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Perturbative unitarity for models with singlet and doublet scalars

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We provide a complete description of unitarity bounds on the gauge-scalar sectors of models with extra $SU(2)$ doublet, neutral singlet, and charged singlet scalars. Such additions are very frequent in models beyond the Standard Model, and, in particular, they are almost universal in models explaining the dark matter problem. We propose a specific classification and minimal set of scattering matrices containing all the relevant information. We also developed a Mathematica implementation of our results, **BounDS**, and we use it to fully study a number of simple cases, comparing with the literature, when available. The Mathematica notebook **BounDS** is provided via a public GitHub repository.

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I. INTRODUCTION

The Standard Model (SM) of particle physics has been extremely successful, culminating in the discovery of a scalar particle consistent with the Higgs boson [1, 2]. Nevertheless, it is not a complete theory, and extensions beyond the Standard Model (BSM) are required. In particular, the SM provides no explanation for dark matter (DM) [3], whose presence is inferred across a wide range of astronomical scales, from kiloparsecs to the size of the observable Universe. Key probes include the rotation curves of spiral galaxies, the study of galaxy clusters, and measurements of the Cosmic Microwave Background [4, 5].

One of the most widely studied scenarios envisions dark matter as consisting of new fundamental particles. The simplest scalar extension introduces a real, neutral $SU(2)$ singlet with a global $\mathbb{Z}_2$ symmetry to ensure stability [6, 7]. Alternatively, a minimal fermionic option is a Weyl fermion, commonly referred to as a sterile neutrino [5]. Other proposed models include the Inert Doublet Model, which extends the Higgs sector with a dark doublet [8]; multi-singlet SM extensions [9]; the Next-to-Minimal Supersymmetric Standard Model [10]; and models featuring scalar or fermionic portals to hidden sectors [11]. Most of these models include extra scalar fields, doublet or singlet under the SM gauge group, and often new discrete symmetries in order to stabilize the dark sector. In exploring such theories, one must start with the theoretical constraints, such as guaranty of a bounded from below potential, assurance of a global minimum, and perturbative unitarity.

In any consistent quantum field theory, probability conservation is encoded in the unitarity of the S -matrix [12, 13]. In scalar extensions motivated by dark matter, this requirement becomes particularly relevant: new scalar fields can induce scattering amplitudes that grow with energy, and enforcing unitarity imposes bounds on their masses and couplings. For $2 \rightarrow 2$ scattering processes, unitarity constraints can be applied to the partial-wave decomposition of the amplitudes, with the strongest high-energy limits coming from the zeroth partial-wave amplitude [12, 13]. In the high-energy limit, the Equivalence Theorem [14] ensures that the scattering of longitudinally polarized gauge bosons can be replaced by the corresponding Goldstone bosons, allowing these scalar amplitudes to capture the dominant contributions to unitarity bounds. Historically, Lee, Quigg, and Thacker applied this method to derive an upper bound on the Higgs boson mass, $M_H^2 \leq 8\pi\sqrt{2}/(3G_F) \approx 1\text{TeV}^2$ [15, 16]. Unitarity has also been applied to the general two-Higgs-doublet model (2HDM) with explicit CP violation [17, 18], to the general N -Higgs-doublet model [19], also with fermions [20], and to constrain large scalar multiplets [21, 22]. More recently, perturbative unitarity analyses have been extended beyond the traditional $2 \rightarrow 2$ limit. For instance, Ref. [23] studies unitarity in general $M \rightarrow N$ scattering processes with $M, N \geq 2$. In particular, in inflationary models featuring a non-minimal coupling with the Ricci scalar, partial-wave unitarity bounds on $2 \rightarrow N$ processes have improved earlier estimates of the energy scale at which new-physics effective interactions invalidate the perturbative expansion [24].

In this work, we extend the analysis of Ref. [19] by constructing an $SU(2) \times U(1)$ -symmetric model with an arbitrary number of scalar fields (doublets, neutral singlets and charged singlets) and by imposing perturbative unitarity constraints in the high-energy limit. Using partial-wave unitarity for tree-level $2 \rightarrow 2$ scattering, we derive bounds on the quartic couplings.

In Section II, we present the most general renormalizable $SU(2) \times U(1)$ -symmetric model in the high-energy limit, composed of an arbitrary number of $SU(2)$ scalar doublets with hypercharge $\frac{1}{2}$ and singlets that are either neutral or carry hypercharge 1. We derive constraints on the Lagrangian couplings imposed by hermiticity and identify additional symmetry relations arising from the interchange of field indices. In Section III, we construct the scattering matrix from second derivatives of the quartic potential, diagonalize it to obtain eigenamplitudes, and impose perturbative unitarity, $|\Lambda| \leq 8\pi$, to constrain the quartic couplings. We propose to classify states by the conserved quantum numbers $|Q, Y, T\rangle$, corresponding to electric charge, hypercharge, and total isospin, respectively. In Appendix A, we illustrate the benefits of labeling states not only by Q and Y , but also by total isospin T , using the Standard Model as a case study in Appendix B. This classification facilitates the construction of scattering matrices and the identification of independent channels, for which we present a minimal basis (meaning that we discuss the selection of basis states that allows all independent eigenvalues to be determined). We provide explicit formulas for the potential and the matrix elements of all scattering processes allowed in our class of models. In Section IV, we present the *Mathematica* notebook **BoundS** that allows for the automatic calculation of the quartic part of the scalar potential and of all scattering matrices in models of this type; the user must only input the particle content, and, if needed,

the transformation properties of the fields under additional flavour and/or generalized CP symmetries¹. We make the notebook publicly available at

<https://github.com/andremlagre/BounDS.git>

In Section V, we apply these methods to particular models by specifying the scalar content and symmetries, and we compare with results in the literature, when available. We present our results in terms of the parameters of the potential. In specific models, it is sometimes possible to write the parameters of the potential in terms of scalar masses and/or mixing angles. That involves deciding on terms of dimension two and/or three, defining the vacuum of the theory, writing all (neutral and charged) scalar mass matrices, diagonalizing them in order to obtain masses and mixing angles in terms of potential parameters, and then inverting those relations. This must be done on a case-by-case basis, and lies beyond the scope of this work. We present our conclusions in Section VI.

In Appendix C we provide further details on our *Mathematica* notebook. Finally, Appendix D addresses the inclusion in the basis of additional quantum numbers, when extra flavour symmetries, such as $\mathbb{Z}_n$ or CP, are present.

II. THE MODEL

A. Particle Content

Consider a gauge theory symmetric under the $SU(2) \times U(1)$ group with an arbitrary number of scalar fields (and their charge conjugates) that are either doublets or singlets of $SU(2)$. We define the electric charge Q as the sum of the third component of isospin T_3 , and the hypercharge Y , and restrict our discussion to models where the electric charge of the scalar particles is either singly charged² or neutral. As a consequence, this theory may have n_D $SU(2)$ scalar doublets with $Y = 1/2$ which we denote by

$$\Phi_i = (\phi_i^+, \phi_i^0)^T, \quad i = 1, \dots, n_D, \quad (1)$$

n_c $SU(2)$ complex scalar singlets with $Y = 1$ which we denote by

$$\varphi_i^+, \quad i = 1, \dots, n_c, \quad (2)$$

and n_n $SU(2)$ real scalar singlets with $Y = 0$ which we denote by

$$\chi_i, \quad i = 1, \dots, n_n. \quad (3)$$

Notice that models with m $SU(2)$ complex scalar singlets with $Y = 0$ are particular cases of models with $n_n = 2m$ $SU(2)$ real scalar singlets with $Y = 0$; hence, we do not treat them separately. The oblique radiative corrections for this class of models has been thoroughly studied in Refs. [25, 26].

B. Scalar potential

In this work, we are interested in constraining the parameters of such models by imposing partial wave unitarity bounds on $2 \rightarrow 2$ scattering processes. We perform all calculations in the high-energy limit where all contributions from propagators vanish, deeming the quartic interactions in the potential the dominant contributions. The most general renormalizable quartic part of the scalar potential of such a theory can be written as:³

$$V \supset V_4 = \lambda_{ab,cd} (\Phi_a^\dagger \Phi_b) (\Phi_c^\dagger \Phi_d) + \alpha_{ab,cd} (\varphi_a^- \varphi_b^+) (\varphi_c^- \varphi_d^+) + \beta_{ab,cd} (\chi_a \chi_b) (\chi_c \chi_d) +$$

¹ These can either be discrete or continuous, abelian or non-abelian.

² In units of the electron charge e .

³ For the simplest model including the $\kappa_{ab,cd}$ terms, see Sec. VH below.

$$\begin{aligned} & \delta_{ab,cd} (\Phi_a^\dagger \Phi_b) (\varphi_c^- \varphi_d^+) + \gamma_{ab,cd} (\Phi_a^\dagger \Phi_b) (\chi_c \chi_d) + \zeta_{ab,cd} (\varphi_a^- \varphi_b^+) (\chi_c \chi_d) + \\ & \kappa_{ab,cd} (\Phi_a^T \sigma_2 \Phi_b) (\varphi_c^- \chi_d) + \kappa_{ab,cd}^* (\Phi_b^\dagger \sigma_2 \Phi_a^*) (\varphi_c^+ \chi_d), \end{aligned} \quad (4)$$

where σ_2 is the 2×2 second Pauli matrix, and a sum over repeated indices is implied.

Although compact, not all couplings in this notation are independent. By swapping fields and requiring the scalar potential to be hermitian, we derive the following relations between quartic couplings:

$$\lambda_{ab,cd} = \lambda_{ba,dc}^* = \lambda_{cd,ab}, \quad (5)$$

$$\alpha_{ab,cd} = \alpha_{ba,dc}^* = \alpha_{cd,ab} = \alpha_{ad,cb}, \quad (6)$$

$$\beta_{ab,cd} = \beta_{(ab,cd)}^* = \beta_{(ab,cd)}, \quad (7)$$

$$\delta_{ab,cd} = \delta_{ba,dc}^* \quad (8)$$

$$\gamma_{ab,cd} = \gamma_{ba,cd}^* = \gamma_{ab,dc}, \quad (9)$$

$$\zeta_{ab,cd} = \zeta_{ba,cd}^* = \zeta_{ab,dc}, \quad (10)$$

$$\kappa_{ab,cd} = -\kappa_{ba,cd}, \quad (11)$$

where (ab, cd) stands for any permutation of the indices. In particular, these relations further imply

$$\lambda_{aa,bb}, \lambda_{ab,ba} \in \mathbb{R}, \quad (12)$$

$$\alpha_{aa,bb} = \alpha_{ab,ba} \in \mathbb{R}, \quad (13)$$

$$\beta_{ab,cd} \in \mathbb{R}, \quad (14)$$

$$\delta_{aa,bb} \in \mathbb{R}, \quad (15)$$

$$\gamma_{aa,cd} \in \mathbb{R}, \quad (16)$$

$$\zeta_{aa,cd} \in \mathbb{R}, \quad (17)$$

$$\kappa_{aa,cd} = 0. \quad (18)$$

III. PARTIAL WAVE UNITARITY BOUNDS

A. Partial wave decomposition

We consider $2 \rightarrow 2$ scattering processes between the scalars of the theory defined in Section II. Let A, B, C , and D be complex scalar fields and let a, b, c , and d be their corresponding flavour indices. At tree-level, the amplitude for the process

$$A_a B_b \rightarrow C_c D_d \quad (19)$$

may have contributions from s , t , and u -channels, and contact interactions. However, as we take the limit where the Mandelstam variables s , t , and u go to infinity, the contributions from s , t , and u -channels vanish, respectively⁴. Consequently, in the high-energy limit, only the quartic interactions involving the external scalars contribute to the tree-level amplitude, allowing us to write it as

$$\mathcal{M}[A_a B_b \rightarrow C_c D_d] = -\frac{\partial^4 V_4}{\partial A_a \partial B_b \partial C_c^* \partial D_d^*}. \quad (20)$$

Any square-integrable function may be expressed as an expansion in a complete set of basis functions. To study perturbative unitarity, it is particularly convenient to choose the basis of Legendre polynomials $P_J(\cos \theta)$, where J is

⁴ For an interesting example see, for example, Appendix A of [27].

the total orbital angular momentum of the final state, and θ the scattering angle. Expanding the amplitude in this basis defines the so-called partial-wave expansion [21, 22]:

$$\mathcal{M}(\cos \theta) = 16\pi \sum_{J=0}^{\infty} a_J (2J+1) P_J(\cos \theta). \quad (21)$$

The numerical coefficients a_J (also known as partial waves) are defined as

$$a_J = \frac{2J+1}{32\pi} \int_{-1}^1 \mathcal{M}(\cos \theta) P_J(\cos \theta) d\cos \theta, \quad (22)$$

and are constrained by tree-level partial-wave unitarity as [21, 22]

$$|a_J| \leq 1, \quad \text{and} \quad |\text{Re}\{a_J\}| \leq \frac{1}{2}. \quad (23)$$

Using the fact that, in the high-energy limit, the amplitude is independent of the scattering angle, and that $P_0(\cos \theta) = 1$, the most stringent partial-wave unitarity bound arises by applying the second condition of Eq. (23) to the zeroth partial wave a_0 . Thus, using Eq. (20), all quartic couplings must satisfy

$$16\pi |a_0| = \left| N_{ab} N_{cd} \frac{\partial^4 V_4}{\partial A_a \partial B_b \partial C_c^* \partial D_d^*} \right| \leq 8\pi, \quad (24)$$

where

$$N_{ij} \equiv \frac{1}{\sqrt{2^{\delta_{ij}}}} \quad (25)$$

is a symmetry factor that accounts for identical particles either in the initial or final state.

B. Scattering matrices

One may go a step further by employing the method of coupled-channel analysis [13, 17, 18]. This approach takes advantage of the fact that partial wave unitarity bounds may be imposed on any specific process, but also to any superposition of states, provided they have the same quantum numbers. By organizing the zeroth partial waves of such scatterings into a coupled-channel matrix, the unitarity condition translates into a bound on its eigenvalues. The most stringent constraint is obtained by requiring that the modulus of the largest eigenvalue to remain below 8π . Once again, note that $J = 0$ partial waves should be multiplied by a $1/\sqrt{2}$ for every pair of identical particles in the initial or final state.

We can classify two-particle initial and final states according to their total electric charge Q and hypercharge Y , following the approach in [19]. In addition to Q and Y , any scattering process involving $SU(2)$ doublets must also conserve total isospin, T . Following the results of Appendix A, we list in Table I the most minimal set of independent two-particle states, labeled by the quantum numbers $|Q, Y, T\rangle$. Throughout we use the notation

$$\phi_{[i}^+ \phi_{j]}^0 \equiv \frac{1}{\sqrt{2}} (\phi_i^+ \phi_j^0 - \phi_j^+ \phi_i^0), \quad (26)$$

$$\Phi_i \Phi_j^* \equiv \frac{1}{\sqrt{2}} (\phi_i^+ \phi_j^- + \phi_i^0 \phi_j^{0*}). \quad (27)$$

Looking at Table I, we notice that the non-redundant sets of states could be classified exclusively in terms of Y and T , as is done for the 2HDM in [17]. On the other hand, Ref. [19] advocated Q and Y when considering all states, since then, as can be seen in Table IX of Appendix A, there are states with different Q for the same (Y, T) . We find it best to make the bridge and consider $|Q, Y, T\rangle$.

Using Eqs. (4)–(18), and (24), the scattering matrix elements for the relevant processes read

$$16\pi a_0 [\varphi_a^+ \varphi_b^+ \rightarrow \varphi_c^+ \varphi_d^+] = 4 N_{ab} N_{cd} \alpha_{ca,db}, \quad (28)$$

$ Q, Y, T\rangle$	State	Conditions	Dimensionality
$ 2, 2, 0\rangle$	$\varphi_i^+ \varphi_j^+$	$i \leq j$	$n_c(n_c + 1)/2$
$ 2, \frac{3}{2}, \frac{1}{2}\rangle$	$\phi_i^+ \varphi_j^+$	—	$n_c n_D$
$ 2, 1, 1\rangle$	$\phi_i^+ \phi_j^+$	$i \leq j$	$n_D(n_D + 1)/2$
$ 1, 1, 0\rangle$	$\{\phi_i^+ \phi_j^0, \varphi_i^+ \chi_j\}$	$\{i < j, \text{—}\}$	$n_D(n_D - 1)/2 + n_c n_n$
$ 1, \frac{1}{2}, \frac{1}{2}\rangle$	$\{\phi_i^+ \chi_j, \phi_i^{0*} \varphi_j^+\}$	—	$n_D(n_c + n_n)$
$ 1, 0, 1\rangle$	$\phi_i^+ \phi_j^{0*}$	—	n_D^2
$ 0, 0, 0\rangle$	$\{\Phi_i \Phi_j^*, \varphi_i^+ \varphi_j^-, \chi_i \chi_j\}$	$\{\text{—}, \text{—}, i \leq j\}$	$n_n(n_n + 1)/2 + n_D^2 + n_c^2$

TABLE I: Basis of two-particle states labeled by $|Q, Y, T\rangle$. This table had been reduced by including only non-redundant sets of states; see Appendix A for more details.

$$16\pi a_0 [\phi_a^+ \varphi_b^+ \rightarrow \phi_c^+ \varphi_d^+] = \delta_{ca,db}, \quad (29)$$

$$16\pi a_0 [\phi_a^+ \phi_b^+ \rightarrow \phi_c^+ \phi_d^+] = N_{ab} N_{cd} (2\lambda_{ca,db} + 2\lambda_{da,cb}), \quad (30)$$

$$16\pi a_0 [\phi_a^+ \phi_b^0 \rightarrow \phi_c^+ \phi_d^0] = (2\lambda_{ca,db} - 2\lambda_{da,cb}), \quad (31)$$

$$16\pi a_0 [\phi_a^+ \phi_b^0 \rightarrow \varphi_c^+ \chi_d] = 2\sqrt{2} i \kappa_{ba,cd}, \quad (32)$$

$$16\pi a_0 [\varphi_a^+ \chi_b \rightarrow \varphi_c^+ \chi_d] = 2\zeta_{ca,bd}, \quad (33)$$

$$16\pi a_0 [\phi_a^+ \chi_b \rightarrow \phi_c^+ \chi_d] = 2\gamma_{ca,bd}, \quad (34)$$

$$16\pi a_0 [\phi_a^+ \chi_b \rightarrow \phi_c^{0*} \varphi_d^+] = 2i\kappa_{ca,db}, \quad (35)$$

$$16\pi a_0 [\phi_a^{0*} \varphi_b^+ \rightarrow \phi_c^{0*} \varphi_d^+] = \delta_{ac,db}, \quad (36)$$

$$16\pi a_0 [\phi_a^+ \phi_b^{0*} \rightarrow \phi_c^+ \phi_d^{0*}] = 2\lambda_{ca,bd}, \quad (37)$$

$$16\pi a_0 [\Phi_a \Phi_b^* \rightarrow \Phi_c \Phi_d^*] = 4\lambda_{ba,cd} + 2\lambda_{ca,bd}, \quad (38)$$

$$16\pi a_0 [\Phi_a \Phi_b^* \rightarrow \varphi_c^+ \varphi_d^-] = \sqrt{2}\delta_{ba,cd}, \quad (39)$$

$$16\pi a_0 [\Phi_a \Phi_b^* \rightarrow \chi_c \chi_d] = 2\sqrt{2} N_{cd} \gamma_{ba,cd}, \quad (40)$$

$$16\pi a_0 [\varphi_a^+ \varphi_b^- \rightarrow \varphi_c^+ \varphi_d^-] = 4\alpha_{ba,cd}, \quad (41)$$

$$16\pi a_0 [\varphi_a^+ \varphi_b^- \rightarrow \chi_c \chi_d] = 2 N_{cd} \zeta_{ba,cd}, \quad (42)$$

$$16\pi a_0 [\chi_a \chi_b \rightarrow \chi_c \chi_d] = 24 N_{ab} N_{cd} \beta_{ab,cd}. \quad (43)$$

Below, we study specific models, showing the scattering matrices $M_{|Q,Y,T\rangle}$ and, when an analytical expression is possible, the corresponding eigenvalues. Note that in Ref. [28] a method is proposed based on principal minors, that forgoes diagonalization. In specific numerical simulations, this is likely to be computationally preferable in all models dealing with large scattering matrices.

IV. MATHEMATICA NOTEBOOK

Deriving partial-wave unitarity bounds for different models is a rather repetitive task. As stated, for a model specified by definite values of n_D , n_c , and n_n , the procedure requires assembling a scattering matrix for each state $|Q, Y, T\rangle$ listed in Table I, computing the corresponding eigenvalues, and verifying that they remain below 8π . In models with additional flavour symmetries, all quartic terms forbidden by those symmetries must also be set to zero.

To optimize this process and reduce the burden on the high-energy physics phenomenology community, we have developed the **Mathematica** tool **BoundS** that automates the calculation. The user simply has to specify the values of n_D , n_c , and n_n , and, if needed, the transformation properties of the fields under additional flavour and/or generalized CP symmetries. **BoundS** carries out the necessary steps to:

- Compute the set of all linearly independent quartic couplings allowed by the symmetries.
- Calculate the 7 independent scattering matrices $M_{|Q,Y,T\rangle}$.
- Block-diagonalize the scattering matrices by swapping rows and columns.
- Output the quartic part of the scalar potential and scattering matrices in L^AT_EX form.
- Output closed expressions for the eigenvalues of the scattering matrices, when possible.

Additional details can be found in Appendix C and the notebook **BounDS** can be downloaded from:

<https://github.com/andremlagre/BounDS.git>

The results of the next section are derived from this notebook.

V. UNITARITY BOUNDS FOR PARTICULAR CASES

We now proceed to study unitarity bounds in specific scenarios by specifying the values of n_D , n_c , and n_n , as well as any additional symmetries. The cases studied explicitly in this work are summarized in Table II.

n_D	n_n	n_c	Symmetries	Ref.
1	0	0	—	[16]
1	1	0	—	
2	0	0	$\mathbb{Z}_2$	[17]
2	0	0	—	[17]
1	2	0	—	[29] generalized
2	1	0	$\mathbb{Z}_2$	
2	2	0	$\mathbb{Z}_2 \otimes \mathbb{Z}'_2$	[30]
2	1	1	—	
3	0	0	$\mathbb{Z}_3$	[19]

TABLE II: Models presented as examples in this article.

A. The Standard Model

In the Standard Model, the scalar sector consists of a single $SU(2)$ doublet Φ_1 . Consequently, the quartic part of the scalar potential simply reads

$$V_4 = \lambda_{11,11} \left(\Phi_1^\dagger \Phi_1 \right)^2. \quad (44)$$

1. Scattering Matrices

For this minimal scalar content, there are only three non-zero scattering matrices. These matrices have rank-1 and take the following form:

$$M_{|1,0,1\rangle} = M_{|2,1,1\rangle} = 2\lambda_{11,11}, \quad (45)$$

$$M_{|0,0,0\rangle} = 6\lambda_{11,11}. \quad (46)$$

2. Unitarity Bounds

Applying the partial wave unitarity condition of Eq. (24) to the eigenvalues of these scattering matrices, we find

$$|2\lambda_{11,11}| \leq 8\pi, \quad |6\lambda_{11,11}| \leq 8\pi \quad \implies \quad \lambda_{11,11} \leq \frac{4\pi}{3}. \quad (47)$$

In order to facilitate the comparison of our results with those of Ref. [19], it is useful to provide a short *dictionary* of notations. Table III lists the correspondence between the coupling employed in this work and the one used in Ref. [19]. This recovers the classic results of [16].

Term	Our Notation	Notation in [19]
$(\Phi_1^\dagger \Phi_1)^2$	$\lambda_{11,11}$	λ

TABLE III: Comparison of coupling notation.

B. 1 Scalar Doublet and 1 Neutral Scalar Singlet

We now consider an extension of the SM where, besides the $SU(2)$ scalar doublet Φ_1 , we add one neutral scalar singlet χ_1 . In this case, the quartic part of the scalar potential takes the form

$$V_4 = \lambda_{11,11} \left(\Phi_1^\dagger \Phi_1 \right)^2 + \beta_{11,11} \chi_1^4 + \gamma_{11,11} \left(\Phi_1^\dagger \Phi_1 \right) \chi_1^2. \quad (48)$$

1. Scattering Matrices

The set of non-zero scattering matrices for this model are given by:

$$M_{|1,0,1\rangle} = M_{|2,1,1\rangle} = 2\lambda_{11,11}, \quad (49)$$

$$M_{|1,\frac{1}{2},\frac{1}{2}\rangle} = 2\gamma_{11,11}, \quad (50)$$

$$M_{|0,0,0\rangle} = \begin{bmatrix} 6\lambda_{11,11} & 2\sqrt{2}\gamma_{11,11} \\ 2\gamma_{11,11} & 12\beta_{11,11} \end{bmatrix}. \quad (51)$$

2. Unitarity Bounds

We apply the partial wave unitarity condition of Eq. (24) to each of the eigenvalues of the zero partial wave amplitude matrix and find

$$\left| 6\beta_{11,11} + 3\lambda_{11,11} \pm \sqrt{9(\lambda_{11,11} - 2\beta_{11,11})^2 + 4\gamma_{11,11}^2} \right| \leq 8\pi, \quad (52)$$

$$|2\lambda_{11,11}| \leq 8\pi, \quad (53)$$

$$|2\gamma_{11,11}| \leq 8\pi. \quad (54)$$

C. The $\mathbb{Z}_2$ -Symmetric 2HDM

We now consider a 2HDM model with a discrete $\mathbb{Z}_2$ symmetry that acts on the two scalar $SU(2)$ doublets, Φ_1 and Φ_2 , as

$$\Phi_1 \rightarrow \Phi_1, \quad \Phi_2 \rightarrow -\Phi_2. \quad (55)$$

This case illustrates the use of our `Mathematica` program in the presence of symmetries. Under this symmetry, the quartic part of the scalar potential reads

$$V_4^{Z_2} = \lambda_{11,11} (\Phi_1^\dagger \Phi_1)^2 + \lambda_{22,22} (\Phi_2^\dagger \Phi_2)^2 + 2\lambda_{11,22} (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + 2\lambda_{12,21} (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) \quad (56)$$

$$+ \lambda_{12,12} (\Phi_1^\dagger \Phi_2)^2 + \lambda_{12,12}^* (\Phi_2^\dagger \Phi_1)^2 \quad (57)$$

$$= \frac{\lambda_1}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2} (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) + \frac{\lambda_5}{2} \left[(\Phi_1^\dagger \Phi_2)^2 + \text{h.c.} \right], \quad (58)$$

where, in Eq. (58) we have employed the standard notation of Ref. [17] and, without loss of generality, choose λ_5 to be real. The mapping between the two notations is given in Table IV.

Term	Our Notation	Notation in [17]
$(\Phi_1^\dagger \Phi_1)^2$	$\lambda_{11,11}$	$\frac{\lambda_1}{2}$
$(\Phi_2^\dagger \Phi_2)^2$	$\lambda_{22,22}$	$\frac{\lambda_2}{2}$
$(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2)$	$2\lambda_{11,22}$	λ_3
$(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1)$	$2\lambda_{12,21}$	λ_4
$(\Phi_1^\dagger \Phi_2)^2$	$\lambda_{12,12}$	$\frac{\lambda_5}{2}$
$(\Phi_2^\dagger \Phi_1)^2$	$\lambda_{12,12}^*$	$\frac{\lambda_5}{2}$

TABLE IV: Comparison of coupling notation.

1. Scattering Matrices

The set of non-zero scattering matrices for this model reads

$$M_{|2,1,1\rangle} = \begin{bmatrix} \lambda_1 & \lambda_5 & 0 \\ \lambda_5 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 + \lambda_4 \end{bmatrix}, \quad (59)$$

$$M_{|1,0,1\rangle} = \begin{bmatrix} \lambda_1 & \lambda_4 & 0 & 0 \\ \lambda_4 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_5 \\ 0 & 0 & \lambda_5 & \lambda_3 \end{bmatrix}, \quad (60)$$

$$M_{|0,0,0\rangle} = \begin{bmatrix} 3\lambda_1 & 2\lambda_3 + \lambda_4 & 0 & 0 \\ 2\lambda_3 + \lambda_4 & 3\lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 + 2\lambda_4 & 3\lambda_5 \\ 0 & 0 & 3\lambda_5 & \lambda_3 + 2\lambda_4 \end{bmatrix}, \quad (61)$$

$$M_{|1,1,0\rangle} = \lambda_3 - \lambda_4. \quad (62)$$

2. Eigenvalues

By imposing the unitarity condition of Eq. (24) to each of the eigenvalues of the zero partial wave amplitude matrix we derive

$$|\lambda_3 \pm \lambda_4| \leq 8\pi, \quad (63)$$

$$|\lambda_3 \pm \lambda_5| \leq 8\pi, \quad (64)$$

$$|\lambda_3 + 2\lambda_4 \pm 3\lambda_5| \leq 8\pi, \quad (65)$$

$$\frac{1}{2} \left| \lambda_1 + \lambda_2 \pm \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_4^2} \right| \leq 8\pi, \quad (66)$$

$$\frac{1}{2} \left| \lambda_1 + \lambda_2 \pm \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_5^2} \right| \leq 8\pi, \quad (67)$$

$$\frac{1}{2} \left| 3\lambda_1 + 3\lambda_2 \pm \sqrt{9(\lambda_1 - \lambda_2)^2 + 4(2\lambda_3 + \lambda_4)^2} \right| \leq 8\pi. \quad (68)$$

We find that our results are in agreement with those derived in [17].

D. The general 2HDM

In this case, we consider the most general scalar potential involving two $SU(2)$ doublets, Φ_1 and Φ_2 . The quartic part of the potential is given by:

$$V_4 = \lambda_{11,11} (\Phi_1^\dagger \Phi_1)^2 + \lambda_{22,22} (\Phi_2^\dagger \Phi_2)^2 + 2\lambda_{11,22} (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + 2\lambda_{12,21} (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) + \left[\lambda_{12,12} (\Phi_1^\dagger \Phi_2)^2 + 2\lambda_{11,12} (\Phi_1^\dagger \Phi_1) (\Phi_1^\dagger \Phi_2) + 2\lambda_{12,22} (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_2) + \text{h.c.} \right] \quad (69)$$

$$= \frac{\lambda_1}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2} (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) + \left[\frac{\lambda_5}{2} (\Phi_1^\dagger \Phi_2)^2 + \lambda_6 (\Phi_1^\dagger \Phi_1) (\Phi_1^\dagger \Phi_2) + \lambda_7 (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_2) + \text{h.c.} \right], \quad (70)$$

where, in Eq. (70) we have employed, once again, the standard notation of Ref. [17]. The mapping between the two notations is given in Table V.

Term	Our Notation	Notation in [17]
$(\Phi_1^\dagger \Phi_1)^2$	$\lambda_{11,11}$	$\frac{\lambda_1}{2}$
$(\Phi_2^\dagger \Phi_2)^2$	$\lambda_{22,22}$	$\frac{\lambda_2}{2}$
$(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2)$	$2\lambda_{11,22}$	λ_3
$(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1)$	$2\lambda_{12,21}$	λ_4
$(\Phi_1^\dagger \Phi_2)^2$	$\lambda_{12,12}$	$\frac{\lambda_5}{2}$
$(\Phi_1^\dagger \Phi_1)(\Phi_1^\dagger \Phi_2)$	$2\lambda_{11,12}$	λ_6
$(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_2)$	$2\lambda_{12,22}$	λ_7

TABLE V: Comparison of coupling notation.

1. Scattering Matrices

The set of non-zero scattering matrices for this model reads

$$M_{|2,1,1\rangle} = \begin{bmatrix} \lambda_1 & \sqrt{2}\lambda_6^* & \lambda_5^* \\ \sqrt{2}\lambda_6 & \lambda_3 + \lambda_4 & \sqrt{2}\lambda_7^* \\ \lambda_5 & \sqrt{2}\lambda_7 & \lambda_2 \end{bmatrix}, \quad (71)$$

$$M_{|1,0,1\rangle} = \begin{bmatrix} \lambda_1 & \lambda_6 & \lambda_6^* & \lambda_4 \\ \lambda_6^* & \lambda_3 & \lambda_5^* & \lambda_7^* \\ \lambda_6 & \lambda_5 & \lambda_3 & \lambda_7 \\ \lambda_4 & \lambda_7 & \lambda_7^* & \lambda_2 \end{bmatrix}, \quad (72)$$

$$M_{|0,0,0\rangle} = \begin{bmatrix} 3\lambda_1 & 3\lambda_6^* & 3\lambda_6 & 2\lambda_3 + \lambda_4 \\ 3\lambda_6 & \lambda_3 + 2\lambda_4 & 3\lambda_5 & 3\lambda_7 \\ 3\lambda_6^* & 3\lambda_5^* & \lambda_3 + 2\lambda_4 & 3\lambda_7^* \\ 2\lambda_3 + \lambda_4 & 3\lambda_7^* & 3\lambda_7 & 3\lambda_2 \end{bmatrix}, \quad (73)$$

$$M_{|1,1,0\rangle} = \lambda_3 - \lambda_4. \quad (74)$$

Our scattering matrices agree with those in [17]. If we set $\lambda_6 = \lambda_7 = 0$ (along with their complex conjugates), we re-obtain, as expected, the results presented in Section VC1.

2. Unitarity Bounds

The eigenvalues of the scattering matrices in this case are, in general, too complex to write down in closed form. However, some can be computed analytically, and unitarity bounds are imposed accordingly,

$$|\lambda_3 - \lambda_4| \leq 8\pi. \quad (75)$$

E. 1 Scalar Doublet and 2 Neutral Scalar Singlets

In this case, the most general quartic part of the scalar potential reads:

$$\begin{aligned} V_4 = & \lambda_{11,11} \left(\Phi_1^\dagger \Phi_1 \right)^2 + \beta_{11,11} \chi_1^4 + \beta_{22,22} \chi_2^4 + 6 \beta_{11,22} \chi_1^2 \chi_2^2 + 4 \beta_{11,12} \chi_1^3 \chi_2 + 4 \beta_{12,22} \chi_1 \chi_2^3 \\ & + \gamma_{11,11} \left(\Phi_1^\dagger \Phi_1 \right) \chi_1^2 + \gamma_{11,22} \left(\Phi_1^\dagger \Phi_1 \right) \chi_2^2 + 2 \gamma_{11,12} \left(\Phi_1^\dagger \Phi_1 \right) \chi_1 \chi_2, \end{aligned} \quad (76)$$

where Φ_1 denotes the scalar doublet and χ_1 and χ_2 are the real scalar singlets. We have used the relations in Eqs. (7) and (9) to simplify the quartic part of the potential.

1. Scattering Matrices

The set of non-zero scattering matrices for this model reads

$$M_{|1,0,1\rangle} = M_{|2,1,1\rangle} = 2\lambda_{11,11}, \quad (77)$$

$$M_{|1,\frac{1}{2},\frac{1}{2}\rangle} = \begin{bmatrix} 2\gamma_{11,11} & 2\gamma_{11,12} \\ 2\gamma_{11,12} & 2\gamma_{11,22} \end{bmatrix}, \quad (78)$$

$$M_{|0,0,0\rangle} = \begin{bmatrix} 6\lambda_{11,11} & 2\sqrt{2}\gamma_{11,11} & 2\sqrt{2}\gamma_{11,12} & 2\sqrt{2}\gamma_{11,22} \\ 2\gamma_{11,11} & 12\beta_{11,11} & 12\sqrt{2}\beta_{11,12} & 12\beta_{11,22} \\ 2\sqrt{2}\gamma_{11,12} & 12\sqrt{2}\beta_{11,12} & 24\beta_{11,22} & 12\sqrt{2}\beta_{12,22} \\ 2\gamma_{11,22} & 12\beta_{11,22} & 12\sqrt{2}\beta_{12,22} & 12\beta_{22,22} \end{bmatrix}. \quad (79)$$

2. Unitarity Bounds

Again, certain eigenvalues of the scattering matrices are too complicated to be expressed in closed form. However, others can be written analytically, and, for those, the corresponding unitarity bounds are then applied, as:

$$\left| \gamma_{11,11} + \gamma_{11,22} \pm \sqrt{4\gamma_{11,12}^2 + (\gamma_{11,11} - \gamma_{11,22})^2} \right| \leq 8\pi, \quad (80)$$

$$|2\lambda_{11,11}| \leq 8\pi, \quad (81)$$

$$(82)$$

In order to compare our results with previous work, we consider the model studied in Ref. [29], where the SM is extended by a neutral complex scalar singlet. In their notation, the quartic part of the scalar potential reads

$$V_4 = \frac{\lambda}{4} (\Phi_1^\dagger \Phi_1)^2 + \frac{\delta_2}{2} (\Phi_1^\dagger \Phi_1) |S|^2 + \frac{d_2}{2} |S|^4, \quad (83)$$

where ⁵ $S = \frac{\chi_1 + i\chi_2}{\sqrt{2}}$. Notice that the case with two real singlets in Eq. (76) is more general than the case with one complex singlet in Eq. (83). But, since the complex scalar singlet can be decomposed into two real scalar singlet fields, χ_1 and χ_2 , we find that our results reproduce those of [29], when we set $\gamma_{11,12} = \beta_{11,12} = \beta_{12,22} = 0$. By matching the terms in Eq. (83) to those in Eq. (76), we derive the mapping between the two notations that is provided in Table VI.

Term	Our Notation	Notation in [29]
$(\Phi_1^\dagger \Phi_1)^2$	$\lambda_{11,11}$	$\frac{\lambda}{4}$
χ_1^4	$\beta_{11,11}$	$\frac{d_2}{16}$
χ_2^4	$\beta_{22,22}$	$\frac{d_2}{16}$
$\chi_1^2 \chi_2^2$	$6\beta_{11,22}$	$\frac{d_2}{8}$
$\chi_1^3 \chi_2$	$4\beta_{11,12}$	0
$\chi_1 \chi_2^3$	$4\beta_{12,22}$	0
$(\Phi_1^\dagger \Phi_1) \chi_1^2$	$\gamma_{11,11}$	$\frac{\delta_2}{4}$
$(\Phi_1^\dagger \Phi_1) \chi_2^2$	$\gamma_{11,22}$	$\frac{\delta_2}{4}$
$(\Phi_1^\dagger \Phi_1) \chi_1 \chi_2$	$2\gamma_{11,12}$	0

TABLE VI: Comparison of coupling notation.

F. 2 Scalar Doublets and 1 Neutral Scalar Singlet with a $\mathbb{Z}_2$ symmetry

We now consider a 2HDM with one additional neutral scalar singlet χ_1 . We impose a $\mathbb{Z}_2$ symmetry that acts on the fields as

$$\Phi_1 \rightarrow \Phi_1, \quad \Phi_2 \rightarrow -\Phi_2, \quad \chi_1 \rightarrow \chi_1. \quad (84)$$

Under this symmetry, and using Eqs. (5), (7), and (9), the scalar potential simplifies to:

$$\begin{aligned} V_4 = & \lambda_{11,11} (\Phi_1^\dagger \Phi_1)^2 + \lambda_{22,22} (\Phi_2^\dagger \Phi_2)^2 + 2\lambda_{11,22} (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + 2\lambda_{12,21} (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) \\ & + \left[\lambda_{12,12} (\Phi_1^\dagger \Phi_2)^2 + \text{h.c.} \right] + \beta_{11,11} \chi_1^4 + \gamma_{11,11} (\Phi_1^\dagger \Phi_1) \chi_1^2 + \gamma_{22,11} (\Phi_2^\dagger \Phi_2) \chi_1^2. \end{aligned} \quad (85)$$

⁵ Equivalently, one could choose $S = \frac{\chi_2 + i\chi_1}{\sqrt{2}}$.

1. Scattering Matrices

The set of non-zero scattering matrices for this model reads

$$M_{|1, \frac{1}{2}, \frac{1}{2}\rangle} = \begin{bmatrix} 2\gamma_{11,11} & 0 \\ 0 & 2\gamma_{22,11} \end{bmatrix}, \quad (86)$$

$$M_{|2,1,1\rangle} = \begin{bmatrix} 2\lambda_{11,11} & 2\lambda_{12,12}^* & 0 \\ 2\lambda_{12,12} & 2\lambda_{22,22} & 0 \\ 0 & 0 & 2\lambda_{11,22} + 2\lambda_{12,21} \end{bmatrix}, \quad (87)$$

$$M_{|1,0,1\rangle} = \begin{bmatrix} 2\lambda_{11,11} & 2\lambda_{12,21} & 0 & 0 \\ 2\lambda_{12,21} & 2\lambda_{22,22} & 0 & 0 \\ 0 & 0 & 2\lambda_{11,22} & 2\lambda_{12,12}^* \\ 0 & 0 & 2\lambda_{12,12} & 2\lambda_{11,22} \end{bmatrix}, \quad (88)$$

$$M_{|0,0,0\rangle} = \begin{bmatrix} 6\lambda_{11,11} & 4\lambda_{11,22} + 2\lambda_{12,21} & 2\sqrt{2}\gamma_{11,11} & 0 & 0 \\ 4\lambda_{11,22} + 2\lambda_{12,21} & 6\lambda_{22,22} & 2\sqrt{2}\gamma_{22,11} & 0 & 0 \\ 2\gamma_{11,11} & 2\gamma_{22,11} & 12\beta_{11,11} & 0 & 0 \\ 0 & 0 & 0 & 2\lambda_{11,22} + 4\lambda_{12,21} & 6\lambda_{12,12} \\ 0 & 0 & 0 & 6\lambda_{12,12}^* & 2\lambda_{11,22} + 4\lambda_{12,21} \end{bmatrix}, \quad (89)$$

$$M_{|1,1,0\rangle} = 2(\lambda_{11,22} - \lambda_{12,21}). \quad (90)$$

2. Unitarity Bounds

Some eigenvalues are too complicated to be written in closed form. Therefore, we show only the results of imposing partial-wave unitarity on the remaining eigenvalues of the zero partial-wave amplitude matrix,

$$|2\gamma_{11,11}| \leq 8\pi, \quad (91)$$

$$|2\gamma_{22,11}| \leq 8\pi, \quad (92)$$

$$2|\lambda_{11,22} \pm \lambda_{12,21}| \leq 8\pi, \quad (93)$$

$$2|\lambda_{11,22} \pm |\lambda_{12,12}|| \leq 8\pi, \quad (94)$$

$$\left| \lambda_{11,11} + \lambda_{22,22} \pm \sqrt{(\lambda_{11,11} - \lambda_{22,22})^2 + 4\lambda_{12,21}^2} \right| \leq 8\pi, \quad (95)$$

$$\left| \lambda_{11,11} + \lambda_{22,22} \pm \sqrt{(\lambda_{11,11} - \lambda_{22,22})^2 + 4|\lambda_{12,12}|^2} \right| \leq 8\pi. \quad (96)$$

G. 2 Scalar Doublets with a $\mathbb{Z}_2$ symmetry + 2 Neutral Scalar Singlets with a $\mathbb{Z}'_2$ symmetry

We now consider a model with two $SU(2)$ doublets and two neutral scalar singlets. We impose two independent $\mathbb{Z}_2$ symmetries that act on these fields as

$$\mathbb{Z}_2 : \quad \Phi_1 \rightarrow \Phi_1 \quad \Phi_2 \rightarrow -\Phi_2, \quad \chi_1 \rightarrow \chi_1, \quad \chi_2 \rightarrow \chi_2, \quad (97)$$

$$\mathbb{Z}'_2: \quad \Phi_1 \rightarrow \Phi_1 \quad \Phi_2 \rightarrow \Phi_2, \quad \chi_1 \rightarrow -\chi_1, \quad \chi_2 \rightarrow -\chi_2, \quad (98)$$

Under these symmetries, the quartic part of the scalar potential becomes:

$$\begin{aligned} V_4 = & \lambda_{11,11} \left(\Phi_1^\dagger \Phi_1 \right)^2 + \lambda_{22,22} \left(\Phi_2^\dagger \Phi_2 \right)^2 + 2\lambda_{11,22} \left(\Phi_1^\dagger \Phi_1 \right) \left(\Phi_2^\dagger \Phi_2 \right) + 2\lambda_{12,21} \left(\Phi_1^\dagger \Phi_2 \right) \left(\Phi_2^\dagger \Phi_1 \right) \\ & + \left[\lambda_{12,12} \left(\Phi_1^\dagger \Phi_2 \right)^2 + \text{h.c.} \right] + \beta_{11,11} \chi_1^4 + \beta_{22,22} \chi_2^4 + 6\beta_{11,22} \chi_1^2 \chi_2^2 + 4\beta_{11,12} \chi_1^3 \chi_2 + 4\beta_{12,22} \chi_2^3 \chi_1 \\ & + \gamma_{11,11} \left(\Phi_1^\dagger \Phi_1 \right) \chi_1^2 + \gamma_{22,11} \left(\Phi_2^\dagger \Phi_2 \right) \chi_1^2 + \gamma_{11,22} \left(\Phi_1^\dagger \Phi_1 \right) \chi_2^2 + 2\gamma_{11,12} \left(\Phi_1^\dagger \Phi_1 \right) \chi_1 \chi_2 \\ & + 2\gamma_{22,12} \left(\Phi_2^\dagger \Phi_2 \right) \chi_1 \chi_2 + \gamma_{22,22} \left(\Phi_2^\dagger \Phi_2 \right) \chi_2^2, \end{aligned} \quad (99)$$

where we have simplified the quartic couplings using the relations in Eqs. (5) and (7).

1. Scattering Matrices

The set of non-zero scattering matrices for this model reads

$$M_{|1, \frac{1}{2}, \frac{1}{2}\rangle} = \begin{bmatrix} 2\gamma_{11,11} & 2\gamma_{11,12} & 0 & 0 \\ 2\gamma_{11,12} & 2\gamma_{11,22} & 0 & 0 \\ 0 & 0 & 2\gamma_{22,11} & 2\gamma_{22,12} \\ 0 & 0 & 2\gamma_{22,12} & 2\gamma_{22,22} \end{bmatrix}, \quad (100)$$

$$M_{|2,1,1\rangle} = \begin{bmatrix} 2\lambda_{11,11} & 2\lambda_{12,12}^* & 0 \\ 2\lambda_{12,12} & 2\lambda_{22,22} & 0 \\ 0 & 0 & 2\lambda_{11,22} + 2\lambda_{12,21} \end{bmatrix}, \quad (101)$$

$$M_{|1,0,1\rangle} = \begin{bmatrix} 2\lambda_{11,11} & 2\lambda_{12,21} & 0 & 0 \\ 2\lambda_{12,21} & 2\lambda_{22,22} & 0 & 0 \\ 0 & 0 & 2\lambda_{11,22} & 2\lambda_{12,12}^* \\ 0 & 0 & 2\lambda_{12,12} & 2\lambda_{11,22} \end{bmatrix}, \quad (102)$$

$$M_{|1,1,0\rangle} = 2(\lambda_{11,22} - \lambda_{12,21}), \quad (103)$$

$$M_{|0,0,0\rangle} = \text{blkdiag}(A, B) \quad (104)$$

where, here and henceforth, $\text{blkdiag}(A, B, \dots)$ refers to a block diagonal matrix, whose entries are the matrices $A, B, \dots$. The matrices A and B in Eq. (104) are given, respectively, by

$$A = \begin{bmatrix} 24\beta_{11,22} & 2\sqrt{2}\gamma_{11,12} & 2\sqrt{2}\gamma_{22,12} & 12\sqrt{2}\beta_{11,12} & 12\sqrt{2}\beta_{12,22} \\ 2\sqrt{2}\gamma_{11,12} & 6\lambda_{11,11} & 4\lambda_{11,22} + 2\lambda_{12,21} & 2\sqrt{2}\gamma_{11,11} & 2\sqrt{2}\gamma_{11,22} \\ 2\sqrt{2}\gamma_{22,12} & 4\lambda_{11,22} + 2\lambda_{12,21} & 6\lambda_{22,22} & 2\sqrt{2}\gamma_{22,11} & 2\sqrt{2}\gamma_{22,22} \\ 12\sqrt{2}\beta_{11,12} & 2\gamma_{11,11} & 2\gamma_{22,11} & 12\beta_{11,11} & 12\beta_{11,22} \\ 12\sqrt{2}\beta_{12,22} & 2\gamma_{11,22} & 2\gamma_{22,22} & 12\beta_{11,22} & 12\beta_{22,22} \end{bmatrix}, \quad (105)$$

$$B = \begin{bmatrix} 2\lambda_{11,22} + 4\lambda_{12,21} & 6\lambda_{12,12} \\ 6\lambda_{12,12}^* & 2\lambda_{11,22} + 4\lambda_{12,21} \end{bmatrix}. \quad (106)$$

2. Unitarity Bounds

As several eigenvalues are too involved to express analytically, we show only the results of imposing partial-wave unitarity on the remaining eigenvalues of the s -wave amplitude matrix, that can be computed explicitly,

$$2 |\lambda_{11,22} \pm \lambda_{12,21}| \leq 8\pi, \quad (107)$$

$$2 |\lambda_{11,22} \pm |\lambda_{12,12}|| \leq 8\pi, \quad (108)$$

$$2 |\lambda_{11,22} + 2\lambda_{12,21} \pm 3|\lambda_{12,12}|| \leq 8\pi, \quad (109)$$

$$\left| \lambda_{11,11} + \lambda_{22,22} \pm \sqrt{(\lambda_{11,11} - \lambda_{22,22})^2 + 4\lambda_{12,21}^2} \right| \leq 8\pi, \quad (110)$$

$$\left| \lambda_{11,11} + \lambda_{22,22} \pm \sqrt{(\lambda_{11,11} - \lambda_{22,22})^2 + 4|\lambda_{12,12}|^2} \right| \leq 8\pi, \quad (111)$$

$$\left| \gamma_{11,11} + \gamma_{11,22} \pm \sqrt{\gamma_{11,11}^2 + 4\gamma_{11,12}^2 - 2\gamma_{11,11}\gamma_{11,22} + \gamma_{11,22}^2} \right| \leq 8\pi, \quad (112)$$

$$\left| \gamma_{22,11} + \gamma_{22,22} \pm \sqrt{\gamma_{22,11}^2 + 4\gamma_{22,12}^2 - 2\gamma_{22,11}\gamma_{22,22} + \gamma_{22,22}^2} \right| \leq 8\pi. \quad (113)$$

We verified our results against those reported in [30] and found them to be in agreement. Table VII summarizes the

Term	Our Notation	Notation of [30]
$(\Phi_1^\dagger \Phi_1)^2$	$\lambda_{11,11}$	$\frac{\lambda_1}{2}$
$(\Phi_2^\dagger \Phi_2)^2$	$\lambda_{22,22}$	$\frac{\lambda_2}{2}$
$(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2)$	$2\lambda_{11,22}$	λ_3
$(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1)$	$2\lambda_{12,21}$	λ_4
$(\Phi_1^\dagger \Phi_2)^2$	$\lambda_{12,12}$	λ_5
$(\Phi_2^\dagger \Phi_1)^2$	$\lambda_{12,12}^*$	λ_5
χ_1^4	$\beta_{11,11}$	$\frac{\lambda_6}{8}$
χ_2^4	$\beta_{22,22}$	$\frac{\lambda_9}{8}$
$\chi_1^2 \chi_2^2$	$6\beta_{11,22}$	$\frac{\lambda_{10}}{4}$
$\chi_1^3 \chi_2$	$4\beta_{11,12}$	$\frac{\lambda_{13}}{6}$
$\chi_1 \chi_2^3$	$4\beta_{12,22}$	$\frac{\lambda_{14}}{6}$
$(\Phi_1^\dagger \Phi_1) \chi_1^2$	$\gamma_{11,11}$	$\frac{\lambda_7}{2}$
$(\Phi_2^\dagger \Phi_2) \chi_1^2$	$\gamma_{22,11}$	$\frac{\lambda_8}{2}$
$(\Phi_1^\dagger \Phi_1) \chi_2^2$	$\gamma_{11,22}$	$\frac{\lambda_{11}}{2}$
$(\Phi_2^\dagger \Phi_2) \chi_2^2$	$\gamma_{22,22}$	$\frac{\lambda_{12}}{2}$
$(\Phi_1^\dagger \Phi_1) \chi_1 \chi_2$	$2\gamma_{11,12}$	$\frac{\lambda_{15}}{2}$
$(\Phi_2^\dagger \Phi_2) \chi_1 \chi_2$	$2\gamma_{22,12}$	$\frac{\lambda_{16}}{2}$

TABLE VII: Comparison of coupling notation.

correspondence between the couplings used in our work and those of [30].

H. 2 Scalar Doublets + 1 Neutral Scalar Singlet + 1 Charged Scalar Singlet

For a model with two $SU(2)$ scalar doublets Φ_1 and Φ_2 , one neutral scalar singlet χ_1 and one charged scalar singlet φ_1^+ , the quartic part of the scalar potential can be written as

$$\begin{aligned}
V_4 = & \lambda_{11,11} \left(\Phi_1^\dagger \Phi_1 \right)^2 + \lambda_{22,22} \left(\Phi_2^\dagger \Phi_2 \right)^2 + 2\lambda_{11,22} \left(\Phi_1^\dagger \Phi_1 \right) \left(\Phi_2^\dagger \Phi_2 \right) + 2\lambda_{12,21} \left(\Phi_1^\dagger \Phi_2 \right) \left(\Phi_2^\dagger \Phi_1 \right) \\
& + \left[\lambda_{12,12} \left(\Phi_1^\dagger \Phi_2 \right)^2 + 2\lambda_{11,12} \left(\Phi_1^\dagger \Phi_1 \right) \left(\Phi_1^\dagger \Phi_2 \right) + 2\lambda_{12,22} \left(\Phi_1^\dagger \Phi_2 \right) \left(\Phi_2^\dagger \Phi_2 \right) + \text{h.c.} \right] \\
& + \alpha_{11,11} \left(\varphi_1^+ \varphi_1^- \right)^2 + \beta_{11,11} \left(\chi_1^2 \right)^2 \\
& + \delta_{11,11} \left(\Phi_1^\dagger \Phi_1 \right) \left(\varphi_1^+ \varphi_1^- \right) + \delta_{12,11} \left(\Phi_1^\dagger \Phi_2 \right) \left(\varphi_1^+ \varphi_1^- \right) + \delta_{12,11}^* \left(\Phi_2^\dagger \Phi_1 \right) \left(\varphi_1^+ \varphi_1^- \right) + \delta_{22,11} \left(\Phi_2^\dagger \Phi_2 \right) \left(\varphi_1^+ \varphi_1^- \right) \\
& + \gamma_{11,11} \left(\Phi_1^\dagger \Phi_1 \right) \left(\chi_1^2 \right) + \gamma_{12,11} \left(\Phi_1^\dagger \Phi_2 \right) \left(\chi_1^2 \right) + \gamma_{12,11}^* \left(\Phi_2^\dagger \Phi_1 \right) \left(\chi_1^2 \right) + \gamma_{22,11} \left(\Phi_2^\dagger \Phi_2 \right) \left(\chi_1^2 \right) + \zeta_{11,11} \left(\varphi_1^+ \varphi_1^- \right) \left(\chi_1^2 \right) \\
& + \left[\kappa_{12,11} \left(\Phi_1^T \sigma_2 \Phi_2 \right) \left(\varphi_1^- \chi_1 \right) - \kappa_{12,11} \left(\Phi_2^T \sigma_2 \Phi_1 \right) \left(\varphi_1^- \chi_1 \right) + \text{h.c.} \right], \tag{114}
\end{aligned}$$

where we have used the relations in Eqs. (5), (8), (9), and (11) to get rid of redundant terms. This is the simplest model featuring the $\kappa_{ab,cd}$ couplings from Eq. (4).

1. Scattering Matrices

The set of non-zero scattering matrices are:

$$M_{|2,2,0\rangle} = 2\alpha_{11,11}, \tag{115}$$

$$M_{|2,\frac{3}{2},\frac{1}{2}\rangle} = \begin{bmatrix} \delta_{11,11} & \delta_{12,11}^* \\ \delta_{12,11} & \delta_{22,11} \end{bmatrix}, \tag{116}$$

$$M_{|1,\frac{1}{2},\frac{1}{2}\rangle} = \begin{bmatrix} \delta_{11,11} & \delta_{12,11} & 0 & -2i\kappa_{12,11}^* \\ \delta_{12,11}^* & \delta_{22,11} & 2i\kappa_{12,11}^* & 0 \\ 0 & -2i\kappa_{12,11} & 2\gamma_{11,11} & 2\gamma_{12,11}^* \\ 2i\kappa_{12,11} & 0 & 2\gamma_{12,11} & 2\gamma_{22,11} \end{bmatrix}, \tag{117}$$

$$M_{|1,1,0\rangle} = \begin{bmatrix} 2\lambda_{11,22} - 2\lambda_{12,21} & 2i\sqrt{2}\kappa_{12,11}^* \\ -2i\sqrt{2}\kappa_{12,11}^* & 2\zeta_{11,11} \end{bmatrix}, \tag{118}$$

$$M_{|1,1,2\rangle} = \begin{bmatrix} 2\lambda_{11,11} & 2\sqrt{2}\lambda_{11,12}^* & 2\lambda_{12,12}^* \\ 2\sqrt{2}\lambda_{11,12} & 2\lambda_{11,22} + 2\lambda_{12,21} & 2\sqrt{2}\lambda_{12,22}^* \\ 2\lambda_{12,12} & 2\sqrt{2}\lambda_{12,22} & 2\lambda_{22,22} \end{bmatrix}, \tag{119}$$

$$M_{|1,0,1\rangle} = \begin{bmatrix} 2\lambda_{11,11} & 2\lambda_{11,12} & 2\lambda_{11,12}^* & 2\lambda_{12,21} \\ 2\lambda_{11,12}^* & 2\lambda_{11,22} & 2\lambda_{12,12}^* & 2\lambda_{12,22}^* \\ 2\lambda_{11,12} & 2\lambda_{12,12} & 2\lambda_{11,22} & 2\lambda_{12,22} \\ 2\lambda_{12,21} & 2\lambda_{12,22} & 2\lambda_{12,22}^* & 2\lambda_{22,22} \end{bmatrix}, \tag{120}$$

$$M_{|0,0,0\rangle} = \begin{bmatrix} 6\lambda_{11,11} & 6\lambda_{11,12}^* & 6\lambda_{11,12} & 4\lambda_{11,22} + 2\lambda_{12,21} & \sqrt{2}\delta_{11,11} & 2\sqrt{2}\gamma_{11,11} \\ 6\lambda_{11,12} & 2\lambda_{11,22} + 4\lambda_{12,21} & 6\lambda_{12,12} & 6\lambda_{12,22} & \sqrt{2}\delta_{12,11} & 2\sqrt{2}\gamma_{12,11} \\ 6\lambda_{11,12}^* & 6\lambda_{12,12}^* & 2\lambda_{11,22} + 4\lambda_{12,21} & 6\lambda_{12,22}^* & \sqrt{2}\delta_{12,11}^* & 2\sqrt{2}\gamma_{12,11}^* \\ 4\lambda_{11,22} + 2\lambda_{12,21} & 6\lambda_{12,22}^* & 6\lambda_{12,22} & 6\lambda_{22,22} & \sqrt{2}\delta_{22,11} & 2\sqrt{2}\gamma_{22,11} \\ \sqrt{2}\delta_{11,11} & \sqrt{2}\delta_{12,11}^* & \sqrt{2}\delta_{12,11} & \sqrt{2}\delta_{22,11} & 4\alpha_{11,11} & \sqrt{2}\zeta_{11,11} \\ 2\gamma_{11,11} & 2\gamma_{12,11}^* & 2\gamma_{12,11} & 2\gamma_{22,11} & \sqrt{2}\zeta_{11,11} & 12\beta_{11,11} \end{bmatrix}. \tag{121}$$

2. Unitarity Bounds

Since certain eigenvalues are too complex to evaluate analytically, unitarity constraints are shown only for the remaining eigenvalues of the zero partial-wave amplitude matrix. These read

$$\left| \zeta_{11,11} + \lambda_{11,22} - \lambda_{12,21} \pm \sqrt{(-\zeta_{11,11} - \lambda_{11,22} + \lambda_{12,21})^2 - 4(\zeta_{11,11}\lambda_{11,22} - \zeta_{11,11}\lambda_{12,21} - 2|\kappa_{12,11}|^2)} \right| \leq 8\pi, \quad (122)$$

$$\frac{1}{2} \left| \delta_{11,11} + \delta_{22,11} \pm \sqrt{(\delta_{11,11} - \delta_{22,11})^2 + 4|\delta_{12,11}|^2} \right| \leq 8\pi, \quad (123)$$

$$|2 \alpha_{11,11}| \leq 8\pi. \quad (124)$$

I. $\mathbb{Z}_3$ -Symmetric 3HDM

Now, we consider a model with three $SU(2)$ doublets Φ_1 , Φ_2 , and Φ_3 . Under a $\mathbb{Z}_3$ symmetry, the three doublets transform as

$$\Phi_1 \rightarrow \Phi_1, \quad \Phi_2 \rightarrow e^{i\frac{2\pi}{3}} \Phi_2, \quad \Phi_3 \rightarrow e^{i\frac{4\pi}{3}} \Phi_3. \quad (125)$$

Under this symmetry, the potential becomes

$$\begin{aligned} V_4 = & \lambda_{11,11} (\Phi_1^\dagger \Phi_1)^2 + \lambda_{22,22} (\Phi_2^\dagger \Phi_2)^2 + \lambda_{33,33} (\Phi_3^\dagger \Phi_3)^2 \\ & + 2\lambda_{11,22} (\Phi_1^\dagger \Phi_1) (\Phi_2^\dagger \Phi_2) + 2\lambda_{11,33} (\Phi_1^\dagger \Phi_1) (\Phi_3^\dagger \Phi_3) + 2\lambda_{22,33} (\Phi_2^\dagger \Phi_2) (\Phi_3^\dagger \Phi_3) \\ & + 2\lambda_{12,21} (\Phi_1^\dagger \Phi_2) (\Phi_2^\dagger \Phi_1) + 2\lambda_{13,31} (\Phi_1^\dagger \Phi_3) (\Phi_3^\dagger \Phi_1) + 2\lambda_{23,32} (\Phi_2^\dagger \Phi_3) (\Phi_3^\dagger \Phi_2) \\ & + 2 \left[\lambda_{12,13} (\Phi_1^\dagger \Phi_2) (\Phi_1^\dagger \Phi_3) + \lambda_{13,23} (\Phi_1^\dagger \Phi_3) (\Phi_2^\dagger \Phi_3) + \lambda_{12,32} (\Phi_1^\dagger \Phi_2) (\Phi_3^\dagger \Phi_2) + \text{h.c.} \right], \end{aligned} \quad (126)$$

where Φ_1, Φ_2 and Φ_3 denote the three scalar doublets.

1. Scattering Matrices

The scattering matrices are all block diagonal. We find

$$M_{[2,1,1]} = \text{blkdiag}(A, B, C), \quad (127)$$

where

$$A = \begin{bmatrix} 2\lambda_{11,33} + 2\lambda_{13,31} & 2\sqrt{2}\lambda_{12,32}^* \\ 2\sqrt{2}\lambda_{12,32} & 2\lambda_{22,22} \end{bmatrix}, \quad (128)$$

$$B = \begin{bmatrix} 2\lambda_{11,11} & 2\sqrt{2}\lambda_{12,13}^* \\ 2\sqrt{2}\lambda_{12,13} & 2\lambda_{22,33} + 2\lambda_{23,32} \end{bmatrix}, \quad (129)$$

$$C = \begin{bmatrix} 2\lambda_{33,33} & 2\sqrt{2}\lambda_{13,23} \\ 2\sqrt{2}\lambda_{13,23}^* & 2\lambda_{11,22} + 2\lambda_{12,21} \end{bmatrix}, \quad (130)$$

$$M_{[1,0,1]} = \text{blkdiag}(D, E, F), \quad (131)$$

where

$$D = \begin{bmatrix} 2\lambda_{22,33} & 2\lambda_{13,23} & 2\lambda_{12,32}^* \\ 2\lambda_{13,23}^* & 2\lambda_{11,33} & 2\lambda_{12,13}^* \\ 2\lambda_{12,32} & 2\lambda_{12,13} & 2\lambda_{11,22} \end{bmatrix}, \quad (132)$$

$$E = \begin{bmatrix} 2\lambda_{11,11} & 2\lambda_{12,21} & 2\lambda_{13,31} \\ 2\lambda_{12,21} & 2\lambda_{22,22} & 2\lambda_{23,32} \\ 2\lambda_{13,31} & 2\lambda_{23,32} & 2\lambda_{33,33} \end{bmatrix}, \quad (133)$$

$$F = \begin{bmatrix} 2\lambda_{22,33} & 2\lambda_{12,32} & 2\lambda_{13,23}^* \\ 2\lambda_{12,32}^* & 2\lambda_{11,22} & 2\lambda_{12,13}^* \\ 2\lambda_{13,23} & 2\lambda_{12,13} & 2\lambda_{11,33} \end{bmatrix}, \quad (134)$$

$$M_{|0,0,0\rangle} = \text{blkdiag}(G, H, I), \quad (135)$$

where

$$G = \begin{bmatrix} 2\lambda_{22,33} + 4\lambda_{23,32} & 6\lambda_{13,23}^* & 6\lambda_{12,32} \\ 6\lambda_{13,23} & 2\lambda_{11,33} + 4\lambda_{13,31} & 6\lambda_{12,13} \\ 6\lambda_{12,32}^* & 6\lambda_{12,13}^* & 2\lambda_{11,22} + 4\lambda_{12,21} \end{bmatrix}, \quad (136)$$

$$H = \begin{bmatrix} 6\lambda_{11,11} & 4\lambda_{11,22} + 2\lambda_{12,21} & 4\lambda_{11,33} + 2\lambda_{13,31} \\ 4\lambda_{11,22} + 2\lambda_{12,21} & 6\lambda_{22,22} & 4\lambda_{22,33} + 2\lambda_{23,32} \\ 4\lambda_{11,33} + 2\lambda_{13,31} & 4\lambda_{22,33} + 2\lambda_{23,32} & 6\lambda_{33,33} \end{bmatrix}, \quad (137)$$

$$I = \begin{bmatrix} 2\lambda_{22,33} + 4\lambda_{23,32} & 6\lambda_{12,32}^* & 6\lambda_{13,23} \\ 6\lambda_{12,32} & 2\lambda_{11,22} + 4\lambda_{12,21} & 6\lambda_{12,13} \\ 6\lambda_{13,23}^* & 6\lambda_{12,13}^* & 2\lambda_{11,33} + 4\lambda_{13,31} \end{bmatrix}, \quad (138)$$

and

$$M_{|1,1,0\rangle} = \begin{bmatrix} 2(\lambda_{11,22} - \lambda_{12,21}) & 0 & 0 \\ 0 & 2(\lambda_{11,33} - \lambda_{13,31}) & 0 \\ 0 & 0 & 2(\lambda_{22,33} - \lambda_{23,32}) \end{bmatrix}. \quad (139)$$

2. Unitarity Bounds

We show the results of partial wave unitarity bounds to the eigenvalues that can be computed analytically,

$$2|\lambda_{11,22} - \lambda_{12,21}| \leq 8\pi, \quad (140)$$

$$2|\lambda_{11,33} - \lambda_{13,31}| \leq 8\pi, \quad (141)$$

$$2|\lambda_{22,33} - \lambda_{23,32}| \leq 8\pi, \quad (142)$$

$$\left| \lambda_{11,11} + \lambda_{22,33} + \lambda_{23,32} \pm \sqrt{(\lambda_{11,11} + \lambda_{22,33} + \lambda_{23,32})^2 - 4(\lambda_{11,11}\lambda_{22,33} + \lambda_{11,11}\lambda_{23,32} - 2|\lambda_{12,13}|^2)} \right| \leq 8\pi, \quad (143)$$

$$\left| \lambda_{11,33} + \lambda_{13,31} + \lambda_{22,22} \pm \sqrt{(\lambda_{11,33} + \lambda_{13,31} + \lambda_{22,22})^2 - 4(\lambda_{11,33}\lambda_{22,22} + \lambda_{13,31}\lambda_{22,22} - 2|\lambda_{12,32}|^2)} \right| \leq 8\pi, \quad (144)$$

$$\left| \lambda_{11,22} + \lambda_{12,21} + \lambda_{33,33} \pm \sqrt{(\lambda_{11,22} + \lambda_{12,21} + \lambda_{33,33})^2 - 4(\lambda_{11,22}\lambda_{33,33} + \lambda_{12,21}\lambda_{33,33} - 2|\lambda_{13,23}|^2)} \right| \leq 8\pi. \quad (145)$$

We compared our results with those presented in [19], and found them to be consistent. Table VIII provides a summary of the correspondence between the quartic couplings used in our study and those in [19].

Term	Our Notation	Notation of [19]
$(\Phi_1^\dagger \Phi_1)^2$	$\lambda_{11,11}$	r_1
$(\Phi_2^\dagger \Phi_2)^2$	$\lambda_{22,22}$	r_2
$(\Phi_3^\dagger \Phi_3)^2$	$\lambda_{33,33}$	r_3
$(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2)$	$2\lambda_{11,22}$	$2r_4$
$(\Phi_1^\dagger \Phi_1)(\Phi_3^\dagger \Phi_3)$	$2\lambda_{11,33}$	$2r_5$
$(\Phi_2^\dagger \Phi_2)(\Phi_3^\dagger \Phi_3)$	$2\lambda_{22,33}$	$2r_6$
$(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1)$	$2\lambda_{12,21}$	$2r_7$
$(\Phi_1^\dagger \Phi_3)(\Phi_3^\dagger \Phi_1)$	$2\lambda_{13,31}$	$2r_8$
$(\Phi_2^\dagger \Phi_3)(\Phi_3^\dagger \Phi_2)$	$2\lambda_{23,32}$	$2r_9$
$(\Phi_1^\dagger \Phi_2)(\Phi_1^\dagger \Phi_3)$	$2\lambda_{12,13}$	$2c_4$
$(\Phi_2^\dagger \Phi_1)(\Phi_3^\dagger \Phi_1)$	$2\lambda_{12,13}^*$	$2c_4^*$
$(\Phi_1^\dagger \Phi_3)(\Phi_2^\dagger \Phi_3)$	$2\lambda_{13,23}$	$2c_{11}$
$(\Phi_3^\dagger \Phi_1)(\Phi_3^\dagger \Phi_2)$	$2\lambda_{13,23}^*$	$2c_{11}^*$
$(\Phi_1^\dagger \Phi_2)(\Phi_3^\dagger \Phi_2)$	$2\lambda_{12,32}$	$2c_{12}$
$(\Phi_2^\dagger \Phi_1)(\Phi_2^\dagger \Phi_3)$	$2\lambda_{12,32}^*$	$2c_{12}^*$

TABLE VIII: Comparison of coupling notation.

VI. OVERVIEW

Almost all models addressing the outstanding issues in the SM include extra $SU(2)$ singlet and/or doublet scalars. In particular, many models addressing the dark matter problem include extra neutral singlet scalars. Such models must be subject to theoretical constraints, even before a simulation starts. Those constraints include boundedness from below, nonexistence of lower lying alternative vacua, and the unitarity bounds on $2 \rightarrow 2$ scattering. In this paper we address fully the perturbative partial-wave unitarity for the tree-level scattering matrix in models with any number of scalar doublets, neutral singlets, and/or charged singlets. Enforcing the correct high-energy behavior, provides bounds on the quartic couplings, freeing us from defining the exact nature of the quadratic and cubic couplings.

In contrast, if one wishes to turn such bounds on restrictions on masses, mixing angles and other directly observable quantities, then one must define the full theory. That is, one must define the quadratic and cubic couplings, a specific vacuum, and the mass matrices must be fully diagonalized. After this, one would strive to invert the relations, turning the unitarity bounds on the quartic couplings into restrictions on combinations of masses and mixing angles. This is possible in simple models, but increasingly more difficult as the number of fields increases. In contrast, although the matrices get larger, our limits on quartic couplings are always applicable, at least numerically.

We classify the states by the conserved quantum numbers Q , Y , and T and show that, once one restricts oneself to the minimal set of states providing all inequivalent bounds, the quantum number Q is redundant. We also discuss examples where the existence of extra symmetries allows for the inclusion of further quantum numbers in the basis, thus greatly simplifying the matrices.

We introduce the **Mathematica** notebook **BoundS** that automatically calculates the quartic part of the potential *and* the scattering matrices and their eigenvalues, for any model with any symmetries (discrete or continuous, abelian or non-abelian). We present results for a variety of particular models, and compare with the literature, when available. Our aim is to help provide complete simulations of models beyond the SM with a necessary and very powerful tool in parameter restriction.

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Appendix A: Redundant scattering matrices

In any scattering process, conserved quantum numbers constrain the possible initial and final states. In a globally symmetric $SU(2) \times U(1)$ quantum field theory, initial states with definite electric charge Q and hypercharge Y can only scatter into states with the same Q and Y . That is because these quantum numbers are protected by the $SU(2) \times U(1)$ symmetry and are, thus, conserved. For this reason, in building $2 \rightarrow 2$ scattering matrices, Ref. [19] labeled two-particle states by $|Q, Y\rangle$. For models like the one defined in Section II, we list in Table. IX all possible two-particle states. However, not all scattering amplitudes are independent. Notice that

$ Q, Y\rangle$	State	Conditions	Dimensionality
$ 2, 2\rangle$	$\varphi_i^+ \varphi_j^+$	$i \leq j$	$n_c(n_c + 1)/2$
$ 2, \frac{3}{2}\rangle$	$\phi_i^+ \varphi_j^+$	—	$n_D n_c$
$ 2, 1\rangle$	$\phi_i^+ \phi_j^+$	$i \leq j$	$n_D(n_D + 1)/2$
$ 1, \frac{3}{2}\rangle$	$\phi_i^0 \varphi_j^+$	—	$n_D n_c$
$ 1, 1\rangle$	$\{\phi_i^+ \phi_j^0, \varphi_i^+ \chi_j\}$	—	$n_D^2 + n_n n_c$
$ 1, \frac{1}{2}\rangle$	$\{\phi_i^+ \chi_j, \phi_i^{0*} \varphi_j^+\}$	—	$n_D(n_n + n_c)$
$ 1, 0\rangle$	$\phi_i^+ \phi_j^{0*}$	—	n_D^2
$ 0, 1\rangle$	$\phi_i^0 \phi_j^0$	$i \leq j$	$n_D(n_D + 1)/2$
$ 0, \frac{1}{2}\rangle$	$\{\phi_i^0 \chi_j, \phi_i^- \varphi_j^+\}$	—	$n_D(n_n + n_c)$
$ 0, 0\rangle$	$\{\phi_i^+ \phi_j^-, \phi_i^0 \phi_j^{0*}, \varphi_i^+ \varphi_j^-, \chi_i \chi_j\}$	$\{—, —, —, i \leq j\}$	$2n_D^2 + n_c^2 + n_n(n_n + 1)/2$

TABLE IX: Basis of two-particle states labeled by $|Q, Y\rangle$. This table includes all states, some of which provide redundant information.

$$\mathcal{M}[\phi_a^+ \varphi_b^+ \rightarrow \phi_c^+ \varphi_d^+] = \mathcal{M}[\phi_a^+ \varphi_b^+ \rightarrow \phi_c^+ \varphi_d^+] = \delta_{ca,db}, \quad (A1)$$

$$\mathcal{M}[\phi_a^+ \phi_b^+ \rightarrow \phi_c^+ \phi_d^+] = \mathcal{M}[\phi_a^0 \phi_b^0 \rightarrow \phi_c^0 \phi_d^0] = 2\lambda_{ca,db} + 2\lambda_{da,cb}, \quad (A2)$$

$$\mathcal{M}[\phi_a^+ \chi_b \rightarrow \phi_c^+ \chi_d] = \mathcal{M}[\phi_a^0 \chi_b \rightarrow \phi_c^0 \chi_d] = 2\gamma_{ca,bd}, \quad (A3)$$

$$\mathcal{M}[\phi_a^+ \chi_b \rightarrow \phi_c^{0*} \varphi_d^+] = \mathcal{M}[\phi_a^0 \chi_b \rightarrow \phi_c^- \varphi_d^+] = 2i\kappa_{ca,db}, \quad (A4)$$

$$\mathcal{M}[\phi_a^{0*} \varphi_b^+ \rightarrow \phi_c^{0*} \varphi_d^+] = \mathcal{M}[\phi_a^- \varphi_b^+ \rightarrow \phi_c^- \varphi_d^+] = \delta_{ac,db}. \quad (A5)$$

Therefore, perturbative unitarity bounds obtained from scatterings involving $|1, \frac{3}{2}\rangle$, $|0, 1\rangle$, and $|0, \frac{1}{2}\rangle$ are redundant because they are identical to those derived from $|2, \frac{3}{2}\rangle$, $|2, 1\rangle$, and $|1, \frac{1}{2}\rangle$, respectively.

In any scattering involving $SU(2)$ doublets, total isospin T must also be conserved. For this reason, two-particle states should be further labeled by $|Q, Y, T\rangle$, with $T = 0, 1$ in this model. Using Clebsch-Gordan coefficients, we can split the two-particle state $\phi_i^+ \phi_j^0$ in $|1, 1\rangle$ into

$$|1, 1, 0\rangle : \quad \phi_i^+ \phi_j^0 \equiv \frac{1}{\sqrt{2}} (\phi_i^+ \phi_j^0 - \phi_j^+ \phi_i^0), \quad (A6)$$

$$|1, 1, 1\rangle : \quad \phi_i^+ \phi_j^0 \equiv \frac{1}{\sqrt{2}} (\phi_i^+ \phi_j^0 + \phi_j^+ \phi_i^0). \quad (\text{A7})$$

In the same manner, we can split the two-particle state $\phi_i^+ \phi_j^-$ in $|0, 0\rangle$ into

$$|0, 0, 0\rangle : \quad \Phi_i \Phi_j^* \equiv \frac{1}{\sqrt{2}} (\phi_i^+ \phi_j^- + \phi_i^0 \phi_j^{0*}), \quad (\text{A8})$$

$$|0, 0, 1\rangle : \quad \overline{\Phi_i \Phi_j^*} \equiv \frac{1}{\sqrt{2}} (\phi_i^+ \phi_j^- - \phi_i^0 \phi_j^{0*}). \quad (\text{A9})$$

These redefinitions of states lead, once again, to redundant scattering matrices because

$$\mathcal{M} [\phi_a^+ \phi_b^+ \rightarrow \phi_c^+ \phi_d^+] = \mathcal{M} [\phi_{(a}^+ \phi_{b)}^0 \rightarrow \phi_{(c}^+ \phi_{d)}^0] = 2\lambda_{ca,db} + 2\lambda_{da,cb}, \quad (\text{A10})$$

$$\mathcal{M} [\phi_a^+ \phi_b^{0*} \rightarrow \phi_c^+ \phi_d^{0*}] = \mathcal{M} [\overline{\Phi_a \Phi_b^*} \rightarrow \overline{\Phi_c \Phi_d^*}] = 2\lambda_{ca,bd}. \quad (\text{A11})$$

We therefore conclude that it is sufficient to apply the partial wave unitarity bounds to the scattering matrices built out of the states listed in Table I, *i.e.* the states

$$|Q, Y, T\rangle = |2, 2, 0\rangle, \left|2, \frac{3}{2}, \frac{1}{2}\right\rangle, |2, 1, 1\rangle, |1, 1, 0\rangle, \left|1, \frac{1}{2}, \frac{1}{2}\right\rangle, |1, 0, 1\rangle, |0, 0, 0\rangle. \quad (\text{A12})$$

Appendix B: The Standard Model example

Labeling states by Q and Y

If we classify the states solely by their electric charge Q and hypercharge Y , the SM requires computing the scattering matrices for the states listed in Table X. The corresponding scattering matrices are

$ Q, Y\rangle$	State	Dimensionality
$ 2, 1\rangle$	$\phi_1^+ \phi_1^+$	1
$ 1, 1\rangle$	$\phi_1^+ \phi_1^0$	1
$ 1, 0\rangle$	$\phi_1^+ \phi_1^{0*}$	1
$ 0, 1\rangle$	$\phi_1^0 \phi_1^0$	1
$ 0, 0\rangle$	$\{\phi_1^+ \phi_1^-, \phi_1^0 \phi_1^{0*}\}$	2

TABLE X: Basis of two-particle states labelled by $|Q, Y\rangle$.

$$M_{|0,1\rangle} = M_{|1,0\rangle} = M_{|1,1\rangle} = M_{|2,1\rangle} = 2\lambda_{11,11}, \quad (\text{B1})$$

$$M_{|0,0\rangle} = \begin{bmatrix} 4\lambda_{11,11} & 2\lambda_{11,11} \\ 2\lambda_{11,11} & 4\lambda_{11,11} \end{bmatrix}. \quad (\text{B2})$$

Including Total Isospin

We now refine the classification by also incorporating the total isospin T . The two-particulate states with definite Q , Y , and T are, thus, the ones listed in Table XI.

Let transformation from the old (Q, Y) basis to the new (Q, Y, T) basis be given by

$$\begin{bmatrix} \Phi_1 \Phi_1^* \\ \overline{\Phi_1 \Phi_1^*} \end{bmatrix} = U \begin{bmatrix} \phi_1^+ \phi_1^- \\ \phi_1^0 \phi_1^{0*} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \phi_1^+ \phi_1^- \\ \phi_1^0 \phi_1^{0*} \end{bmatrix}. \quad (\text{B3})$$

$ Q, Y, T\rangle$	State	Dimensionality
$ 2, 1, 1\rangle$	$\phi_1^+ \phi_1^+$	1
$ 1, 1, 1\rangle$	$\phi_{(1)}^+ \phi_1^0$	1
$ 1, 0, 1\rangle$	$\phi_1^+ \phi_1^{0*}$	1
$ 0, 0, 1\rangle$	$\bar{\Phi}_1 \Phi_1^*$	1
$ 0, 0, 0\rangle$	$\Phi_1 \Phi_1^*$	1
$ 0, 1, 1\rangle$	$\phi_1^0 \phi_1^0$	1

TABLE XI: Basis of two-particle states labelled by $|Q, Y, T\rangle$.

As a result, the scattering matrix for the states $|0, 0\rangle$, gets diagonalized as

$$(M_{|0,0\rangle})_{\text{new}} = U(M_{|0,0\rangle})_{\text{old}} U^\dagger = \begin{bmatrix} 6\lambda_{11,11} & 0 \\ 0 & 2\lambda_{11,11} \end{bmatrix}. \quad (\text{B4})$$

Similarly, for the $|1, 1\rangle$ states, only the symmetric combination survives

$$\phi_{(1)}^+ \phi_1^0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1^+ \phi_1^0 \\ \phi_1^0 \phi_1^+ \end{bmatrix}, \quad (\text{B5})$$

meaning

$$(M_{|1,1\rangle})_{\text{new}} = 2\lambda_{11,11}. \quad (\text{B6})$$

Collecting all cases, the scattering eigenvalues are

$$M_{|0,0,0\rangle} = 6\lambda_{11,11}, \quad (\text{B7})$$

$$M_{|0,0,1\rangle} = M_{|1,1,0\rangle} = M_{|2,1\rangle} = M_{|1,0\rangle} = M_{|0,1\rangle} = 2\lambda_{11,11}. \quad (\text{B8})$$

The eigenvalues, and therefore the unitarity bounds, follow directly. This demonstrates the advantage of organizing states in the $|Q, Y, T\rangle$ basis: the scattering matrices simplify and, in this case, diagonalize naturally.

Appendix C: BounDS

We have developed **BounDS**, a **Mathematica** notebook, that automates the process of writing the quartic potential and deriving partial wave unitarity bounds for models within the class of models defined in Section II. Notice that ScannerS [29, 31] has a tool to calculate the scattering matrices, but the user must introduce the correct potential. Here, the user just introduces the number of fields and their symmetries; our program calculates the correct quartic potential and then the relevant scattering matrices. As stated, the notebook can be downloaded from

<https://github.com/andremilagre/BounDS.git>

The notebook **BounDS** is divided into three parts. To illustrate how it works, we consider the $U(1)$ -symmetric 2HDM. In step 1, the user specifies the number of fields to include.

```
nD=2;
nC=0;
nn=0;
```

If needed, the user can also impose additional symmetries on the fields by specifying the number of symmetries **nSym**, and the way the symmetries act on the fields. The latter is done by populating the vector **Sym[]**:

```

nSym = 1;

Sym[1] = {
  Φ[1] -> Φ[1],
  Φ[2] -> Exp[I a] Φ[2]
};

Assume = {a ∈ Reals};

```

The list `Assume` contains information about group theoretical parameters, such as the space they are defined in and relationships between them. For these parameters, the user *must* use Latin characters. Furthermore, the function `Conjugate[]` may be used for symmetries that require conjugation of a field. `Bounds` has been successfully tested with a wide range of symmetry groups, including both discrete and continuous, abelian and non-abelian cases.

After evaluating the cells in step 1, the user can simply run all the cells in step 2. Under the hood, this module does the following operations:

- Lists the minimal set of linearly independent quartic couplings allowed by the symmetries.
- Assembles the 7 independent scattering matrices defined in Section III B.
- Block-diagonalizes the scattering matrices by swapping rows and columns.

Finally, step 3 is dedicated to visualizing and analyzing the output. By calling `Potential4`, the user can output the quartic part of the most general scalar potential allowed by the symmetries of the model. Anywhere in the code, the `Mathematica` function `TeXForm` can be called to get the $\text{\LaTeX}$ source code:

```
Potential4 //TeXForm
```

$$V_4 = \lambda_{11,11} (\Phi_1^\dagger \Phi_1)^2 + 2\lambda_{11,22} \Phi_2^\dagger \Phi_2 \Phi_1^\dagger \Phi_1 + 2\lambda_{12,21} \Phi_1^\dagger \Phi_2 \Phi_2^\dagger \Phi_1 + \lambda_{22,22} (\Phi_2^\dagger \Phi_2)^2. \quad (\text{C1})$$

Furthermore, the user may access the basis vector and the corresponding scattering matrix by specifying the value of Q , Y , and T , and evaluating the functions `Basis[Q, Y, T]` and `ScatteringMatrix[Q, Y, T]`, respectively:

```

Q = 0;
Y = 0;
T = 0;

Basis[Q, Y, T] //TeXForm
ScatteringMatrix[Q, Y, T] //TeXForm

```

$$\left(\Phi_2 \Phi_2^\dagger, \Phi_1 \Phi_1^\dagger, \Phi_2 \Phi_1^\dagger, \Phi_1 \Phi_2^\dagger \right), \quad (\text{C2})$$

$$\begin{bmatrix} 6\lambda_{22,22} & 2(2\lambda_{11,22} + \lambda_{12,21}) & 0 & 0 \\ 2(2\lambda_{11,22} + \lambda_{12,21}) & 6\lambda_{11,11} & 0 & 0 \\ 0 & 0 & 2(\lambda_{11,22} + 2\lambda_{12,21}) & 0 \\ 0 & 0 & 0 & 2(\lambda_{11,22} + 2\lambda_{12,21}) \end{bmatrix}. \quad (\text{C3})$$

Note that the ordering of the two-particle states in the basis vector may differ from the expected convention. This is an artifact of the block-diagonalization routine. However, the corresponding eigenvalues are unaffected by this reordering.

Finally, the list of eigenvalues for a given scattering matrix can be obtained by calling `EigenList[Q, Y, T]`, again, with definite values for Q , Y , and T :

```

Q = 0;
Y = 0;
T = 0;

EigenList[Q, Y, T] //FullSimplify //TeXForm

```

$$\begin{aligned}
& 2(\lambda_{11,22} + 2\lambda_{12,21}), \\
& 2(\lambda_{11,22} + 2\lambda_{12,21}), \\
& 3\lambda_{11,11} + 3\lambda_{22,22} - \sqrt{4(2\lambda_{11,22} + \lambda_{12,21})^2 + 9(\lambda_{11,11} - \lambda_{22,22})^2}, \\
& 3\lambda_{11,11} + 3\lambda_{22,22} + \sqrt{4(2\lambda_{11,22} + \lambda_{12,21})^2 + 9(\lambda_{11,11} - \lambda_{22,22})^2}.
\end{aligned} \tag{C4}$$

Appendix D: $\mathbb{Z}_2$ -Symmetric 2HDM

In a process involving complex scalars that are singlets under the SM gauge group, there are CP-even and CP-odd components that scatter independently. Furthermore, if there are additional flavour symmetries, each two-particle state must also be labelled by its corresponding charge S_α , following the idea of [17]. Therefore, we can label all states by,

$$|Q, Y, T, \mathbf{CP}, \mathbf{S}_1, \mathbf{S}_2, \dots\rangle. \tag{D1}$$

Let us consider the $\mathbb{Z}_2$ -symmetric 2HDM with the two-particle states labeled by $|Q, Y, T, \mathbb{Z}_2\rangle$. The basis of states is explicitly given in Table XII, where, in the last two lines, we have used the definition in Eq. (27).

$ Q, Y, T, \mathbb{Z}_2\rangle$	State	Dimensionality
$ 2, 1, 1, +1\rangle$	$\{\phi_1^+ \phi_1^+, \phi_2^+ \phi_2^+\}$	2
$ 2, 1, 1, -1\rangle$	$\phi_1^+ \phi_2^+$	1
$ 1, 1, 0, -1\rangle$	$\phi_{[1}^+ \phi_{2]}^0$	1
$ 1, 0, 1, +1\rangle$	$\{\phi_1^+ \phi_1^{0*}, \phi_2^+ \phi_2^{0*}\}$	2
$ 1, 0, 1, -1\rangle$	$\{\phi_1^+ \phi_2^{0*}, \phi_2^+ \phi_1^{0*}\}$	2
$ 0, 0, 0, +1\rangle$	$\{\Phi_1 \Phi_1^*, \Phi_2 \Phi_2^*\}$	2
$ 0, 0, 0, -1\rangle$	$\{\Phi_1 \Phi_2^*, \Phi_2 \Phi_1^*\}$	2

TABLE XII: Basis of two-particle states labelled by $|Q, Y, T, \mathbb{Z}_2\rangle$.

The scattering matrices are thus

$$M_{|2, 1, 1, +1\rangle} = \begin{bmatrix} 2\lambda_{11,11} & 2\lambda_{12,12} \\ 2\lambda_{12,12} & 2\lambda_{22,22} \end{bmatrix}, \tag{D2}$$

$$M_{|2, 1, 1, -1\rangle} = 2(\lambda_{11,22} + \lambda_{12,21}), \tag{D3}$$

$$M_{|1, 1, 0, -1\rangle} = 2(\lambda_{11,22} - \lambda_{12,21}), \tag{D4}$$

$$M_{|1, 0, 1, +1\rangle} = \begin{bmatrix} 2\lambda_{11,11} & 2\lambda_{12,21} \\ 2\lambda_{12,21} & 2\lambda_{22,22} \end{bmatrix}, \tag{D5}$$

$$M_{|1, 0, 1, -1\rangle} = \begin{bmatrix} 2\lambda_{11,22} & 2\lambda_{12,12} \\ 2\lambda_{12,12} & 2\lambda_{11,22} \end{bmatrix}, \tag{D6}$$

$$M_{|0, 0, 0, +1\rangle} = \begin{bmatrix} 6\lambda_{11,11} & 2(2\lambda_{11,22} + \lambda_{12,21}) \\ 2(2\lambda_{11,22} + \lambda_{12,21}) & 6\lambda_{22,22} \end{bmatrix}, \tag{D7}$$

$$M_{|0, 0, 0, -1\rangle} = \begin{bmatrix} 2(\lambda_{11,22} + 2\lambda_{12,21}) & 6\lambda_{12,12} \\ 6\lambda_{12,12} & 2(\lambda_{11,22} + 2\lambda_{12,21}) \end{bmatrix}. \tag{D8}$$

Using the substitutions in Table IV we recover the well-known unitarity bounds for the $\mathbb{Z}_2$ -symmetric 2HDM, shown in Eqs. (63)–(68).

We find that, by labelling states with flavour symmetries in addition to charge, hypercharge, and total isospin, the scattering matrices simplify further and reduce in dimensionality compared to those in Section VC1. This reduction

greatly facilitates the subsequent calculations, as shown in this appendix.

-
- [1] **ATLAS** Collaboration, G. Aad et al., *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett. B* **716** (2012) 1–29, [[arXiv:1207.7214](#)].
 - [2] **CMS** Collaboration, S. Chatrchyan et al., *Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC*, *Phys. Lett. B* **716** (2012) 30–61, [[arXiv:1207.7235](#)].
 - [3] **Particle Data Group** Collaboration, C. Patrignani et al., *Review of Particle Physics*, *Chin. Phys. C* **40** (2016), no. 10 100001.
 - [4] **Planck** Collaboration, N. Aghanim et al., *Planck 2018 results. VI. Cosmological parameters*, *Astron. Astrophys.* **641** (2020) A6, [[arXiv:1807.06209](#)]. [Erratum: *Astron. Astrophys.* 652, C4 (2021)].
 - [5] M. Cirelli, A. Strumia, and J. Zupan, *Dark Matter*, [[arXiv:2406.01705](#)].
 - [6] C. P. Burgess, M. Pospelov, and T. ter Veldhuis, *The Minimal model of nonbaryonic dark matter: A Singlet scalar*, *Nucl. Phys. B* **619** (2001) 709–728, [[hep-ph/0011335](#)].
 - [7] **GAMBIT** Collaboration, P. Athron et al., *Status of the scalar singlet dark matter model*, *Eur. Phys. J. C* **77** (2017), no. 8 568, [[arXiv:1705.07931](#)].
 - [8] A. Ilnicka, M. Krawczyk, and T. Robens, *Inert Doublet Model in light of LHC Run I and astrophysical data*, *Phys. Rev. D* **93** (2016), no. 5 055026, [[arXiv:1508.01671](#)].
 - [9] M. Gonçalves, M. Mühlleitner, R. Santos, and T. Trindade, *Dark Matter in Multi-Singlet Extensions of the Standard Model*, [[arXiv:2505.07753](#)].
 - [10] U. Ellwanger, C. Hugonie, and A. M. Teixeira, *The Next-to-Minimal Supersymmetric Standard Model*, *Phys. Rept.* **496** (2010) 1–77, [[arXiv:0910.1785](#)].
 - [11] G. Arcadi, F. Costa, A. Goudelis, and O. Lebedev, *Higgs portal dark matter freeze-in at stronger coupling: observational benchmarks*, *JHEP* **07** (2024) 044, [[arXiv:2405.03760](#)].
 - [12] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, 3, 2014.
 - [13] H. E. Logan, *Lectures on perturbative unitarity and decoupling in Higgs physics*, [[arXiv:2207.01064](#)].
 - [14] H. G. J. Veltman, *The Equivalence Theorem*, *Phys. Rev. D* **41** (1990) 2294.
 - [15] B. W. Lee, C. Quigg, and H. B. Thacker, *The Strength of Weak Interactions at Very High-Energies and the Higgs Boson Mass*, *Phys. Rev. Lett.* **38** (1977) 883–885.
 - [16] B. W. Lee, C. Quigg, and H. B. Thacker, *Weak Interactions at Very High-Energies: The Role of the Higgs Boson Mass*, *Phys. Rev. D* **16** (1977) 1519.
 - [17] I. F. Ginzburg and I. P. Ivanov, *Tree-level unitarity constraints in the most general 2hdm*, [[hep-ph/0508020](#)].
 - [18] S. Kanemura and K. Yagyu, *Unitarity bound in the most general two Higgs doublet model*, *Phys. Lett. B* **751** (2015) 289–296, [[arXiv:1509.06060](#)].
 - [19] M. P. Bento, H. E. Haber, J. C. Romão, and J. P. Silva, *Multi-higgs doublet models: physical parametrization, sum rules and unitarity bounds*, *JHEP* (2017) [[arXiv:1708.09408](#)].
 - [20] M. P. Bento, H. E. Haber, J. C. Romão, and J. P. Silva, *Multi-Higgs doublet models: the Higgs-fermion couplings and their sum rules*, *JHEP* **10** (2018) 143, [[arXiv:1808.07123](#)].
 - [21] K. Hally, H. E. Logan, and T. Pilkington, *Constraints on large scalar multiplets from perturbative unitarity*, *Phys. Rev. D* **85** (2012) 095017, [[arXiv:1202.5073](#)].
 - [22] A. Milagre and L. Lavoura, *Unitarity constraints on large multiplets of arbitrary gauge groups*, *Nucl. Phys. B* **1004** (2024) 116542, [[arXiv:2403.12914](#)].
 - [23] L. C. Bresciani, G. Levati, and P. Paradisi, *Amplitudes and partial wave unitarity bounds*, [[arXiv:2504.12855](#)].
 - [24] T. Steingasser, M. P. Hertzberg, and D. I. Kaiser, *Precision Unitarity Calculations in Inflationary Models*, [[arXiv:2505.20386](#)].
 - [25] W. Grimus, L. Lavoura, O. M. Ogreid, and P. Osland, *A Precision constraint on multi-Higgs-doublet models*, *J. Phys. G* **35** (2008) 075001, [[arXiv:0711.4022](#)].
 - [26] W. Grimus, L. Lavoura, O. M. Ogreid, and P. Osland, *The Oblique parameters in multi-Higgs-doublet models*, *Nucl. Phys. B* **801** (2008) 81–96, [[arXiv:0802.4353](#)].
 - [27] M. P. Bento, “Sum rules and unitarity in multi-Higgs doublet models.” Web page: <https://fenix.tecnico.ulisboa.pt/cursos/meft/dissertacao/1972678479053628>, 2017. Master’s Thesis, IST - University of Lisbon.
 - [28] M. P. Bento, J. C. Romão, and J. P. Silva, *Unitarity bounds for all symmetry-constrained 3HDMs*, *JHEP* **08** (2022) 273, [[arXiv:2204.13130](#)].

- [29] M. Mühlleitner, M. O. P. Sampaio, R. Santos, and J. Wittbrodt, *ScannerS: parameter scans in extended scalar sectors*, *Eur. Phys. J. C* **82** (2022), no. 3 198, [[arXiv:2007.02985](#)].
- [30] R. Boto, T. P. Rebelo, J. C. Romão, and J. P. Silva, *Machine Learning in the 2HDM2S model for Dark Matter*, [arXiv:2509.01677](#).
- [31] R. Coimbra, M. O. P. Sampaio, and R. Santos, *ScannerS: Constraining the phase diagram of a complex scalar singlet at the LHC*, *Eur. Phys. J. C* **73** (2013) 2428, [[arXiv:1301.2599](#)].