $K_{\mathfrak{A}'}$ is $\mathbb{R}$ -Cartier then we say that $\mathfrak{A}'/U$ is a *log birational model* of $\mathfrak{A}/U$.

Definition 3.2 (Minimal models). Let $\mathfrak{A}/U$ be an adjoint foliated structure with ambient variety X and $\mathfrak{A}'/U$ a log birational model of $\mathfrak{A}/U$ with ambient variety X' and associated birational map $\phi : X \dashrightarrow X'$, such that $K_{\mathfrak{A}'}$ is nef/ U . Let t be the parameter of $\mathfrak{A}$.

- (1) We say that $\mathfrak{A}'/U$ is a *bs-weak lc model* or *weak lc model in the sense of Birkar-Shokurov* of $\mathfrak{A}/U$, if for any prime divisor D on X which is exceptional over X' ,

$$a(D, \mathfrak{A}) \leq a(D, \mathfrak{A}').$$

We also say that ϕ is a *bs-weak lc model* of $\mathfrak{A}/U$.

- (2) We say that $\mathfrak{A}'/U$ is a *bs-minimal model* or *minimal model in the sense of Birkar-Shokurov* of $\mathfrak{A}/U$, if for any prime divisor D on X which is exceptional over X' ,

$$a(D, \mathfrak{A}) < a(D, \mathfrak{A}').$$

We also say that ϕ is a *bs-minimal model* of $\mathfrak{A}/U$.

- (3) We say that $\mathfrak{A}'/U$ is a *bs-semi-ample model* or *semi-ample model in the sense of Birkar-Shokurov* of $\mathfrak{A}/U$ if it is a bs-weak lc model of $\mathfrak{A}/U$ and $K_{\mathfrak{A}'}$ is semi-ample/ U . We also say that ϕ is a *bs-semi-ample model* of $\mathfrak{A}/U$.
- (4) We say that $\mathfrak{A}'/U$ is a *bs-good minimal model* or *good minimal model in the sense of Birkar-Shokurov* of $\mathfrak{A}/U$ if it is a bs-minimal model of $\mathfrak{A}/U$ and $K_{\mathfrak{A}'}$ is semi-ample/ U . We also say that ϕ is a *bs-good minimal model* of $\mathfrak{A}/U$.

If, in addition, the induced birational map $X \dashrightarrow X'$ does not extract any divisor, then we remove the initial “bs-” or the phrase “in the sense of Birkar-Shokurov” in the previous definitions.

- (5) We say that $\mathfrak{A}'/U$ is a *log minimal model* of $\mathfrak{A}/U$ if it is a bs-minimal model of $\mathfrak{A}/U$, $\mathfrak{A}'$ is $\mathbb{Q}$ -factorial qdlt if $t < 1$, and $\mathfrak{A}'$ is $\mathbb{Q}$ -factorial ACSS if $t = 1$. We also say that ϕ is a *log minimal model* of $\mathfrak{A}/U$.
- (6) We say that $\mathfrak{A}'/U$ is a *good log minimal model* of $\mathfrak{A}/U$ if it is a log minimal model of $\mathfrak{A}$ and $K_{\mathfrak{A}'}$ is semi-ample/ U . We also say that ϕ is a *good log minimal model* of $\mathfrak{A}/U$.

Lemma 3.3. *Let $\mathfrak{A}/U$ be an adjoint foliated structure and let $\mathfrak{A}'/U$ be a bs-weak lc model of $\mathfrak{A}/U$. Let X, X' be the ambient varieties of $\mathfrak{A}$ and $\mathfrak{A}'$ respectively and $\phi : X \dashrightarrow X'$ the associated birational map. Let $p : W \rightarrow X$ and $q : W \rightarrow X'$ be birational morphisms such that $q = \phi \circ p$. Assume that*

$$p^* K_{\mathfrak{A}} = q^* K_{\mathfrak{A}'} + E,$$

then $E \geq 0$ and is exceptional/ X' .

Proof. For any prime divisor D that is an irreducible component of E ,

$$\text{mult}_D E = a(D, \mathfrak{A}') - a(D, \mathfrak{A}).$$

Therefore, if D is not exceptional/ X , then:

- If D is not exceptional/ X' , then $\text{mult}_D E = 0$ by Definition 3.1.
- If D is exceptional/ X' , then $\text{mult}_D E \geq 0$ by Definition 3.2(1).

Therefore, $p_*E \geq 0$. Since $K_{\mathfrak{A}'}$ is nef/ U , $q^*K_{\mathfrak{A}'}$ is nef/ X , hence E is anti-nef/ X . By the negativity lemma, $E \geq 0$.

If E is not exceptional/ X' , then there exists a component D of E that is not exceptional/ X' . If D is not exceptional/ X , then $\text{mult}_D E = 0$ by Definition 3.1, a contradiction. Thus D is exceptional over X . In particular, ϕ extracts D . Since $\mathfrak{A}'/U$ is a log birational model of $\mathfrak{A}$,

$$a(D, \mathfrak{A}') = a(D, \mathfrak{A}),$$

which implies that $\text{mult}_D E = 0$, a contradiction. $\square$

Lemma 3.4. *Let $\mathfrak{A}/U$ be an adjoint foliated structure. Let $\mathfrak{A}_1/U$ and $\mathfrak{A}_2/U$ be two bs-weak lc models of $\mathfrak{A}/U$. Let X, X_1, X_2 be the ambient varieties of $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ respectively with induced birational maps $\phi : X_1 \dashrightarrow X_2$. Let $h_1 : W \rightarrow X_1$ and $h_2 : W \rightarrow X_2$ be two birational morphisms such that $\phi \circ h_1 = h_2$. Then:*

- (1) $h_1^*K_{\mathfrak{A}_1} = h_2^*K_{\mathfrak{A}_2}$.
- (2) If $K_{\mathfrak{A}_2}$ is semi-ample/ U , then $K_{\mathfrak{A}_1}$ is semi-ample/ U .
- (3) If $K_{\mathfrak{A}_2}$ is ample/ U , then ϕ is a morphism.

Proof. Let $\phi_1 : X \dashrightarrow X_1$ and $\phi_2 : X \dashrightarrow X_2$ be the induced birational maps. Possibly replacing W with a higher model, we may assume that the induced birational map $h : W \rightarrow X$ is a morphism. Let

$$E_i := h^*K_{\mathfrak{A}} - h_i^*K_{\mathfrak{A}_i}$$

for $i \in \{1, 2\}$. By Lemma 3.3, $E_i \geq 0$ and is exceptional over X_i for $i \in \{1, 2\}$. Thus $h_{1,*}(E_2 - E_1) \geq 0$ and $E_1 - E_2$ is nef/ X_1 , and $h_{2,*}(E_1 - E_2) \geq 0$ and $E_2 - E_1$ is nef/ X_2 . By the negativity lemma, $E_2 - E_1 \geq 0$ and $E_1 - E_2 \geq 0$. Thus $E_1 = E_2$, which implies (1). (2) immediately follows from (1). By (1), if $K_{\mathfrak{A}_2}$ is ample/ U , then $h_2 : W \rightarrow X_2$ is the ample model/ U of $h^*K_{\mathfrak{A}_1}$, hence ϕ is the ample model/ U of $K_{\mathfrak{A}_1}$. Since $K_{\mathfrak{A}_1}$ is semi-ample/ U , ϕ is a morphism. This implies (3). $\square$

Lemma 3.5. *Let r be a positive real number. Let $\mathfrak{A}_1/U$ and $\mathfrak{A}_2/U$ be two adjoint foliated structures with the same ambient variety X and such that*

$$K_{\mathfrak{A}_2} \equiv_U rK_{\mathfrak{A}_1}.$$

Let $\mathfrak{A}'_1/U$ be a weak lc model (resp. minimal model) of $\mathfrak{A}_1/U$ with ambient variety X' and induced birational map $\phi : X \dashrightarrow X'$. Let $\mathfrak{A}'_2 := \phi_\mathfrak{A}'_1$. Then:*

- (1) $\mathfrak{A}'_2/U$ is a weak lc model (resp. minimal model) of $\mathfrak{A}_2/U$.
- (2) If $\mathfrak{A}'_1/U$ is a semi-ample model (resp. good minimal model) of $\mathfrak{A}_1/U$ and

$$K_{\mathfrak{A}_2} \sim_{\mathbb{R}, U} rK_{\mathfrak{A}_1},$$

then $\mathfrak{A}'_2/U$ is a semi-ample model (resp. good minimal model) of $\mathfrak{A}_2/U$.

Proof. Let $p : W \rightarrow X$ and $q : W \rightarrow X'$ be a resolution of indeterminacy. By Lemma 3.3,

$$p^*K_{\mathfrak{A}_1} = q^*K_{\mathfrak{A}'_1} + E$$

for some $\mathbb{R}$ -divisor $E \geq 0$ that is exceptional/ X' . Then

$$p^*K_{\mathfrak{A}_2} \equiv_U r q^*K_{\mathfrak{A}'_1} + rE,$$

so that

$$K_{\mathfrak{A}'_2} = q_*p^*K_{\mathfrak{A}_2} \equiv q_*(r q^*K_{\mathfrak{A}'_1} + rE) = rK_{\mathfrak{A}'_1}$$

is nef/ U . Moreover, if $K_{\mathfrak{A}'_1}$ is semi-ample/ U and $K_{\mathfrak{A}_2} \sim_{\mathbb{R},U} rK_{\mathfrak{A}_1}$, then

$$K_{\mathfrak{A}'_2} \sim_{\mathbb{R},U} rK_{\mathfrak{A}'_1}$$

is semi-ample/ U .

We have

$$p^*K_{\mathfrak{A}_2} \equiv_U q^*K_{\mathfrak{A}'_2} + rE.$$

Therefore, for any prime divisor D on X which is exceptional over X' ,

$$a(D, \mathfrak{A}'_2) - a(D, \mathfrak{A}_2) = -\text{mult}_D p_*(rE) = r(a(D, \mathfrak{A}'_1) - a(D, \mathfrak{A}_1)).$$

Thus, $a(D, \mathfrak{A}_2) \leq$ (resp. $<$) $a(D, \mathfrak{A}'_2)$ if and only if $a(D, \mathfrak{A}_1) \leq$ (resp. $<$) $a(D, \mathfrak{A}'_1)$ and the lemma follows immediately from the definitions. $\square$

Theorem 3.6 (Very exceptional MMP). *Let $\mathfrak{A}/U$ be an lc algebraically integrable adjoint foliated structure with ambient variety X and parameter t . Assume that X is potentially klt, and $K_{\mathfrak{A}} \sim_{\mathbb{R},U} E \geq 0$ for some very exceptional/ U (cf. [Bir12, Definition 3.1]) $\mathbb{R}$ -divisor E . Then we may run a $K_{\mathfrak{A}}$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor and any such MMP terminates with a good minimal model $\mathfrak{A}'/U$ such that $K_{\mathfrak{A}'} \sim_{\mathbb{R},U} 0$.*

Proof. Let $\phi_i : \mathfrak{A}_i \dashrightarrow \mathfrak{A}_{i+1}$ be a step of a $K_{\mathfrak{A}}$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor A . For each i , let X_i the ambient variety of $\mathfrak{A}_i$, let A_i and E_i be the images of A and E on X_i respectively, and let

$$\lambda_i := \inf\{s \geq 0 \mid K_{\mathfrak{A}_i} + sA_i \text{ is nef}\}$$

be the scaling numbers. By [Cas+25, Theorem 8.1], either this MMP terminates with $\mathfrak{A}_n/U$ for some $n \geq 0$, or $\lim_{i \rightarrow +\infty} \lambda_i = 0$. In the latter case, we let n be a positive integer such that ϕ_i is a flip for any $i \geq n$, hence

$$K_{\mathfrak{A}_n} = \lim_{i \rightarrow +\infty} (\psi_{i,n})_*(K_{\mathfrak{A}_i} + \lambda_i A_i)$$

is movable/ U , where $\psi_{i,n} : X_i \dashrightarrow X_n$ is the induced birational map. In either case, $K_{\mathfrak{A}_n} \sim_{\mathbb{R}} E_n \geq 0$ is movable/ U and very exceptional/ U , hence $E_n = 0$ by [Bir12, Lemma 3.4]. Thus, we have $\mathfrak{A}' = \mathfrak{A}_n$. $\square$

Definition 3.7 (Foliated log smooth model). Let $\mathfrak{A}/U := (X, \mathcal{F}, B, \mathbf{M}, t)/U$ be an lc algebraically integrable adjoint foliated structure and $h : X' \rightarrow X$ a foliated log resolution of $\mathfrak{A}$. Let $E \geq 0$ be an h -exceptional $\mathbb{R}$ -divisor and let $\mathfrak{A}' := (h^*\mathfrak{A}, E)$. We say that $\mathfrak{A}'$ is a *foliated log smooth model* of $\mathfrak{A}$ if

- (1) $\mathfrak{A}'$ is foliated log smooth and lc, and
- (2) for any h -exceptional prime divisor D that is not an lc place of $\mathfrak{A}$, D is a component of E .

Lemma 3.8. *Let $\mathfrak{A}/U$ be an lc algebraically integrable adjoint foliated structure with ambient variety X . Let $\mathfrak{A}_W$ be a foliated log smooth model of $\mathfrak{A}$ with ambient variety W .*

Then any bs-weak lc model (resp. bs-minimal model, bs-semi-ample model, bs-good minimal model, log minimal model, good log minimal model) of $\mathfrak{A}_W/U$ is a bs-weak lc model (resp. bs-minimal model, bs-semi-ample model, bs-good minimal model, log minimal model, good log minimal model) of $\mathfrak{A}/U$.

Proof. We let $h : W \rightarrow X$ be the induced birational morphism. We may write

$$K_{\mathfrak{A}_W} = h^*K_{\mathfrak{A}} + E$$

for some $E \geq 0$ that is h -exceptional, and $D \subset \text{Supp } E$ for any h -exceptional prime divisor D that is not an lc place of $\mathfrak{A}$.

Claim 3.9. *Let $\mathfrak{A}'/U$ be a bs-weak lc model of $\mathfrak{A}_W/U$. Then*

$$a(D, \mathfrak{A}) \leq a(D, \mathfrak{A}')$$

for any prime divisor D over X .

Proof. Let X' be the ambient variety of $\mathfrak{A}'$, $\phi_W : W \dashrightarrow X'$ be the induced birational map, and let $p : V \rightarrow W$ and $q : V \rightarrow X'$ be a common resolution such that $q = \phi_W \circ p$. By Lemma 3.3,

$$p^* K_{\mathfrak{A}_W} = q^* K_{\mathfrak{A}'} + F$$

for some $F \geq 0$ that is exceptional over X' . Then we have

$$p^* h^* K_{\mathfrak{A}} = q^* K_{\mathfrak{A}'} + F - p^* E,$$

so

$$p^* E - F \sim_{\mathbb{R}, X} q^* K_{\mathfrak{A}'}$$

is nef/ X . Since $h_* p_*(F - p^* E) = h_* p_* F \geq 0$, by the negativity lemma, $F \geq p^* E$. Thus $a(D, \mathfrak{A}) \leq a(D, \mathfrak{A}')$ for any prime divisor D over X . $\square$

Proof of Lemma 3.8 continued. Let t be the parameter of $\mathfrak{A}$. First we prove the bs-weak lc model case. Let $\mathfrak{A}'/U$ be a bs-weak lc model of $\mathfrak{A}_W/U$ with ambient variety X' and induced birational map $\phi_W : W \dashrightarrow X'$. By Claim 3.9, we only need to show that $\mathfrak{A}'/U$ is a log birational model of $\mathfrak{A}/U$. Let $\phi : X \dashrightarrow X'$ be the induced morphism and let

$$\tilde{\mathfrak{A}}' := (\phi_* \mathfrak{A}, \text{Exc}(\phi^{-1})^{\text{nin}} + (1 - t) \text{Exc}(\phi^{-1})^{\text{inv}}).$$

Then we only need to show that $\mathfrak{A}' = \tilde{\mathfrak{A}}'$. Since $\mathfrak{A}'/U$ is a bs-weak lc model of $\mathfrak{A}_W/U$, we have

$$\mathfrak{A}' = ((\phi_W)_* \mathfrak{A}_W, \text{Exc}(\phi_W^{-1})^{\text{nin}} + (1 - t) \text{Exc}(\phi_W^{-1})^{\text{inv}}).$$

Let D be a prime divisor on X' . Let $\tilde{B}'$ and B' be the boundaries of $\tilde{\mathfrak{A}}'$ and $\mathfrak{A}'$ respectively. There are three cases:

Case 1. D is not exceptional over X . In this case,

$$-\text{mult}_D \tilde{B}' = a(D, \tilde{\mathfrak{A}}') = a(D, \mathfrak{A}) = a(D, \mathfrak{A}_W) = a(D, \mathfrak{A}') = -\text{mult}_D B',$$

so $\text{mult}_D B' = \text{mult}_D \tilde{B}'$.

Case 2. D is exceptional over W . In this case, D is a component of $\text{Exc}(\phi_W^{-1})$ and a component of $\text{Exc}(\phi^{-1})$, hence

$$\text{mult}_D \tilde{B}' = t\epsilon_{\mathcal{F}}(D) + (1 - t) = \text{mult}_D B'.$$

Case 3. D is exceptional over X but not exceptional over W . In this case,

$$-\text{mult}_D B' = a(D, \mathfrak{A}') = a(D, \mathfrak{A}_W).$$

Since $E \geq 0$, $a(D, \mathfrak{A}_W) \leq a(D, \mathfrak{A})$. By Claim 3.9, $a(D, \mathfrak{A}) \leq a(D, \mathfrak{A}')$. Thus

$$-\text{mult}_D B' = a(D, \mathfrak{A}) = a(D, \mathfrak{A}') = a(D, \mathfrak{A}_W).$$

By Definition 3.7(4), $a(D, \mathfrak{A}) = -t\epsilon_{\mathcal{F}}(D) - (1 - t)$, which implies that

$$\text{mult}_D B' = t\epsilon_{\mathcal{F}}(D) + (1 - t) = \text{mult}_D (\text{Exc}(\phi^{-1})^{\text{nin}} + (1 - t) \text{Exc}(\phi^{-1})^{\text{inv}}) = \text{mult}_D \tilde{B}'.$$

Thus $B' = \tilde{B}'$, so $\mathfrak{A}'/U$ is a log birational model of $\mathfrak{A}/U$, and we are done for the bs-weak lc model case.

Next we prove the bs-minimal model case. Suppose that $\mathfrak{A}'/U$ be a bs-minimal model of $\mathfrak{A}_W/U$. For any prime divisor D on X which is exceptional over X' , $h_*^{-1}D$ is a prime divisor on W which is exceptional over X' . Thus

$$a(D, \mathfrak{A}) = a(D, \mathfrak{A}_W) < a(D, \mathfrak{A}').$$

The bs-minimal model case immediately follows from the bs-weak lc model case.

The bs-semi-ample model, bs-good minimal model, log minimal model, and good log minimal model cases follow immediately from the bs-weak lc model and the bs-minimal model cases. $\square$

Lemma 3.10. *Let $\mathfrak{A}/U$ be an lc algebraically integrable adjoint foliated structure and $\mathfrak{A}'/U$ a bs-weak lc model of $\mathfrak{A}/U$. Let $\mathfrak{A}_W$ be a foliated log smooth model of $\mathfrak{A}$. Let X, X', W be the ambient varieties of $\mathfrak{A}, \mathfrak{A}', \mathfrak{A}_W$ respectively, and assume that the induced birational map $\phi_W : W \dashrightarrow X'$ is a morphism.*

Then we may run a $K_{\mathfrak{A}_W}$ -MMP/ X' with scaling of an ample/ X' $\mathbb{R}$ -divisor which terminates with a good minimal model $\mathfrak{A}_Y/X'$ of $\mathfrak{A}_W/X'$ such that

$$K_{\mathfrak{A}_Y} = q^* K_{\mathfrak{A}'}.$$

where Y is the ambient variety of $\mathfrak{A}_Y$ and $q : Y \rightarrow X'$ is the induced morphism. In particular, $\mathfrak{A}_Y/U$ is a log minimal model of $\mathfrak{A}_W/U$.

Proof. Let t be the parameter of $\mathfrak{A}$. If $t = 1$ then we are done by [LMX24, Lemma 4.13], so we may assume that $t < 1$.

Let $h : W \rightarrow X$ be the induced birational morphism. We have

$$K_{\mathfrak{A}_W} = h^* K_{\mathfrak{A}} + E$$

for some $E \geq 0$ that is exceptional/ X . By Lemma 3.3, we have

$$h^* K_{\mathfrak{A}} = \phi_W^* K_{\mathfrak{A}'} + F$$

where $F \geq 0$ is exceptional/ X' . Thus

$$K_{\mathfrak{A}_W} \sim_{\mathbb{R}, X'} F + E.$$

Claim 3.11. *E is exceptional/ X' .*

Proof. Let D be a component of E . By Definition 3.7(2), $a(D, \mathfrak{A}) > -t\epsilon_{\mathcal{F}}(D) - (1-t)$ and D is exceptional/ X .

Assume that D is not exceptional over X' . Since $\mathfrak{A}'/U$ is a log birational model of $\mathfrak{A}/U$ and $\mathfrak{A}$ is lc, $a(D, \mathfrak{A}') = -t\epsilon_{\mathcal{F}}(D) - (1-t)$. Since $F \geq 0$, $a(D, \mathfrak{A}) \leq a(D, \mathfrak{A}')$. Thus $a(D, \mathfrak{A}) = -t\epsilon_{\mathcal{F}}(D) - (1-t)$, hence D is not a component of E , a contradiction. $\square$

Proof of Lemma 3.10 continued. By Claim 3.11, $F+E$ is exceptional over X' . By Theorem 3.6, we may run a $K_{\mathfrak{A}_W}$ -MMP/ X' with scaling of an ample/ X' divisor, which terminates with a good minimal model $\mathfrak{A}_Y/X'$ of $\mathfrak{A}_W/X'$ such that $K_{\mathfrak{A}_Y} \sim_{\mathbb{R}, X'} 0$. Since $t < 1$, by [Cas+24, Proposition 3.5], $\mathfrak{A}_Y$ is $\mathbb{Q}$ -factorial qdlt, and $a(D, \mathfrak{A}_W) < a(D, \mathfrak{A}_Y)$ for any prime divisor D on W that is exceptional/ Y . By the negativity lemma,

$$K_{\mathfrak{A}_Y} = q^* K_{\mathfrak{A}'}.$$

The lemma follows. $\square$

Lemma 3.12. *Let $\mathfrak{A}/U$ be an lc algebraically integrable adjoint foliated structure. If $\mathfrak{A}/U$ has a bs-weak lc model (resp. bs-semi-ample model), then $\mathfrak{A}/U$ has a log minimal model (resp. good log minimal model).*

Proof. By Lemma 3.4 we only need to prove the bs-weak lc model case. The lemma follows immediately from Lemmas 3.8 and 3.10. $\square$

Lemma 3.13. *Let $\mathfrak{A}/U$ be an lc algebraically integrable adjoint foliated structure with ambient variety X . Let $f : Y \rightarrow X$ be a birational morphism, $E \geq 0$ an f -exceptional $\mathbb{R}$ -divisor, and $\mathfrak{A}_Y := (f^*\mathfrak{A}, E)$. Assume that $\mathfrak{A}_Y$ is lc. Then:*

- (1) *Any bs-weak lc model of $\mathfrak{A}/U$ is a bs-weak lc model of $\mathfrak{A}_Y/U$.*
- (2) *If $\mathfrak{A}/U$ has a bs-weak lc model (resp. bs-semi-ample model), then $\mathfrak{A}_Y/U$ has a log minimal model (resp. good log minimal model).*

Proof. (1) Let $\mathfrak{A}'/U$ be a bs-weak lc model of $\mathfrak{A}/U$, $\phi : X \dashrightarrow X'$ the induced birational map, and $\phi_Y := \phi \circ f$. Let $p : W \rightarrow Y$ and $q : W \rightarrow X'$ be a resolution of indeterminacy, and let $h := f \circ p$. By Lemma 3.3,

$$h^*K_{\mathfrak{A}} = q^*K_{\mathfrak{A}'} + F$$

for some $F \geq 0$ that is exceptional over X' . Thus

$$p^*K_{\mathfrak{A}_Y} = q^*K_{\mathfrak{A}'} + p^*E + F.$$

Thus $a(D, \mathfrak{A}_Y) \leq a(D, \mathfrak{A}')$ for any prime divisor D over X' . In particular, if $a(D, \mathfrak{A}') = -\epsilon_{\mathcal{F}}(D)$, then $a(D, \mathfrak{A}_Y) = -\epsilon_{\mathcal{F}}(D)$.

Let t be the parameter of $\mathfrak{A}$. Since $\mathfrak{A}'/U$ is a log birational model of $\mathfrak{A}/U$ and $\mathfrak{A}$ is lc, we have

$$\mathfrak{A}' = (\phi_*\mathfrak{A}, \text{Exc}(\phi^{-1})^{\text{nin}} + (1-t)\text{Exc}(\phi^{-1})^{\text{inv}}).$$

Let

$$\tilde{\mathfrak{A}}' = ((\phi_Y)_*\mathfrak{A}_Y, \text{Exc}(\phi_Y^{-1})^{\text{nin}} + (1-t)\text{Exc}(\phi_Y^{-1})^{\text{inv}}).$$

Let B' and $\tilde{B}'$ be the boundaries of $\mathfrak{A}'$ and $\tilde{\mathfrak{A}}'$ respectively. For any prime divisor D on X' , there are two cases:

Case 1. D is not exceptional over X . In this case,

$$-\text{mult}_D B' = a(D, \mathfrak{A}') = a(D, \mathfrak{A}) = a(D, \mathfrak{A}_Y) = a(D, \mathfrak{A}') = -\text{mult}_D \tilde{B}',$$

so $\text{mult}_D B' = \text{mult}_D \tilde{B}'$.

Case 2. D is exceptional over X . In this case,

$$a(D, \mathfrak{A}') = -\text{mult}_D B' = -t\epsilon_{\mathcal{F}}(D) - (1-t).$$

Since $a(D, \mathfrak{A}_Y) \leq a(D, \mathfrak{A}')$ and $\mathfrak{A}_Y$ is lc, $a(D, \mathfrak{A}_Y) = -t\epsilon_{\mathcal{F}}(D) - (1-t)$. Therefore, if D is not exceptional over Y , then

$$\text{mult}_D \tilde{B}' = -a(D, \mathfrak{A}_Y) = \epsilon_{\mathcal{F}}(D) = \text{mult}_D B',$$

and if D is exceptional over Y , then

$$\text{mult}_D \tilde{B}' = \text{mult}_D (\text{Exc}(\phi_Y^{-1})^{\text{nin}} + (1-t)\text{Exc}(\phi_Y^{-1})^{\text{inv}}) = t\epsilon_{\mathcal{F}}(D) + (1-t) = \text{mult}_D B'.$$

Thus $B' = \tilde{B}'$, hence $\mathfrak{A}'/U$ is a log birational model of $\mathfrak{A}_Y/U$. Since $K_{\mathfrak{A}'}$ is nef/ U , and $a(D, \mathfrak{A}_Y) \leq a(D, \mathfrak{A}')$ for any prime divisor D over X' , $\mathfrak{A}'/U$ is a bs-weak lc model of $\mathfrak{A}_Y/U$, and we get (1).

(2) It follows from (1), Lemma 3.12, and Lemma 3.4. $\square$

Lemma 3.14. *Let $\mathfrak{A}/U$ be an lc adjoint foliated structure with ambient variety X . Let $f : Y \rightarrow X$ be a birational morphism which only extracts lc places of $\mathfrak{A}$ and $\mathfrak{A}_Y := f_*\mathfrak{A}$.*

Then any bs-weak lc model (resp. bs-minimal model) of $\mathfrak{A}_Y/U$ is a bs-weak lc model (resp. bs-minimal model) of $\mathfrak{A}/U$.

Proof. Let $\mathfrak{A}'/U$ be a bs-weak lc model (resp. bs-minimal model) of $\mathfrak{A}_Y/U$ and let X' be its ambient variety. Then for any prime divisor D on X that is exceptional/ X' , we have

$$a(D, \mathfrak{A}) = a(D, \mathfrak{A}_Y) \leq (\text{resp. } <) a(D, \mathfrak{A}').$$

For any prime divisor D on X' that is exceptional/ X , if D is exceptional/ Y , then D is an lc place of $\mathfrak{A}'$, hence an lc place of $\mathfrak{A}$. If D is not exceptional/ Y , then $\text{center}_Y D$ is a prime divisor that is contracted by f , so D is an lc place of $\mathfrak{A}$. Therefore, $\mathfrak{A}'/U$ is a log birational model of $\mathfrak{A}/U$. The lemma follows. $\square$

Lemma 3.15. *Let $\mathfrak{A}/U$ be an adjoint foliated structure with ambient variety X and let $\phi : X \dashrightarrow X'$ be a $K_{\mathfrak{A}}$ -negative map/ U with $\mathfrak{A}' := \phi_*\mathfrak{A}$. Then:*

- (1) *Any (bs-)minimal model of $\mathfrak{A}/U$ is a (bs-)minimal model of $\mathfrak{A}'/U$.*
- (2) *Any (bs-)minimal model of $\mathfrak{A}'/U$ is a (bs-)minimal model of $\mathfrak{A}/U$.*

Proof. (1) Let $\mathfrak{A}''/U$ be a bs-minimal model of $\mathfrak{A}/U$ with ambient variety X'' .

For any prime divisor D on X' that is exceptional over X'' , we have

$$a(D, \mathfrak{A}') = a(D, \mathfrak{A}) < a(D, \mathfrak{A}'').$$

For any prime divisor D on X'' that is exceptional/ X' , if D is not exceptional/ X , then by [LMX24, Lemma 2.25], $\sigma_D(X/U, K_{\mathfrak{A}}) > 0$. Let $p : W \rightarrow X$, $q : W \rightarrow X''$ be a resolution of indeterminacy of the induced birational map $\psi : X \dashrightarrow X''$. By Lemma 3.3,

$$p^*K_{\mathfrak{A}} = q^*K_{\mathfrak{A}''} + E$$

for some $E \geq 0$ that is exceptional/ X'' , so

$$0 < \sigma_D(X/U, K_{\mathfrak{A}}) = \sigma_{p_*^{-1}D}(W/U, q^*K_{\mathfrak{A}''} + E) = \sigma_{p_*^{-1}D}(E) = 0,$$

which is not possible. Therefore, any prime divisor D on X'' that is exceptional/ X' is also exceptional/ X , so $\mathfrak{A}''/U$ is a log birational model of $\mathfrak{A}'/U$, and if ψ does not extract any divisor, then the induced birational map $X' \dashrightarrow X''$ does not extract any divisor. (1) follows.

(2) Let $\mathfrak{A}''/U$ be a bs-minimal model of $\mathfrak{A}'/U$. Then for any prime divisor D on X that is exceptional/ X'' , if D is not exceptional/ X' , then

$$a(D, \mathfrak{A}) = a(D, \mathfrak{A}') < a(D, \mathfrak{A}''),$$

and if D is exceptional/ X' , then

$$a(D, \mathfrak{A}') = a(D, \mathfrak{A}) < a(D, \mathfrak{A}'').$$

Moreover, since $\phi_*\mathfrak{A} = \mathfrak{A}'$ and $\mathfrak{A}''/U$ is a log birational model of $\mathfrak{A}'/U$, $\mathfrak{A}''/U$ is a log birational model of $\mathfrak{A}/U$. Hence $\mathfrak{A}''/U$ is a bs-minimal model of $\mathfrak{A}/U$. Moreover, if the induced birational map $X' \dashrightarrow X''$ does not extract any divisor, then the induced birational map $X \dashrightarrow X''$ does not extract any divisor. The lemma follows. $\square$

Lemma 3.16. *Let $\mathfrak{A}/U$ be an adjoint foliated structure with ambient variety X . Let $\phi_i : X_i \dashrightarrow X_{i+1}$, $X_0 := X$ be a sequence of steps of a $K_{\mathfrak{A}}$ -MMP/ U with scaling of some $\mathbb{R}$ -divisor C . Let $\mathfrak{A}_i, C_i$ be the images of $\mathfrak{A}, C$ on X_i respectively and let*

$$\lambda_i := \inf\{s \geq 0 \mid K_{\mathfrak{A}_i} + sC_i \text{ is nef}/U\}$$

be the scaling numbers of this MMP. Assume this MMP does not terminate. Let $\lambda := \lim_{i \rightarrow +\infty} \lambda_i$. Then for any $i \gg 0$, we have the following.

- (1) ϕ_i is a flip.
- (2) If $\lambda = 0$, then $K_{\mathfrak{A}_i}$ is movable/ U .
- (3) Assume that $\lambda = 0$ and $\mathfrak{A}/U$ has a bs-minimal model (resp. minimal model) $\mathfrak{A}'/U$ with ambient variety X' . Let $\psi_i : X_i \dashrightarrow X'$ be the induced birational map. Then ψ_i does not contract any divisor (resp. ψ_i is small).

Proof. (1) Let $h : Y \rightarrow X$ be a resolution of X , n the number of h -exceptional prime divisors, and $\rho := \dim N^1(Y/U) \otimes \mathbb{R} - n$. By [AHK07, Lemma 1.6], for any i , the induced birational map $X \dashrightarrow X_i$ contracts at most $\rho - 1$ prime divisors, and (1) follows.

(2) Let $\phi_{i,j} : X_i \dashrightarrow X_j$ be the induced birational maps. By (1), if $i \gg 0$ then

$$K_{\mathfrak{A}_i} = \lim_{j \rightarrow +\infty} (f_{i,j}^{-1})_*(K_{\mathfrak{A}_j} + \lambda_j C_j)$$

is movable/ U .

(3) Let $p : W \rightarrow X_i$ and $q : W \rightarrow X'$ be a resolution of indeterminacy of ψ_i . By Lemma 3.15, $\mathfrak{A}'/U$ is a bs-minimal model (resp. minimal model) of $\mathfrak{A}_i/U$. By Lemma 3.3,

$$F := p^* K_{\mathfrak{A}_i} - q^* K_{\mathfrak{A}'} \geq 0$$

is exceptional/ X' . By (2) and [LX25, Lemma 3.4],

$$0 = N_\sigma(K_{\mathfrak{A}_i}/U) = p_* N_\sigma(p^* K_{\mathfrak{A}_i}/U) = p_* N_\sigma((q^* K_{\mathfrak{A}'} + F)/U) = p_* F.$$

Since $a(D, \mathfrak{A}_i) < a(D, \mathfrak{A}')$ for any prime divisor D on X_i that is exceptional/ X' , $\text{Supp}(p_* F)$ contains all prime divisors on X_i that are exceptional/ X' , hence ψ_i does not contract any divisor. In particular, if $\mathfrak{A}'/U$ is a minimal model of $\mathfrak{A}/U$, then ψ_i is small. $\square$

4. EXISTENCE OF MINIMAL MODELS AND MMP WITH SCALING

The goal of this section is to prove that, for lc algebraically integrable adjoint foliated structures, existence of a minimal model, even in the sense of Birkar-Shokurov, implies the termination of MMP with scaling. This result can be seen as an analogue of [Bir12, Theorem 1.9], [HL22, Theorem 4.1], [LT22, Theorem 4.1], [TX24, Theorem 2.19] for pairs or generalized pairs. Our proof generally follows the same ideas in these references with minor differences as we do not need to introduce an auxiliary ample divisor whose components span the Neron-Severi cone. The main result of this section is the following:

Theorem 4.1. *Let $\mathfrak{A}(0)/U := (X, \mathcal{F}, B_0^{\text{nin}} + (1 - t_0)B_0^{\text{inv}}, \mathbf{M}_0, t_0)/U$ and $\mathfrak{A}(1)/U := (X, \mathcal{F}, B_1^{\text{nin}} + (1 - t_1)B_1^{\text{inv}}, \mathbf{M}_1, t_1)/U$ be two lc algebraically integrable adjoint foliated structures. Let $\mathfrak{A}(s) := s\mathfrak{A}(1) + (1 - s)\mathfrak{A}(0)$ for any $s \in \mathbb{R}$ and let $C := K_{\mathfrak{A}_1} - K_{\mathfrak{A}_0}$. Assume the following.*

- (1) $B_1 \geq B_0$, $\mathbf{M}_1 - \mathbf{M}_0$ is nef/ U , and $t_1 \geq t_0$.
- (2) $\phi_i : X_i \dashrightarrow X_{i+1}$ is a sequence of steps of a $K_{\mathfrak{A}(0)}$ -MMP/ U with scaling of C with $X_0 := X$, $\mathfrak{A}_i(s)$, C_i the images of $\mathfrak{A}(s)$, C on X_i respectively for any $s \in [0, 1]$ and any i , and

$$\lambda_i := \inf\{s \geq 0 \mid K_{\mathfrak{A}_i(s)} \text{ is nef}/U\}$$

the scaling numbers of this MMP. In particular, we assume that $\lambda_0 \leq 1$, $K_{\mathfrak{A}(\lambda_0)}$ is nef, and there exists a $K_{\mathfrak{A}_i(0)}$ -negative extremal ray/ U that is $K_{\mathfrak{A}_i(\lambda_i)}$ -trivial unless $\lambda_i = 0$.

(3) $K_{\mathfrak{A}_i(\lambda_i)}$ is NQC/U for any $i \gg 0$.

(4) $\mathfrak{A}(0)/U$ has a $KNQC$ bs-weak lc model $\mathfrak{A}_Y/U$.

Then either this MMP terminates or $\lim_{i \rightarrow +\infty} \lambda_i > 0$.

We need to prove several lemmas before proving Theorem 4.1.

Lemma 4.2. *Let $\mathfrak{A}_i/U := (X, \mathcal{F}, B_i^{\text{inv}} + (1 - t_i)B_i^{\text{inv}}, \mathbf{M}_i, t_i)/U$, $i \in \{1, 2\}$ be two algebraically integrable adjoint foliated structures such that $B_2 \geq B_1$, $\mathbf{M}_2 - \mathbf{M}_1$ is nef/ U , and $t_2 \geq t_1$. If $\mathfrak{A}_2$ is $\mathbb{Q}$ -factorial qdlt, then $\mathfrak{A}_1$ is $\mathbb{Q}$ -factorial qdlt.*

Proof. We first reduce to the case when $t_1 = t_2$. If $t_1 < t_2$, then

$$\mathfrak{A}'_1 := (X, \mathcal{F}, B_2^{\text{inv}} + (1 - t_1)B_2^{\text{inv}}, \mathbf{M}_2, t_1) = \frac{t_1}{t_2}\mathfrak{A}_2 + \frac{t_2 - t_1}{t_2}(X, B_2, \mathbf{M}_2)$$

is lc, and any lc place of $\mathfrak{A}'_1$ is an lc place of $(X, B_2, \mathbf{M}_2)$. Moreover, since $\mathfrak{A}_2$ is qdlt, $t_2 < 1$ and $(X, B_2, \mathbf{M}_2)$ is qdlt. Thus $\mathfrak{A}'_1$ is qdlt. Possibly replacing $\mathfrak{A}_2$ with $\mathfrak{A}'_1$, we may assume that $t_1 = t_2$.

Let $\mathbf{N} := \mathbf{M}_2 - \mathbf{M}_1$. Let E be an lc place of $\mathfrak{A}_1$. We let $h : Y \rightarrow X$ be a birational morphism such that $\mathbf{M}_1, \mathbf{M}_2$ descend to Y and E is on Y . Since $B_2 \geq B_1$, $\mathbf{M}_2 - \mathbf{M}_1$ is nef, and $\mathfrak{A}_2$ is lc, we have that E is an lc place of $\mathfrak{A}_2$, $\text{mult}_E(B_2 - B_1) = 0$, and $\text{mult}_E(h^*\mathbf{N}_X - \mathbf{N}_Y) = 0$. Since $\mathfrak{A}_2$ is qdlt, E is an lc place of $(X, B_2, \mathbf{M}_2)$. We have

$$a(E, X, B_1, \mathbf{M}_1) = a(E, X, B_2, \mathbf{M}_2) + \text{mult}_E(B_2 - B_1) + \text{mult}_E(h^*\mathbf{N}_X - \mathbf{N}_Y) = -1.$$

Thus E is an lc place of $(X, B_1, \mathbf{M}_1)$. By definition, $\mathfrak{A}_1$ is qdlt. $\square$

Lemma 4.3. *Let $\mathfrak{A}/U = (X, \mathcal{F}, B, \mathbf{M}, t)/U$ be an lc algebraically integrable adjoint foliated structure such that X is potentially klt and $K_{\mathfrak{A}}$ is NQC/U . Write $K_{\mathfrak{A}} = \sum a_i M_i$ where each M_i is nef/ U Cartier and each $a_i > 0$. Let $l := \frac{2 \dim X + 1}{\min\{a_i\}}$.*

Let $\tilde{\mathfrak{A}}/U = (X, \mathcal{F}, \tilde{B}, \tilde{\mathbf{M}}, \tilde{t})/U$ be an lc algebraically integrable adjoint foliated structure and let $\mathfrak{A}(s) := s\tilde{\mathfrak{A}} + (1 - s)\mathfrak{A}$ for any $s \in [0, 1]$. Then for any $s \in [0, \frac{1}{l+1}]$, any sequence of steps of a $K_{\mathfrak{A}(s)}$ -MMP/ U is $K_{\mathfrak{A}}$ -trivial.

Proof. We may assume that $s > 0$. We have

$$K_{\mathfrak{A}(s)} = s \left(K_{\tilde{\mathfrak{A}}} + \left(\frac{1}{s} - 1 \right) K_{\mathfrak{A}} \right)$$

so any sequence of steps of a $K_{\mathfrak{A}(s)}$ -MMP/ U is a sequence of steps of a $(K_{\tilde{\mathfrak{A}}} + rK_{\mathfrak{A}})$ -MMP/ U for some $r \geq l$. Assume that a sequence of steps $\phi : X \dashrightarrow X'$ of a $K_{\mathfrak{A}(s)}$ -MMP/ U is $K_{\mathfrak{A}}$ -trivial. Let $\mathfrak{A}'(a) := \phi_*\mathfrak{A}(a)$ for any $a \in [0, 1]$ and $M'_i := \phi_*M_i$ for each i . Then by the contraction theorem, M'_i is nef/ U and Cartier for any i . Moreover, ϕ is also $K_{\mathfrak{A}(1)}$ -negative, so $\mathfrak{A}'(1)$ is lc.

For any step $X' \xrightarrow{f} T \leftarrow X''$ of a $K_{\mathfrak{A}'(s)}$ -MMP/ U where f is the contraction of an extremal ray, let R be the $K_{\mathfrak{A}'(s)}$ -negative extremal ray contraction by f . Then R is also $K_{\mathfrak{A}'(1)}$ -negative. By [Cas+24, Theorem 1.3], R is spanned by a rational extremal ray C such that $0 > K_{\mathfrak{A}'(1)} \cdot C \geq -2 \dim X$. If $K_{\mathfrak{A}'(0)} \cdot C > 0$, then

$$\begin{aligned} 0 > K_{\mathfrak{A}'(s)} \cdot C &= s \left(K_{\mathfrak{A}'(1)} + \left(\frac{1}{s} - 1 \right) K_{\mathfrak{A}'(0)} \right) \cdot C \geq s \left(-2 \dim X + l \sum a_i M'_i \cdot C \right) \\ &\geq s \left(-2 \dim X + \frac{2 \dim X + 1}{\min\{a_i\}} \cdot \min\{a_i\} \right) > 0, \end{aligned}$$

which is not possible. Thus f is $K_{\mathfrak{A}'(0)}$ -trivial, and we are done by induction on the number of steps of the MMP. $\square$

Lemma 4.4 (Lifting the MMP). *Let $\mathfrak{A}/U$ be an lc algebraically integrable adjoint foliated structure with ambient variety X and parameter t . Let C be an $\mathbb{R}$ -divisor on X and let $\phi_i : X_i \xrightarrow{f_i} T_i \xleftarrow{f_i^+} X_{i+1}$, $X_0 := X$ be a sequence of steps of a $K_{\mathfrak{A}}$ -MMP/ U and $\mathfrak{A}_i, C_i$ the images of $\mathfrak{A}, C$ on X_i respectively, where f_i is the contraction of an extremal ray. Let $h : X' \rightarrow X$ be a $\mathbb{Q}$ -factorial ACSS modification of $\mathfrak{A}$ if $t = 1$ and a $\mathbb{Q}$ -factorial qdlt modification of $\mathfrak{A}$ if $t < 1$. Let $\mathfrak{A}' := h^*\mathfrak{A}$.*

Assume that either X is potentially klt, or Theorem 4.1 holds when X is $\mathbb{Q}$ -factorial klt. Then there exists a sequence of birational maps/ U $\phi'_i : X'_i \dashrightarrow X'_{i+1}$, $X'_0 := X'$ and birational morphisms $h_i : X'_i \rightarrow X_i$, $h_0 := h$ satisfying the following. Let $\mathfrak{A}'_i$ be the image of $\mathfrak{A}'$ on X'_i , then for each i :

- (1) h_i is a $\mathbb{Q}$ -factorial ACSS modification of $\mathfrak{A}_i$ if $t = 1$ and is a $\mathbb{Q}$ -factorial qdlt modification of $\mathfrak{A}_i$ when $t < 1$, with $h_i^*\mathfrak{A}_i = \mathfrak{A}'_i$.
- (2) $h_{i+1} \circ \phi'_i = \phi_i \circ h_i$.
- (3) ϕ'_i is a $K_{\mathfrak{A}'_i}$ -MMP/ T_i that is not the identity morphism, and $\mathfrak{A}'_{i+1}/T_i$ is a good log minimal model of $\mathfrak{A}'_i/T_i$.
- (4) $\{\phi'_i\}$ is a sequence of steps of a $K_{\mathfrak{A}'}$ -MMP/ U .
- (5) If $\{\phi_i\}$ is a sequence of steps of a $K_{\mathfrak{A}}$ -MMP/ U with scaling of C with scaling numbers

$$\lambda_i := \inf\{s \geq 0 \mid K_{\mathfrak{A}_i} + sC_i \text{ is nef}/U\},$$

then:

- (a) ϕ'_i is a sequence of steps of a $K_{\mathfrak{A}'}$ -MMP/ U with scaling of $C' := h^*C$ with scaling numbers all equal to λ_i .
- (b) $\{\phi'_i\}$ is a sequence of steps of a $K_{\mathfrak{A}'}$ -MMP/ U with scaling of C' .

Proof. If $t = 1$, then we are done by [LMX24, Propositions 8.2, A.45]. Note that [LMX24] further requires that h is “strict and super” but the same lines of the arguments work in our setting. In the rest of the proof, we assume that $t < 1$.

Since (4) follows from (3) and (5.b) follows from (5.a), we only need to prove (1)(2)(3) and (5.a).

Let n be a non-negative integer. We prove the proposition by induction on n under the induction hypothesis that we have already constructed $\mathfrak{A}'_i/U$ and h_i for any $i \leq n$ and $\phi_{i,Y}$ for any $i \leq n-1$ which satisfy (1)(2)(3)(5). When $n = 0$, this follows from our assumption, so we may assume that $n > 0$. We need to construct ϕ'_n, h_{n+1} , and $\mathfrak{A}'_{n+1}/U$.

When X is potentially klt, we let (X, Δ) be a klt pair,

$$\mathfrak{A}(\epsilon) := \epsilon(X, \Delta) + (1 - \epsilon)\mathfrak{A}$$

for any $\epsilon \in [0, 1]$, and $\mathfrak{A}_i(\epsilon)$ the image of $\mathfrak{A}(\epsilon)$ on X_i for any i . Then $\mathfrak{A}(\epsilon)$ is klt for any $\epsilon \in (0, 1]$, and the induced birational map $X \dashrightarrow X_n$ is also a sequence of steps of a $K_{\mathfrak{A}(\epsilon)}$ -MMP for any $0 \leq \epsilon \ll 1$. Thus $\mathfrak{A}_n(\epsilon)$ is klt for any $0 < \epsilon \ll 1$. Moreover, ϕ_n is a step of a $K_{\mathfrak{A}_n(\epsilon)}$ -MMP/ U for any $0 < \epsilon \ll 1$. Let $\mathfrak{A}'_n(\epsilon) := h_n^*\mathfrak{A}_n(\epsilon)$ for any $\epsilon \in [0, 1]$. Then $\mathfrak{A}'_n(\epsilon)$ is $\mathbb{Q}$ -factorial sub-klt for any $0 < \epsilon \ll 1$. Since h_n only extracts lc places of $\mathfrak{A}_n$ and $t < 1$, $\mathfrak{A}'_n(\epsilon)$ is $\mathbb{Q}$ -factorial klt for any $0 < \epsilon \ll 1$.

We let H_n be a supporting function of the extremal ray/ U contracted by f_n and let

$$L_n := H_n - K_{\mathfrak{A}_n},$$

such that $L_n = \lambda_n C_n$ if we are running an MMP/ U with scaling of C_n . Then L_n is ample/ T_n .

We run a $K_{\mathfrak{A}_n}$ -MMP/ T_n with scaling of an ample divisor A_n . Note that $\mathfrak{A}_{n+1}/T_n$ is a good minimal model of $\mathfrak{A}'_n/T_n$. When X is potentially klt, this MMP is also a $K_{\mathfrak{A}'_n(\epsilon)}$ -MMP/ T_n with scaling of μA_n for some $0 < \epsilon \ll 1$. Thus by [Cas+25, Theorem 2.1.1], and by Theorem 4.1 when X is not potentially klt, this MMP terminates with a good minimal model $\mathfrak{A}'_{n+1}(\epsilon)/T_n$ of $\mathfrak{A}'_n(\epsilon)/T_n$ with ambient variety X_{n+1} and induced birational map $\phi'_n : X'_n \dashrightarrow X'_{n+1}$.

Let $\mathfrak{A}'_{n+1} := (\phi'_n)_* \mathfrak{A}'_n$. By Lemma 3.5, $\mathfrak{A}'_{n+1}/T_n$ is a good log minimal model of $\mathfrak{A}'_n/T_n$. This implies (3).

Since ϕ_n is the ample model/ T_n of $K_{\mathfrak{A}_n}$, there exists an induced birational morphism $h_{n+1} : X'_{n+1} \rightarrow X_n$ such that $h_{n+1} \circ \phi'_n = \phi_n \circ h_n$. This implies (2). Moreover, $K_{\mathfrak{A}'_{n+1}} = h_{n+1}^* K_{\mathfrak{A}_{n+1}}$, so h_{n+1} is a $\mathbb{Q}$ -factorial qdlt modification of $\mathfrak{A}_i$. This implies (1).

Moreover, this MMP is also an MMP/ U with scaling of L_n as $K_{\mathfrak{A}'_n} + h_n^* L_n \sim_{\mathbb{R}, T_n} 0$ and $K_{\mathfrak{A}_n} + h_n^* L_n = h_n^* H_n$ is nef/ U . This implies (5.a) and we are done. $\square$

Remark 4.5. We are going to prove Theorem 4.1 in the following way: we are going to use Lemma 4.4 for the $\mathbb{Q}$ -factorial klt case to prove Theorem 4.1 when X is $\mathbb{Q}$ -factorial klt. This in turn imply Lemma 4.4 unconditionally. Using the same lines of the proof, we can prove the general case of Theorem 4.1.

Proof of Theorem 4.1. We prove the Theorem contradiction. By Lemma 3.12 we may assume that $\mathfrak{A}_Y/U$ is a $\mathbb{Q}$ -factorial log minimal model of $\mathfrak{A}(0)/U$. We may assume that this MMP does not terminates and $\lim_{i \rightarrow +\infty} \lambda_i = 0$. Let $\varphi : X \dashrightarrow Y$ be the induced birational map.

By Remark 4.5, we may assume that one of the following conditions holds:

- X is $\mathbb{Q}$ -factorial klt.
- The $\mathbb{Q}$ -factorial klt case of Theorem 4.1 holds.

Step 1. Write $\mathfrak{A}_i(s) := (X_i, \mathcal{F}_i, B_i(s), \mathbf{M}(s), t(s))$ for any i and any $s \in [0, 1]$. By Lemma 3.16, possibly truncating the MMP, we may assume that

- (i) ϕ_i is a flip for every i and $K_{\mathfrak{A}(0)}$ is movable/ U .

Possibly replacing $\mathfrak{A}(s)$ with $\mathfrak{A}_{i+1}(s)$ for any $s \in [0, 1]$ and some $i+1$ such that $\lambda_i > \lambda_{i+1}$, and then replacing $\mathfrak{A}(1)$ with $\mathfrak{A}(\lambda_1)$, we may assume that

- (ii) $\lambda_1 = 1$, and
- (iii) $\mathfrak{A}(1 + \epsilon)$ is lc for some $\epsilon > 0$. In particular, if $t_0 < 1$, then $t_1 < 1$.

Write $\mathfrak{A}_Y(0) := (Y, \mathcal{F}_Y, B_Y, \mathbf{M}, t_0)/U$. Let $p : W \rightarrow X$ and $q : W \rightarrow Y$ be a resolution of indeterminacy of φ so that p is a foliated log resolution of $(X, \mathcal{F}, \text{Supp } B_0 \cup \text{Supp } B_1, \mathbf{M})$ and q is a foliated log resolution of $(Y, \mathcal{F}, \text{Supp } B_Y, \mathbf{M})$. By Lemma 3.16, φ does not contract any divisor.

Step 2. In this step we construct a model Y' . Let

$$\mathfrak{A}_W(s) := (p_*^{-1} \mathfrak{A}(s), \text{Exc}(p)^{\text{inv}} + (1 - st_1 - (1 - s)t_0) \text{Exc}(p)^{\text{inv}})$$

Then $\mathfrak{A}_W(s)$ is a foliated log smooth model of $\mathfrak{A}(s)$ for any $s \in [0, 1 + \epsilon]$. By Lemma 3.10, we may run a $K_{\mathfrak{A}_W(0)}$ -MMP/ Y with scaling of an ample divisor which terminates with a good log minimal model $\mathfrak{A}_{Y'}(0)/Y$ of $\mathfrak{A}_W(0)/Y$ such that $K_{\mathfrak{A}_{Y'}(0)} \sim_{\mathbb{R}, Y} 0$. Let Y' be the ambient variety of $\mathfrak{A}_{Y'}(0)$, $\psi : W \dashrightarrow Y'$ the induced birational map, and $\mathfrak{A}_{Y'}(s) := \psi_* \mathfrak{A}_W(s)$ for any s .

Step 3. In this step we construct a model X' . Let $F(s) := K_{\mathfrak{A}_W(s)} - p^*K_{\mathfrak{A}(s)}$ for any s . Since $K_{\mathfrak{A}_W(1)} \sim_{\mathbb{R}, X} F(1) \geq 0$ is exceptional/ X , by Theorem 3.6, we may run a $K_{\mathfrak{A}_W(1)}$ -MMP/ X which terminates with a good log minimal model $\mathfrak{A}'(1)/X$ of $\mathfrak{A}(1)/X$ with ambient variety X' and associated with birational map $\xi : W \dashrightarrow X'$ such that $K_{\mathfrak{A}'(1)} \sim_{\mathbb{R}, X} 0$. Let $\mathfrak{A}'(s) := \xi_*\mathfrak{A}_W(s)$ for any s . We have $K_{\mathfrak{A}'(1)} = h^*K_{\mathfrak{A}(1)}$ where $h : X' \rightarrow X$ is the induced birational morphism.

By (iii), $\mathfrak{A}'(1)$ is $\mathbb{Q}$ -factorial qdlt when $t_0 < 1$ and is $\mathbb{Q}$ -factorial ACSS when $t_0 = 1$. By Lemma 4.2, for any $s \in [0, 1]$, $\mathfrak{A}'(s)$ is $\mathbb{Q}$ -factorial qdlt when $t_0 < 1$ and is $\mathbb{Q}$ -factorial ACSS when $t_0 = 1$.

The divisors contracted by ξ are exactly the prime divisors W that are irreducible components of $F(1)$, which are exactly prime divisors on W that are not lc places of $\mathfrak{A}(1)$. Since $\mathfrak{A}(1 + \epsilon)$ is lc and

$$\mathfrak{A}(1) = \frac{1}{1 + \epsilon}\mathfrak{A}(1 + \epsilon) + \frac{\epsilon}{1 + \epsilon}\mathfrak{A}(0),$$

any lc place of $\mathfrak{A}(1)$ is an lc place of $\mathfrak{A}(0)$. By (1) and [Cas+24, Proposition 3.3], any lc place of $\mathfrak{A}(0)$ is an lc place of $\mathfrak{A}(1)$.

Thus the divisors contracted by ξ are exactly the prime divisors on W that are not lc places of $\mathfrak{A}(0)$. In particular, h is a $\mathbb{Q}$ -factorial ACSS modification of $\mathfrak{A}(0)$ if $t_0 = 1$ and is a $\mathbb{Q}$ -factorial qdlt modification of $\mathfrak{A}(0)$ if $t_0 < 1$.

Step 4. We show that $\mathfrak{A}_{Y'}(0)/U$ is a minimal model of $\mathfrak{A}'(0)/U$.

First we show that the induced birational map $\varphi' : X' \dashrightarrow Y'$ does not extract any divisor. Suppose that there exists a prime divisor D on Y' that is exceptional/ X' . Then D is exceptional/ X , so D is an lc place of $\mathfrak{A}(0)$. Since $\text{center}_W D$ is a divisor and ξ only contracts prime divisors that are not lc places of $\mathfrak{A}(0)$, D is not contracted by ξ , a contradiction.

Now for any prime divisor D on X' that is exceptional/ Y' , we have

$$a(D, \mathfrak{A}'(0)) = a(D, \mathfrak{A}_W(0)) < a(D, \mathfrak{A}_{Y'}(0)).$$

Thus $\mathfrak{A}_{Y'}(0)/U$ is a minimal model of $\mathfrak{A}'(0)/U$.

Step 5. In this step we construct models X'_i .

By Lemma 4.4, there exists a sequence of steps of a $K_{\mathfrak{A}'(0)}$ -MMP/ U with scaling of $C' := h^*C = K_{\mathfrak{A}'(1)} - K_{\mathfrak{A}'(0)}$, $\phi'_i : X'_i \dashrightarrow X'_{i+1}$, $X'_0 := X'$ satisfying the following. Let $\mathfrak{A}'_i(s), C'_i$ be the images of $\mathfrak{A}'(s), C'$ on X'_i respectively for any i, s and

$$\lambda'_i := \inf\{s \geq 0 \mid K_{\mathfrak{A}'_i(s)} \text{ is nef}/U\}$$

the scaling numbers. Then there exist a strictly increasing sequence n_i so that the induced birational maps $h_i : X'_{n_i} \rightarrow X_i$ are morphisms, each h_i is a $\mathbb{Q}$ -factorial ACSS modification of $\mathfrak{A}_i(0)$ if $t = 1$, a $\mathbb{Q}$ -factorial qdlt modification of $\mathfrak{A}_i(0)$ if $t < 1$, $\mathfrak{A}'_{n_i}(0) = h_i^*\mathfrak{A}_i(0)$, and $\lambda'_j = \lambda_i$ when $n_i \leq j \leq n_{i+1} - 1$. In particular, $\lim_{i \rightarrow +\infty} \lambda'_i = 0$.

Step 6. In this step we construct models $X(\delta, \delta')$ for $0 < \delta, \delta' \ll 1$. Let L be an ample divisor on W and $\mathbf{L} := \bar{L}$. By our construction, ψ is a sequence of steps of a $(K_{\mathfrak{A}_W(\delta)} + \delta'L)$ -MMP/ Y for any $0 \leq \delta, \delta' \ll 1$.

In particular, for any $0 \leq \delta, \delta' \ll 1$, $(\mathfrak{A}_{Y'}(\delta), \delta'\mathbf{L})$ is $\mathbb{Q}$ -factorial qdlt if $t < 1$, and is $\mathbb{Q}$ -factorial ACSS if $t = 1$. Since $K_{\mathfrak{A}_{Y'}(0)}$ is NQC/ U , by Lemma 4.3, for any $0 < \delta, \delta' \ll 1$, any sequence of steps of a $(K_{\mathfrak{A}_{Y'}(\delta)} + \delta'\mathbf{L}_{Y'})$ -MMP/ U is $K_{\mathfrak{A}_{Y'}(0)}$ -trivial. By [Cas+25, Lemma 7.2] and [CHLX23, Lemma 16.1.1], for any $0 < \delta, \delta' \ll 1$, there exists an lc algebraically

integrable adjoint foliated structure $\mathfrak{A}_{Y'}(\delta, \delta')$ with ambient variety Y' , and an ample/ U $\mathbb{R}$ -divisor $H(\delta, \delta')$ on Y' , such that

$$K_{\mathfrak{A}_{Y'}(\delta, \delta')} + H(\delta, \delta') \sim_{\mathbb{R}, U} K_{\mathfrak{A}_{Y'}(\delta)} + \delta' \mathbf{L}_{Y'}.$$

By [Cas+25, Theorem 8.1], we may run a $(K_{\mathfrak{A}_{Y'}(\delta, \delta')} + H(\delta, \delta'))$ -MMP/ U which terminates with a good minimal model $\mathfrak{A}(\delta, \delta')/U$ of $(\mathfrak{A}_{Y'}(\delta, \delta'), \overline{H(\delta, \delta')})/U$ with ambient variety $X(\delta, \delta')$ and induced birational map $Y' \dashrightarrow X(\delta, \delta')$. Let $\mathfrak{A}_{X(\delta, \delta')}(s)$ be the image of $\tilde{\mathfrak{A}}_{Y'}(s, \delta')$ on $X(\delta, \delta')$ for any s . Then

$$K_{\mathfrak{A}(\delta, \delta')} \sim_{\mathbb{R}, U} K_{\mathfrak{A}_{X(\delta, \delta')}(\delta)} + \delta' \mathbf{L}_{X(\delta, \delta')}$$

is semi-ample/ U and

$$K_{\mathfrak{A}_{X(\delta, \delta')}(0)}$$

is nef/ U as $Y' \dashrightarrow X(\delta, \delta')$ if $K_{\mathfrak{A}_{Y'}(0)}$ -trivial. Moreover, since $K_{\mathfrak{A}_{Y'}(\delta)}$ and $\delta' \mathbf{L}_{Y'}$ are movable/ U , $K_{\mathfrak{A}_{Y'}(\delta, \delta')} + H(\delta, \delta')$ is movable/ U , so the induced birational map $Y' \dashrightarrow X(\delta, \delta')$ is small.

Step 7. By our construction, ξ is a sequence of steps of a $(K_{\mathfrak{A}_W(1)} + \delta L)$ -MMP/ X for any $0 < \delta \ll 1$. Since $\mathfrak{A}_{Y'}(0)/U$ is a minimal model of $\mathfrak{A}'(0)/U$, by Lemma 3.15, $\mathfrak{A}_{Y'}(0)/U$ is a minimal model of $\mathfrak{A}'_i(0)/U$ for any i . By Lemma 3.16 and our assumption, we may let $i \gg 0$ be a positive integer such that

- $1 \gg \lambda_i > \lambda_{i+1}$,
- $K_{\mathfrak{A}'_{n_i}(0)}$ is movable/ U ,
- the induced birational map $f : X'_{n_i} \dashrightarrow Y'$ is small,
- $K_{\mathfrak{A}_i(\lambda_i)}$ is NQC/ U , and
- $K_{\mathfrak{A}_X(\lambda_i, \delta')} + \delta' \mathbf{L}_{X(\lambda_i, \delta')}$ is semi-ample/ U , $K_{\mathfrak{A}_X(\lambda_i, 0)}$ is nef/ U , and the induced birational map $Y' \dashrightarrow X(\lambda_i, \delta')$ is small for any $0 < \delta' \ll 1$.

By our construction, $h_i^* K_{\mathfrak{A}_i(\lambda_i)} = K_{\mathfrak{A}'_{n_i}(\lambda_i)}$ is NQC/ U . Since the induced birational map $X' \dashrightarrow X_{n_i}$ is a $K_{\mathfrak{A}'(\lambda_i)}$ -MMP/ U , $X' \dashrightarrow X_{n_i}$ is also a sequence of steps of a $(K_{\mathfrak{A}'(\lambda_i)} + \delta \mathbf{L}_{X'})$ -MMP/ U for any $0 \leq \delta \ll 1$.

By [Cas+25, Lemma 7.2] and [CHLX23, Lemma 16.1.1], for any $0 < \delta \ll 1$, there exists an adjoint foliated structure $(\mathfrak{A}'(1), \mathbf{N}(\delta))/U$ for some nef/ U $\mathbf{b}$ -divisor $\mathbf{N}(\delta)$ and an ample $\mathbb{R}$ -divisor $P(\delta)$ on X' , such that

$$K_{\mathfrak{A}'(1)} + \mathbf{N}(\delta)_{X'} + P(\delta) \sim_{\mathbb{R}, U} K_{\mathfrak{A}'(1)} + \delta \mathbf{L}_{X'},$$

$(\mathfrak{A}'(1), \mathbf{N}(\delta))$ is $\mathbb{Q}$ -factorial ACSS if $t = 1$ and $\mathbb{Q}$ -factorial qdlt if $t < 1$. Let $\mathbf{P}(\delta) := \overline{P(\delta)}$. For any $0 < \delta \ll 1$, by [Cas+25, Lemma 7.2] and [CHLX23, Lemma 16.1.1] again, there exists an lc adjoint foliated structure $\mathfrak{B}(\delta)/U$ with ambient variety X'_{n_i} and an ample/ U $\mathbb{R}$ -divisor $H(\delta)$ on X'_{n_i} , such that

$$K_{\mathfrak{B}(\delta)} + H(\delta) \sim_{\mathbb{R}, U} K_{\mathfrak{A}'_{n_i}(\lambda_i)} + \mathbf{N}(\delta)_{X'_{n_i}} + \mathbf{P}(\delta)_{X'_{n_i}} \sim_{\mathbb{R}, U} K_{\mathfrak{A}'_{n_i}(\lambda_i)} + \delta \mathbf{L}_{X'_{n_i}}.$$

By Lemma 4.3, for any $0 < \delta \ll 1$, any sequence of steps of a $(K_{\mathfrak{A}'_{n_i}(\lambda_i)} + \delta \mathbf{L}_{X'_{n_i}})$ -MMP/ U is $K_{\mathfrak{A}'_{n_i}(\lambda_i)}$ -trivial.

Pick $0 < \tau \ll 1$ that is general in $\mathbb{R}/\mathbb{Q}$. By [Cas+25, Theorem 8.1], we may run a $(K_{\mathfrak{B}(\tau)} + H(\tau))$ -MMP/ U which terminates with a good minimal model $(\mathfrak{B}_T(\tau), \overline{H(\tau)})/U$ of $(\mathfrak{B}(\tau), \overline{H(\tau)})/U$ associated with birational map $X'_{n_i} \dashrightarrow T$. Let $\mathfrak{A}_T(s)$ be the image of $\mathfrak{A}'_{n_i}(s)$ on T for any s , then

$$K_{\mathfrak{A}_T(\lambda_i)} + \tau \mathbf{L}_T$$

is semi-ample/ U and

$$K_{\mathfrak{A}_T(\lambda_i)}$$

is nef/ U . Moreover, since $K_{\mathfrak{A}'_{n_i}(\lambda_i)}$ and L are movable/ U , the induced birational map $X'_{n_i} \dashrightarrow T$ is small.

Moreover, we have that

$$K_{\mathfrak{A}_{X(\lambda_i, \tau)}(\lambda_i)} + \tau \mathbf{L}_{X(\lambda_i, \tau)}$$

is semi-ample/ U and

$$K_{\mathfrak{A}_{X(\lambda_i, \tau)}(0)}$$

is nef/ U , and the induced birational map $Y' \dashrightarrow X(\lambda_i, \tau)$ is small. Thus the induced birational map $\rho : T \dashrightarrow X(\delta, \tau)$ is small.

Step 8. Since ρ is small, $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(\lambda_i)} + \tau \mathbf{L}_{X(\lambda_i, \tau)}$ and $K_{\mathfrak{A}_T(\lambda_i)} + \tau \mathbf{L}_T$ are crepant. Since τ is general in $\mathbb{R}/\mathbb{Q}$, $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(\lambda_i)}$ and $K_{\mathfrak{A}_T(\lambda_i)}$ are crepant, hence $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(\lambda_i)}$ is nef/ U . Since $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(0)}$ is nef/ U and $\lambda_i > \lambda_{i+1} > 0$, $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(\lambda_{i+1})}$ is nef/ U .

We have that $K_{\mathfrak{A}'_{n_{i+1}}(\lambda_i)}$ and $K_{\mathfrak{A}'_{n_i}(\lambda_i)}$ are crepant, $K_{\mathfrak{A}'_{n_i}(\lambda_i)}$ and $K_{\mathfrak{A}_T(\lambda_i)}$ are crepant, and $K_{\mathfrak{A}_T(\lambda_i)}$ and $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(\lambda_i)}$ are crepant. Thus $K_{\mathfrak{A}'_{n_{i+1}}(\lambda_i)}$ and $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(\lambda_i)}$ are crepant. Since $K_{\mathfrak{A}'_{n_{i+1}}(\lambda_{i+1})}$ and $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(\lambda_{i+1})}$ are nef/ U and the induced birational map $X'_{n_{i+1}} \dashrightarrow X(\lambda_i, \tau)$ is small, by applying the negativity lemma twice, $K_{\mathfrak{A}'_{n_{i+1}}(\lambda_{i+1})}$ and $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(\lambda_{i+1})}$ are crepant. Since $\lambda_i > \lambda_{i+1}$, $K_{\mathfrak{A}'_{n_{i+1}}(s)}$ and $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(s)}$ are crepant for any s . In particular, $K_{\mathfrak{A}'_{n_{i+1}}(0)}$ and $K_{\mathfrak{A}_{X(\lambda_i, \tau)}(0)}$ are crepant, hence $K_{\mathfrak{A}'_{n_{i+1}}(0)}$ is nef. Thus the MMP $\{\phi'_i\}$ terminates, hence the MMP $\{\phi_i\}$ terminates, a contradiction. We conclude the proof. $\square$

Theorem 4.6. *Let $\mathfrak{A}/U$ be an lc algebraically integrable adjoint foliated structure such that X is potentially klt. Assume that $\mathfrak{A}/U$ has a KNQC bs-weak lc model $\mathfrak{A}_Y/U$. Then any sequence of steps of a $K_{\mathfrak{A}}$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor terminates with a minimal model of $\mathfrak{A}/U$.*

Proof. Let $\phi_i : \mathfrak{A}_i \dashrightarrow \mathfrak{A}_{i+1}$, $\mathfrak{A}_0 := \mathfrak{A}$ be a sequence of steps of a $K_{\mathfrak{A}}$ -MMP/ U with scaling of an ample/ U $\mathbb{R}$ -divisor A . Let X_i be the ambient variety of $\mathfrak{A}_i$, A_i the image of A on X_i , and

$$\lambda_i := \inf\{s \geq 0 \mid K_{\mathfrak{A}_i} + sA_i \text{ is nef}/U\}$$

the scaling numbers. We may assume that this MMP does not terminate and let $\lim_{i \rightarrow +\infty} \lambda_i = \lambda$. There are two possibilities:

Case 1. $\lambda < \lambda_i$ for any i . In this case, by [Cas+25, Theorem 8.1], $\lambda = 0$ and $K_{\mathfrak{A}_i} + \lambda_i A_i$ is semi-ample/ U for any i . The theorem follows from Theorem 4.1 in this case.

Case 2. $\lambda = \lambda_i > 0$ for $i \gg 0$. In this case, $(\mathfrak{A}_i, \lambda \bar{A})/U$ is a good minimal model of $(\mathfrak{A}, \lambda \bar{A})/U$ for any $i \gg 0$. Let $h_i : X'_i \rightarrow X_i$ be a small $\mathbb{Q}$ -factorialization of X_i for each i and let $\mathfrak{A}'_i := h_i^*(\mathfrak{A}_i, \lambda \bar{A})$ for any i . Then $\mathfrak{A}'_i/U$ are $\mathbb{Q}$ -factorial good minimal models of $(\mathfrak{A}, \lambda \bar{A})/U$ for any i . By [Cas+25, Theorem 2.5.2, Lemma 3.29], the induced birational map $\phi'_{i,j} : X'_i \dashrightarrow X'_j$ is an isomorphism for some $0 \ll i < j$. This is not possible because $\phi'_{i,j}$ is a non-trivial sequence of steps of a $K_{h_i^* \mathfrak{A}_i}$ -MMP. $\square$

Remark 4.7. In Theorem 4.1, by letting $t_0 = t_1 = 0$, we indeed obtain the generalized pair analogue of [Bir12, Theorem 1.9]. Note that the NQC case was proven in [TX24, Theorem 2.19] but the non-NQC case remains open.

Proof of Theorem 1.2. By [Cas+25, Theorem 2.5.4], $(X, \mathcal{F}, t)$ has a KNQC minimal model. The theorem follows from Theorem 4.6. $\square$

5. FINITENESS OF AMPLE MODELS

5.1. Basic properties of ample models.

Lemma 5.1. *Let $\pi : X \rightarrow U$ be a projective morphism between normal quasi-projective varieties. Let D_1, D_2 be two $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisors on X .*

- (1) *For any birational map/ U $\phi : X \dashrightarrow X'$, if ϕ is a minimal model/ U (resp. weak lc model/ U) of D_1 and D_2 , then ϕ is also a minimal model/ U (resp. weak lc model/ U) of $tD_1 + (1-t)D_2$ for any $t \in [0, 1]$.*
- (2) *For any birational map/ U $\phi : X \dashrightarrow X'$, if ϕ is a minimal model/ U of D_1 and a weak lc model/ U of D_2 , then ϕ is a minimal model of $tD_1 + (1-t)D_2$ for any $t \in [0, 1)$.*

Proof. Let $D'_i := \phi_* D_i$. Let $p : W \rightarrow X$ and $q : W \rightarrow X'$ be a resolution of indeterminacy of ϕ and write $p^* D_i = q^* D'_i + E_i$ for $i \in \{1, 2\}$. Then $E_i \geq 0$, and $\text{Supp}(p_* E_i)$ contains all prime divisors on X that are exceptional/ X' if ϕ is a minimal model of D_i . We have

$$p^*(tD_1 + (1-t)D_2) = q^*(tD'_1 + (1-t)D'_2) + (tE_1 + (1-t)E_2)$$

and (1) and (2) follow. $\square$

Lemma 5.2. *Let $\pi : X \rightarrow U$ be a projective morphism between normal quasi-projective varieties and let D_1, D_2 be two $\mathbb{R}$ -divisors on X such that $tD_1 + (1-t)D_2$ is semi-ample/ U for any $t \in (0, 1)$. Then:*

- (1) *There exists a contraction/ U $f : X \rightarrow Z$ such that $tD_1 + (1-t)D_2 = f^* H_t$ for some ample/ U $\mathbb{R}$ -divisor H_t for any $t \in (0, 1)$.*
- (2) *If D_1 is semi-ample/ U , then there exists a contraction/ U $g : Z \rightarrow Y$ such that $D_1 = (g \circ f)^* A$ for some ample/ U $\mathbb{R}$ -divisor A .*

Proof. Let $f_t : X \rightarrow Z_t$ be the contraction induced by $tD_1 + (1-t)D_2$, then the curves contracted by f_t are exactly the curves C such that $D_1 \cdot C = 0$ and $D_2 \cdot C = 0$, hence $f_t = f_{t'}$ for any $t, t' \in (0, 1)$. (1) follows by letting $f := f_t$.

Moreover, when D_1 is semi-ample/ U , since all curves C contracted by f_t satisfies that $D_1 \cdot C = 0$, we have $D_1 = f^* D_Z$ for some $\mathbb{R}$ -divisor D_Z on Z . When D_1 is semi-ample/ U , D_Z is semi-ample/ U , so there exists a contraction/ U $g : Z \rightarrow Y$ such that $D_Z = g^* A$ for some ample/ U $\mathbb{R}$ -divisor A . (2) follows. $\square$

Lemma 5.3. *Let $\pi : X \rightarrow U$ be a projective morphism between normal quasi-projective varieties and let D_1, D_2 be two $\mathbb{Q}$ -divisors on X such that $tD_1 + (1-t)D_2$ is semi-ample/ U for some irrational number $t \in (0, 1)$. Let $f : X \rightarrow Z$ be the contraction induced by $tD_1 + (1-t)D_2$. Then for any $s \in (t - \epsilon, t + \epsilon)$,*

$$sD_1 + (1-s)D_2 = f^* H_s$$

for some ample/ U $\mathbb{R}$ -divisor H_s . In particular, $sD_1 + (1-s)D_2$ is semi-ample/ U .

Proof. By [HLS24, Lemma 5.3], $sD_1 + (1-s)D_2 \sim_{\mathbb{R}, Z} 0$ for any $s \in \mathbb{R}$. We have

$$sD_1 + (1-s)D_2 = f^* H_s$$

for some $\mathbb{R}$ -divisor H_s for any $s \in \mathbb{R}$, and H_t is ample/ U . Thus H_s is ample/ U for any $|s - t| \ll 1$. $\square$

5.2. Proof of the main theorems.

Theorem 5.4. *Assume that $\mathfrak{A}(t)/U := (X, \mathcal{F}, B^{\text{inv}} + (1-t)B^{\text{inv}}, \mathbf{M}, t)/U$ is an algebraically integrable adjoint foliated structure for any $t \in [0, 1]$. Let $0 \leq s_0 < s_1 \leq 1$ be two real numbers such that*

- $\mathfrak{A}(t)/U$ is lc and has a bs-semi-ample model for any $t \in [s_0, s_1]$, and
- $\mathfrak{A}(s_1)/U$ has a KNQC bs-weak lc model.

Then there exist finitely many real numbers

$$s_0 =: t_1 < t_2 < \cdots < t_n := s_1$$

satisfying the following. Let $\Gamma := \{\{t_i\}, (t_i, t_{i+1}) \mid 1 \leq i \leq n-1\}$ if $\mathfrak{A}(s_1)/U$ does not have bs-semi-ample model and let $\Gamma := \{\{t_i\}, (t_j, t_{j+1}) \mid 1 \leq i \leq n, 1 \leq j \leq n-1\}$ if $\mathfrak{A}(s_1)/U$ has a bs-semi-ample model.

Then for any $\mathcal{P} \in \Gamma$, there exists a birational map/ U $\phi_{\mathcal{P}} : X \dashrightarrow X_{\mathcal{P}}$ and a rational map/ U $\psi_{\mathcal{P}} : X \dashrightarrow Z_{\mathcal{P}}$ satisfying the following.

- (1) *For any $\mathcal{P} \in \Gamma$ and $t \in \mathcal{P}$, $\phi_{\mathcal{P}}$ is a $\mathbb{Q}$ -factorial bs-good minimal model of $\mathfrak{A}(t)/U$.*
- (2) *For any $\mathcal{P} \in \Gamma$ and $t \in \mathcal{P}$, $\psi_{\mathcal{P}}$ is the ample model/ U of $K_{\mathfrak{A}(t)}$.*
- (3) *For any $\mathcal{P}, \mathcal{Q} \in \Gamma$ such that $\mathcal{Q} \subset \partial \mathcal{P}$, there exist a unique contraction $\mu_{\mathcal{P}, \mathcal{Q}} : Z_{\mathcal{P}} \rightarrow Z_{\mathcal{Q}}$ such that $\mu_{\mathcal{P}, \mathcal{Q}} \circ \psi_{\mathcal{P}} = \psi_{\mathcal{Q}}$.*

Moreover:

- (4) *If X is $\mathbb{Q}$ -factorial klt, then $\phi_{\mathcal{P}}$ does not extract any divisor. In particular, for any $\mathcal{P} \in \Gamma$ and $t \in \mathcal{P}$, $\phi_{\mathcal{P}}$ is a $\mathbb{Q}$ -factorial good minimal model of $\mathfrak{A}(t)/U$.*
- (5) *If B is a $\mathbb{Q}$ -divisor, $\mathbf{M}$ is a $\mathbb{Q}$ - $\mathbf{b}$ -divisor, and s_0, s_1 are rational numbers, then we may choose t_i so that they are rational numbers.*

Proposition 5.5. *Theorem 5.4 holds if $\mathfrak{A}(s_1)/U$ has a bs-semi-ample model.*

Proof. First we prove (1) and reduce (4) to (1).

Pick $t_0 \in (s_0, s_1)$ and let $h : W \rightarrow X$ be a $\mathbb{Q}$ -factorial qdlt modification of $\mathfrak{A}(t_0)$. Then for any prime h -exceptional divisor E , E is an lc place of $\mathfrak{A}(t_0)$, hence an lc place of $\mathfrak{A}(t)$ for any $t \in [0, 1]$ as $\mathfrak{A}(s_0)$ and $\mathfrak{A}(s_1)$ are lc. We let $\mathfrak{A}_W(t) := h^* \mathfrak{A}(t)$ for any $t \in [0, 1]$. By Lemma 3.14, possibly replacing $\mathfrak{A}(t)$ with $\mathfrak{A}_W(t)$ for any t , we may assume that X is $\mathbb{Q}$ -factorial klt.

Assume that for any $t \in [s_0, s_1]$, there exists an open subset $V_t \ni t$ of $[s_0, s_1]$ and rational maps $\phi_t^- : X \dashrightarrow X_t^-$, $\phi_t^+ : X \dashrightarrow X_t^+$, and $\phi_t : X \dashrightarrow X_t$ satisfying the following: For any $s \in V_t$, if $s < t$ (resp. $s = t$, $s > t$), then ϕ_t^- (resp. ϕ_t , ϕ_t^+) is a $\mathbb{Q}$ -factorial good minimal model/ U of $\mathfrak{A}(s)/U$. Then $\{V_t\}_{t \in [s_0, s_1]}$ is an open covering of $[s_0, s_1]$, hence there exist finitely many real numbers

$$s_0 =: t_1 < t_2 < \cdots < t_n := s_1,$$

such that $\{V_{t_i}\}_{1 \leq i \leq n}$ is an open covering of $[s_0, s_1]$, and that $V_{t_i} \not\subset V_{t_j}$ for an $i \neq j$. In particular, $V_{t_i} \cap V_{t_{i+1}} \neq \emptyset$, hence $\phi_{t_i}^+ = \phi_{t_{i+1}}^-$ for any i . These t_i satisfy our requirements. Therefore, we only need to find V_t and $\phi_t^-, \phi_t, \phi_t^+$ as above for any $t \in [s_0, s_1]$.

Fix $t \in [s_0, s_1]$. By Theorem 4.6, we may run a $K_{\mathfrak{A}(t)}$ -MMP/ U with scaling of an ample divisor which terminates with a $\mathbb{Q}$ -factorial good minimal model $\mathfrak{A}'(t)/U$ of $\mathfrak{A}(t)/U$. Let $\phi_t : X \dashrightarrow X' := X_t$ be the induced birational map and let $\mathfrak{A}'(s) := (\phi_t)_* \mathfrak{A}(s)$ for any $s \in [0, 1]$. Then X' is klt, and there exists a contraction/ U $f : X' \rightarrow Z$ and an ample/ U $\mathbb{R}$ -divisor H on Z such that

$$K_{\mathfrak{A}'(t)} = f^* H.$$

There exists $\epsilon_1 > 0$, such that for any $s \in I_1 := [t - \epsilon_1, t + \epsilon_1] \cap [s_0, s_1]$, ϕ_t is also a sequence of steps of a $K_{\mathfrak{A}(s)}$ -MMP/ U , hence ϕ_t is $K_{\mathfrak{A}(s)}$ -negative and $K_{\mathfrak{A}'(s)}$ is lc. By Lemma 3.15, $\mathfrak{A}'(s)/U$ has a good minimal model for any $s \in I_1$. By Lemma 4.3, there exists $\epsilon_2 \in (0, \epsilon_1)$, such that for any $s \in I_2 := [t - \epsilon_2, t + \epsilon_2] \cap [s_0, s_1]$, any sequence of steps of a $K_{\mathfrak{A}'(s)}$ -MMP/ U is $K_{\mathfrak{A}'(t)}$ -trivial, hence a sequence of steps of a $K_{\mathfrak{A}'(s)}$ -MMP/ Z .

Let $t_- := \max\{t - \epsilon_2, s_0\}$ and $t_+ := \min\{t + \epsilon_2, s_1\}$. By Theorem 4.6, we may run a $K_{\mathfrak{A}'(t_-)}$ -MMP/ U (resp. $K_{\mathfrak{A}'(t_+)}$ -MMP/ U) which terminates with a good minimal model $\mathfrak{A}_-(t_-)/U$ (resp. $\mathfrak{A}_+(t_+)/U$) of $\mathfrak{A}'(t_-)/U$ (resp. $\mathfrak{A}'(t_+)/U$) with induced birational map $\phi_t^- : X' \dashrightarrow X_t^-$ (resp. $\phi_t^+ : X' \dashrightarrow X_t^+$). ϕ_t^-, ϕ_t^+ are maps/ Z . Let $\mathfrak{A}_-(s) := (\phi_t^-)_* \mathfrak{A}'(s)$ and $\mathfrak{A}_+(s) := (\phi_t^+)_* \mathfrak{A}'(s)$ for any $s \in [0, 1]$. Then for $s \in [t_-, t]$ (resp. $(t, t_+]$), $K_{\mathfrak{A}'(s)} \sim_{\mathbb{R}, Z} \mu_s K_{\mathfrak{A}_-(t_-)}$ (resp. $\sim_{\mathbb{R}, Z} \mu_s K_{\mathfrak{A}_+(t_+)}$) for some $\mu_s > 0$. Thus ϕ_t^- (resp. ϕ_t^+) is also a sequence of steps of a $K_{\mathfrak{A}_-(s)}$ -MMP/ Z and $K_{\mathfrak{A}_-(s)}$ (resp. $K_{\mathfrak{A}_+(s)}$) is nef/ Z . Since any sequence of steps of a $K_{\mathfrak{A}'(s)}$ -MMP/ U is a sequence of steps of a $K_{\mathfrak{A}'(s)}$ -MMP/ Z , $K_{\mathfrak{A}_-(s)}$ (resp. $K_{\mathfrak{A}_+(s)}$) is nef/ U , so $\mathfrak{A}_-(s)/U$ (resp. $\mathfrak{A}_+(s)/U$) is a minimal model of $\mathfrak{A}'(s)/U$. By Lemma 3.15, $\mathfrak{A}_-(s)/U$ (resp. $\mathfrak{A}_+(s)/U$) is a minimal model of $\mathfrak{A}(s)/U$. By Lemma 3.4, $K_{\mathfrak{A}_-(s)}$ (resp. $K_{\mathfrak{A}_+(s)}$) is semi-ample/ U . We may let $V_t := (t_-, t_+)$ if $t \notin \{s_0, s_1\}$, $V_{s_0} = [s_0, s_{0,+})$, and $V_{s_1} := (s_{1,-}, s_1]$. (1) follows by our construction of $\phi_t^-, \phi_t, \phi_t^+$.

(2) follows from (1), the uniqueness of ample model, and Lemma 5.2(1). (3) follows from Lemma 5.2(2).

Finally, we prove (5). Assume that $t_i \notin \mathbb{Q}$ for some i , then $i \notin \{1, n\}$. Let $\phi_+ := \phi_{(t_i, t_{i+1})}$, $\phi_- := \phi_{(t_{i-1}, t_i)}$, $X^+ := X_{(t_i, t_{i+1})}$, and $X^- := X_{(t_{i-1}, t_i)}$.

By comparing discrepancies and since nef is a closed condition, ϕ_+ and ϕ_- are $\mathbb{Q}$ -factorial bs-weak lc models of $\mathfrak{A}(t_i)/U$. For any $t \in [0, 1]$, write

$$\mathfrak{A}_+(t) := \left((\phi_+)_* \mathfrak{A}(s), \sum_D (t\epsilon_{\mathcal{F}}(D) + (1-t))D \right)$$

where the sum runs through all prime divisors D on X^+ that are exceptional/ X and

$$\mathfrak{A}_-(t) := \left((\phi_-)_* \mathfrak{A}(t), \sum_D (t\epsilon_{\mathcal{F}}(D) + (1-t))D \right)$$

where the sum runs through all prime divisors D on X^- that are exceptional/ X .

For any prime divisor D on X that is exceptional/ X^+ (resp. X^-), we have that $a(D, \mathfrak{A}(t_i)) \leq a(D, \mathfrak{A}_+(t_i))$ (resp. $a(D, \mathfrak{A}(t_i)) \leq a(D, \mathfrak{A}_-(t_i))$). Assume that $a(D, \mathfrak{A}(t_i)) = a(D, \mathfrak{A}_+(t_i))$ (resp. $a(D, \mathfrak{A}(t_i)) = a(D, \mathfrak{A}_-(t_i))$). Since t_i is irrational, we have that $a(D, \mathfrak{A}(t)) = a(D, \mathfrak{A}_+(t))$ (resp. $a(D, \mathfrak{A}(t)) = a(D, \mathfrak{A}_-(t))$) for any $t \in [0, 1]$, which is not possible because $a(D, \mathfrak{A}(t)) < a(D, \mathfrak{A}_+(t))$ (resp. $a(D, \mathfrak{A}(t)) < a(D, \mathfrak{A}_-(t))$) when $t \in (t_i, t_{i+1})$ (resp. $t \in (t_{i-1}, t_i)$). Therefore, $a(D, \mathfrak{A}(t_i)) < a(D, \mathfrak{A}_+(t_i))$ (resp. $a(D, \mathfrak{A}(t_i)) < a(D, \mathfrak{A}_-(t_i))$). Thus ϕ_+ and ϕ_- are both bs-good minimal models of $\mathfrak{A}(t_i)$.

By Lemma 5.3, $K_{\mathfrak{A}_+(t)}$ and $K_{\mathfrak{A}_-(t)}$ are semi-ample/ U for any $|t - t_i| \ll 1$. Since there are only finitely many prime divisors D on X that are exceptional/ X^+ or X^- , there exist rational numbers t_i^-, t_i^+ such that $t_i^- < t_i < t_i^+$, and $\mathfrak{A}_+(t)/U$ and $\mathfrak{A}_-(t)/U$ are $\mathbb{Q}$ -factorial good minimal models of $\mathfrak{A}(t_i)/U$ for any $t \in [t_i^-, t_i^+]$. By Lemma 5.3 again, for any $|t - t_i| \ll 1$, $\mathfrak{A}_+(t)/U$ and $\mathfrak{A}_-(t)/U$ have the same ample model. By the uniqueness of ample models, $\psi_{\{t_i\}} = \psi_{(t_{i-1}, t_i)} = \psi_{(t_i, t_{i+1})}$. Now we may replace Γ by

$$\{\{t_j\}, \{t_i^-\}, \{t_i^+\}, (t_k, t_{k+1}), (t_{i-1}, t_i^-), (t_i^-, t_i^+), (t_i^+, t_{i+1}) \mid j \neq i, k \neq i, i-1\}.$$

let

$$\begin{aligned}\psi_{(t_{i-1}, t_i^-)} &:= \psi_{\{t_i^-\}} := \psi_{(t_i^-, t_i^+)} := \psi_{\{t_i^+\}} := \psi_{(t_i^+, t_{i+1})} := \psi_{t_i}, \\ \phi_{(t_{i-1}, t_i^-)} &:= \phi_{\{t_i^-\}} := \phi_{(t_i^-, t_i^+)} := \phi_{\{t_i^+\}} := \phi_{(t_{i-1}, t_i)},\end{aligned}$$

and

$$\phi_{(t_i^+, t_{i+1})} := \phi_{(t_i, t_{i+1})}.$$

We are done by induction on the number of irrational numbers which appear in $\{t_i\}_{i=1}^n$. $\square$

Lemma 5.6. *Let $\mathfrak{A}(t)/U := (X, \mathcal{F}, B^{\text{inv}} + (1-t)B^{\text{inv}}, \mathbf{M}, t)/U$ be an algebraically integrable adjoint foliated structure for any $t \in [0, 1]$. Assume that*

- (1) $\mathfrak{A}(t_0)/U$ is lc and has a KNQC bs-weak lc model for some $t_0 \in (0, 1]$,
- (2) either $t_0 < 1$, or $\mathfrak{A}(t)$ is lc for some $t \in [0, 1)$, and
- (3) there exists a strictly increasing sequence $\{t_i\}_{i=1}^{+\infty}$ such that $\lim_{i \rightarrow +\infty} t_i = t_0$ and $\mathfrak{A}(t_i)/U$ has a KNQC bs-weak lc model.

Then there exists $\epsilon_0 > 0$ and a birational map $\phi : X \dashrightarrow X'$ satisfying the following: For any $t \in (t_0 - \epsilon_0, t_0)$, ϕ is a $\mathbb{Q}$ -factorial KNQC bs-minimal model of $\mathfrak{A}(t)/U$. Moreover, if X is $\mathbb{Q}$ -factorial klt, then ϕ does not extract any divisor.

Proof. By [Cas+24, Proposition 3.3], $\mathfrak{A}(t)$ is lc for any $t \in [0, t_0]$. Let $h : W \rightarrow X$ be a $\mathbb{Q}$ -factorial qdlt modification of $\mathfrak{A}(t)$ for some $t \in (0, t_0)$. Then any prime h -exceptional divisor D is an lc place of $\mathfrak{A}(t)$, hence an lc place of $\mathfrak{A}(s)$ for any $s \in [0, 1]$ as $\mathfrak{A}(0)$ and $\mathfrak{A}(t_0)$ are lc. We let $\mathfrak{A}_W(t) := h^*\mathfrak{A}(t)$ for any $t \in [0, 1]$. By Lemmas 3.13 and 3.14, we may replace $\mathfrak{A}(t)$ with $\mathfrak{A}_W(t)$ and assume that X is $\mathbb{Q}$ -factorial klt.

By Theorem 4.6, we may run a $K_{\mathfrak{A}(t_0)}$ -MMP/ U which terminates with a $\mathbb{Q}$ -factorial KNQC minimal model $\mathfrak{A}'(t_0)/U$ of $\mathfrak{A}(t_0)/U$ with induced birational map $\phi : X \dashrightarrow X'$. Let $\mathfrak{A}'(t) := \phi_*\mathfrak{A}(t)$ for any $t \in [0, 1]$. Then there exists $t_0 > \epsilon_1 > 0$, such that ϕ is also a sequence of steps of a $K_{\mathfrak{A}(t_0-\epsilon)}$ -MMP/ U for any $\epsilon \in [0, \epsilon_1]$. By Lemma 3.15, any minimal model of $\mathfrak{A}'(t_0 - \epsilon)/U$ is a minimal model of $\mathfrak{A}(t_0 - \epsilon)/U$. By Lemma 4.3, there exists $0 < \epsilon_2 < \epsilon_1$, such that for any $\epsilon \in [0, \epsilon_2]$, any sequence of steps of a $K_{\mathfrak{A}'(t_0-\epsilon)}$ -MMP/ U is $K_{\mathfrak{A}'(t_0)}$ -trivial.

Pick $i \gg 0$ such that $t_i \in (t_0 - \epsilon_2, t_0)$. By Theorem 4.6, we may run a $K_{\mathfrak{A}'(t_i)}$ -MMP/ U which terminates with a $\mathbb{Q}$ -factorial KNQC minimal model $\mathfrak{A}''(t_i)/U$ of $\mathfrak{A}'(t_i)/U$ with induced birational map $\psi : X' \dashrightarrow X''$. Let $\mathfrak{A}''(t) := \psi_*\mathfrak{A}'(t)$ for any $t \in [0, 1]$. Since ψ is $K_{\mathfrak{A}'(t_0)}$ -trivial, ψ is also a sequence of steps of $K_{\mathfrak{A}'(t)}$ -MMP/ U for any $t \in [t_i, t_0]$. Moreover, since $K_{\mathfrak{A}''(t_0)}$ and $K_{\mathfrak{A}''(t_i)}$ are both NQC/ U , $K_{\mathfrak{A}''(t)}$ is NQC/ U for any $t \in [t_i, t_0]$. Thus $\mathfrak{A}''(t)/U$ is a $\mathbb{Q}$ -factorial KNQC minimal model of $\mathfrak{A}'(t)/U$, hence a $\mathbb{Q}$ -factorial KNQC minimal model of $\mathfrak{A}(t)/U$. We may take $\epsilon_0 := t_0 - t_i$. $\square$

Proof of Theorem 5.4. By Proposition 5.5, we may assume that $\mathfrak{A}(s_1)/U$ does not have a bs-semi-ample model.

(1) By Lemmas 5.6 and 3.4, there exists $s_1 - s_0 > \epsilon > 0$ and a birational map $\phi : X \dashrightarrow X'$, such that $s_1 - \epsilon$ is rational, and for any $t \in (s_1 - \epsilon, s_1)$, ϕ is a $\mathbb{Q}$ -factorial bs-good minimal model of $\mathfrak{A}(t)/U$. By Lemma 5.2(1), there exists a contraction $f : X' \rightarrow Z$ such that the induced birational map $\psi : f \circ \phi$ is the ample model/ U of $K_{\mathfrak{A}(t)}$.

Since $\mathfrak{A}(s_1 - \epsilon_0)/U$ has a bs-semi-ample model, by Proposition 5.5, there exist finitely many real numbers

$$s_0 =: t_1 < t_2 < \cdots < t_{n-1} =: s_1 - \epsilon$$

satisfying the following. Let $\Gamma' := \{\{t_i\}, (t_j, t_{j+1}) \mid 1 \leq i \leq n, 1 \leq j \leq n-1\}$. Then for any $\mathcal{P} \in \Gamma'$, there exists a birational map/ U $\phi_{\mathcal{P}} : X \dashrightarrow X_{\mathcal{P}}$, and a rational map/ U $\psi_{\mathcal{P}} : X \dashrightarrow Z_{\mathcal{P}}$ satisfying the following.

- For any $t \in \mathcal{P}$, $\phi_{\mathcal{P}}$ is a $\mathbb{Q}$ -factorial bs-good minimal model of $\mathfrak{A}(t)/U$.
- For any $\mathcal{P} \in \Gamma'$ and $t \in \mathcal{P}$, $\psi_{\mathcal{P}}$ is the ample model of $\mathfrak{A}(t)/U$.
- For any $\mathcal{P}, \mathcal{Q} \in \Gamma'$ such that $\mathcal{Q} \subset \partial\mathcal{P}$, there exist a unique contraction $\mu_{\mathcal{P}, \mathcal{Q}} : Z_{\mathcal{P}} \rightarrow Z_{\mathcal{Q}}$ such that $\mu_{\mathcal{P}, \mathcal{Q}} \circ \psi_{\mathcal{P}} = \psi_{\mathcal{Q}}$.
- If X is $\mathbb{Q}$ -factorial klt, then $\phi_{\mathcal{P}}$ does not extract any divisor. In particular, for any $\mathcal{P} \in \Gamma'$ and $t \in \mathcal{P}$, $\phi_{\mathcal{P}}$ is a $\mathbb{Q}$ -factorial good minimal model of $\mathfrak{A}(t)/U$.
- If B is a $\mathbb{Q}$ -divisor, $\mathbf{M}$ is a $\mathbb{Q}$ - $\mathbf{b}$ -divisor, and s_0 is a rational number, then we may choose t_i so that they are rational numbers.

We may let $t_n := s_1$, $\Gamma := \Gamma' \cup \{(s_1 - \epsilon, s_1)\}$, $\phi_{(s_1 - \epsilon, s_1)} := \phi$, and $\psi_{(s_1 - \epsilon, s_1)} := \psi$. Then (1)(2)(4)(5) follow. By comparing discrepancies and since nef/ U is a closed condition, ϕ is a $\mathbb{Q}$ -factorial bs-semi-ample model of $\mathfrak{A}(s_1 - \epsilon)/U$. Thus (3) follows from Lemma 5.2. $\square$

Proof of Theorem 1.1. Since X is klt, possibly replacing X with a small $\mathbb{Q}$ -factorialization, we may assume that X is $\mathbb{Q}$ -factorial. Since $\mathfrak{A}_{\lambda}$ is of general type, by [Cas+25, Theorem 5.1], $K_{\mathcal{F}}$ is pseudo-effective. Let $h : (X', \mathcal{F}', B') \rightarrow (X, \mathcal{F}, 0)$ be a $\mathbb{Q}$ -factorial ACSS modification of $\mathcal{F}$ such that $(X', \mathcal{F}', B'; G)/Z$ is ACSS associated with contraction $f : X' \rightarrow Z$. Since $\mathcal{F}$ is canonical, h is an isomorphism over the generic point of Z and $B' = 0$. Let F be a general fiber of f , then F is klt and $K_F = h^*K_{\mathcal{F}}|_F = h^*K_X|_F$ is big, so F has a good minimal model. By [HJLL24, Theorem 3.15], $\mathcal{F}'$ has a minimal model/ Z . By the cone theorem [ACSS21, Theorem 3.9], $\mathcal{F}'$ has a minimal model. By Lemma 3.15, $\mathcal{F}$ has a minimal model, which is KNQC as $K_{\mathcal{F}}$ is a $\mathbb{Q}$ -divisor.

Pick $0 < \delta \ll 1$ such that $\lambda - \delta \in \mathbb{Q}$ and let $\epsilon := \max\{0, \lambda - \delta\}$. Then $\mathfrak{A}_t$ is klt and of general type for any $t \in [\epsilon, 1)$, hence $(X, \mathcal{F}, t)$ has a good minimal model for any $t \in [\epsilon, 1)$ by [Cas+25, Theorem 2.1.1]. The theorem follows from Theorem 5.4. $\square$

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DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, 180 QUEEN’S GATE, LONDON SW7 2AZ, UK

Email address: p.cascini@imperial.ac.uk

DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, NO. 5 YIHEYUAN ROAD, HAIDIAN DISTRICT, PEKING 100871, CHINA

Email address: liujiahao@math.pku.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SAN DIEGO, 9500 GILMAN DRIVE # 0112, LA JOLLA, CA 92093-0112, USA

Email address: f2meng@ucsd.edu

DIPARTIMENTO DI MATEMATICA “F. ENRIQUES”, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA SALDINI 50, 20133 MILANO (MI), ITALY

Email address: roberto.svaldi@unimi.it

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SAN DIEGO, 9500 GILMAN DRIVE # 0112, LA JOLLA, CA 92093-0112, USA

Email address: l6xie@ucsd.edu