

Constraint Qualification for Generic Parameter Families of Constraints in Optimization

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Abstract

Constraint qualifications (CQs) are central to the local analysis of constrained optimization. In this paper, we completely determine the validity of the four classical CQs—LICQ, MFCQ, ACQ, and GCQ—for constraint map-germs that arise in generic four-parameter families. Our approach begins by proving that all four CQs are invariant under the action of the group $\mathcal{K}[G]$ and under the operation of reduction. As a consequence, the verification of CQ-validity for a generic constraint reduces to checking CQ-validity on the $\mathcal{K}[G]$ -normal forms of fully reduced map-germs. Such normal forms have been classified in our recent work. In the present paper, we verify which CQs hold in each germ appearing in the classification tables from that work. This analysis provides a complete picture of the generic landscape of the four classical CQs. Most notably, we find that there exist numerous generic map-germs for which GCQ holds while all stronger CQs fail, showing that the gap between GCQ and the other qualifications is not an exceptional phenomenon but arises generically.

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1 Introduction

A constrained optimization problem asks for the minimization of objective functions subject to a collection of equality and inequality constraints. Formally, one seeks minimizers (or more generally, “Pareto solutions”, for its definition, see e.g., [20]) of functions $f_1(x), \dots, f_p(x)$ under the conditions $g_1(x), \dots, g_q(x) \leq 0$ and $h_1(x) = \dots = h_r(x) = 0$. A fundamental tool for characterizing solutions is the Karush–Kuhn–Tucker (KKT) condition, which requires the existence of Lagrange multipliers such that first-order stationarity (the gradients of the objective and active constraints are balanced) and complementary slackness (inactive constraints have zero corresponding multipliers) hold [6, 5, 21, 16, 20]. In unconstrained optimization, this reduces to the familiar first-order condition $\nabla f(x) = 0$, which every local minimizer satisfies. In constrained problems, by contrast, the existence of multipliers does not automatically follow from local minimality. This fact is precisely what motivates constraint qualifications (CQs): they are assumptions placed only on the constraint system, ensuring the existence of multipliers and thus the validity of the KKT condition at all local minimizers.

In this paper we focus on four classical CQs that have been most widely studied in optimization theory and applied practice (see Definitions 3.1–3.4 for the precise definitions of them):

1. **Linear Independence Constraint Qualification (LICQ)** [13]: requires linear independence of the gradients of all active constraints. It implies uniqueness of the associated Lagrange multipliers.
2. **Mangasarian–Fromovitz Constraint Qualification (MFCQ)** [17]: weaker than LICQ, guaranteeing nonemptiness and boundedness of the multiplier set [8]. It also underlies stability results of feasible sets [10].
3. **Abadie Constraint Qualification (ACQ)** [1]: equivalent to metric regularity in the differentiable convex inequality setting [14].
4. **Guignard Constraint Qualification (GCQ)** [11]: the weakest among the four; it is necessary and sufficient for the KKT condition to hold at local minimizers of any objection functions [9].

These conditions form a strict hierarchy [22]:

$$\text{LICQ} \implies \text{MFCQ} \implies \text{ACQ} \implies \text{GCQ}. \quad (1.A)$$

In practice, LICQ and MFCQ are frequently invoked because they can be checked by simple rank conditions or directional arguments. By contrast, ACQ and GCQ, despite their generality, require analyzing tangent cones and are therefore much harder to verify directly. It is also known that each implication above is *strict*: none of the reverse implications hold in general. Concrete counterexamples witnessing the failure of the converses ($\text{MFCQ} \not\Rightarrow \text{LICQ}$, $\text{ACQ} \not\Rightarrow \text{MFCQ}$, $\text{GCQ} \not\Rightarrow \text{ACQ}$) are well documented in the literature (see, e.g., [22]).

However, the available constructions tend to be rather bespoke and leave open whether such separations occur *generically*—that is, in typical parameterized constraint families encountered in practice. In Appendix A, we investigate genericity of some of the counterexamples in the literature and show that some of them are far from generic. One of our aims is to address this gap by identifying generic constraints in which a stronger CQ fails while a weaker one still holds. As shown by our later results, such gaps indeed arise generically; see Theorem 4.2 and Tables 4, 5, and 6 for details.

The aim of this work is to determine, for each point of a feasible set arising from a generic parameter family of constraints, whether or not a given constraint qualification (CQ) holds. CQ-validity is a local property: whether a CQ holds at a point depends only on the germ of the constraint at that point. In [12], we classify the *full reductions* of constraint map-germs appearing in generic four parameter families up to $\mathcal{K}[G]$ -equivalence (see Section 2 for the definitions of (full) reduction and $\mathcal{K}[G]$ -equivalence). *As we establish below, the four CQs under study are invariant under $\mathcal{K}[G]$ -equivalence and reduction; hence the classification of [12] is the natural vehicle for deciding CQ-validity.* We first prove that reduction commutes with $\mathcal{K}[G]$ -equivalence (Lemmas 2.1 and 2.3) and that LICQ, MFCQ, ACQ, and GCQ are invariant under $\mathcal{K}[G]$ -equivalence and reduction (Theorems 3.1 and 3.2). Consequently, it suffices to check, for each reduced normal-form listed in [12], which of the four CQs hold. We carry out this verification and compile complete validity tables across all generic classes (Theorem 4.2). In particular, we find that GCQ is satisfied in a broader range of the generic classes than any of the other stronger CQs (LICQ, MFCQ, ACQ), underscoring its role as an essential fallback condition that guarantees the existence of Lagrange multipliers when those stronger qualifications fail to hold. These results collectively provide a generic picture of how and when each CQ is satisfied.

Beyond the immediate theoretical contributions, we anticipate our framework will support applications in the following ways:

- **Generic examples for CQ research:** By exhibiting constraint systems that satisfy, say, MFCQ but violate LICQ, or that satisfy ACQ but not MFCQ, etc., one can confirm the well-known strictness relations or investigate potential new CQs lying in the gaps. The genericity of these families underscores that these degeneracies are neither rare nor pathological in practical contexts.
- **Benchmarking and algorithmic impact:** Optimization algorithms often assume or exploit certain CQs. Having a collection of representative examples—classified by which CQs hold—can facilitate more rigorous testing of algorithmic reliability and performance.
- **Path to efficient recognition of ACQ and GCQ:** In principle, verifying whether ACQ or GCQ holds can be challenging because they require analysis of tangent cones. In upcoming work, we plan to design recognition algorithms that use the singularity classification of [12] to decide systematically which class a given constraint system belongs to. Once that class is identified, our results immediately specify whether ACQ or GCQ is satisfied. This can significantly simplify the process of CQ checking in software for large-scale or complex problems.

We note certain limitations in this study. First, our results restrict to problems involving C^∞ smooth equality and inequality constraints, leaving out non-smooth or more abstract constraint structures (such as cone constraints). Second, we focus exclusively on CQs that

are invariant under the action of $\mathcal{K}[G]$ studied in [12], thereby excluding conditions like Pseudonormality [7], Slater's CQ (specific to convex settings) [23] or linearity conditions [6, 5, 21, 16]. Third, we target CQs that appear in a *generic* sense as determined by the equivalences in [12], leaving aside specialized constraints like the constant rank CQ (CRCQ). However, we remark that the condition CRCQ often implies MFCQ for suitably reformulated problems [15], so our omission of CRCQ does not substantially affect the generic analysis.

The rest of this paper is organized as follows. In Section 2, we briefly recall the necessary background on standard CQs and the main classification results from [12]. Then, in Section 3, we introduce the mathematical definition for each CQ, and show that these are invariant under $\mathcal{K}[G]$ -equivalence and reduction. In Section 4, we verify, for each class, which of the standard CQs (LICQ, MFCQ, ACQ, GCQ) are satisfied and discuss the resulting hierarchy in detail.

2 Preliminaries: constraints, $\mathcal{K}[G]$ -equivalence, reduction, and classification of constraint map-germs

In this section, we recall the basic notions of constraint map-germs, feasible set-germs, and the group $\mathcal{K}[G]$, following [12]. We then discuss the notion of reduction of a constraint map-germ and state Theorem 5.1 from [12], which classifies constraint map-germs appearing in generic parameter families of constraints with up to 4-parameters.

The precise definition of constraint map-germ and feasible set is as follows.

Definition 2.1 (Constraint map-germ and feasible set). Let $g = (g_1, \dots, g_q): (\mathbb{R}^n, 0) \rightarrow \mathbb{R}^q$ and $h = (h_1, \dots, h_r): (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^r, 0)$ be smooth map-germs at $0 \in \mathbb{R}^n$. We call (g, h) a *constraint map-germ* at 0, where g_i represent *inequality* constraint functions and h_j represent *equality* constraint functions. The corresponding *feasible set-germ* $M(g, h)$ is the germ at 0 of the set

$$M(g, h) = \{x \in (\mathbb{R}^n, 0) \mid g_i(x) \leq 0 \text{ for all } 1 \leq i \leq q, \ h_j(x) = 0 \text{ for all } 1 \leq j \leq r\}.$$

If $g_i(0) > 0$ for some i , then $M(g, h)$ is empty. On the other hand, if $g_i(0) < 0$ for some i , then g_i does not affect the feasible set-germ near 0 and may be removed from the constraint list without changing $M(g, h)$.

We next introduce $\mathcal{K}[G]$ -equivalence. Following [12], let $\mathcal{K}$ denote Mather's group [18] and let $\mathcal{K}[G]$ be the subgroup of $\mathcal{K}$ consisting of those coordinate changes that preserve the inequality/equality structure of constraints. Concretely, a pair (Φ, Ψ) belongs to $\mathcal{K}[G]$ if $\Phi: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ is a diffeomorphism-germ of the source, $\Psi: (\mathbb{R}^n, 0) \rightarrow G$ is a smooth map-germ into a target group G of block matrices of the form

$$G = \left\{ \begin{pmatrix} C & B \\ O_{r,q} & A \end{pmatrix} \mid C \in G_{gp}, \ B \in M_{q,r}(\mathbb{R}), \ A \in GL(r, \mathbb{R}) \right\},$$

where $G_{gp} = G_d \rtimes P_q$, the semidirect product of G_d and the group of $q \times q$ permutation matrices P_q , $O_{r,q}$ is the $r \times q$ zero matrix, $M_{q,r}(\mathbb{R})$ is the set of $q \times r$ matrices, and $GL(r, \mathbb{R})$ is the set of $r \times r$ regular matrices. For a constraint map-germ (g, h) , the action of $(\Phi, \Psi) \in \mathcal{K}[G]$ is given by

$$(\Phi, \Psi) \cdot (g(x), h(x)) = \left(C(x) g(\Phi^{-1}(x)) + B(x) h(\Phi^{-1}(x)), A(x) h(\Phi^{-1}(x)) \right),$$

where $C(x), B(x), A(x)$ denote the respective block components of $\Psi(x)$. Two constraint map-germs (g, h) and (g', h') are $\mathcal{K}[G]$ -equivalent if they lie in the same $\mathcal{K}[G]$ -orbit, i.e., there exists $(\Phi, \Psi) \in \mathcal{K}[G]$ such that

$$(g, h) = (\Phi, \Psi) \cdot (g', h').$$

By construction, $\mathcal{K}[G]$ -equivalence preserves the feasible set-germs: if (g, h) is $\mathcal{K}[G]$ -equivalent to (g', h') , then $M(g, h)$ is diffeomorphic to $M(g', h')$ via the map Φ .

The concept of *reduction* deals with simplifying a constraint map-germ by removing inactive (strictly satisfied) inequalities and by restricting it to the submanifold defined by certain equality constraints. Again, we summarize the definition from [12]:

Definition 2.2 (Reduction). Let $(g, h): (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^q \times \mathbb{R}^r, 0)$ be a constraint map-germ with nonempty feasible set $M(g, h)$,

$$(k) = (k_1, \dots, k_{q-s}) \subset \{1, \dots, q\}$$

be a subset of indices corresponding to $q - s$ inequality components $g_{k_i}(0)$ that are all < 0 (i.e., inactive), and

$$(i) = (i_1, \dots, i_{r-\ell}) \subset \{1, \dots, r\}$$

be a subset for which $dh_{i_1}(0), \dots, dh_{i_{r-\ell}}(0)$ are linearly independent. Let $\iota_{(i)}: (\mathbb{R}^{n-r+\ell}, 0) \rightarrow (\mathbb{R}^n, 0)$ be an immersion-germ to the submanifold $(h_{i_1}, \dots, h_{i_{r-\ell}})^{-1}(0)$ near 0. Then the *reduction* of (g, h) relative to (k) and (i) is given by the map-germ

$$(g, h)_{\iota_{(i)}, (k)} := (g_{\iota_{(i)}, (k)}, h_{\iota_{(i)}}) := \left(g_1 \circ \iota_{(i)}, \dots, g_q \circ \iota_{(i)}, h_1 \circ \iota_{(i)}, \dots, h_r \circ \iota_{(i)} \right),$$

The reduction $(g_{\iota_{(i)}, (k)}, h_{\iota_{(i)}})$ of (g, h) is called a *full reduction* if $g_{\iota_{(i)}, (k)}(0) = 0$ and the rank of $dh_{\iota_{(i)}, 0}$ is zero.

Specifically, if the resulting constraint satisfies that its all the inequality constraints are active and the rank of $dh_{\iota_{(i)}, 0}$ is zero, we call the reduction full reduction.

Note that the feasible set-germ of (g, h) is preserved by this operation (up to diffeomorphism).

2.1 Relation between reduction and $\mathcal{K}[G]$ -equivalence

In this section, we clarify how the reduction of a constraint map-germ interacts with the group-action by $\mathcal{K}[G]$. First, we show that if two constraint map-germs have the same numbers of active inequality and equality constraints, and their respective reductions are $\mathcal{K}[G]$ -equivalent, then the original (unreduced) germs are also $\mathcal{K}[G]$ -equivalent (Lemma 2.1). Second, we show that conversely when both germs are finitely $\mathcal{K}[G]$ -determined, $\mathcal{K}[G]$ -equivalence of the unreduced germs forces $\mathcal{K}[G]$ -equivalence of any pair of reductions that retain the same numbers of constraints (Lemma 2.3). These results imply that one may freely work with fully reduced representatives when classifying constraint map-germs under $\mathcal{K}[G]$ without losing information. Any $\mathcal{K}[G]$ -orbit is uniquely determined by, and can be recovered from, the orbit of its reduction.

Lemma 2.1. *Let (g, h) and (g', h') be two constraint map-germs with the equal number of inequality and equality constraints. Then, (g, h) and (g', h') are $\mathcal{K}[G]$ -equivalent if their reductions $(g, h)_{\iota_{(i)}, (k)}$ and $(g', h')_{\iota_{(i')}, (k')}$ are $\mathcal{K}[G]$ -equivalent.*

Proof. Suppose $(g, h)_{\iota_{(i)}, (k)}$ and $(g', h')_{\iota_{(i')}, (k')}$ are $\mathcal{K}[G]$ -equivalent. In what follows, we write $h = (h_R, h_S)$, $h' = (h'_R, h'_S)$, $g = (g_R, g_S)$, $g' = (g'_R, g'_S)$, where $h_R = (h_{i_1}, \dots, h_{i_{r-\ell}})$, $h'_R = (h'_{i'_1}, \dots, h'_{i'_{r-\ell}})$, h_S, h'_S are remaining part of h and h' , respectively, and we take g_R, g_S, g'_R, g'_S in the same way. By the definition of constraint map-germs, $h(0) = h'(0) = 0$ holds. Without loss of generality, by appropriate permutation of components, we can assume (g, h) and (g', h') can be written as above. By choosing coordinates of $\mathbb{R}^n$ appropriately, we can suppose $h_{i_j}(x) = x_j$ for $j = 1, \dots, r - \ell$ and $\iota_{(i)}(x_{r-\ell+1}, \dots, x_n) = (0, \dots, 0, x_{r-\ell+1}, \dots, x_n)$. By the assumption, there is a diffeomorphism-germ $\tilde{\phi}: (\mathbb{R}^{n-r+\ell}, 0) \rightarrow (\mathbb{R}^{n-r+\ell}, 0)$ and map-germs $\tilde{C}_{22}: (\mathbb{R}^{n-r+\ell}, 0) \rightarrow G_{gp}$, $\tilde{B}_{22}: (\mathbb{R}^{n-r+\ell}, 0) \rightarrow M_{q-s, \ell}(\mathbb{R})$, and $\tilde{A}_{22}: (\mathbb{R}^{n-r+\ell}, 0) \rightarrow GL(\mathbb{R}, \ell)$ such that

$$\begin{pmatrix} \tilde{C}_{22} & \tilde{B}_{22} \\ O & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} g_S \circ \iota_{(i)}(x_{r-\ell+1}, \dots, x_n) \\ h_S \circ \iota_{(i)}(x_{r-\ell+1}, \dots, x_n) \end{pmatrix} = \begin{pmatrix} g'_S \circ \iota_{(i')} \circ \phi(x_{r-\ell+1}, \dots, x_n) \\ h'_S \circ \iota_{(i')} \circ \phi(x_{r-\ell+1}, \dots, x_n) \end{pmatrix}$$

holds. We define $\Psi: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such as

$$\Psi(x) = x_1 d(h'_{i'_1})_0 + \dots + x_{r-\ell} d(h'_{i'_{r-\ell}})_0 + \iota_{(i')} \circ \phi(x_{r-\ell+1}, \dots, x_n).$$

Then, Ψ is a diffeomorphism-germ. Then,

$$\begin{aligned} g'_S \circ \Psi(x) &= g'_S \circ \Psi(0, \dots, 0, x_{r-\ell+1}, \dots, x_n) + \sum_{j=1}^{r-\ell} x_j \tilde{g}_j(x) \\ &= g'_S \circ \iota_{(i')} \circ \phi(x_{r-\ell+1}, \dots, x_n) + \sum_{j=1}^{r-\ell} x_j \tilde{g}_j(x) \end{aligned}$$

holds by Hadamard's theorem by choosing $\tilde{g}_j$ appropriately. In the similar manner,

$$h'_R \circ \Psi(x) = h'_R \circ \Psi(0, \dots, 0, x_{r-\ell+1}, \dots, x_n) + \sum_{j=1}^{r-\ell} x_j \left(\tilde{h}'_R \right)_j(x) = \sum_{j=1}^{r-\ell} x_j \left(\tilde{h}'_R \right)_j(x),$$

by noting that $h'_R \circ \iota_{(i')} \circ \phi = 0$ and again choosing $\left(\tilde{h}'_R \right)_j$ appropriately. In addition,

$$\begin{aligned} h'_S \circ \Psi(x) &= h'_S \circ \Psi(0, \dots, 0, x_{r-\ell+1}, \dots, x_n) + \sum_{j=1}^{r-\ell} x_j \left(\tilde{h}'_S \right)_j(x), \\ &= h'_S \circ \iota_{(i')} \circ \phi(x_{r-\ell+1}, \dots, x_n) + \sum_{j=1}^{r-\ell} x_j \left(\tilde{h}'_S \right)_j(x) \end{aligned}$$

holds. By noting that $h_R(x) = (x_1, \dots, x_{r-\ell})$, there exist $\tilde{A}_{11}, \tilde{A}_{21}$ and $\tilde{B}_{21}$ such that

$$\begin{aligned} \begin{pmatrix} g'_R \circ \Psi(x) \\ g'_S \circ \Psi(x) \\ h'_R \circ \Psi(x) \\ h'_S \circ \Psi(x) \end{pmatrix} &= \begin{pmatrix} g'_R \circ \Psi(x) \\ g'_S \circ \iota_{(i')} \circ \phi(x_{r-\ell+1}, \dots, x_n) \\ 0 \\ h'_S \circ \iota_{(i')} \circ \phi(x_{r-\ell+1}, \dots, x_n) \end{pmatrix} + \begin{pmatrix} O & O & O & O \\ O & O & \tilde{B}_{21} & O \\ O & O & \tilde{A}_{11} & O \\ O & O & \tilde{A}_{21} & O \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ h_R(x) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} I & O & O & O \\ O & \tilde{C}_{22} & \tilde{B}_{21} & \tilde{B}_{22} \\ O & O & \tilde{A}_{11} & O \\ O & O & \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} g'_R \circ \Psi(x) \\ g_S \circ \iota_{(i)}(x_{r-\ell+1}, \dots, x_n) \\ h_R(x) \\ h_S \circ \iota_{(i)}(x_{r-\ell+1}, \dots, x_n) \end{pmatrix} \end{aligned}$$

holds. We can also choose $(\tilde{g}_S)_j, (\tilde{h}_S)_j$ so that the equalities

$$\begin{aligned} g_S \circ \iota_{(i)}(x_{r-\ell+1}, \dots, x_n) &= g_S(0, \dots, 0, x_{r-\ell+1}, \dots, x_n) \\ &= g_S(x_1, \dots, x_n) - \sum_{j=1}^{r-\ell} x_j (\tilde{g}_S)_j(x) \end{aligned}$$

and

$$\begin{aligned} h_S \circ \iota_{(i)}(x_{r-\ell+1}, \dots, x_n) &= h_S(0, \dots, 0, x_{r-\ell+1}, \dots, x_n) \\ &= h_S(x_1, \dots, x_n) - \sum_{j=1}^{r-\ell} x_j (\tilde{h}_S)_j(x) \end{aligned}$$

hold. Since all the components of $g_R(0)$ is less than 0, we can define the diagonal matrix $\tilde{C}_{11}$ such that $(\tilde{C}_{11})_{j,j} = g'_{k'_j} \circ \Psi(x) / g_{k_j}(x)$ holds for each $j \in \{1, \dots, s\}$. We then obtain

$$\begin{pmatrix} g'_R \circ \Psi(x) \\ g'_S \circ \Psi(x) \\ h'_R \circ \Psi(x) \\ h'_S \circ \Psi(x) \end{pmatrix} = \begin{pmatrix} \tilde{C}_{11} & O & O & O \\ O & \tilde{C}_{22} & \tilde{B}'_{21} & \tilde{B}_{22} \\ O & O & \tilde{A}_{11} & O \\ O & O & \tilde{A}'_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} g_R(x) \\ g_S(x) \\ h_R(x) \\ h_S(x) \end{pmatrix},$$

where $\tilde{B}'_{21}$ and $\tilde{A}'_{21}$ are modified accordingly. The image of $\begin{pmatrix} \tilde{C}_{11} & O \\ O & \tilde{C}_{22} \end{pmatrix}$ is in G_{gp} by the construction. Moreover, the matrix $\begin{pmatrix} \tilde{A}_{11} & O \\ \tilde{A}'_{21} & \tilde{A}_{22} \end{pmatrix}$ is in $GL(\mathbb{R}, r)$ since the image of $\tilde{A}_{22}$ is in $GL(\mathbb{R}, \ell)$, $d(h'_R \circ \Psi)_0 = \tilde{A}_{11}(0) \cdot d(h_R)_0$ and its rank is $r - \ell$. This proves that the two constraint map-germs (g, h) and (g', h') are $\mathcal{K}[G]$ -equivalent. $\square$

The converse of this theorem holds if one of the constraint map-germs (g, h) and (g', h') is finitely $\mathcal{K}[G]$ -determined. (In that case, of course both of the constraint map-germs are finitely $\mathcal{K}[G]$ -determined.) To show this, let us introduce some necessary terminologies. Let $\mathcal{E}_n = \{f | f: (\mathbb{R}^n, 0) \rightarrow \mathbb{R}\}$ be the ring of function-germs. For a constraint map-germ $(g, h): (\mathbb{R}^n, 0) \rightarrow \mathbb{R}^{q+r}$ whose feasible set contains the origin (this condition is equivalent to $g(0) \leq 0$), we define an $\mathbb{R}$ -algebra

$$Q(h) = \mathcal{E}_n / \langle h_1, \dots, h_r \rangle_{\mathcal{E}_n}$$

by following Mather [19] where $\langle h_1, \dots, h_r \rangle_{\mathcal{E}_n}$ is the ideal in $\mathcal{E}_n$ generated by $h_1, \dots, h_r$. In addition, we define $Q_k(h) = Q(h) / \left(\langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1} \cdot Q(h) \right)$ for $k \in \mathbb{N}$. Note that

$$\langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1} \cdot Q(h) \cong \left(\langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1} + \langle h_1, \dots, h_r \rangle_{\mathcal{E}_n} \right) / \langle h_1, \dots, h_r \rangle_{\mathcal{E}_n}$$

holds. Then, the third isomorphism theorem implies that

$$Q_k(h) \cong \mathcal{E}_n / \left(\langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1} + \langle h_1, \dots, h_r \rangle_{\mathcal{E}_n} \right).$$

Lemma 2.2. *Let $(g, h), (g', h') : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}^{q+r}$ be two constraint map-germs whose feasible set-germs are non-empty and let $k \in \mathbb{N}$. Then, the corresponding k -jets $j^k(g, h)$ and $j^k(g', h')$ are in the same orbit under the action of $\mathcal{K}[G]^k$ if and only if there exist an $\mathbb{R}$ -algebra isomorphism $\phi : Q_k(h) \rightarrow Q_k(h')$, a permutation $\sigma : \{1, \dots, q\} \rightarrow \{1, \dots, q\}$ and $u_j \in \mathcal{E}_n$ with $u_j(0) > 0$ such that $\phi \circ \pi_h^k(g_j)$ is equal to $\pi_{h'}^k(u_j g'_{\sigma(j)})$, where $\pi_h^k : \mathcal{E}_n \rightarrow Q_k(h)$ is the projection.*

Proof. Proof of “if” part: Since $Q_k(h) \cong Q_k(h')$ holds, h and h' have the same rank at the origin. We denote it by $r - \ell$ for $\ell \in \{0, \dots, r\}$. Without loss of generality, we can suppose

$$h(x_1, \dots, x_n) = (x_1, \dots, x_{r-\ell}, h_{r-\ell+1}(x_{r-\ell+1}, \dots, x_n), \dots, h_r(x_{r-\ell+1}, \dots, x_n)),$$

and

$$h'(x_1, \dots, x_n) = (x_1, \dots, x_{r-\ell}, h'_{r-\ell+1}(x_{r-\ell+1}, \dots, x_n), \dots, h'_r(x_{r-\ell+1}, \dots, x_n)).$$

Let $\phi : Q_k(h) \rightarrow Q_k(h')$, $\sigma : \{1, \dots, q\} \rightarrow \{1, \dots, q\}$ and $u_j \in \mathcal{E}_n$ satisfy the assumption. For $r - \ell + 1 \leq j \leq n$, we take a polynomial $p_j(X)$ with variables $X_{r-\ell+1}, \dots, X_n$ so that $\phi(\pi_h^k(x_j))$ is equal to $p_j(\pi_{h'}^k(x_{r-\ell+1}), \dots, \pi_{h'}^k(x_n))$. Define $\psi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ by

$$x_j \circ \psi = \begin{cases} x_j & (1 \leq j \leq r - \ell) \\ p_j(x_{r-\ell+1}, \dots, x_n) & (r - \ell + 1 \leq j \leq n) \end{cases}.$$

Then, ψ is invertible, since ϕ is an isomorphism and thus the matrix $\left(\frac{\partial p_j}{\partial x_l}\right)_{r-\ell+1 \leq j, l \leq n}$ is invertible. Furthermore the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}_n & \xrightarrow{\psi^*} & \mathcal{E}_n \\ \downarrow \pi_h^k & & \downarrow \pi_{h'}^k \\ Q_k(h) & \xrightarrow{\phi} & Q_k(h') \end{array}$$

Hence, replacing h' by $h' \circ \psi^{-1}$ and g' by $g' \circ \psi^{-1}$, we may suppose that

$$\langle h_1, \dots, h_r \rangle_{\mathcal{E}_n} + \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1} = \langle h'_1, \dots, h'_r \rangle_{\mathcal{E}_n} + \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1}, \quad (2.A)$$

and

$$g_j = u_j g'_{\sigma(j)} \mod \langle h'_1, \dots, h'_r \rangle_{\mathcal{E}_n} + \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1}$$

for $j \in \{1, \dots, q\}$. Replacing h' by another map-germ having the same k -jet, we may suppose that

$$\langle h_1, \dots, h_r \rangle_{\mathcal{E}_n} = \langle h'_1, \dots, h'_r \rangle_{\mathcal{E}_n} \quad (2.B)$$

by the following argument. By Eq. (2.A), there exist map-germs $\tilde{A}, \tilde{B} : (\mathbb{R}^n, 0) \rightarrow M_{r,r}(\mathbb{R})$ and $\tilde{h}, \tilde{h}' \in \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1} \mathcal{E}_n^r$ such that

$$\begin{aligned} h &= \tilde{A}h' + \tilde{h} \\ h' &= \tilde{B}h + \tilde{h}' \end{aligned}$$

hold. By using Lemma in [18, §.2], there exists a matrix $\tilde{C} \in M_{r,r}(\mathbb{R})$ such that

$$\tilde{C} \left(I_r - \tilde{A}(0) \tilde{B}(0) \right) + \tilde{B}(0)$$

is regular, where I_r is $r \times r$ unit matrix. Put $D(x) = \tilde{C} \left(I_r - \tilde{A}(x) \tilde{B}(x) \right) + \tilde{B}(x)$. Then,

$$\begin{aligned} Dh &= \tilde{C} \left(I_r - \tilde{A} \tilde{B} \right) h + \tilde{B} h \\ &= \tilde{C} \left(h - \tilde{A} \tilde{B} h \right) + \tilde{B} h \\ &= \tilde{C} \left(h - \tilde{A} \left(h' - \tilde{h}' \right) \right) + h' - \tilde{h}' \\ &= \tilde{C} \left(\tilde{h} + \tilde{A} \tilde{h}' \right) + h' - \tilde{h}' \\ &= h' + \left(\tilde{C} \left(\tilde{h} + \tilde{A} \tilde{h}' \right) - \tilde{h}' \right). \end{aligned}$$

Since $\tilde{C} \left(\tilde{h} + \tilde{A} \tilde{h}' \right) - \tilde{h}' \in \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1} \mathcal{E}_n^r$ holds, h' has the same k -jet as $h' + \left(\tilde{C} \left(\tilde{h} + \tilde{A} \tilde{h}' \right) - \tilde{h}' \right)$. By replacing h' by the latter germ, we obtain Eq. (2.B). Therefore, there exist map-germs $C: (\mathbb{R}^n, 0) \rightarrow G_{gp}$, $B: (\mathbb{R}^n, 0) \rightarrow M_{q,r}(\mathbb{R})$ and $A: (\mathbb{R}^n, 0) \rightarrow GL(r, \mathbb{R})$ such that

$$\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} C & B \\ O & A \end{pmatrix} \begin{pmatrix} g' \\ h' \end{pmatrix} \mod \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1}$$

holds.

Proof of “only if” part: By the assumption, there exist a diffeomorphism-germ $\phi: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ and map-germs $C: (\mathbb{R}^n, 0) \rightarrow G_{gp}$, $B: (\mathbb{R}^n, 0) \rightarrow M_{q,r}(\mathbb{R})$ and $A: (\mathbb{R}^n, 0) \rightarrow GL(r, \mathbb{R})$ such that

$$\begin{pmatrix} g \\ h \end{pmatrix} \circ \psi = \begin{pmatrix} C & B \\ O & A \end{pmatrix} \begin{pmatrix} g' \\ h' \end{pmatrix} \mod \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1}$$

holds. Define a homomorphism of $\mathbb{R}$ -algebra $\overline{\psi^*}: Q_k(h) \rightarrow Q_k(h')$ as $\overline{\psi^*}(\pi_h^k(f)) = \pi_{h'}^k(f \circ \psi)$ for $f \in \mathcal{E}_n$. $\overline{\psi^*}$ is well-defined. It is because if $\pi_h^k(f_1) = \pi_h^k(f_2)$ holds, there exists $c_j \in \mathcal{E}_n$ for $j \in \{1, \dots, r\}$ such that

$$f_1 - f_2 = \sum_{j=1}^r c_j h_j \mod \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1}$$

holds. By composing it with ψ , we obtain

$$f_1 \circ \psi - f_2 \circ \psi = \sum_{j=1}^r (c_j \circ \psi) (h_j \circ \psi) \mod \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1}$$

since $\psi^* \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1} \subset \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1}$ holds. Since $h_j \circ \psi = \sum_{l=1}^r A_{jl} h'_l$ holds, we obtain $\pi_{h'}^k(f_1 \circ \psi) = \pi_{h'}^k(f_2 \circ \psi)$. It is easy to check that $\overline{\psi^*}$ is an isomorphism of $\mathbb{R}$ -algebra. Since the image of C is in G_{gp} , there exists a permutation $\sigma: \{1, \dots, q\} \rightarrow \{1, \dots, q\}$ such that

$$g_j \circ \psi = u_j \cdot g'_{\sigma(j)} + \sum_{l=1}^r B_{jl} h'_l \mod \langle x_1, \dots, x_n \rangle_{\mathcal{E}_n}^{k+1}$$

holds for some $u_j \in \mathcal{E}_n$ such that $u_j(0) > 0$ for all $j \in \{1, \dots, q\}$. This implies that

$$\overline{\psi^*} \left(\pi_h^k(g_j) \right) = \pi_{h'}^k(g_j \circ \psi) = \pi_{h'}^k \left(u_j \cdot g'_{\sigma(j)} \right).$$

This proves the lemma. $\square$

Lemma 2.3. *Let (g, h) and (g', h') be two constraint map-germs whose feasible set-germs are non-empty. If (g, h) and (g', h') are finitely $\mathcal{K}[G]$ -determined and $\mathcal{K}[G]$ -equivalent, their reductions $(g, h)_{\iota_{(i)}, (k)}$ and $(g', h')_{\iota_{(i')}, (k')}$ having the equal number of inequality and equality constraints are $\mathcal{K}[G]$ -equivalent.*

Proof. By Proposition 2.1 (2) in [12], (g, h) and (g', h') have a finite $\mathcal{K}[G]$ -codimension. By Lemma 3.3 in [12], their reductions have equal or smaller $\mathcal{K}[G]$ -codimension. Again by Proposition 2.1 (2) in [12], their reductions are finitely $\mathcal{K}[G]$ -determined. Thus, we can take $k \in \mathbb{N}$ so that $(g, h)_{\iota_{(i)}, (k)}$ and $(g', h')_{\iota_{(i')}, (k')}$ are both k - $\mathcal{K}[G]$ -determined.

By Lemma 2.2, there exist an isomorphism of $\mathbb{R}$ -algebra $\phi: Q_k(h) \cong Q_k(h')$, a permutation $\sigma: \{1, \dots, q\} \rightarrow \{1, \dots, q\}$ and some units $u_j \in \mathcal{E}_n$ satisfying $u_j(0) > 0$ such that $\phi(\pi_h^k(g_j)) = \pi_{h'}^k(u_j g'_{\sigma(j)})$ holds for all $j \in \{1, \dots, q\}$. Define $\overline{\iota_{(i)}^*}: Q_k(h) \rightarrow Q_k(\iota_{(i)}^* h)$ as $\overline{\iota_{(i)}^*}(\pi_h^k(f)) = \pi_{\iota_{(i)}^* h}^k(f \circ \iota_{(i)})$, then it is easy to check that this is a well-defined isomorphism of $\mathbb{R}$ -algebra. Define $\tilde{\phi} := \overline{\iota_{(i')}^*} \circ \phi \circ \overline{\iota_{(i)}^*}^{-1}: Q_k(\iota_{(i)}^* h) \rightarrow Q_k(\iota_{(i')}^* h')$. Then, this gives an isomorphism of $\mathbb{R}$ -algebra from $Q_k(h_{\iota_{(i)}})$ to $Q_k(h'_{\iota_{(i')}})$ since $Q_k(h_{\iota_{(i)}}) \cong Q_k(\iota_{(i)}^* h)$ and $Q_k(h'_{\iota_{(i')}}) \cong Q_k(\iota_{(i')}^* h')$ hold. Take any $j \in \{1, \dots, q\}$. Then,

$$\begin{aligned} \tilde{\phi} \left(\pi_{\iota_{(i)}^* h}^k(g_j \circ \iota_{(i)}) \right) &= \overline{\iota_{(i')}^*} \circ \phi \circ \overline{\iota_{(i)}^*}^{-1} \left(\pi_{\iota_{(i)}^* h}^k(g_j \circ \iota_{(i)}) \right) = \overline{\iota_{(i')}^*} \circ \phi \left(\pi_h^k(g_j) \right) \\ &= \overline{\iota_{(i')}^*} \left(\pi_{h'}^k(u_j g'_{\sigma(j)}) \right) = \pi_{\iota_{(i')}^* h'}^k \left((u_j \circ \iota_{(i')}) (g'_{\sigma(j)} \circ \iota_{(i')}) \right) \end{aligned}$$

holds. Since $u_j \circ \iota_{(i')}(0) = u_j(0) > 0$ holds, $j^k(g, h)_{\iota_{(i)}, (k)}$ and $j^k(g', h')_{\iota_{(i')}, (k')}$ are $\mathcal{K}[G]^k$ -equivalent by Lemma 2.2. Since they are k - $\mathcal{K}[G]$ -determined, this proves the lemma. $\square$

Lemma 2.4. *The $\mathcal{K}[G]$ -codimension of $(g, h)_{\iota_{(i)}, (k)}$ is equal to that of (g, h) , and the same is true for the $\mathcal{K}[G]_e$ -codimension if $\mathcal{K}[G]$ ($\mathcal{K}[G]_e$)-codimension is finite.*

Proof. If (g, h) is a submersion, the $\mathcal{K}[G]$ -codimensions and $\mathcal{K}[G]_e$ -codimensions of (g, h) and its reduction are all equal to 0, in particular the statement holds. In what follows, we assume that (g, h) is not a submersion. The $\mathcal{K}[G]_e$ -codimension is the sum of the $\mathcal{K}[G]$ -codimension and $-n + q + r$ by Proposition 2.2 [12] and $-n + q + r$ is invariant under reduction. (Note that n, q, r are respectively the number of variables, active inequality constraints, and active equality constraints.) It is thus enough to show the statement for the $\mathcal{K}[G]_e$ -codimension.

Lemma 2.3 implies that $\mathcal{K}[G]$ -action to (g, h) does not change $\mathcal{K}[G]$ -class of its reduction. Therefore, we can assume $(k) = (s + 1, \dots, q)$, $(i) = (1, \dots, r - \ell)$, and the following hold:

$$\begin{aligned} h(x) &= (x_1, \dots, x_{r-\ell}, h_{r-\ell+1}(0, \dots, 0, x_{r-\ell+1}, \dots, x_n), \dots, h_r(0, \dots, 0, x_{r-\ell+1}, \dots, x_n)), \\ g(x) &= (g_1(0, \dots, 0, x_{r-\ell+1}, \dots, x_n), \dots, g_q(0, \dots, 0, x_{r-\ell+1}, \dots, x_n)), \\ \iota_{(i)}(y) &= (0, y) \quad (\text{where } y \in \mathbb{R}^{n-r+\ell}). \end{aligned}$$

By the definition, the tangent space $TK[G]_e(g, h)$ is equal to

$$t(g, h)(\mathcal{E}_n^n) + h^* \mathcal{M}_r \mathcal{E}_n^{q+r} + \langle g_1 e_1, \dots, g_q e_q \rangle_{\mathcal{E}_n}.$$

The image $t(g, h)(\mathcal{E}_n^n)$ has the following generating set as an $\mathcal{E}_n$ -module:

$$\left\{ \left(\sum_{i=1}^q \frac{\partial g_i}{\partial x_j} e_i + \sum_{i=1}^r \frac{\partial h_i}{\partial x_j} e_{i+q} \right) \mid j = 1, \dots, n \right\} \\ = \{e_{q+1}, \dots, e_{q+r-\ell}\} \cup \left\{ \left(\sum_{i=1}^q \frac{\partial g_i}{\partial x_j} e_i + \sum_{i=r-\ell+1}^r \frac{\partial h_i}{\partial x_j} e_{i+q} \right) \mid j = r - \ell + 1, \dots, n \right\}.$$

Furthermore, $\langle g_1 e_1, \dots, g_q e_q \rangle_{\mathcal{E}_n}$ contains $\langle e_{s+1}, \dots, e_q \rangle_{\mathcal{E}_n}$ since $g_{s+1}, \dots, g_q \in \mathcal{E}_n$ are units. Since $h^* \mathcal{M}_r$ contains $x_1, \dots, x_{r-\ell}$, the tangent space $TK[G]_e(g, h)$ contains

$$\langle x_1, \dots, x_{r-\ell} \rangle_{\mathcal{E}_n} \mathcal{E}_n^{q+r} + \langle e_{s+1}, \dots, e_{q+r-\ell} \rangle_{\mathcal{E}_n},$$

which is the kernel of the map $\iota_{(i)}^* \circ \rho : \mathcal{E}_n^{q+r} \rightarrow \mathcal{E}_{n-r+\ell}^{s+\ell}$, where $\rho : \mathcal{E}_n^{q+r} \rightarrow \mathcal{E}_n^{s+\ell}$ is the projection removing the $s+1, \dots, q+r-\ell$ -th components. We thus obtain:

$$\mathcal{E}_n^{q+r} / TK[G]_e(g, h) \\ \cong \left(\frac{\mathcal{E}_n^{q+r}}{\langle x_1, \dots, x_{r-\ell} \rangle_{\mathcal{E}_n} \mathcal{E}_n^{q+r} + \langle e_{s+1}, \dots, e_{q+r-\ell} \rangle_{\mathcal{E}_n}} \right) / \left(\frac{TK[G]_e(g, h)}{\langle x_1, \dots, x_{r-\ell} \rangle_{\mathcal{E}_n} \mathcal{E}_n^{q+r} + \langle e_{s+1}, \dots, e_{q+r-\ell} \rangle_{\mathcal{E}_n}} \right).$$

It is easy to see that the map $\iota_{(i)}^* \circ \rho$ sends $TK[G]_e(g, h)$ to $TK[G]_e(g, h)_{\iota_{(i)}, (k)}$. Thus, $\mathcal{E}_n^{q+r} / TK[G]_e(g, h)$ is isomorphic to $\mathcal{E}_{n-r+\ell}^{s+\ell} / TK[G]_e(g, h)_{\iota_{(i)}, (k)}$. $\square$

2.2 Classification of generic constraint map-germs

We now recall the main theorem in [12], which classifies constraint map-germs appearing in generic parameter families of constraints with up to four parameters. Compared with Tables 1, 2, and 3 in [12], the normal forms presented here have been slightly modified by eliminating signs that can be removed under the $\mathcal{K}[G]$ -action. These modifications are not essential and preserve the completeness of the classification. For the definitions of the $\mathcal{K}[G]$ -codimension and $\mathcal{K}[G]$ -determinacy, see [12].

Theorem 2.1 ([12, Theorem 5.2]). *Suppose $n \gg q, r$. Let N be an n -manifold without boundary, $b \leq 4$, and $U \subset \mathbb{R}^b$ be an open subset. The set consisting of constraint mappings $(g, h) \in C^\infty(N \times U, \mathbb{R}^{q+r})$ with the following conditions is residual in $C^\infty(N \times U, \mathbb{R}^{q+r})$.*

1. *For any $u \in U$ and $\bar{x} \in M(g_u, h_u)$, the corank of $(dh_u)_{\bar{x}}$ is at most 1.*
2. *For any $u \in U$ and $\bar{x} \in M(g_u, h_u)$ at which there is no active inequality constraint (i.e., there is no $k \in \{1, \dots, q\}$ with $g_k(\bar{x}, u) = 0$), a full reduction of the germ $(g, h) : (N \times U, (\bar{x}, u)) \rightarrow \mathbb{R}^{q+r}$ is $\mathcal{K}[G]$ -equivalent to either the trivial family of the constant map-germ, or a versal unfolding of one of the germs in Table 1 with the $\mathcal{K}[G]_e$ -codimension at most b .*

In what follows, we will assume that (g_u, h_u) has an active inequality constraint at $x \in M(g_u, h_u)$.

3. For any $u \in U$ and $\bar{x} \in M(g_u, h_u)$ with $\text{corank}((dh_u)_{\bar{x}}) = 0$, a full reduction of the germ $(g_u, h_u) : (N, \bar{x}) \rightarrow \mathbb{R}^{q+r}$ is $\mathcal{K}[G]$ -equivalent to either a submersion-germ, or one of the germs in Table 2 with stratum $\mathcal{K}[G]_e$ -codimension at most b . Furthermore, if a full reduction of (g_u, h_u) is $\mathcal{K}[G]$ -equivalent to the germ of neither type (6) nor type (10), a full reduction of $(g, h) : (N \times U, (\bar{x}, u)) \rightarrow \mathbb{R}^{q+r}$ is a versal unfolding of (g_u, h_u) .
4. For any $(\bar{x}, u) \in N \times U$ with $\text{corank}((dh_u)_{\bar{x}}) = 1$, a full reduction of the germ $(g_u, h_u) : (N, \bar{x}) \rightarrow \mathbb{R}^{q+r}$ is $\mathcal{K}[G]$ -equivalent to one of the germs in Table 3 with stratum $\mathcal{K}[G]_e$ -codimension at most b (in particular the number of active inequality constraints is at most 3). Furthermore, if a full reduction of (g_u, h_u) is $\mathcal{K}[G]$ -equivalent to the germ of neither type (4) nor type (8), a full reduction of $(g, h) : (N \times U, (\bar{x}, u)) \rightarrow \mathbb{R}^{q+r}$ is a versal unfolding of (g_u, h_u) .

type	jet	range	$\mathcal{K}$ -determinacy	$\mathcal{K}[G]_e$ -cod.
$(1, k)$	$x_1^k + \sum_{j=2}^n \epsilon_j x_j^2$	$2 \leq k \leq 5$	k	$k - 1$
(2)	$x_1^3 + \epsilon_2 x_1 x_2^2 + x_3^2 + \sum_{j=4}^n \epsilon_j x_j^2$		3	4

Table 1: The $\mathcal{K}[G]$ -equivalent classes of map-germs without inequality constraints appearing as a full reduction of a generic four-parameter family of constraint mappings, where $\epsilon_j \in \{1, -1\}$.

type	$\tilde{g}(x_{l+1}, \dots, x_n)$	l	range	$\mathcal{K}[G]$ -determinacy	(stratum) $\mathcal{K}[G]_e$ -cod.
(1, k)	$\epsilon_q x_q^k + \sum_{j=q+1}^n \epsilon_j x_j^2$	$q-1$	$2 \leq k \leq 5$	k	$k-1$
(2)	$x_q^3 + \epsilon_{q+1} x_q x_{q+1}^2 + \sum_{j=q+2}^n \epsilon_j x_j^2$				
(3, k)	$\epsilon_{q-1} x_{q-1}^k + \sum_{j=q}^n \epsilon_j x_j^2$	$q-2$	$2 \leq k \leq 4$ $k=3, 4$	k	k
(4, k)	$\epsilon_q x_q^k + x_{q-1} x_q + \sum_{j=q+1}^n \epsilon_j x_j^2$				
(5)	$\epsilon_{q-1} x_{q-1}^2 + x_q^3 + \sum_{j=q+1}^n \epsilon_j x_j^2$				
(6)	$\sum_{j=1}^2 \delta_j x_{q-j}^2 + \alpha x_{q-2} x_{q-1} + \sum_{j=q}^n \epsilon_j x_j^2$	$q-3$	$\alpha \in \mathbb{R}, \delta_j = \pm 1, (*)$	2	3
(7)	$\epsilon_{q-2} (x_{q-2}^3 + \epsilon'_{q-1} x_{q-1}^2)^2 + \epsilon_{q-1} x_{q-1}^3 + \sum_{j=q}^n \epsilon_j x_j^2$				
(8)	$\epsilon_{q-2} x_{q-2}^3 + \epsilon_{q-1} x_{q-1}^2 + \epsilon'_{q-1} x_{q-2} x_{q-1} + \sum_{j=q}^n \epsilon_j x_j^2$				
(9)	$x_q^3 + \epsilon_{01} x_{q-1} x_q + \epsilon_{02} x_{q-2} x_q$ $\epsilon_{12} x_{q-2} x_{q-1} + \sum_{j=q+1}^n \epsilon_j x_j^2$	$q-4$	$\alpha_{ij} \in \mathbb{R}, \delta_j = \pm 1, (**)$	3	4
(10)	$\sum_{j=1}^3 \delta_j x_{q-4+j}^2$ $+ \sum_{1 \leq i < j \leq 3} \alpha_{ij} x_{q-4+i} x_{q-4+j}$ $\epsilon_0 x_{q-3} x_{q-2} x_{q-1} + \sum_{j=q}^n \epsilon_j x_j^2$				

Table 2: Normal forms $\left(x_1, \dots, x_{q-1}, \sum_{j=1}^{l_1} x_j - \sum_{j=l_1+1}^l x_j + \tilde{g}(x_{l+1}, \dots, x_n)\right)$ of map-germs without equality constraints appearing as a full reduction of a generic four-parameter family of constraint mappings, where $\epsilon_j, \epsilon'_j, \epsilon_{ij} \in \{1, -1\}$, $0 \leq l_1 \leq \lceil \frac{l}{2} \rceil$, $(*)$ (for type (6)) is the condition $4\delta_1 \delta_2 - \alpha^2 \neq 0$, and $(**)$ (for type (10)) is the following condition:

$$4\delta_i \delta_j - \alpha_{ij}^2 \neq 0 \quad (i, j \in \{1, 2, 3\}, i \neq j), \quad 4\delta_1 \delta_2 \delta_3 + \alpha_{12} \alpha_{13} \alpha_{23} - \delta_3 \alpha_{12}^2 - \delta_2 \alpha_{13}^2 - \delta_3 \alpha_{23}^2 \neq 0.$$

type	h	q	range	$\mathcal{K}[G]$ - det.	(stratum) $\mathcal{K}[G]_e$ -cod.
$(1, k)$	$x_1^k + \sum_{j=2}^n \epsilon_j x_j^2$	1	$2 \leq k \leq 4$	k	k
(2)	$x_2^3 + x_1^2 + \sum_{j=3}^n \epsilon_j x_j^2$			3	4
$(3, k)$	$x_2^k + \epsilon_1 x_1 x_2 + \sum_{j=3}^n \epsilon_j x_j^2$		$3 \leq k \leq 4$	k	k
(4)	$\delta_1 x_1^2 + \delta_2 x_2^2 + \alpha x_1 x_2 + \sum_{j=3}^n \epsilon_j x_j^2$	2	$\alpha \in \mathbb{R}, \delta_j = \pm 1, (*)$	2	3
(5)	$x_1^3 + \epsilon_1 x_2^2 + \epsilon_2 x_1 x_2 + \sum_{j=3}^n \epsilon_j x_j^2$			3	4
(6)	$(x_1 + \epsilon_1 x_2)^2 + \epsilon_2 x_2^3 + \sum_{j=3}^n \epsilon_j x_j^2$			3	4
(7)	$x_3^3 + \epsilon_1 x_2 x_3 + \epsilon_2 x_3 x_1 + \epsilon_3 x_1 x_2$ $+ \sum_{j=4}^n \epsilon_j x_j^2$			3	4
(8)	$\sum_{j=1}^3 \delta_j x_j^2 + \sum_{1 \leq i < j \leq 3} \alpha_{ij} x_i x_j$ $+ \epsilon_1 x_1 x_2 x_3 + \sum_{j=4}^n \epsilon_j x_j^2$	3	$\alpha_{ij} \in \mathbb{R}, \delta_j = \pm 1, (**)$	3	4

Table 3: Normal forms $(g_1(x), \dots, g_q(x), h(x)) = (x_1, \dots, x_q, h(x))$ of map-germs with equality/inequality constraints appearing as a full reduction of a generic four-parameter family of constraint mappings, where $\epsilon_j \in \{1, -1\}$, $(*)$ and $(**)$ are the same conditions as those in Table 2.

3 Definitions and properties of constraint qualifications

In this section, we provide precise definitions of the four classical CQs and prove that these are invariant under $\mathcal{K}[G]$ -equivalence (Theorem 3.1) and reductions (Theorem 3.2). Although CQs were originally defined for constraint mappings, we will deal with map-germs $(g, h) : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}^{q+r}$ since all the CQs discussed in this paper are local properties. In what follows, we assume that the set-germ $M(g, h)$ is not empty (i.e., $g_j(x) \leq 0$ for any $j \in \{1, \dots, q\}$). Let $I = \{j \in \{1, \dots, q\} | g_j(0) = 0\}$ be the set of indices of the active inequality constraints and $g_I = (g_j)_{j \in I}$.

Definition 3.1 (LICQ [13]). A constraint (g, h) satisfies *linear independence constraint qualification* (LICQ) if the Jacobi matrix of (g_I, h) has corank 0.

Definition 3.2 (Mangasarian-Fromovitz, MFCQ [17]). A constraint (g, h) satisfies *Mangasarian-Fromovitz constraint qualification* (MFCQ) if the Jacobi matrix of h has corank 0 and there exists a vector $d \in \mathbb{R}^n$ such that $dg_{j,0}(d) < 0$ holds for all $j \in I$ and $dh_{j,0}(d) = 0$ holds for all $j \in \{1, \dots, r\}$. We call such a vector d an *MF-vector* of the constraint in what follows.

Let

$$C^+(g, h) = \left\{ d \in \mathbb{R}^n \mid \exists \{x_l\}_l \subset M(g, h), \lim_{l \rightarrow \infty} x_l = 0, \exists \{t_l\}_l \subset \mathbb{R}_{>0}, d = \lim_{l \rightarrow \infty} t_l x_l \right\}$$

be the tangent cone of the feasible set-germ $M(g, h)$, and

$$L^+(g, h) = \{d \in \mathbb{R}^n \mid \forall j \in I, dg_{j,0}(d) \leq 0, dh_0(d) = 0\}$$

be the linearized cone of (g, h) . Note that when we consider a constraint map-germ without (in)equality constraints, we denote its tangent cone by $C^+(g)$ or $C^+(h)$, and the same for the linearized cone. The reader can refer to [4] for basic properties of tangent/linearized cones.

Definition 3.3 (Abadie, ACQ [1]). A constraint (g, h) satisfies *Abadie constraint qualification (ACQ)* if $C^+(g, h) = L^+(g, h)$.

For a subset $X \subset \mathbb{R}^n$, let $X^\circ = \{v \in \mathbb{R}^n | \forall d \in X, v \cdot d \leq 0\}$, which is called the *polar* of X .

Definition 3.4 (Guignard, GCQ [11]). A constraint (g, h) satisfies *Guignard constraint qualification (GCQ)* if $C^+(g, h)^\circ = L^+(g, h)^\circ$.

Theorem 3.1. *Above constraint qualifications LICQ, MFCQ, ACQ, GCQ are invariant under the action of $\mathcal{K}[G]$.*

Proof. Let (g, h) be a constraint map-germ, $\phi: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a diffeomorphism-germ, and $C: (\mathbb{R}^n, 0) \rightarrow G_{qp}$, $B: (\mathbb{R}^n, 0) \rightarrow M_{q,r}(\mathbb{R})$, and $A: (\mathbb{R}^n, 0) \rightarrow GL(r, \mathbb{R})$ be map-germs. We define (g', h') as follows:

$$\begin{pmatrix} g' \circ \phi(x) \\ h' \circ \phi(x) \end{pmatrix} = \begin{pmatrix} C(x) & | & B(x) \\ \hline O_{r,q} & | & A(x) \end{pmatrix} \begin{pmatrix} g(x) \\ h(x) \end{pmatrix}. \quad (3.A)$$

In what follows, we assume that (g, h) satisfies the CQs and show that so does (g', h') . Since inactive inequality constraints at the origin are irrelevant for the CQs, we assume that all the inequality constraints are active, i.e., $g(0) = 0$ in this proof.

LICQ: Since $\mathcal{K}[G]$ is a subgroup of $\mathcal{K}$ and the action of $\mathcal{K}$ preserves the rank of the Jacobi matrix, it is obvious that (g', h') also satisfies LICQ.

MFCQ: By the assumption, h and h' are $\mathcal{K}$ -equivalent. Since dh_0 has corank 0, dh'_0 has corank 0 as well. Since the constraint (g, h) satisfies MFCQ at the origin, there exists an MF-vector $d \in \mathbb{R}^n$ such that $dg_0(d) < 0$ and $dh_0(d) = 0$. By differentiating both hand sides of Eq. (3.A) by x and taking the inner product with the vector d , we obtain

$$\begin{aligned} (dg'_0 \circ d\phi_0(d), dh'_0 \circ d\phi_0(d)) &= (C(0)dg_0(d) + B(0)dh_0(d), A(0)dh_0(d)) \\ &= (C(0)dg_0(d), 0), \end{aligned}$$

where we used $dh_0(d) = 0$. In addition, $C(0)dg_0(d) < 0$ holds because $dg_0(d) < 0$ and $C(0)$ is a generalized permutation matrix. This proves $dg'_0(d\phi_0(d)) < 0$ and $dh'_0(d\phi_0(d)) = 0$, which implies that $d\phi_0(d)$ is an MF-vector of the constraint (g', h') . This proves that MFCQ is invariant under the action of $\mathcal{K}[G]$.

ACQ: Since $C^+(g', h') \subset L^+(g', h')$ always holds, it is enough to show $C^+(g', h') \supset L^+(g', h')$. If $d \in L^+(g', h')$, then $(d\phi_0)^{-1}(d) \in L^+(g, h)$ holds by using the similar argument in case of MFCQ. Since $C^+(g, h) = L^+(g, h)$ holds, $(d\phi_0)^{-1}(d) \in C^+(g, h)$ holds as well. By definition, there exists sequences $\{x_l\}_l \subset M(g, h)$ and $\{t_l\}_l \subset \mathbb{R}_{>0}$ such that $\lim_{l \rightarrow \infty} x_l = 0$ and $(d\phi_0)^{-1}(d) = \lim_{l \rightarrow \infty} t_l x_l$ holds. In that case, the sequence $\{\phi(x_l)\}_l \subset M(g', h')$ satisfies $\lim_{l \rightarrow \infty} \phi(x_l) = 0$ and

$$\begin{aligned} \lim_{l \rightarrow \infty} t_l \phi(x_l) &= \lim_{l \rightarrow \infty} t_l (\phi(x_l) - \phi(0)) \\ &= \lim_{l \rightarrow \infty} \left(t_l d\phi_0(x_l) + t_l O(\|x_l\|^2) \right) \\ &= d\phi_0 \left(\lim_{l \rightarrow \infty} t_l x_l \right) = d. \end{aligned}$$

This implies that $d \in C^+(g', h')$.

GCQ: The argument in case of ACQ shows $d\phi_0(C^+(g, h)) = C^+(g', h')$ and $d\phi_0(L^+(g, h)) = L^+(g', h')$. This implies that

$$\begin{aligned}
C^+(g', h')^\circ &= d\phi_0(C^+(g, h))^\circ \\
&= \{v \in \mathbb{R}^n \mid \forall d \in C^+(g, h), \ v \cdot d\phi_0(d) \leq 0\} \\
&= \{v \in \mathbb{R}^n \mid \forall d \in C^+(g, h), \ (d\phi_0)^*(v) \cdot d \leq 0\} \\
&= ((d\phi_0)^*)^{-1}(C^+(g, h)^\circ) \\
&= ((d\phi_0)^*)^{-1}(L^+(g, h)^\circ) \\
&= d\phi_0(L^+(g, h))^\circ \\
&= L^+(g', h')^\circ,
\end{aligned}$$

where $(d\phi_0)^*$ is the adjoint of $d\phi_0$. □

We next discuss invariance of CQs under reductions. Let $(g, h) : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}^q \times \mathbb{R}^r$ be a constraint map-germ, and $(g, h)_{\iota_{(i)}, (k)}$ be a reduction of (g, h) relative to $(k) = (k_1, \dots, k_{q-s})$ and $(i) = (i_1, \dots, i_{r-\ell})$.

Theorem 3.2. *A constraint map-germ (g, h) satisfies LICQ, MFCQ, ACQ, and GCQ at the origin if and only if its reduction $(g, h)_{\iota_{(i)}, (k)}$ satisfies LICQ, MFCQ, ACQ, and GCQ, respectively.*

In order to prove the theorem, we first prepare several lemma. Let $MF(g, h) \subset \mathbb{R}^n$ be the set of MF-vectors of (g, h) , that is,

$$MF(g, h) = \{d \in \mathbb{R}^n \mid dh_0(d) = 0, dg_{j,0}(d) < 0 \text{ for } \forall j \text{ with } g_j(0) = 0\}.$$

Lemma 3.1. *The following equalities hold:*

$$L^+(g, h) = d(\iota_{(i)})_0 \left(L^+ \left((g, h)_{\iota_{(i)}, (k)} \right) \right) \quad (3.B)$$

$$MF(g, h) = d(\iota_{(i)})_0 \left(MF \left((g, h)_{\iota_{(i)}, (k)} \right) \right) \quad (3.C)$$

$$C^+(g, h) = d(\iota_{(i)})_0 \left(C^+ \left((g, h)_{\iota_{(i)}, (k)} \right) \right) \quad (3.D)$$

hold.

Proof. Proof of Eq. (3.B) ($\supset$): Take any $d \in L^+ \left((g, h)_{\iota_{(i)}, (k)} \right)$. Then, by definition $d(g_j \circ \iota_{(i)})_0(d) \leq 0$ for all $j \in \{1, \dots, q\}$ with $g_j(0) = 0$, and $d(h_j \circ \iota_{(i)})_0(d) = 0$ for all $j \in \{1, \dots, r\}$. The latter implies that $d(h_j)_0(d(\iota_{(i)})_0(d)) = 0$ holds for j in the same set. Along with the fact that $\text{Im } d(\iota_{(i)})_0 = \text{Ker } d(h_{i_1}, \dots, h_{i_{r-\ell}})_0$, we obtain $d(h_j)_0(d(\iota_{(i)})_0(d)) = 0$ for all $j \in \{1, \dots, r\}$. This proves that $d(\iota_{(i)})_0(d) \in L^+(g, h)$ and thus $L^+(g, h) \supset d(\iota_{(i)})_0 \left(L^+ \left((g, h)_{\iota_{(i)}, (k)} \right) \right)$ holds since d was taken arbitrarily.

Proof of Eq. (3.B) ($\subset$): Take any $d \in L^+(g, h)$. Then,

$$d \in \text{Ker } d(h_1, \dots, h_r) \subset \text{Ker } d(h_1, \dots, h_r) = \text{Im } d(\iota_{(i)})_0$$

holds and thus there exists $d' \in \mathbb{R}^{n-r-\ell}$ such that $d = d(\iota_{(i)})_0(d')$ holds. Then, $0 \geq d(g_j)_0(d) = d(g_j \circ \iota_{(i)})_0(d')$ holds for all $j \in \{1, \dots, q\}$ and $0 = d(h_j)_0(d) = d(h_j \circ \iota_{(i)})_0(d')$ holds for all $j \in \{1, \dots, q\}$. This implies that d' is contained in $L^+((g, h)_{\iota_{(i)}, (k)})$, and thus d is contained in $d(\iota_{(i)})_0(L^+((g, h)_{\iota_{(i)}, (k)}))$.

Proof of Eq. (3.C) is omitted since it is quite similar to that of Eq. (3.B).

Proof of Eq. (3.D) ($\supset$): Take any $d \in C^+((g, h)_{\iota_{(i)}, (k)})$. By definition, there exist sequences $\{x_j\}_j$ in $M((g, h)_{\iota_{(i)}, (k)})$ and $\{t_j\}_j$ in $\mathbb{R}_{>0}$ such that $x_j \rightarrow 0$ and $t_j x_j \rightarrow d$ as $j \rightarrow \infty$. If j is sufficiently large, $\iota_{(i)}(x_j) \in M(g, h)$ holds. In addition, $\iota_{(i)}(x_j) \rightarrow 0$ and $t_j \cdot \iota_{(i)}(x_j) \rightarrow d(\iota_{(i)})_0(d)$ holds. The latter holds since

$$\begin{aligned} t_j \cdot \iota_{(i)}(x_j) &= \iota_{(i)}(0) + t_j \cdot d(\iota_{(i)})_0(x_j) + t_j O(\|x_j\|^2), \\ &= d(\iota_{(i)})_0(t_j x_j) + \|t_j x_j\| O(\|x_j\|), \\ &\rightarrow d(\iota_{(i)})_0(d) \quad (j \rightarrow \infty). \end{aligned}$$

This proves $C^+(g, h) \supset d(\iota_{(i)})_0(C^+((g, h)_{\iota_{(i)}, (k)}))$.

Proof of Eq. (3.D) ($\subset$): Take any $d \in C^+(g, h)$. By definition, there exist sequences $\{x_j\}$ in $M(g, h)$ and $\{t_j\}_j$ in $\mathbb{R}_{>0}$ such that $x_j \rightarrow 0$ and $t_j x_j \rightarrow d$ as $j \rightarrow \infty$. Since

$$x_j \in M(g, h) \subset (h_{i_1}, \dots, h_{i_{r-\ell}})^{-1}(0) = \text{Im } \iota_{(i)}$$

holds for each j , there exists a sequence $x'_j \in (\mathbb{R}^{n-r+\ell}, 0)$ such that $x_j = \iota_{(i)}(x'_j)$ holds. Since $x_j \rightarrow 0$ as $j \rightarrow \infty$, $\iota_{(i)}(x'_j) \rightarrow 0$ as $j \rightarrow \infty$ holds. Since $\iota_{(i)}$ is a homeomorphism to its image, $x'_j \rightarrow 0$ as $j \rightarrow \infty$ follows. Since

$$t_j x_j = t_j \cdot \iota_{(i)}(x'_j) = t_j d(\iota_{(i)})_0(x'_j) + t_j O(\|x_j\|^2) = d(\iota_{(i)})_0(t_j x'_j) + \|t_j x_j\| O(\|x_j\|)$$

holds for each j , we obtain $d = \lim_{j \rightarrow \infty} t_j x_j = \lim_{j \rightarrow \infty} d(\iota_{(i)})_0(t_j x'_j)$. This implies that

$$\begin{aligned} (d(\iota_{(i)})_0)^{-1}(d) &= (d(\iota_{(i)})_0)^{-1} \left(\lim_{j \rightarrow \infty} d(\iota_{(i)})_0(t_j x'_j) \right) \\ &= \lim_{j \rightarrow \infty} (d(\iota_{(i)})_0)^{-1} (d(\iota_{(i)})_0(t_j x'_j)) \\ &= \lim_{j \rightarrow \infty} t_j x'_j \end{aligned}$$

holds. By definition, $d' = \lim_{j \rightarrow \infty} t_j x'_j \in C^+((g, h)_{\iota_{(i)}, (k)})$ holds, this proves $C^+(g, h) \subset d(\iota_{(i)})_0(C^+((g, h)_{\iota_{(i)}, (k)}))$. $\square$

Lemma 3.2. *The following equalities hold:*

$$\begin{aligned} (d(\iota_{(i)})_0)^* (L^+(g, h)^\circ) &= L^+((g, h)_{\iota_{(i)}, (k)})^\circ, \\ (d(\iota_{(i)})_0)^* (C^+(g, h)^\circ) &= C^+((g, h)_{\iota_{(i)}, (k)})^\circ. \end{aligned}$$

Proof. The first equality follows from the following equalities:

$$\begin{aligned}
\left(d(\iota_{(i)})_0\right)^* (L^+(g, h)^\circ) &= \left(d(\iota_{(i)})_0\right)^* \left(\left(d(\iota_{(i)})_0 \left(L^+ \left((g, h)_{\iota_{(i)}, (k)} \right) \right) \right)^\circ \right) \quad (\because \text{Eq. (3.B)}) \\
&= \left(d(\iota_{(i)})_0\right)^* \left(\left(\left(d(\iota_{(i)})_0 \right)^* \right)^{-1} \left(L^+ \left((g, h)_{\iota_{(i)}, (k)} \right)^\circ \right) \right) \\
&= L^+ \left((g, h)_{\iota_{(i)}, (k)} \right)^\circ,
\end{aligned}$$

where the second equality can be shown in the same way as that in the proof of Theorem 3.1 for GCQ, while the third one holds since $\left(d(\iota_{(i)})_0\right)^*$ is surjective. $\square$

Proof of Theorem 3.2. LICQ: We first observe that $\left(d(g_I, h)_{\iota_{(i)}, (k)}\right)_0$ is the composition of the injection $(d\iota_{(i)})_0$ and the restriction $d(g_I, h)|_{\text{Ker } d(h_{i_1}, \dots, h_{i_{r-\ell}})_0}$. Since $\text{Ker } d(g_I, h)$ is contained in $\text{Ker } d(h_{i_1}, \dots, h_{i_{r-\ell}})_0$, we obtain

$$\begin{aligned}
\text{rank} \left(d(g_I, h)_{\iota_{(i)}, (k)} \right)_0 &= (n - r + \ell) - \dim \text{Ker } d(g_I, h)_0|_{\text{Ker } d(h_{i_1}, \dots, h_{i_{r-\ell}})_0} \\
&= (n - r + \ell) - \dim \text{Ker } d(g_I, h)_0 \\
&= (n - r + \ell) - (n - \text{rank } d(g_I, h)_0) \\
&= \text{rank } d(g_I, h)_0 - r + \ell.
\end{aligned}$$

Since the germ (g, h) (resp. $(g_I, h)_{\iota_{(i)}, (k)}$) satisfies LICQ if and only if $\text{rank } d(g_I, h)_0 = s + r$ (resp. $\text{rank} \left(d(g_I, h)_{\iota_{(i)}, (k)} \right)_0 = s + \ell$), this proves the claim.

MFCQ: The Jacobi matrix of h has corank 0 if and only if that of its reduction $h_{\iota_{(i)}}$ has corank 0 (Lemma 3.1 in [12]). By Eq. (3.C) in Lemma 3.1, (g, h) has an MF-vector if and only if its reduction has.

ACQ: The claim immediately follows from Lemma 3.1.

GCQ: Lemma 3.2 implies that if (g, h) satisfies GCQ, its reduction $(g, h)_{\iota_{(i)}, (k)}$ satisfies GCQ. Therefore, in what follows, we show “only if” part.

Suppose the reduction $(g, h)_{\iota_{(i)}, (k)}$ satisfies GCQ. By Lemma 3.2, the following equality holds:

$$L^+(g, h)^\circ + \text{Ker} \left(d(\iota_{(i)})_0 \right)^* = C^+(g, h)^\circ + \text{Ker} \left(d(\iota_{(i)})_0 \right)^*.$$

Since $\text{Ker} \left(d(\iota_{(i)})_0 \right)^* = \left(\text{Im } d(\iota_{(i)})_0 \right)^\perp$ and $\text{Im } d(\iota_{(i)})_0 = \text{Ker } d(h_{i_1}, \dots, h_{i_{r-\ell}})_0 \supset L^+(g, h) \supset C^+(g, h)$,

$$\text{Ker} \left(d(\iota_{(i)})_0 \right)^* \subset \left(\text{Im } d(\iota_{(i)})_0 \right)^\circ \subset L^+(g, h)^\circ \subset C^+(g, h)^\circ$$

hold. We thus obtain:

$$L^+(g, h)^\circ = L^+(g, h)^\circ + \text{Ker} \left(d(\iota_{(i)})_0 \right)^* = C^+(g, h)^\circ + \text{Ker} \left(d(\iota_{(i)})_0 \right)^* = C^+(g, h)^\circ.$$

This completes the proof of “only if” part. $\square$

4 Verification of constraint qualifications for $\mathcal{K}[G]$ -classes of generic constraints

Recall that the purpose of this paper is to determine, for a *generic* constraint map(-germ), when each of the four classical CQs (LICQ, MFCQ, ACQ, GCQ) hold. By the results of the previous section (Theorems 3.1 and 3.2), these four CQs are invariant under $\mathcal{K}[G]$ -equivalence and reduction. Combining this with the classification in [12]—which lists the *full reductions* of generic constraint map-germs up to $\mathcal{K}[G]$ -equivalence—our task reduces to checking CQ-validity for the normal forms displayed in Tables 1–3. We now carry out this verification. Note that the implication in Eq. (1.A) is known for these constraint qualifications.

4.1 LICQ and MFCQ

First, among the normal forms, only the regular class (i.e., $g(x) = (x_1, \dots, x_q)$ and $h(x) = (x_{q+1}, \dots, x_{q+r})$) satisfies LICQ. In what follows, we discuss which classes satisfy MFCQ. By definition, MFCQ is violated if the Jacobi matrix of an equality constraint has positive corank. In particular no classes in Tables 1 and 3 satisfy MFCQ. Therefore, we consider germs in Table 2. MFCQ-validity depends only on the 1-jet, and each germ in Table 2 has the following 1-jet for some $l_1 \in \{0, 1, \dots, \lceil \frac{l}{2} \rceil\}$ and $l \in \{0, \dots, q-1\}$:

$$j^1 g(0) = \left(x_1, \dots, x_{q-1}, \sum_{j=1}^{l_1} x_j - \sum_{j=l_1+1}^l x_j \right). \quad (4.A)$$

If $0 < l_1$ holds, the constraint map-germ g satisfies MFCQ. This can be shown as follows: Suppose that is the case. Set

$$d = \left(\underbrace{-l, -1, \dots, -1}_q, \underbrace{0, \dots, 0}_{n-q} \right) \in \mathbb{R}^n.$$

Then, $dg_{i,0}(d) = -1 < 0$ for $i \in \{2, \dots, q-1\}$ and $dg_{q,0}(d) \leq (l-1) - l = -1 < 0$ hold. This proves the claim. If $l_1 = 0$ and $d \in \mathbb{R}^n$ satisfies $dg_{1,0}(d), \dots, dg_{l,0}(d) < 0$, then $d_1, \dots, d_l < 0$ and thus $dg_{q,0}(d) = -\sum_{i=1}^l d_i > 0$. Thus, g does not satisfy MFCQ. In summary, we obtain the following theorem.

Theorem 4.1. *No germ in Tables 1 and 3 satisfies MFCQ. A germ in Table 2 satisfies MFCQ if and only if $l_1 > 0$, where l_1 is the parameter in the normal form in the caption of the table (or Eq. (4.A)).*

Remark 4.1. For a general constraint map-germ (g, h) with corank $dh_0 = 0$, suppose that the 1-jet of a full reduction of (g, h) is $\mathcal{K}[G]^1$ -equivalent to that in Eq. (4.A) for some l, l_1 . Then, (g, h) satisfies MFCQ if and only if $l_1 > 0$.

4.2 ACQ and GCQ

Next, we compute tangent cones of feasible-set germs in Table 1, Table 2 (violating MFCQ), and Table 3 and confirms if ACQ and GCQ hold for each class. Note that we assume n is sufficiently large so that each normal form has a quadratic part (cf. Theorem 2.1).

In order to determine tangent cones, we need the following lemmas.

Lemma 4.1. Let $g = (x_1, \dots, x_q)$ and $h(x) = Q(x) + R(x)$, where $Q(x)$ is a quadratic polynomial and $R(x)$ is a polynomial consisting of terms with degree larger than 2.

1. $C^+(g, h)$ is contained in $\overline{C}(q, Q) := \{d \in \mathbb{R}^n \mid d_1, \dots, d_q \leq 0, Q(d) = 0\}$.
2. Let R_r be the degree r homogeneous part of R . The element $d \in \overline{C}(q, Q)$ is contained in $C^+(g, h)$ if either $R_r(d) = 0$ for any $r \geq 3$, or there exists $v \in \mathbb{R}^n$ satisfying the following conditions:
 - $v_j \leq 0$ for any $j \in \{1, \dots, q\}$ with $d_j = 0$,
 - if the inner product $v \cdot \nabla Q(d)$ is not 0, its sign is opposite to that of $R_{r_0}(d)$, where $r_0 = \min\{r \geq 0 \mid R_r(d) \neq 0\}$,
 - if $v \cdot \nabla Q(d) = 0$, the product ${}^t v \text{Hess}(Q)v$ is not 0 and its sign is opposite to that of $R_{r_0}(d)$, where $\text{Hess}(Q) = \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}\right)_{i,j}$ is the Hessian matrix of Q .

We denote by $\underline{C}(q, Q, R)$ the set of $d \in \overline{C}(q, Q)$ satisfying the condition in the second statement.

Proof. Take $d \in C^+(g, h)$. There exist $\{x^{(m)}\}_{m \in \mathbb{N}} \subset M(g, h)$ and $\{t_m\}_{m \in \mathbb{N}} \subset \mathbb{R}_{>0}$ such that $\lim_{m \rightarrow \infty} x^{(m)} = 0$ and $d = \lim_{m \rightarrow \infty} t_m x^{(m)}$ hold. Since the j -th component $x_j^{(m)}$ of $x^{(m)}$ is less than or equal to 0 for $j = 1, \dots, q$, so is the limit $d_j = \lim_{m \rightarrow \infty} t_m x_j^{(m)}$. Let $e \geq 3$ be the lowest order of the term of R . Since $x^{(m)}$ is contained in $M(g, h)$, $h(x^{(m)}) = Q(x^{(m)}) + R(x^{(m)})$ is equal to 0. We thus obtain:

$$Q(t_m x^{(m)}) + t_m^2 R(x^{(m)}) = 0.$$

By taking the limit $m \rightarrow \infty$, we obtain $Q(d) = 0$ since

$$\left| t_m^2 R(x^{(m)}) \right| = O\left(\left| t_m x^{(m)} \right|^2\right) \cdot O\left(\left| x^{(m)} \right|^{e-2}\right)$$

holds and $\lim_{m \rightarrow \infty} \left| t_m x^{(m)} \right|^2 = |d|^2$ whereas $\lim_{m \rightarrow \infty} \left| x^{(m)} \right|^{e-2} = 0$ holds. Thus, d is contained in $\overline{C}(q, Q)$.

Take $d \in \overline{C}(q, Q)$. Suppose that $R_r(d) = 0$ for any $r \geq 3$. The following holds for any $m \gg 0$:

$$h(m^{-1}d) = Q(m^{-1}d) + \sum_{r \geq 3} R_r(m^{-1}d) = m^{-2}Q(d) + \sum_{r \geq 3} m^{-r}R_r(d) = 0.$$

Thus, d is contained in $C^+(g, h)$ since $\lim_{m \rightarrow \infty} m^{-1}d = 0$ and $d = \lim_{m \rightarrow \infty} m \cdot m^{-1}d$. In what follows, we assume that $R_r(d) \neq 0$ for some $r \geq 3$. We take a vector $v \in \mathbb{R}^n$ satisfying the conditions in the second statement for d . By the Taylor's theorem, the following equalities hold:

$$\begin{aligned} & h\left(m^{-1}d + m^{-5/4}v\right) \\ &= h\left(m^{-1}d\right) + (v \cdot \nabla h\left(m^{-1}d\right)) m^{-5/4} \\ & \quad + \frac{1}{2}({}^t v \text{Hess}(h)(m^{-1}d)v) m^{-5/2} + O(m^{-15/4}) \quad (m \rightarrow \infty) \\ &= h\left(m^{-1}d\right) + (v \cdot \nabla Q(d)) m^{-9/4} \\ & \quad + \frac{1}{2}({}^t v \text{Hess}(Q)v) m^{-5/2} + O(m^{-3}) \quad (m \rightarrow \infty). \end{aligned}$$

If $v \cdot \nabla Q(d)$ is not 0, its sign is opposite to that of $R_{r_0}(d)$, and thus the sign of $h(m^{-1}d + m^{-5/4}v)$ is opposite to that of $h(m^{-1}d) = m^{-r_0}R_{r_0}(d) + O(m^{-r_0-1})$ for $m \gg 0$. By the intermediate value theorem, there exists $\theta_m \in (0, 1)$ with $h(m^{-1}d + m^{-5/4}\theta_m v) = 0$. Let $x^{(m)} = m^{-1}d + m^{-5/4}\theta_m v$. Since $v_j \leq 0$ for $j \in \{1, \dots, q\}$ with $d_j = 0$, $x^{(m)}$ is contained in $M(g, h)$ for $m \gg 0$. It is easy to check that $\lim_{m \rightarrow \infty} x^{(m)} = 0$ and $\lim_{m \rightarrow \infty} mx^{(m)} = d$. Hence, d is contained in $C^+(g, h)$. If $v \cdot \nabla Q(d) = 0$, ${}^t v \text{Hess}(Q)v$ is not zero and its sign is opposite to that of $R_{r_0}(d)$. We can thus deduce that d is contained in $C^+(g, h)$ in the same way. $\square$

In what follows, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote $x' = (x_{l+1}, \dots, x_n) \in \mathbb{R}^{n-l+1}$.

Lemma 4.2. *Let $g(x) = (x_1, \dots, x_{q-1}, \tilde{g}(x) = -\sum_{j=1}^l x_j + Q(x') + R(x'))$, where $Q(x')$ is a quadratic polynomial and $R(x')$ is a polynomial consisting of terms with degree larger than 2.*

1. $C^+(g)$ is contained in $\overline{D}(l, Q)$, where

$$\overline{D}(l, Q) = \{d \in \mathbb{R}^n \mid d_1 = \dots = d_l = 0, d_{l+1} \leq 0, \dots, d_{q-1} \leq 0, Q(d') \leq 0\}.$$

2. The element $d \in \overline{D}(l, Q)$ is contained in $C^+(g)$ if one of the following conditions holds:

- $Q(d') < 0$,
- $R_r(d') = 0$ for any $r \geq 3$, where R_r is the degree r homogeneous part of R ,
- $R_{r_0}(d') < 0$, where $r_0 = \min\{r \geq 0 \mid R_r(d') \neq 0\}$,
- there exists $v = (0, \dots, 0, v_{l+1}, \dots, v_n) \in \mathbb{R}^n$ satisfying the following conditions:
 - (a) $v_j \leq 0$ for any $j \in \{l+1, \dots, q-1\}$ with $d_j = 0$,
 - (b) if the inner product $v' \cdot \nabla Q(d')$ is not 0, it is less than 0,
 - (c) if $v' \cdot \nabla Q(d') = 0$, the product ${}^t v' \text{Hess}(Q)v'$ is less than 0.

Proof. Let $d \in C^+(g)$. Since $C^+(g)$ is contained in $L^+(g)$, the (in)equalities $d_1 = \dots = d_l = 0$ and $d_{l+1}, \dots, d_{q-1} \leq 0$ hold. Take $\{x^{(m)}\}_{m \in \mathbb{N}} \subset M(g)$ and $\{t_m\}_{m \in \mathbb{N}} \subset \mathbb{R}_{>0}$ so that $\lim_{m \rightarrow \infty} x^{(m)} = 0$ and $d = \lim_{m \rightarrow \infty} t_m x^{(m)}$ hold. Since $x^{(m)} \in M(g)$, the following inequality holds:

$$0 \geq \tilde{g}(x^{(m)}) \geq Q(x_{l+1}^{(m)}, \dots, x_n^{(m)}) + R(x_{l+1}^{(m)}, \dots, x_n^{(m)}).$$

By multiplying t_m^2 to this inequality and taking the limit $m \rightarrow \infty$, we obtain $Q(d') \leq 0$. Thus, d is contained in $\overline{D}(l, Q)$.

Let $d \in \overline{D}(l, Q)$. If $Q(d') < 0$, $m^{-1}d$ is contained in $M(g)$ for $m \gg 0$ since

$$\tilde{g}(m^{-1}d) = Q(m^{-1}d') + R(m^{-1}d') = m^{-2}Q(d') + O(m^{-3}) < 0 \quad (m \rightarrow \infty).$$

Since $\lim_{m \rightarrow \infty} m^{-1}d = 0$ and $\lim_{m \rightarrow \infty} m \cdot m^{-1}d = d$, d is contained in $C^+(g)$. We can also deduce that $d \in C^+(g)$ if either $R_{r_0}(d') < 0$ or $R_r(d') = 0$ for any $r \geq 3$ in the same manner. In what follows, we assume that $Q(d') = 0$ and there exists $v \in \mathbb{R}^{n-l}$ satisfying the conditions in Lemma 4.2. We will show that $m^{-1}d + m^{-5/4}v$ is contained in $M(g)$ for $m \gg 0$. For $j \leq l$, the j -th component of $m^{-1}d + m^{-5/4}v$ (which is $g_j(m^{-1}d + m^{-5/4}v)$) is equal to 0 since $d_j = v_j = 0$. For $l+1 \leq j \leq q-1$, the j -th component of $m^{-1}d + m^{-5/4}v$ (which is

$g_j(m^{-1}d + m^{-5/4}v)$ is less than 0 for $m \gg 0$ if $d_j < 0$. If $d_j = 0$, v_j is less than or equal to 0 by the assumption on v , and thus the j -th component of $m^{-1}d + m^{-5/4}v$ is also less than or equal to 0 for $m \gg 0$. We can obtain the following equality in the same way as in the proof of Lemma 4.1:

$$\tilde{g}(m^{-1}d + m^{-5/4}v) = (v' \cdot \nabla Q(d'))m^{-9/4} + \frac{1}{2}({}^t v' \text{Hess}(Q)v')m^{-5/2} + O(m^{-3}) \quad (m \rightarrow \infty).$$

If $v' \cdot \nabla Q(d')$ is not 0, it is less than 0 by the assumption, and thus $\tilde{g}(m^{-1}d + m^{-5/4}v)$ is also less than 0 for $m \gg 0$. If $v' \cdot \nabla Q(d') = 0$, ${}^t v' \text{Hess}(Q)v'$ is less than 0 by the assumption, and thus $\tilde{g}(m^{-1}d + m^{-5/4}v)$ is also less than 0 for $m \gg 0$. We can eventually deduce that $m^{-1}d + m^{-5/4}v$ is contained in $M(g)$ for $m \gg 0$. Hence d is contained in $C^+(g)$ since $\lim_{m \rightarrow \infty} m^{-1}d + m^{-5/4}v = 0$ and $\lim_{m \rightarrow \infty} m(m^{-1}d + m^{-5/4}v) = d$. $\square$

Corollary 4.1. *Let $g, \tilde{g}, Q, R$ be the same as those in Lemma 4.2. Suppose that $Q(x')$ is equal to $P(x') + \sum_{j=l+s+1}^n \epsilon_j x_j^2$ for some quadratic polynomial P with variables $x_{l+1}, \dots, x_{l+s}$ and $\epsilon_j \in \{1, -1\}$, and $R(x')$ is a homogeneous polynomial with variables $x_{l+1}, \dots, x_{l+s}$. Then, $\overline{D}(l, Q) \setminus \{(0, \dots, 0, d_{l+1}, \dots, d_{l+s}, 0, \dots, 0) \mid R(d') > 0\}$ is contained in $C^+(g)$, and $C^+(g) = \overline{D}(l, Q)$ if $\epsilon_j = -1$ for some $j \geq l + s + 1$.*

Proof. The tangent cone $C^+(g)$ is contained in $\overline{D}(l, Q)$ by Lemma 4.2. Let $d \in \overline{D}(l, Q)$. Since $R(x') = R_{r_0}(x')$ by the assumption on R , d is contained in $C^+(g)$ if $R(d') \leq 0$ by Lemma 4.2. If $d_j \neq 0$ for some $j \geq l + s + 1$, the vector $v = -\epsilon_j d_j e_j$ satisfies the conditions in Lemma 4.2. Indeed, $v_1 = \dots = v_{q-1} = 0$ (in particular v satisfies the condition (a) in Lemma 4.2), $v' \cdot \nabla Q(d') = -2\epsilon_j d_j^2 < 0$. We thus obtain $\overline{D}(l, Q) \setminus \{(0, \dots, 0, d_{l+1}, \dots, d_{l+s}, 0, \dots, 0) \mid R(d') > 0\} \subset C^+(g)$. If $d_{l+s+1} = \dots = d_n = 0$ and $\epsilon_j = -1$ for some $j \geq l + s + 1$, the vector $v = e_j$ satisfies the conditions in Lemma 4.2. Indeed, $v_1 = \dots = v_{q-1} = 0$, $v' \cdot \nabla Q(d') = v' \cdot \nabla P(d') = 0$ since $P(x')$ is a polynomial with variables $x_{l+1}, \dots, x_{l+s}$, and ${}^t v' \text{Hess}(Q)v' = 2\epsilon_j = -2 < 0$. Hence $C^+(g)$ is equal to $\overline{D}(l, Q)$ if $\epsilon_j = -1$ for some $j \geq l + s + 1$. $\square$

4.2.1 ACQ and GCQ in Table 1

In this case, the linearized cones of normal forms in Table 1 are $\mathbb{R}^n$ since the gradients of the normal forms are zero.

Proposition 4.1. *ACQ does not hold for any germ in Table 1.*

Proof. Let h be any germ in Table 1. By Lemma 4.1, the tangent cone $C^+(h)$ is contained in $\overline{C}(0, Q) = \{d \in \mathbb{R}^n \mid Q(d) = 0\}$, where Q is the quadratic part of h . One can easily check that $Q(e_n) \neq 0$, in particular $C^+(h) \subsetneq \mathbb{R}^n = L^+(h)$ for any h in Table 1. $\square$

The germ of type $(1, k)$ in Table 1

Let $k \geq 2$, $h = x_1^k + \sum_{j=2}^n \epsilon_j x_j^2$, where $\epsilon_j \in \{1, -1\}$. For $k \geq 3$, let $Q = \sum_{j=2}^n \epsilon_j x_j^2$ and $R = x_1^k$.

Proposition 4.2. *If $k = 2$, the tangent cone $C^+(h)$ is equal to $\overline{C}(0, Q)$.*

Proof. We can put $R(x) = 0$ in this case, and thus the proposition holds by Lemma 4.1. $\square$

Proposition 4.3. Assume $k \geq 3$. The tangent cone $C^+(h)$ is equal to $\{0\}$ if k is even and all the ϵ_j s are 1. If k is odd and all the ϵ_j s have the same sign δ , $C^+(h)$ is equal to $\overline{C}(0, Q) \setminus \{d \in \mathbb{R}^n | \delta d_1 > 0, d_2 = \dots = d_n = 0\}$. $C^+(h)$ is equal to $\overline{C}(0, Q)$ in all the other cases.

Proof. In the first case, it is easy to see that $M(g, h)$ is equal to $\{0\}$, and thus its tangent cone is also $\{0\}$.

In all the cases, $C^+(h)$ is contained in $\overline{C}(0, Q)$ by Lemma 4.1. The element $d \in \overline{C}(0, Q)$ is contained in $C^+(h)$ if $d_1 = 0$ holds since $R(d) = d_1^k = 0$ in this case. Therefore, we consider the case $d_1 \neq 0$. If $\nabla Q(d) \neq 0$, we can choose $v = -\text{sign}(R(d)) \cdot \nabla Q(d)$ so that the sign of $v \cdot \nabla Q(d)$ is opposite to that of $R(d)$ and thus d is contained in $C^+(h)$ by Lemma 4.1. Therefore, we obtain

$$\overline{C}(0, Q) \setminus \{d \in \mathbb{R}^n | d_1 \neq 0, d_2 = \dots = d_n = 0\} \subset C^+(h).$$

Suppose k is odd and all the ϵ_j s have the same sign δ . Take any $d \in \mathbb{R}^n$ such that $d_1 \neq 0, d_2 = \dots = d_n = 0$. Then, $\nabla Q(d) = 0$ holds. In what follows, we will show that d is contained in $C^+(h)$ if and only if $\delta d_1 < 0$ holds. First, suppose $\delta d_1 < 0$. Then, the sign of ${}^t e_2 \text{Hess}(Q) e_2 = 2\delta$ is opposite to that of $R(d) = d_1^k$. Therefore, d is contained in $C^+(h)$ by Lemma 4.1. Conversely, for any $d \in C^+(h)$, there exist sequences $\{t_m\} \subset \mathbb{R}_{>0}$ and $\{x^{(m)}\} \subset M(h)$ such that $d = \lim_{m \rightarrow \infty} t_m x^{(m)}$ holds. Since $\delta t_m^k (x_1^{(m)})^k = -t_m^k \sum_{j=2}^n (x_j^{(m)})^2 \leq 0$ holds, by taking $m \rightarrow \infty$ in the both sides of the inequality implies that $\delta d_1^k \leq 0$. Therefore, we obtain $C^+(h) = \overline{C}(0, Q) \setminus \{d \in \mathbb{R}^n | \delta d_1 > 0, d_2 = \dots = d_n = 0\}$ in case if k is odd and all the signs of ϵ_j is δ .

Suppose k is even and all the ϵ_j s are -1 . Take any $d \in \mathbb{R}^n$ such that $d_1 \neq 0, d_2 = \dots = d_n = 0$. Then, $\nabla Q(d) = 0$ holds. In that case, d is contained in $C^+(h)$ since the sign of ${}^t e_2 \text{Hess}(Q) e_2 = -2$ is opposite to that of $R(d) = d_1^k$. Therefore, $C^+(h) = \overline{C}(0, Q)$ holds in this case.

Suppose $\{\epsilon_2, \dots, \epsilon_n\} = \{1, -1\}$. In this case, we can suppose $\epsilon_2 = 1$ and $\epsilon_3 = -1$ without loss of generality. Take any $d \in \mathbb{R}^n$ such that $d_1 \neq 0, d_2 = \dots = d_n = 0$. Then, $\nabla Q(d) = 0$ holds. In that case, d is contained in $C^+(h)$ by Lemma 4.1 since ${}^t e_2 \text{Hess}(Q) e_2 = 2$ and ${}^t e_3 \text{Hess}(Q) e_3 = -2$ hold and thus v can be chosen to e_2 or e_3 so that ${}^t v \text{Hess}(Q) v$ has the opposite sign to $R(d)$. Therefore, $C^+(h) = \overline{C}(0, Q)$ holds in this case. $\square$

Proposition 4.4. GCQ holds for h if and only if $k = 2$ and one of ϵ_j s is -1 or $k \geq 3$ and $\{\epsilon_2, \dots, \epsilon_n\} = \{1, -1\}$.

Proof. It is easy to check that $L^+(h)^\circ$ is equal to $\{0\}$.

Proof of “if” part: First, suppose $k = 2$ and $\epsilon_2 = -1$. In this case, $C^+(h)$ is equal to $\overline{C}(0, Q)$ by Proposition 4.2. Take any $w \in C^+(h)^\circ = \overline{C}(0, Q)^\circ$. For $j \in \{1, \dots, n\}$, either $\pm e_1 \pm e_j$ or $\pm e_2 \pm e_j$ is contained in $\overline{C}(0, Q)$. Therefore, $w \cdot (\pm e_1 \pm e_j) = \pm w_1 \pm w_j \leq 0$ or $w \cdot (\pm e_2 \pm e_j) = \pm w_2 \pm w_j \leq 0$ hold for $j \in \{1, \dots, n\}$. This implies that $w_j = 0$ for all $j \in \{1, \dots, n\}$. This proves $w \in L^+(h)^\circ$ and thus $C^+(h)^\circ \subset L^+(h)^\circ$. Since $C^+(h)^\circ \supset L^+(h)^\circ$ always holds, this proves that GCQ holds in this case.

Second, suppose $k \geq 3$ and $\{\epsilon_2, \dots, \epsilon_n\} = \{1, -1\}$. In this case, $C^+(h)$ is equal to $\overline{C}(0, Q)$ by Proposition 4.3. Without loss of generality, we can assume $\epsilon_2 = 1$ and $\epsilon_3 = -1$. Take any $w \in C^+(h)^\circ = \overline{C}(0, Q)^\circ$. Since $\pm e_1 \in \overline{C}(0, Q)$ holds, $w_1 = 0$ holds. For $j \in \{2, \dots, n\}$, either $\pm e_2 \pm e_j$ or $\pm e_3 \pm e_j$ is contained in $\overline{C}(0, Q)$. Therefore, $w \cdot (\pm e_2 \pm e_j) = \pm w_2 \pm w_j \leq 0$ or $w \cdot (\pm e_3 \pm e_j) = \pm w_3 \pm w_j \leq 0$ hold for $j \in \{2, \dots, n\}$. This implies that $w_j = 0$ for all $j \in$

$\{2, \dots, n\}$. This proves $w \in L^+(h)^\circ$ and thus $C^+(h)^\circ \subset L^+(h)^\circ$. Since $C^+(h)^\circ \supset L^+(h)^\circ$ always holds, this proves that GCQ holds in this case.

Proof of “only if” part: We prove this by proving the contraposition. First, suppose $k = 2$ and all of ϵ_j s are 1. In this case, $M(h) = \{0\}$ and thus $C^+(h) = \{0\}$ holds. Therefore, $C^+(h)^\circ = \mathbb{R}^n \neq L^+(h)^\circ$ and thus GCQ is violated.

Second, suppose $k \geq 3$ and all of ϵ_j s have the same sign δ . In this case, $\overline{C}(0, Q) = \{d \in \mathbb{R}^n | d_2 = \dots = d_n = 0\}$ holds and thus $\{0\} = L^+(h)^\circ \subsetneq \overline{C}(0, Q)^\circ = \{w \in \mathbb{R}^n | w_1 = 0\}$. By Lemma 4.1, $C^+(h)$ is contained in $\overline{C}(0, Q)$, and thus $\overline{C}(0, Q)^\circ \subset C^+(h)^\circ$. Therefore, GCQ is violated. $\square$

The germ of type (2) in Table 1

Let $h = x_1^3 + \epsilon_2 x_1 x_2^2 + x_3^2 + \sum_{j=4}^n \epsilon_j x_j^2$, where $\epsilon_j \in \{1, -1\}$, $Q = x_3^2 + \sum_{j=4}^n \epsilon_j x_j^2$ and $R = x_1^3 + \epsilon_2 x_1 x_2^2$.

Proposition 4.5. *The tangent cone $C^+(h)$ is equal to*

$$\overline{C}(0, Q) \setminus \{d \in \mathbb{R}^n | d_1^3 + \epsilon_2 d_1 d_2^2 > 0, d_3 = \dots = d_n = 0\}$$

if all of $\epsilon_4, \dots, \epsilon_n$ are 1, and $C^+(h) = \overline{C}(0, Q)$ otherwise.

Proof. In all the cases, the element $d \in \overline{C}(0, Q)$ is contained in $C^+(h)$ if $d_1^3 + \epsilon_2 d_1 d_2^2 = 0$ holds since $R(d) = 0$ in this case. Therefore, we consider the case $d_1^3 + \epsilon_2 d_1 d_2^2 \neq 0$. If $\nabla Q(d) \neq 0$, we can choose $v = -\text{sign}(R(d)) \cdot \nabla Q(d)$ so that the sign of $v \cdot \nabla Q(d)$ is opposite to that of $R(d)$ and thus d is contained in $C^+(h)$ by Lemma 4.1. Therefore, we obtain

$$\overline{C}(0, Q) \setminus \{d \in \mathbb{R}^n | d_1^3 + \epsilon_2 d_1 d_2^2 \neq 0, d_3 = \dots = d_n = 0\} \subset C^+(h).$$

Suppose all of $\epsilon_4, \dots, \epsilon_n$ are 1. Take any $d \in \mathbb{R}^n$ such that $d_1^3 + \epsilon_2 d_1 d_2^2 \neq 0, d_3 = \dots = d_n = 0$. Then, $\nabla Q(d) = 0$ holds. In that case, d is contained in $C^+(h)$ if and only if $d_1^3 + \epsilon_2 d_1 d_2^2 < 0$ holds. First, suppose $d_1^3 + \epsilon_2 d_1 d_2^2 < 0$ holds. Then, the sign of ${}^t e_3 \text{Hess}(Q) e_3 = 2$ is opposite to that of $R(d)$. Therefore, d is contained in $C^+(h)$ by Lemma 4.1. Conversely, for any $d \in C^+(h)$, there exist sequences $\{t_m\} \subset \mathbb{R}_{>0}$ and $\{x^{(m)}\} \subset M(h)$ such that $d = \lim_{m \rightarrow \infty} t_m x^{(m)}$ holds. Since

$$t_m^3 \left(x_1^{(m)}\right)^3 + \epsilon_2 t_m \left(x_1^{(m)}\right) \cdot t_m^2 \left(x_2^{(m)}\right)^2 = -t_m^3 \sum_{j=2}^n \left(x_j^{(m)}\right)^2 \leq 0$$

holds, by taking $m \rightarrow \infty$ in the both sides of the inequality implies that $d_1^3 + \epsilon_2 d_1 d_2^2 \leq 0$. This proves

$$C^+(h) = \overline{C}(0, Q) \setminus \{d \in \mathbb{R}^n | d_1^3 + \epsilon_2 d_1 d_2^2 > 0, d_3 = \dots = d_n = 0\}$$

in case all of $\epsilon_4, \dots, \epsilon_n$ are 1.

Suppose one of $\epsilon_4, \dots, \epsilon_n$ is -1 . Without loss of generality, we can assume $\epsilon_4 = -1$. Take any $d \in \mathbb{R}^n$ such that $d_1^3 + \epsilon_2 d_1 d_2^2 \neq 0, d_3 = \dots = d_n = 0$. Then, $\nabla Q(d) = 0$ holds. In that case, d is contained in $C^+(h)$ since ${}^t e_3 \text{Hess}(Q) e_3 = 2$ and ${}^t e_4 \text{Hess}(Q) e_4 = -2$ hold and v can be chosen to e_3 or e_4 so that ${}^t v \text{Hess}(Q) v$ has the opposite sign to that of $R(d)$. Therefore, $C^+(h) = \overline{C}(0, Q)$ holds in this case. $\square$

Proposition 4.6. *GCQ holds for h if and only if one of ϵ_j s is -1 .*

Proof. It is easy to check that $L^+(h)^\circ$ is equal to $\{0\}$.

Proof of “if” part: Without loss of generality, we can assume $\epsilon_4 = -1$. Take any $w \in \overline{C}(0, Q)^\circ$. Since $\pm e_1, \pm e_2$ is contained in $\overline{C}(0, Q)$, $w_1 = w_2 = 0$ holds. For $j \in \{3, \dots, n\}$, either $\pm e_3 \pm e_j$ or $\pm e_4 \pm e_j$ is contained in $\overline{C}(0, Q)$ and thus $w_3 = \dots = w_n = 0$ holds as well. This proves $\overline{C}(0, Q)^\circ = \{0\} = L^+(h)^\circ$. By Proposition 4.5, $C^+(h)$ is equal to $\overline{C}(0, Q)$, and thus GCQ holds for h .

Proof of “only if” part: We prove this by proving the contraposition. Suppose all of $\epsilon_4, \dots, \epsilon_n$ are 1. Then, $\overline{C}(0, Q) = \{d \in \mathbb{R}^n | d_3 = \dots = d_n = 0\}$. Since $C^+(h) \subset \overline{C}(0, Q)$ holds, $\overline{C}(0, Q)^\circ \subset C^+(h)^\circ$ holds. Since e_3 is contained in $\overline{C}(0, Q)^\circ$ but not contained in $L^+(h)^\circ$, GCQ is violated. $\square$

Example 4.1. Suppose $h(x) = x_1^3 + x_2^2$. In this case, the constraint $h(x) = 0$ does not satisfy GCQ at the origin. The theorem by Gould and Tolle [9] implies that there exists an objective function f such that the KKT condition does not hold at a local minimum of f subject to $h = 0$. In fact, if we set $f(x) = -x_1$, then, $x = (0, 0)$ is the minimum of the above optimization problem but the KKT condition, i.e., there is no constant $u \in \mathbb{R}$ such that $df_0 = u \cdot dh_0$ holds.

4.2.2 ACQ and GCQ in Table 2

First of all, ACQ and GCQ hold for a germ in Table 2 with $l_1 > 0$ since MFCQ holds for such a germ. For this reason, we will discuss germs with $l_1 = 0$ below. The linearized cone $L^+(g)$ is equal to $\{d \in \mathbb{R}^n | d_1 = \dots = d_l = 0, d_{l+1}, \dots, d_{q-1} \leq 0\}$ for a germ g in Table 2. In what follows, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote $x' = (x_{l+1}, \dots, x_n) \in \mathbb{R}^{n-l+1}$. Note that the integer l may vary depending on the context. Nevertheless, we adopt a unified notation x' for simplicity, with the understanding that l is determined by the context in each case.

The germ of type $(1, k)$ in Table 2

Let $k \geq 2$, $g = \left(x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-1} x_j + \epsilon_q x_q^k + \sum_{j=q+1}^n \epsilon_j x_j^2\right)$, where $\epsilon_j \in \{1, -1\}$.

Proposition 4.7. *If $k = 2$, the tangent cone $C^+(g)$ is equal to $\overline{D}(q-1, Q)$.*

Proof. We can put $R(x') = 0$ in this case, and thus the proposition holds by Lemma 4.2. $\square$

Proposition 4.8. *Suppose that k is larger than or equal to 3. The tangent cone $C^+(g)$ is equal to $\overline{D}(q-1, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n | \epsilon_q d_q^k > 0\}$ if $\epsilon_{q+1} = \dots = \epsilon_n = 1$, and $C^+(g) = \overline{D}(q-1, Q)$ otherwise.*

Proof. We can put $Q = \sum_{j=q+1}^n \epsilon_j x_j^2$ and $R = \epsilon_q x_q^k$ in this case. By Corollary 4.1 (with $P(x') = 0$ and $s = 1$), $\overline{D}(q-1, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) | \epsilon_q d_q^k > 0\}$ is contained in $C^+(g)$, and $C^+(g) = \overline{D}(q-1, Q)$ if $\epsilon_j = -1$ for some $j \geq q+1$. If $\epsilon_{q+1} = \dots = \epsilon_n = +1$, any $x \in M(g)$ satisfies the following inequality:

$$0 \geq -\sum_{j=1}^{q-1} x_j + \epsilon_q x_q^k + \sum_{j=q+1}^n x_j^2 \Leftrightarrow \epsilon_q x_q^k \leq \sum_{j=1}^{q-1} x_j - \sum_{j=q+1}^n x_j^2 \leq 0.$$

Thus, any $d \in C^+(g)$ also satisfies the inequality $\epsilon_q d_q^k \leq 0$ when $\epsilon_{q+1} = \dots = \epsilon_n = 1$. $\square$

Proposition 4.9. *ACQ holds for g if and only if one of the following conditions holds:*

1. $k = 2$ and $\epsilon_q = \dots = \epsilon_n = -1$,
2. $k \geq 3$ and $\epsilon_{q+1} = \dots = \epsilon_n = -1$.

Proof. Suppose that the condition in the proposition does not hold, say $\epsilon_j = 1$ for $j \geq q$ (resp. $j \geq q+1$) if $k = 2$ (resp. $k \geq 3$). It is easy to check that e_j is contained in $L^+(g)$ but not in $\overline{D}(q-1, Q)$, in particular $C^+(g) \subset \overline{D}(q-1, Q) \subsetneq L^+(g)$. Suppose conversely that the condition in the proposition hold. By Propositions 4.7 and 4.8, the tangent cone $C^+(g)$ is equal to $\overline{D}(q-1, Q)$, which is equal to $\{d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-1} = 0\} = L^+(g)$ by the assumption. $\square$

Proposition 4.10. *GCQ holds for g if and only if one of the following conditions holds:*

1. $k = 2$ and one of $\epsilon_q, \dots, \epsilon_n$ is -1 ,
2. $k \geq 3$ and one of $\epsilon_{q+1}, \dots, \epsilon_n$ is -1 ,

Proof. It is easy to check that $L^+(g)^\circ$ is equal to $\{w \in \mathbb{R}^n \mid w_q = \dots = w_n = 0\}$.

Proof of “if” part: By Propositions 4.7 and 4.8, the conditions implies that $C^+(g)$ is equal to $\overline{D}(q-1, Q)$. Since $C^+(g) \subset L^+(g)$ always holds, it is enough to show $\overline{D}(q-1, Q)^\circ \subset L^+(g)^\circ$.

Suppose that the condition 1. in Proposition 4.10 holds. Without loss of generality, we can assume $\epsilon_q = -1$. Take any $w \in \overline{D}(q-1, Q)^\circ$. Since the vector $\pm e_q$ is contained in $\overline{D}(q-1, Q)$, $w \cdot (\pm e_q) = \pm w_q$ is less than or equal to 0, implying $w_q = 0$. Since the vector $\pm e_q \pm e_j$ is contained in $\overline{D}(q-1, Q)$ for any $j \in \{q+1, \dots, n\}$, $w \cdot (\pm e_q \pm e_j) = \pm w_j$ is less than or equal to 0, implying $w_j = 0$. We thus obtain $w \in L^+(g)^\circ$.

Suppose that the condition 2. in Proposition 4.10 holds. Without loss of generality, we can assume $\epsilon_{q+1} = -1$. Take any $w \in \overline{D}(q-1, Q)^\circ$. Since $\pm e_q$ is contained in $\overline{D}(q-1, Q)$, $w \cdot (\pm e_q) = \pm w_q$ is less than or equal to 0, implying $w_q = 0$. Since the vector $\pm e_{q+1}$ is contained in $\overline{D}(q-1, Q)$, $w \cdot (\pm e_{q+1}) = \pm w_{q+1}$ is less than or equal to 0, implying $w_{q+1} = 0$. Since the vector $\pm e_{q+1} \pm e_j$ is contained in $\overline{D}(q-1, Q)$ for any $j \in \{q+2, \dots, n\}$, $w \cdot (\pm e_{q+1} \pm e_j) = \pm w_j$ is less than or equal to 0, implying $w_j = 0$. We thus obtain $w \in L^+(g)^\circ$.

Proof of “only if” part: We show the contraposition of the statement. If $k = 2$ and $\epsilon_q = \dots = \epsilon_n = 1$, Proposition 4.7 implies that $C^+(g)$ is equal to

$$\overline{D}(q-1, Q) = \left\{ d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-1} = 0, \sum_{j=q}^n d_j^2 = 0 \right\} = \{0\}.$$

In particular, $C^+(g)^\circ = \mathbb{R}^n \neq L^+(g)^\circ$. If $k \geq 3$ and $\epsilon_{q+1} = \dots = \epsilon_n = 1$, Proposition 4.8 implies that $C^+(g)$ is equal to

$$\begin{aligned} & \overline{D}(q-1, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid \epsilon_q d_q^k > 0\} \\ & = \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid \epsilon_q d_q^k \leq 0\}. \end{aligned}$$

In particular, $C^+(g)^\circ$ contains e_{q+1} , which is not contained in $L^+(g)^\circ$. $\square$

The germ of type (2) in Table 2

Let $g = (x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-1} x_j + x_q^3 + \epsilon_{q+1} x_q x_{q+1}^2 + \sum_{j=q+2}^n \epsilon_j x_j^2)$, where $\epsilon_j \in \{1, -1\}$, $Q = \sum_{j=q+2}^n \epsilon_j x_j^2$, and $R = x_q^3 + \epsilon_{q+1} x_q x_{q+1}^2$.

Proposition 4.11. *The tangent cone $C^+(g)$ is equal to $\overline{D}(q-1, Q) \setminus \{(0, \dots, 0, d_q, d_{q+1}, 0, \dots, 0) \in \mathbb{R}^n \mid d_q^3 + \epsilon_{q+1} d_q d_{q+1}^2 > 0\}$ if $\epsilon_{q+2} = \dots = \epsilon_n = 1$, and $C^+(g) = \overline{D}(q-1, Q)$ otherwise.*

Proof. By Corollary 4.1 (with $P(x') = 0$ and $s = 2$), $\overline{D}(q-1, Q) \setminus \{(0, \dots, 0, d_q, d_{q+1}, 0, \dots, 0) \mid d_q^3 + \epsilon_{q+1} d_q d_{q+1}^2 > 0\}$ is contained in $C^+(g)$, and $C^+(g) = \overline{D}(q-1, Q)$ if $\epsilon_j = -1$ for some $j \geq q+2$. If $\epsilon_{q+2} = \dots = \epsilon_n = +1$, any $x \in M(g)$ satisfies the following inequality:

$$0 \geq -\sum_{j=1}^{q-1} x_j + x_q^3 + \epsilon_{q+1} x_q x_{q+1}^2 + \sum_{j=q+2}^n x_j^2 \Leftrightarrow x_q^3 + \epsilon_{q+1} x_q x_{q+1}^2 \leq \sum_{j=1}^{q-1} x_j - \sum_{j=q+2}^n x_j^2 \leq 0.$$

Thus, any $d \in C^+(g)$ also satisfies the inequality $d_q^3 + \epsilon_{q+1} d_q d_{q+1}^2 \leq 0$ when $\epsilon_{q+2} = \dots = \epsilon_n = 1$. $\square$

Proposition 4.12. *ACQ holds for g if and only if $\epsilon_{q+2} = \dots = \epsilon_n = -1$,*

Proof. Suppose that the condition in the proposition does not hold, say $\epsilon_j = 1$ for $j \geq q+2$. It is easy to check that e_j is contained in $L^+(g)$ but not in $\overline{D}(q-1, Q)$, in particular $C^+(g) \subset \overline{D}(q-1, Q) \subsetneq L^+(g)$. Suppose conversely that the condition in the proposition hold. By Proposition 4.11, the tangent cone $C^+(g)$ is equal to $\overline{D}(q-1, Q)$, which is equal to $\{d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-1} = 0\} = L^+(g)$ by the assumption. $\square$

Proposition 4.13. *GCQ holds for g if and only if one of $\epsilon_{q+2}, \dots, \epsilon_n$ is -1 .*

Proof. It is easy to check that $L^+(g)^\circ$ is equal to $\{w \in \mathbb{R}^n \mid w_q = \dots = w_n = 0\}$.

Proof of “if” part: Without loss of generality, we can assume $\epsilon_{q+2} = -1$. By Propositions 4.11, the condition implies that $C^+(g)$ is equal to $\overline{D}(q-1, Q)$. Since $C^+(g) \subset L^+(g)$ always holds, it is enough to show $\overline{D}(q-1, Q)^\circ \subset L^+(g)^\circ$. Take any $w \in \overline{D}(q-1, Q)^\circ$. Since the vector $\pm e_q, \pm e_{q+1}$ is contained in $\overline{D}(q-1, Q)$, $w_q = 0$ and $w_{q+1} = 0$ hold. Since the vector $\pm e_{q+2}$ is contained in $\overline{D}(q-1, Q)$, $w_{q+2} = 0$ holds. Since the vector $\pm e_{q+2} \pm e_j$ is contained in $\overline{D}(q-1, Q)$ for any $j \in \{q+3, \dots, n\}$, $w \cdot (\pm e_{q+2} \pm e_j) = \pm w_j$ is less than or equal to 0, implying $w_j = 0$. We thus obtain $w \in L^+(g)^\circ$.

Proof of “only if” part: We show the contraposition of the statement, that is, if all of $\epsilon_{q+2}, \dots, \epsilon_n$ are 1, GCQ is violated. Under the assumption,

$$\overline{D}(q-1, Q) = \left\{ d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-1} = 0, \sum_{j=q+2}^n d_j^2 = 0 \right\}$$

and thus $\overline{D}(q-1, Q)^\circ = \{w \in \mathbb{R}^n \mid w_q = w_{q+1} = 0\}$ holds. Therefore, $C^+(g)^\circ \supset \overline{D}(q-1, Q)^\circ \not\subset L^+(g)^\circ$ holds and thus GCQ is violated. $\square$

The germ of type (3, k) in Table 2

Let $k \geq 2$, $g = (x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-2} x_j + \epsilon_{q-1} x_{q-1}^k + \sum_{j=q}^n \epsilon_j x_j^2)$, where $\epsilon_j \in \{1, -1\}$.

Proposition 4.14. *If $k = 2$, the tangent cone $C^+(g)$ is equal to $\overline{D}(q-2, Q)$.*

Proof. We can put $R(x') = 0$ in this case, and thus the proposition holds by Lemma 4.2. $\square$

Proposition 4.15. *Suppose that k is larger than or equal to 3. The tangent cone $C^+(g)$ is equal to $\overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_{q-1}, 0, \dots, 0) \in \mathbb{R}^n \mid \epsilon_{q-1}d_{q-1}^k > 0\}$ if $\epsilon_q = \dots = \epsilon_n = 1$, and $C^+(g) = \overline{D}(q-2, Q)$ otherwise.*

Proof. We can put $Q = \sum_{j=q}^n \epsilon_j x_j^2$ and $R = \epsilon_{q-1} x_{q-1}^k$ in this case. By Corollary 4.1 (with $P(x') = 0$ and $s = 1$), $\overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_{q-1}, 0, \dots, 0) \mid \epsilon_{q-1}d_{q-1}^k > 0\}$ is contained in $C^+(g)$, and $C^+(g) = \overline{D}(q-2, Q)$ if $\epsilon_j = -1$ for some $j \geq q$. If $\epsilon_q = \dots = \epsilon_n = +1$, any $x \in M(g)$ satisfies the following inequality:

$$0 \geq -\sum_{j=1}^{q-2} x_j + \epsilon_{q-1} x_{q-1}^k + \sum_{j=q}^n x_j^2 \Leftrightarrow \epsilon_{q-1} x_{q-1}^k \leq \sum_{j=1}^{q-2} x_j - \sum_{j=q}^n x_j^2 \leq 0.$$

Thus, any $d \in C^+(g)$ also satisfies the inequality $\epsilon_{q-1}d_{q-1}^k \leq 0$ when $\epsilon_q = \dots = \epsilon_n = 1$. $\square$

Proposition 4.16. *ACQ holds for g if and only if one of the following conditions holds:*

1. $k = 2$ and $\epsilon_{q-1} = \dots = \epsilon_n = -1$,
2. $k \geq 3$ and $\epsilon_q = \dots = \epsilon_n = -1$.

Proof. Suppose that the condition in the proposition does not hold, say $\epsilon_j = 1$ for $j \geq q-1$ (resp. $j \geq q$) if $k = 2$ (resp. $k \geq 3$). It is easy to check that $-e_j$ is contained in $L^+(g)$ but not in $\overline{D}(q-2, Q)$, in particular $C^+(g) \subset \overline{D}(q-2, Q) \subsetneq L^+(g)$. Suppose conversely that the condition in the proposition hold. By Propositions 4.14 and 4.15, the tangent cone $C^+(g)$ is equal to $\overline{D}(q-2, Q)$, which is equal to $\{d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-2} = 0, d_{q-1} \leq 0\} = L^+(g)$ by the assumption. $\square$

Proposition 4.17. *GCQ holds for g if and only if one of $\epsilon_q, \dots, \epsilon_n$ is -1 .*

Proof. It is easy to check that $L^+(g)^\circ$ is equal to $\{w \in \mathbb{R}^n \mid w_{q-1} \geq 0, w_q = \dots = w_n = 0\}$.

Proof of “if” part: Without loss of generality, we can assume $\epsilon_q = -1$. By Propositions 4.14 and 4.15, $C^+(g)$ is equal to $\overline{D}(q-2, Q)$. Take any $w \in \overline{D}(q-2, Q)^\circ$. Since $\pm e_q$ is contained in $\overline{D}(q-2, Q)$, $w_q = 0$ holds. Since $-e_{q-1} \pm e_q$ is contained in $\overline{D}(q-2, Q)$, $w_{q-1} \geq 0$ holds. Since $\pm e_q \pm e_j$ is contained in $\overline{D}(q-2, Q)$ for $j \in \{q+1, \dots, n\}$, $w_j = 0$ holds. This proves $w \in L^+(g)^\circ$.

Proof of “only if” part: We show the contraposition of the statement, that is, if all of $\epsilon_{q+2}, \dots, \epsilon_n$ are 1, GCQ is violated. Under the assumption and $k \geq 3$,

$$\overline{D}(q-2, Q) = \left\{ d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-2} = 0, d_{q-1} \leq 0, \sum_{j=q}^n d_j^2 = 0 \right\}$$

and thus $\overline{D}(q-2, Q)^\circ = \{w \in \mathbb{R}^n \mid w_{q-1} \geq 0\}$ holds. Therefore, $C^+(g)^\circ \supset \overline{D}(q-2, Q)^\circ \not\supseteq L^+(g)^\circ$ holds and thus GCQ is violated.

Under the assumption and $k = 2$,

$$\overline{D}(q-2, Q) = \left\{ d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-2} = 0, d_{q-1} \leq 0, \epsilon_{q-1}d_{q-1}^2 + \sum_{j=q}^n d_j^2 \leq 0 \right\}$$

holds. Take any $d \in \overline{D}(q-2, Q)$. Then, $d_q^2 \leq \sum_{j=q}^n d_j^2 \leq |d_{q-1}^2|$ and thus $|d_q| \leq |d_{q-1}|$ and $(e_{q-1} + e_q) \cdot d = d_{q-1} + d_q \leq 0$ hold. Thus, $w = e_{q-1} + e_q$ is contained in $\overline{D}(q-2, Q)^\circ$. Since w is not contained in $L^+(g)^\circ$, GCQ is violated. $\square$

The germ of type $(4, k)$ in Table 2

Let $k \geq 3$, $g = (x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-2} x_j + \epsilon_q x_q^k + x_{q-1} x_q + \sum_{j=q+1}^n \epsilon_j x_j^2)$, where $\epsilon_j \in \{1, -1\}$, $Q = x_{q-1} x_q + \sum_{j=q+1}^n \epsilon_j x_j^2$, and $R = \epsilon_q x_q^k$.

Proposition 4.18. *The tangent cone $C^+(g)$ is equal to $\overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q < 0, \epsilon_q d_q^k > 0\}$ if $\epsilon_{q+1} = \dots = \epsilon_n = 1$, and $C^+(g) = \overline{D}(q-2, Q)$ otherwise.*

Proof. By Corollary 4.1 (with $P(x') = x_{q-1} x_q$ and $s = 2$), the set

$$\overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_{q-1}, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid \epsilon_q d_q^k > 0\}$$

is contained in $C^+(g)$, and $C^+(g) = \overline{D}(q-2, Q)$ if $\epsilon_j = -1$ for some $j \geq q+1$.

In what follows, we assume $\epsilon_{q+1} = \dots = \epsilon_n = +1$. Let $d = (0, \dots, 0, d_{q-1}, d_q, 0, \dots, 0) \in \overline{D}(q-2, Q)$ with $\epsilon_q d_q^k > 0$. Since $d \in \overline{D}(q-2, Q)$, d_{q-1} and $Q(d') = d_{q-1} d_q$ are less than or equal to 0. If $d_{q-1} < 0$, $d_{q-1} d_q$ is also less than 0 since d_q is not 0. The vector $m^{-1}d$ is contained in $M(g)$ for $m \gg 0$ since $g_j(m^{-1}d) = m^{-1}d_j \leq 0$ for $j = 1, \dots, q-1$ and

$$\tilde{g}(m^{-1}d) = \epsilon_q m^{-k} d_q^k + m^{-2} d_{q-1} d_q < 0.$$

Hence $d = \lim_{m \rightarrow \infty} m \cdot m^{-1}d$ is contained in $C^+(g)$. If $d_{q-1} = 0$ and $d_q > 0$, the vector $x^{(m)} := m^{-1}d_q e_q - m^{-3/2}d_q e_{q-1}$ is contained in $M(g)$ for $m \gg 0$ since

$$g_j(x^{(m)}) = \begin{cases} 0 & (j \leq q-2) \\ -m^{-3/2}d_q < 0 & (j = q-1), \end{cases}$$

and $\tilde{g}(x^{(m)}) = m^{-k} \epsilon_q d_q^k - m^{-5/2} d_q^2 < 0$ for $m \gg 0$. Thus, $d = \lim_{m \rightarrow \infty} m \cdot x^{(m)}$ is contained in $C^+(g)$. We eventually obtain:

$$\overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q < 0, \epsilon_q d_q^k > 0\} \subset C^+(g).$$

Suppose that $d = (0, \dots, 0, d_{q-1}, d_q, 0, \dots, 0)$ with $d_q < 0$ and $\epsilon_q d_q^k > 0$ is contained in $C^+(g)$. Take $x^{(m)} \in M(g)$ and $t_m > 0$ so that $\lim_{m \rightarrow \infty} x^{(m)} = 0$ and $\lim_{m \rightarrow \infty} t_m x^{(m)} = d$. The following inequality holds:

$$\begin{aligned} 0 &\geq -\sum_{j=1}^{q-2} x_j^{(m)} + \epsilon_q \left(x_q^{(m)}\right)^k + x_{q-1}^{(m)} x_q^{(m)} + \sum_{j=q+1}^n (x_j^{(m)})^2 \\ &\Leftrightarrow \epsilon_q \left(x_q^{(m)}\right)^k + x_{q-1}^{(m)} x_q^{(m)} \leq \sum_{j=1}^{q-2} x_j^{(m)} - \sum_{j=q+1}^n (x_j^{(m)})^2 \leq 0. \end{aligned}$$

Since $\lim_{m \rightarrow \infty} t_m x^{(m)} = d$, $d_q < 0$ and $\epsilon_q d_q^k > 0$, $\epsilon_q \left(t_m x_q^{(m)}\right)^k$ is larger than 0 and $\left(t_m x_{q-1}^{(m)}\right) \left(t_m x_q^{(m)}\right) \geq 0$ for $m \gg 0$. Thus, $\epsilon_q \left(x_q^{(m)}\right)^k + x_{q-1}^{(m)} x_q^{(m)}$ is larger than 0, contradicting the inequality above. Hence we obtain:

$$\overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q < 0, \epsilon_q d_q^k > 0\} = C^+(g). \quad \square$$

Proposition 4.19. *ACQ does not hold for g .*

Proof. The vector $-e_{q-1} - \epsilon_q e_q$ is contained in $L^+(g)$ but not in $\overline{D}(q-2, Q)$, in particular $C^+(g) \subset \overline{D}(q-1, Q) \subsetneq L^+(g)$. $\square$

Proposition 4.20. *GCQ holds for g if and only if either of the following holds:*

1. *one of $\epsilon_{q+1}, \dots, \epsilon_n$ is -1 .*
2. *$\epsilon_{q+1} = \dots = \epsilon_n = 1$ and $\epsilon_q = (-1)^{k+1}$.*

Proof. It is easy to check that $L^+(g)^\circ$ is equal to $\{w \in \mathbb{R}^n \mid w_{q-1} \geq 0, w_q = \dots = w_n = 0\}$. In what follows, we will show that $\overline{D}(q-2, Q)^\circ \subset L^+(g)^\circ$ holds. Take any $w \in \overline{D}(q-2, Q)^\circ$. Since $-e_{q-1}$ is contained in $\overline{D}(q-2, Q)$, $w_{q-1} \geq 0$ holds. Since $\pm e_q$ is contained in $\overline{D}(q-2, Q)$, $w_q = 0$ holds. For $j \in \{q+1, \dots, n\}$ and $s > 0$, $-\frac{1}{s}e_{q-1} + \epsilon_j s e_q \pm e_j$ is contained in $\overline{D}(q-2, Q)$, $-\frac{1}{s}w_{q-1} \pm w_j \leq 0$ holds. This s can be chosen arbitrarily large and thus $w_j = 0$ holds for $j \in \{q+1, \dots, n\}$. This proves $w \in L^+(g)^\circ$.

If one of $\epsilon_{q+1}, \dots, \epsilon_n$ is -1 , $C^+(g) = \overline{D}(q-2, Q)$ holds and GCQ hold in that case.

If $\epsilon_{q+1} = \dots = \epsilon_n = 1$ holds,

$$C^+(g) = \overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q < 0, \epsilon_q d_q^k > 0\}$$

holds. If $\epsilon_q = (-1)^{k+1}$, $C^+(g) = \overline{D}(q-2, Q)$ holds and GCQ holds in that case. If $\epsilon_q = (-1)^k$,

$$C^+(g) = \overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q < 0\}$$

holds. In what follows, we will show that $w = -e_q$ is contained in $C^+(g)^\circ$. Take any $d \in C^+(g)$. If one of $d_{q+1}, \dots, d_n$ is non-zero,

$$d_{q-1}d_q \leq - \sum_{j=q+1}^n d_j^2 < 0$$

holds. This implies that $d_{q-1} < 0$ and thus $d_q > 0$ holds. Therefore, $w \cdot d = -d_q < 0$ holds. If $d_{q+1} = \dots = d_n = 0$ holds, $d_q \geq 0$ and thus $w \cdot d = -d_q \leq 0$. This proves $w \in C^+(g)^\circ$. Since w is not contained in $L^+(g)^\circ$, GCQ is violated in this case. $\square$

The germ of type (5) in Table 2

Let $g = \left(x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-2} x_j + \epsilon_{q-1} x_{q-1}^2 + x_q^3 + \sum_{j=q+1}^n \epsilon_j x_j^2\right)$, where $\epsilon_j \in \{1, -1\}$, $Q = \epsilon_{q-1} x_{q-1}^2 + \sum_{j=q+1}^n \epsilon_j x_j^2$, and $R = x_q^3$.

Proposition 4.21. *The tangent cone $C^+(g)$ is equal to $\overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q^3 > 0\}$ if $\epsilon_{q-1} = \epsilon_{q+2} = \dots = \epsilon_n = 1$, and $C^+(g) = \overline{D}(q-2, Q)$ otherwise.*

Proof. By Corollary 4.1 (with $P(x') = \epsilon_{q-1} x_{q-1}^2$ and $s = 2$), the set

$$\overline{D}(q-2, Q) \setminus \{(0, \dots, 0, d_{q-1}, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q^3 > 0\}$$

is contained in $C^+(g)$, and $C^+(g) = \overline{D}(q-2, Q)$ if $\epsilon_j = -1$ for some $j \geq q+1$.

In what follows, we assume $\epsilon_{q+1} = \dots = \epsilon_n = +1$. Let $d = (0, \dots, 0, d_{q-1}, d_q, 0, \dots, 0) \in \overline{D}(q-2, Q)$ with $d_q^3 > 0 \Leftrightarrow d_q > 0$. Since $d \in \overline{D}(q-2, Q)$, $d_{q-1} \leq 0$ and $Q(d') = \epsilon_{q-1}d_{q-1}^2 \leq 0$. If $\epsilon_{q-1} = -1$ and $d_{q-1} < 0$, the vector $v = d_{q-1}e_{q-1}$ satisfies the conditions in Lemma 4.2. Indeed, $v_1 = \dots = v_{q-2} = 0$, $v_{q-1} = d_{q-1} < 0$ (in particular v satisfies the condition (a) in Lemma 4.2), and $v' \cdot \nabla Q(d') = -2d_{q-1}^2 < 0$. If $\epsilon_{q-1} = -1$ and $d_{q-1} = 0$, the vector $v = -e_{q-1}$ satisfies the conditions in Lemma 4.2. Indeed, $v_1 = \dots = v_{q-2} = 0$, $v_{q-1} = -1 < 0$, $v' \cdot \nabla Q(d') = 0$, and ${}^t v' \text{Hess}(Q)v' = -2 < 0$. We thus obtain $C^+(g) = \overline{D}(q-2, Q)$ if $\epsilon_{q-1} = -1$.

If $\epsilon_{q-1} = 1$, any $x \in M(g)$ satisfies the following inequality:

$$0 \geq -\sum_{j=1}^{q-2} x_j + x_{q-1}^2 + x_q^3 + \sum_{j=q+1}^n x_j^2 \Leftrightarrow x_q^3 \leq \sum_{j=1}^{q-1} x_j - x_{q-1}^2 - \sum_{j=q+1}^n x_j^2 \leq 0.$$

Thus, any $d \in C^+(g)$ also satisfies the inequality $d_q^3 \leq 0$. $\square$

Proposition 4.22. *ACQ holds for g if and only if $\epsilon_{q-1} = \epsilon_{q+1} = \dots = \epsilon_n = -1$.*

Proof. Suppose that the condition in the proposition does not hold, say $\epsilon_j = 1$ for $j = q-1$ or $j \geq q+1$. It is easy to check that $-e_j$ is contained in $L^+(g)$ but not in $\overline{D}(q-2, Q)$, in particular $C^+(g) \subset \overline{D}(q-2, Q) \subsetneq L^+(g)$. Suppose conversely that the condition in the proposition hold. By Proposition 4.21, the tangent cone $C^+(g)$ is equal to $\overline{D}(q-2, Q)$, which is equal to $\{d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-2} = 0, d_{q-1} \leq 0\} = L^+(g)$ by the assumption. $\square$

Proposition 4.23. *GCQ holds for g if and only if one of $\epsilon_{q+1}, \dots, \epsilon_n$ is -1 .*

Proof. It is easy to check that $L^+(g)^\circ$ is equal to $\{w \in \mathbb{R}^n \mid w_{q-1} \geq 0, w_q = \dots = w_n = 0\}$.

Proof of “if” part: Without loss of generality, we can assume $\epsilon_{q+1} = -1$. By Proposition 4.21, $C^+(g)$ is equal to $\overline{D}(q-2, Q)$. Take any $w \in C^+(g)^\circ$. Since $\pm e_q$ is contained in $\overline{D}(q-2, Q)$, $w_q = 0$ holds. Since $\pm e_{q+1}$ is contained in $\overline{D}(q-2, Q)$, $w_{q+1} = 0$ holds. Since $\pm e_{q+1} \pm e_j$ is contained in $\overline{D}(q-2, Q)$ for $j \in \{q+2, \dots, n\}$, $w_j = 0$ holds. Since $-e_{q-1} \pm e_{q+1}$ is contained in $\overline{D}(q-2, Q)$, $w_{q-1} \geq 0$ holds. This proves $w \in L^+(g)^\circ$.

Proof of “only if” part: We show the contraposition of the statement, that is, if all of $\epsilon_{q+1}, \dots, \epsilon_n$ are 1, GCQ is violated. Under the assumption,

$$\overline{D}(q-2, Q) = \left\{ d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-2} = 0, d_{q-1} \leq 0, \epsilon_{q-1}d_{q-1}^2 + \sum_{j=q}^n d_j^2 \leq 0 \right\}$$

holds. Take any $d \in \overline{D}(q-2, Q)$. Then, $d_q^2 \leq \sum_{j=q}^n d_j^2 \leq |d_{q-1}^2|$, in particular $|d_q| \leq |d_{q-1}|$, and thus $(e_{q-1} + e_q) \cdot d = d_{q-1} + d_q \leq 0$ hold. Thus, $w = e_{q-1} + e_q$ is contained in $\overline{D}(q-2, Q)^\circ$. Since w is not contained in $L^+(g)^\circ$, GCQ is violated. $\square$

The germ of type (6) in Table 2

Let $g = (x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-3} x_j + \sum_{j=1}^2 \delta_j x_{q-j}^2 + \alpha x_{q-2} x_{q-1} + \sum_{j=q}^n \epsilon_j x_j^2)$, where $\epsilon_j, \delta_j \in \{1, -1\}$ and $\alpha \in \mathbb{R}$ satisfying $4\delta_1 \delta_2 - \alpha^2 \neq 0$.

Proposition 4.24. *The tangent cone $C^+(g)$ is equal to $\overline{D}(q-3, Q)$.*

Proof. We can put $R(x') = 0$ in this case, and thus the proposition holds by Lemma 4.2. $\square$

Proposition 4.25. *ACQ holds for g if and only if $\delta_1 = \delta_2 = -1$, $\alpha < 2$, and $\epsilon_q = \dots = \epsilon_n = -1$.*

Proof. If $\delta_j = 1$ for $j = 1, 2$, $-e_{q-j}$ is contained in $L^+(g)$ but not in $\overline{D}(q-3, Q)$. If $\delta_1 = \delta_2 = -1$ and $\alpha > 2$, $-e_{q-1} - e_q$ is contained in $L^+(g)$ but not in $\overline{D}(q-3, Q)$. If $\epsilon_j = 1$ for $j \geq q$, $-e_j$ is contained in $L^+(g)$ but not in $\overline{D}(q-3, Q)$. In each case, $C^+(g) = \overline{D}(q-3, Q)$ is a proper subset of $L^+(g)$.

Suppose that the condition in the proposition hold. For $d \in L^+(g) = \{d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-3} = 0, d_{q-2}, d_{q-1} \leq 0\}$, the value $Q(d)$ is estimated as follows:

$$Q(d) = -(d_{q-1} - d_{q-2})^2 + (\alpha - 2)d_{q-1}d_{q-2} - \sum_{j=q}^n d_j^2 \leq 0.$$

Thus, d is contained in $\overline{D}(q-3, Q) = C^+(g)$. □

Proposition 4.26. *GCQ holds for g if and only if one of $\epsilon_q, \dots, \epsilon_n$ is -1 .*

Proof. It is easy to check that $L^+(g)^\circ$ is equal to

$$\{w \in \mathbb{R}^n \mid w_{q-2} \geq 0, w_{q-1} \geq 0, w_q = \dots = w_n = 0\}.$$

Proof of “if” part: Without loss of generality, we can assume $\epsilon_q = -1$. Take any $w \in C^+(g)^\circ$. Since $\pm e_q \in C^+(g)$, $w_q = 0$ holds. Since $\pm e_q \pm e_j \in C^+(g)$ holds for $j \in \{q+1, \dots, n\}$, $w_j = 0$ holds. Since $-e_j \pm e_q \in C^+(g)$ holds for $j \in \{q-2, q-1\}$, $w_j \geq 0$ holds for $j \in \{q-2, q-1\}$. This proves $w \in L^+(g)^\circ$.

Proof of “only if” part: We show the contraposition of the statement, that is, if all of $\epsilon_q, \dots, \epsilon_n$ are 1, GCQ is violated. We take $P \in O(2)$ so that the following equality holds:

$$\delta_1 x_1^2 + \delta_2 x_2^2 + \alpha x_1 x_2 = (x_1, x_2)^t P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \neq 0$ is the eigenvalues of $\begin{pmatrix} \delta_1 & \alpha/2 \\ \alpha/2 & \delta_2 \end{pmatrix}$. The following then holds for $d \in C^+(g)$:

$$\begin{aligned} & \delta_1 d_{q-1}^2 + \delta_2 d_{q-2}^2 + \alpha d_{q-1} d_{q-2} + \sum_{j=q}^n d_j^2 \leq 0 \\ \Rightarrow & \sum_{j=q}^n d_j^2 \leq \left| (d_{q-1}, d_{q-2})^t P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P \begin{pmatrix} d_{q-1} \\ d_{q-2} \end{pmatrix} \right| \\ \Rightarrow & \sum_{j=q}^n d_j^2 \leq \max\{|\lambda_1|, |\lambda_2|\} \|(d_{q-1}, d_{q-2})\|^2. \end{aligned}$$

Let $R = \sqrt{\max\{|\lambda_1|, |\lambda_2|\}}$. For any $d \in C^+(g)$, the inner product $(R(e_{q-1} + e_{q-2}) + e_q) \cdot d$ is estimated as follows:

$$\begin{aligned} & (R(e_{q-1} + e_{q-2}) + e_q) \cdot d \\ &= R(d_{q-1} + d_{q-2}) + d_q \\ &\leq -R(|d_{q-1}| + |d_{q-2}|) + |d_q| \quad (\because d_{q-1}, d_{q-2} \leq 0) \end{aligned}$$

$$\begin{aligned}
&\leq -R \|(d_{q-1}, d_{q-2})\| + |d_q| && \left(\cdot \|(d_{q-1}, d_{q-2})\| = \sqrt{d_{q-1}^2 + d_{q-2}^2} \leq |d_{q-1}| + |d_{q-2}| \right) \\
&\leq -R \sqrt{\frac{\sum_{j=q}^n d_j^2}{\max\{|\lambda_1|, |\lambda_2|\}}} + |d_q| \\
&\leq \left(-\frac{R}{\sqrt{\max\{|\lambda_1|, |\lambda_2|\}}} + 1 \right) |d_q| = 0.
\end{aligned}$$

Thus, $R(e_{q-1} + e_{q-2}) + e_q$ is contained in $C^+(g)^\circ$. However, it is not in $L^+(g)^\circ$, and thus GCQ is violated. $\square$

The germ of type (7) in Table 2

Let $g = (x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-3} x_j + \epsilon_{q-2}(x_{q-2} + \epsilon'_{q-1}x_{q-1})^2 + \epsilon_{q-1}x_{q-1}^3 + \sum_{j=q}^n \epsilon_j x_j^2)$, where $\epsilon_j, \epsilon'_{q-1} \in \{1, -1\}$, $Q = \epsilon_{q-2}(x_{q-2} + \epsilon'_{q-1}x_{q-1})^2 + \sum_{j=q}^n \epsilon_j x_j^2$, and $R = \epsilon_{q-1}x_{q-1}^3$.

Proposition 4.27. *The tangent cone $C^+(g)$ is equal to $\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_{q-1}, 0, \dots, 0) \in \mathbb{R}^n \mid d_{q-1} < 0\}$ if $\epsilon_{q-1} = -1$ and $\epsilon_{q-2} = \epsilon_q = \dots = \epsilon_n = 1$, and $C^+(g) = \overline{D}(q-3, Q)$ otherwise.*

Proof. By Corollary 4.1 (with $P(x') = \epsilon_{q-2}(x_{q-2} + \epsilon'_{q-1}x_{q-1})^2$ and $s = 2$), $\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_{q-2}, d_{q-1}, 0, \dots, 0) \in \mathbb{R}^n \mid \epsilon_{q-1}d_{q-1}^3 > 0\}$ is contained in $C^+(g)$, and $C^+(g) = \overline{D}(q-3, Q)$ if $\epsilon_j = -1$ for some $j \geq q$. In particular, $C^+(g) = \overline{D}(q-3, Q)$ if $\epsilon_{q-1} = 1$ since any $d \in \overline{D}(q-3, Q)$ satisfies $d_{q-1} \leq 0$.

In what follows, we assume $\epsilon_{q-1} = -1$ and $\epsilon_q = \dots = \epsilon_n = +1$. Let $d = d_{q-2}e_{q-2} + d_{q-1}e_{q-1} \in \overline{D}(q-3, Q)$ with $\epsilon_{q-1}d_{q-1}^3 > 0 \Leftrightarrow d_{q-1} < 0$. If $\epsilon_{q-2} = -1$ and $d_{q-2} + \epsilon'_{q-1}d_{q-1} \neq 0$, $Q(d')$ is equal to $\epsilon_{q-2}(d_{q-2} + \epsilon'_{q-1}d_{q-1})^2 < 0$, and thus d is contained in $C^+(g)$ by Lemma 4.2. If $\epsilon_{q-2} = -1$ and $d_{q-2} + \epsilon'_{q-1}d_{q-1} = 0$, the vector $v = -e_{q-1}$ satisfies the conditions in Lemma 4.2. Indeed, $v_1 = \dots = v_{q-2} = 0$, $v_{q-1} = -1 < 0$, $v' \cdot \nabla Q(d') = 0$, and ${}^t v' \text{Hess}(Q)v' = -2 < 0$. We thus obtain $C^+(g) = \overline{D}(q-2, Q)$ if $\epsilon_{q-2} = -1$.

If $\epsilon_{q-2} = 1$, any $x \in M(g)$ satisfies the following inequality:

$$\begin{aligned}
0 &\geq -\sum_{j=1}^{q-3} x_j + (x_{q-2} + \epsilon'_{q-1}x_{q-1})^2 - x_{q-1}^3 + \sum_{j=q}^n x_j^2 \\
&\Leftrightarrow x_{q-1}^3 \geq -\sum_{j=1}^{q-1} x_j + (x_{q-2} + \epsilon'_{q-1}x_{q-1})^2 + \sum_{j=q}^n x_j^2 \geq 0.
\end{aligned}$$

Thus, any $d \in C^+(g)$ also satisfies the inequality $d_{q-1}^3 \geq 0 \Leftrightarrow d_{q-1} \geq 0$. $\square$

Proposition 4.28. *ACQ holds for g if and only if $\epsilon_{q-2} = \epsilon_q = \dots = \epsilon_n = -1$,*

Proof. Suppose that the condition in the proposition does not hold, say $\epsilon_j = 1$ for $j = q-2$ or $j \geq q$. It is easy to check that $-e_j$ is contained in $L^+(g)$ but not in $\overline{D}(q-3, Q)$, in particular $C^+(g) \subset \overline{D}(q-3, Q) \subsetneq L^+(g)$. Suppose conversely that the condition in the proposition holds. By Proposition 4.27, the tangent cone $C^+(g)$ is equal to $\overline{D}(q-3, Q)$, which is equal to $\{d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-3} = 0, d_{q-2}, d_{q-1} \leq 0\} = L^+(g)$ by the assumption. $\square$

Proposition 4.29. *GCQ holds for g if and only if one of $\epsilon_q, \dots, \epsilon_n$ is -1 .*

Proof. It is easy to check that $L^+(g)^\circ$ is equal to

$$\{w \in \mathbb{R}^n \mid w_{q-2} \geq 0, w_{q-1} \geq 0, w_q = \dots = w_n = 0\}.$$

Proof of “if” part: Without loss of generality, we can assume $\epsilon_q = -1$. Take any $w \in C^+(g)^\circ$. Since $\pm e_q \in C^+(g)$, $w_q = 0$ holds. Since $\pm e_q \pm e_j \in C^+(g)$ holds for $j \in \{q+1, \dots, n\}$, $w_j = 0$ holds. Since $-e_j \pm e_q \in C^+(g)$ holds for $j \in \{q-2, q-1\}$, $w_j \geq 0$ holds for $j \in \{q-2, q-1\}$. This proves $w \in L^+(g)^\circ$.

Proof of “only if” part: We show the contraposition of the statement, that is, if all of $\epsilon_q, \dots, \epsilon_n$ are 1, GCQ is violated. Since the eigenvalues of $\begin{pmatrix} \epsilon_{q-2} & \epsilon_{q-2}\epsilon'_{q-1} \\ \epsilon_{q-2}\epsilon'_{q-1} & \epsilon_{q-2} \end{pmatrix}$ are 0 and $2\epsilon_{q-2}$, we can obtain $\sum_{j=q}^n d_j^2 \leq 2\|(d_{q-2}, d_{q-1})\|^2$ for $d \in C^+(g)$ in the same way as in the proof of Proposition 4.26. Let $w = \sqrt{2}(e_{q-2} + e_{q-1}) + e_q$. As in the proof of Proposition 4.26, we obtain the following inequality for any $d \in C^+(g)$:

$$w \cdot d = \sqrt{2}(d_{q-2} + d_{q-1}) + d_q \leq \left(-\frac{\sqrt{2}}{\sqrt{|2\epsilon_{q-2}|}} + 1 \right) |d_q| = 0.$$

This proves $w \in C^+(g)^\circ$. Since w is not contained in $L^+(g)^\circ$, GCQ is violated. $\square$

The germ of type (8) in Table 2

Let $g = (x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-3} x_j + \epsilon_{q-2}x_{q-2}^3 + \epsilon_{q-1}x_{q-1}^2 + \epsilon'_{q-1}x_{q-2}x_{q-1} + \sum_{j=q}^n \epsilon_j x_j^2)$, where $\epsilon_j, \epsilon'_{q-1} \in \{1, -1\}$, $Q = \epsilon_{q-1}x_{q-1}^2 + \epsilon'_{q-1}x_{q-2}x_{q-1} + \sum_{j=q}^n \epsilon_j x_j^2$, and $R = \epsilon_{q-2}x_{q-2}^3$.

Proposition 4.30. *The tangent cone $C^+(g)$ is equal to $\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_{q-2}, 0, \dots, 0) \in \mathbb{R}^n \mid d_{q-2} < 0\}$ if $\epsilon_{q-2} = -1$ and $\epsilon'_{q-1} = \epsilon_q = \dots = \epsilon_n = 1$, and $C^+(g) = \overline{D}(q-3, Q)$ otherwise.*

Proof. By Corollary 4.1 (with $P(x') = \epsilon_{q-1}x_{q-1}^2 + \epsilon'_{q-1}x_{q-2}x_{q-1}$ and $s = 2$), $\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_{q-2}, d_{q-1}, 0, \dots, 0) \in \mathbb{R}^n \mid \epsilon_{q-2}d_{q-2}^3 > 0\}$ is contained in $C^+(g)$, and $C^+(g) = \overline{D}(q-3, Q)$ if $\epsilon_j = -1$ for some $j \geq q$. In particular, $C^+(g) = \overline{D}(q-3, Q)$ if $\epsilon_{q-2} = 1$ since any $d \in \overline{D}(q-3, Q)$ satisfies $d_{q-2} \leq 0$.

In what follows, we assume $\epsilon_{q-2} = -1$ and $\epsilon_q = \dots = \epsilon_n = +1$. Let $d = d_{q-2}e_{q-2} + d_{q-1}e_{q-1} \in \overline{D}(q-3, Q)$ with $\epsilon_{q-2}d_{q-2}^3 > 0 \Leftrightarrow d_{q-2} < 0$. Since $d \in \overline{D}(q-3, Q)$, $Q(d') = d_{q-1}(\epsilon_{q-1}d_{q-1} + \epsilon'_{q-1}d_{q-2})$ is less than or equal to 0. By Lemma 4.2, d is contained in $C^+(g)$ if d_{q-1} and $\epsilon_{q-1}d_{q-1} + \epsilon'_{q-1}d_{q-2}$ are not equal to 0. If $\epsilon_{q-1}d_{q-1} + \epsilon'_{q-1}d_{q-2} = 0$, d_{q-1} is not 0 since $d_{q-2} < 0$, and ϵ_{q-1} is not equal to ϵ'_{q-1} since $d_{q-2}, d_{q-1} < 0$. The gradient $\nabla Q(d')$ is calculated as follows:

$$\nabla Q(d') = (\epsilon'_{q-1}d_{q-1}, \epsilon'_{q-1}d_{q-2} + 2\epsilon_{q-1}d_{q-1}, 0, \dots, 0) = (\epsilon'_{q-1}d_{q-1}, \epsilon_{q-1}d_{q-1}, 0, \dots, 0).$$

In particular, both the 1st and 2nd components of $\nabla Q(d')$ are not 0, and their signs are opposite. Thus, there exists $v \in \mathbb{R}^n$ with $v_j \leq 0$ for $j \leq q-1$ and $v' \cdot \nabla Q(d') < 0$. By Lemma 4.2, d is contained in $C^+(g)$. If $d_{q-1} = 0$ and $\epsilon'_{q-1} = -1$, the vector $v = -e_{q-1}$ satisfies the conditions in Lemma 4.2. Indeed, $v_1 = \dots = v_{q-2} = 0$, $v_{q-1} = -1 < 0$, $v' \cdot \nabla Q(d') = d_{q-2} < 0$.

So far we have shown that

$$\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_{q-2}, 0, \dots, 0) \in \mathbb{R}^n \mid d_{q-2} < 0\} \subset C^+(g),$$

and $C^+(g) = \overline{D}(q-3, Q)$ if $\epsilon'_{q-1} = -1$. We will show that $d = (0, \dots, 0, d_{q-2}, 0, \dots, 0)$ ($d_{q-2} < 0$) is not contained in $C^+(g)$ if $\epsilon'_{q-1} = 1$. Suppose contrary that d is in $C^+(g)$. Take $t_m > 0$ and $x^{(m)} \in M(g)$ so that $\lim_{m \rightarrow \infty} x^{(m)} = 0$ and $\lim_{m \rightarrow \infty} t_m \cdot x^{(m)} = d$. The following inequality holds:

$$\begin{aligned} 0 &\geq -\sum_{j=1}^{q-3} x_j^{(m)} - \left(x_{q-2}^{(m)}\right)^3 + \epsilon_{q-1} \left(x_{q-1}^{(m)}\right)^2 + x_{q-2}^{(m)} x_{q-1}^{(m)} + \sum_{j=q}^n (x_j^{(m)})^2 \\ &\Leftrightarrow \left(x_{q-2}^{(m)}\right)^3 \geq -\sum_{j=1}^{q-3} x_j^{(m)} + \frac{x_{q-1}^{(m)}}{t_m} \left(\epsilon_{q-1} t_m x_{q-1}^{(m)} + t_m x_{q-2}^{(m)}\right) + \sum_{j=q}^n (x_j^{(m)})^2. \end{aligned}$$

Since $\lim_{m \rightarrow \infty} t_m \cdot x^{(m)} = d$, $t_m x_{q-1}^{(m)}$ and $t_m x_{q-2}^{(m)}$ tend to 0 and d_{q-2} , respectively, as $m \rightarrow \infty$. Hence $\epsilon_{q-1} t_m x_{q-1}^{(m)} + t_m x_{q-2}^{(m)}$ is less than 0 for $m \gg 0$. We thus obtain $x_{q-2}^{(m)} \geq 0$, contradicting $0 > d_{q-2} = \lim_{m \rightarrow \infty} t_m \cdot x_{q-2}^{(m)}$. We eventually obtain:

$$\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_{q-2}, 0, \dots, 0) \in \mathbb{R}^n \mid d_{q-2} < 0\} = C^+(g)$$

if $\epsilon'_{q-1} = 1$. □

Proposition 4.31. *ACQ holds for g if and only if $\epsilon_{q-1} = \epsilon'_{q-1} = \epsilon_q = \dots = \epsilon_n = -1$,*

Proof. If $\epsilon_j = 1$ for $j \geq q-1$, the vector $-e_j$ is contained in $L^+(g)$ but not in $\overline{D}(q-3, Q)$. If $\epsilon'_{q-1} = -1$, the vector $-2e_{q-2} - e_{q-1}$ is contained in $L^+(g)$ but not in $\overline{D}(q-3, Q)$. In each case, $C^+(g) \subset \overline{D}(q-3, Q)$ is a proper subset of $L^+(g)$.

Suppose that the condition in the proposition hold. By Proposition 4.30, the tangent cone $C^+(g)$ is equal to $\overline{D}(q-3, Q)$, which is equal to $\{d \in \mathbb{R}^n \mid d_1 = \dots = d_{q-3} = 0, d_{q-2}, d_{q-1} \leq 0\} = L^+(g)$ by the assumption. □

Proposition 4.32. *GCQ holds for g if and only if one of $\epsilon_q, \dots, \epsilon_n$ is -1 .*

Proof. It is easy to check that $L^+(g)^\circ$ is equal to

$$\{w \in \mathbb{R}^n \mid w_{q-2} \geq 0, w_{q-1} \geq 0, w_q = \dots = w_n = 0\}.$$

Proof of “if” part: Without loss of generality, we can assume $\epsilon_q = -1$. Take any $w \in C^+(g)^\circ$. Since $\pm e_q \in C^+(g)$, $w_q = 0$ holds. Since $\pm e_q \pm e_j \in C^+(g)$ holds for $j \in \{q+1, \dots, n\}$, $w_j = 0$ holds. Since $-e_j \pm e_q \in C^+(g)$ holds for $j \in \{q-2, q-1\}$, $w_j \geq 0$ holds for $j \in \{q-2, q-1\}$. This proves $w \in L^+(g)^\circ$.

Proof of “only if” part: We show the contraposition of the statement, that is, if all of $\epsilon_q, \dots, \epsilon_n$ are 1, GCQ is violated. Let $\lambda_1, \lambda_2 \neq 0$ be the eigenvalues of $\begin{pmatrix} 0 & \epsilon'_{q-1}/2 \\ \epsilon'_{q-1}/2 & \epsilon_{q-1} \end{pmatrix}$. We can show that $w = R(e_{q-2} + e_{q-1}) + e_q$ is contained in $C^+(g)^\circ$ for $R = \sqrt{\max\{|\lambda_1|, |\lambda_2|\}}$ in the same way as in the proof of Proposition 4.26. Since w is not contained in $L^+(g)^\circ$, GCQ is violated. □

The germ of type (9) in Table 2

Let $g = \left(x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-3} x_j + x_q^3 + \epsilon_{01} x_q x_{q-1} + \epsilon_{02} x_q x_{q-2} + \epsilon_{12} x_{q-1} x_{q-2} + \sum_{j=q+1}^n \epsilon_j x_j^2\right)$, where $\epsilon_j, \epsilon_{ij} \in \{1, -1\}$, $Q = \epsilon_{01} x_q x_{q-1} + \epsilon_{02} x_q x_{q-2} + \epsilon_{12} x_{q-1} x_{q-2} + \sum_{j=q+1}^n \epsilon_j x_j^2$, and $R = x_q^3$.

Proposition 4.33. *The tangent cone $C^+(g)$ is equal to $\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q > 0\}$ if $\epsilon_{01} = \epsilon_{02} = -1$ and $\epsilon_{q+1} = \dots = \epsilon_n = 1$, and $C^+(g) = \overline{D}(q-3, Q)$ otherwise.*

Proof. By Corollary 4.1 (with $P(x') = \epsilon_{01}x_qx_{q-1} + \epsilon_{02}x_qx_{q-2} + \epsilon_{12}x_{q-1}x_{q-2}$ and $s = 3$), $\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_{q-2}, d_{q-1}, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q^3 > 0\}$ is contained in $C^+(g)$, and $C^+(g) = \overline{D}(q-3, Q)$ if $\epsilon_j = -1$ for some $j \geq q+1$.

In what follows, we assume $\epsilon_{q+1} = \dots = \epsilon_n = +1$. Let $d = d_{q-2}e_{q-2} + d_{q-1}e_{q-1} + d_qe_q \in \overline{D}(q-3, Q)$ with $d_q^3 > 0 \Leftrightarrow d_q > 0$. Since $d \in \overline{D}(q-3, Q)$, $Q(d')$ is less than or equal to 0. By Lemma 4.2, d is contained in $C^+(g)$ if $Q(d') < 0$. Suppose that $Q(d') = \frac{1}{2}({}^t d' \text{Hess}(Q) d')$ is equal to 0. If $(\nabla Q(d'))_3 = \epsilon_{01}d_{q-1} + \epsilon_{02}d_{q-2}$ is not 0, the vector $v = d - \text{sign}(\epsilon_{01}d_{q-1} + \epsilon_{02}d_{q-2})e_q$ satisfies the conditions in Lemma 4.2, where $\text{sign}(\epsilon_{01}d_{q-1} + \epsilon_{02}d_{q-2})$ is $+1$ (resp. -1) if $\epsilon_{01}d_{q-1} + \epsilon_{02}d_{q-2}$ is positive (resp. negative). Indeed, $v_1 = \dots = v_{q-1} = 0$, and

$$v' \cdot \nabla Q(d') = {}^t d' \text{Hess}(Q) d' - \text{sign}(\epsilon_{01}d_{q-1} + \epsilon_{02}d_{q-2})(\nabla Q(d'))_3 < 0.$$

If $(\nabla Q(d'))_3 = \epsilon_{01}d_{q-1} + \epsilon_{02}d_{q-2} = 0$, either $d_{q-1} = d_{q-2} = 0$ or ϵ_{01} is not equal to ϵ_{02} . If $(d_{q-2}, d_{q-1}) \neq (0, 0)$ and the latter condition holds, d_{q-1} is equal to d_{q-2} , in particular both are not equal to 0. Since $\text{Hess}(Q)$ is regular and $d' \neq 0$, either the 1st or the 2nd component of $\nabla(Q)(d') = \text{Hess}(Q)d'$ is not 0. If $(\nabla Q(d'))_1 \neq 0$, the vector $v = d - \varepsilon \text{sign}((\nabla Q(d'))_1)e_{q-2}$ satisfies the conditions in Lemma 4.2 for $0 < \varepsilon \ll 1$. Indeed, $v_1 = \dots = v_{q-3} = 0$, $v_{q-1} = d_{q-1} < 0$, $v_{q-2} = d_{q-2} - \varepsilon \text{sign}((\nabla Q(d'))_1) < 0$ for $\varepsilon \ll 1$, and

$$v' \cdot \nabla Q(d') = {}^t d' \text{Hess}(Q) d' - \varepsilon \text{sign}((\nabla Q(d'))_1)(\nabla Q(d'))_1 < 0.$$

By Lemma 4.2, d is contained in $C^+(g)$. One can also show that $d \in C^+(g)$ if $(\nabla Q(d'))_2 \neq 0$ in the same manner.

So far, we have shown the following inclusion:

$$\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q > 0\} \subset C^+(g).$$

Let $d = (0, \dots, 0, d_q, 0, \dots, 0)$ with $d_q > 0$. Since $\nabla Q(d') = (\epsilon_{02}d_q, \epsilon_{01}d_q, 0, \dots, 0)$, there exists $v \in \mathbb{R}^n$ with $v_j \leq 0$ for $j \leq q-1$ and $v' \cdot \nabla Q(d') < 0$ if either ϵ_{02} or ϵ_{01} is 1. Thus, $C^+(g) = \overline{D}(q-3, Q)$ unless $\epsilon_{02} = \epsilon_{01} = -1$. Suppose that $\epsilon_{02} = \epsilon_{01} = -1$ and d is contained in $C^+(g)$. Take $t_m > 0$ and $x^{(m)} \in M(g)$ so that $\lim_{m \rightarrow \infty} x^{(m)} = 0$ and $\lim_{m \rightarrow \infty} t_m \cdot x^{(m)} = d$. The following inequality holds:

$$\begin{aligned} 0 &\geq -\sum_{j=1}^{q-3} x_j^{(m)} + \left(x_q^{(m)}\right)^3 - x_q^{(m)}x_{q-2}^{(m)} - x_q^{(m)}x_{q-1}^{(m)} + \epsilon_{12}x_{q-1}^{(m)}x_{q-2}^{(m)} + \sum_{j=q+1}^n (x_j^{(m)})^2 \\ &\Leftrightarrow \left(x_q^{(m)}\right)^3 \leq \sum_{j=1}^{q-3} x_j^{(m)} + \frac{x_{q-2}^{(m)}}{t_m} \left(t_m x_q^{(m)} - \epsilon_{12}t_m x_{q-1}^{(m)}\right) + x_q^{(m)}x_{q-1}^{(m)} - \sum_{j=q+1}^n (x_j^{(m)})^2. \end{aligned}$$

Since $\lim_{m \rightarrow \infty} t_m \cdot x^{(m)} = d$, $t_m x_{q-1}^{(m)}$ and $t_m x_q^{(m)}$ tend to 0 and d_q , respectively, as $m \rightarrow \infty$. Hence both $t_m x_q^{(m)} - \epsilon_{12}t_m x_{q-1}^{(m)}$ and $x_q^{(m)}$ are larger than 0 for $m \gg 0$. We thus obtain $x_q^{(m)} \leq 0$ when $m \gg 0$, contradicting the fact $x_q^{(m)} > 0$. We eventually obtain:

$$\overline{D}(q-3, Q) \setminus \{(0, \dots, 0, d_q, 0, \dots, 0) \in \mathbb{R}^n \mid d_q > 0\} = C^+(g)$$

if $\epsilon_{01} = \epsilon_{02} = -1$. □

Proposition 4.34. *ACQ does not hold for g .*

Proof. The vector $\epsilon_{01}e_q - e_{q-1}$ is contained in $L^+(g)$ but not in $\overline{D}(q-3, Q)$, in particular $C^+(g) \subset \overline{D}(q-3, Q) \subsetneq L^+(g)$. $\square$

Proposition 4.35. *GCQ holds for g if and only if either*

1. *one of $\epsilon_{q+1}, \dots, \epsilon_n$ is -1 , or*
2. *ϵ_{01} or ϵ_{02} is 1 .*

Proof. It is easy to check that $L^+(g)^\circ$ is equal to

$$\{w \in \mathbb{R}^n | w_{q-2} \geq 0, w_{q-1} \geq 0, w_q = \dots = w_n = 0\}.$$

First we show that $\overline{D}(q-3, Q)^\circ \subset L^+(g)^\circ$. Take any $w \in \overline{D}(q-3, Q)^\circ$. Since $\pm e_q \in \overline{D}(q-3, Q)^\circ$, $w_q = 0$ holds. Since $-e_j \in \overline{D}(q-3, Q)^\circ$ holds for $j \in \{q-2, q-1\}$, $w_j \geq 0$ holds for $j \in \{q-2, q-1\}$. Since $-\frac{1}{s}e_{q-1} + \epsilon_{01}\epsilon_j s e_q \pm e_j \in \overline{D}(q-3, Q)^\circ$ holds for $s > 0$ and $j \in \{q+1, \dots, n\}$, $-\frac{1}{s}w_{q-1} \pm w_j \leq 0$ holds. Since this holds for arbitrarily large $s > 0$, we obtain $w_j = 0$ for $j \in \{q+1, \dots, n\}$. This proves $w \in L^+(g)^\circ$.

Therefore, if $C^+(g) = \overline{D}(q-3, Q)$ holds, GCQ holds. By Proposition 4.33, this holds if either one of $\epsilon_{01}, \epsilon_{02}$ is 1 , or one of $\epsilon_{q+1}, \dots, \epsilon_n$ is -1 . In what follows, we consider the case that $\epsilon_{01} = \epsilon_{02} = -1$ and $\epsilon_{q+1} = \dots = \epsilon_n = 1$ and prove that GCQ is violated in this case.

Suppose $\epsilon_{12} = 1$. We claim that $w = e_q$ is contained in $C^+(g)^\circ$. Take any $d \in C^+(g)$. Then,

$$-d_q d_{q-1} - d_q d_{q-2} + d_{q-1} d_{q-2} + \sum_{j=q+1}^n d_j^2 \leq 0$$

holds. If $d_{q-1} + d_{q-2} = 0$, $d_{q+1} = \dots = d_n = 0$ holds and thus $d_q < 0$ holds by Proposition 4.33. In this case, $w \cdot d = d_q < 0$ holds. If $d_{q-1} + d_{q-2} < 0$,

$$d_q \leq \frac{-d_{q-1} d_{q-2} - \sum_{j=q+1}^n d_j^2}{-d_{q-1} - d_{q-2}} < 0$$

holds and thus $w \cdot d = d_q < 0$ holds. Since w is not contained in $L^+(g)^\circ$, this proves that GCQ is violated in this case.

Suppose $\epsilon_{12} = -1$. We claim that $w = e_{q-2} + e_{q-1} + e_q$ is contained in $C^+(g)^\circ$. Take any $d \in C^+(g)$. Then,

$$-d_q d_{q-1} - d_q d_{q-2} - d_{q-1} d_{q-2} + \sum_{j=q+1}^n d_j^2 \leq 0$$

holds. If $d_{q-1} + d_{q-2} = 0$, $d_{q+1} = \dots = d_n = 0$ holds and thus $d_q < 0$ holds by Proposition 4.33. In this case, $w \cdot d = d_q < 0$ holds. If $d_{q-1} + d_{q-2} < 0$,

$$d_{q-2} + d_{q-1} + d_q \leq \frac{-d_{q-2}^2 - d_{q-1}^2 - d_{q-1} d_{q-2} - \sum_{j=q+1}^n d_j^2}{-d_{q-1} - d_{q-2}} < 0$$

holds and thus $w \cdot d = d_{q-2} + d_{q-1} + d_q < 0$ holds. Since w is not contained in $L^+(g)^\circ$, this proves that GCQ is violated in this case. $\square$

The germ of type (10) in Table 2

We put

$$g = \left(x_1, \dots, x_{q-1}, -\sum_{j=1}^{q-4} x_j + \sum_{j=1}^3 \delta_j x_{q-4+j}^2 + \sum_{1 \leq i < j \leq 3} \alpha_{ij} x_{q-4+i} x_{q-4+j} \right. \\ \left. + \epsilon_0 x_{q-3} x_{q-2} x_{q-1} + \sum_{j=q}^n \epsilon_j x_j^2 \right),$$

where $\alpha \in \mathbb{R}$ and $\delta_j \in \{1, -1\}$ satisfy the condition $(**)$ in Table 2 and $\epsilon_j \in \{1, -1\}$, $Q = +\sum_{j=1}^3 \delta_j x_{q-4+j}^2 + \sum_{1 \leq i < j \leq 3} \alpha_{ij} x_{q-4+i} x_{q-4+j} + \sum_{j=q}^n \epsilon_j x_j^2$, and $R = \epsilon_0 x_{q-3} x_{q-2} x_{q-1}$.

Proposition 4.36. *The tangent cone $C^+(g)$ is equal to $\overline{D}(q-4, Q)$.*

Proof. By Corollary 4.1 (with $P(x') = \sum_{j=1}^3 \delta_j x_{q-4+j}^2 + \sum_{1 \leq i < j \leq 3} \alpha_{ij} x_{q-4+i} x_{q-4+j}$ and $s = 3$), $\overline{D}(q-4, Q) \setminus \{(0, \dots, 0, d_{q-3}, d_{q-2}, d_{q-1}, 0, \dots, 0) \in \mathbb{R}^n \mid \epsilon_0 d_{q-3} d_{q-2} d_{q-1} > 0\}$ is contained in $C^+(g)$, and $C^+(g) = \overline{D}(q-4, Q)$ if $\epsilon_j = -1$ for some $j \geq q$. Since $d_{q-3}, d_{q-2}, d_{q-1} \leq 0$ for any $d \in \overline{D}(q-4, Q)$, $C^+(g) = \overline{D}(q-4, Q)$ if $\epsilon_0 = 1$.

In what follows, we assume $\epsilon_0 = -1$ and $\epsilon_q = \dots = \epsilon_n = +1$. Let $d = d_{q-3} e_{q-3} + d_{q-2} e_{q-2} + d_{q-1} e_{q-1} \in \overline{D}(q-4, Q)$ with $d_{q-3} d_{q-2} d_{q-1} < 0 \Leftrightarrow d_{q-3}, d_{q-2}, d_{q-1} \neq 0$. Since $d \in \overline{D}(q-4, Q)$, $Q(d')$ is less than or equal to 0. By Lemma 4.2, d is contained in $C^+(g)$ if $Q(d') < 0$. Suppose that $Q(d') = \frac{1}{2}({}^t d' \text{Hess}(Q) d')$ is equal to 0. The matrix $\text{Hess}(Q)$ is regular since δ_j and α_{ij} satisfy the condition $(**)$ in Table 2. Since $d' \neq 0$, one of the components of $\nabla(Q)(d') = \text{Hess}(Q) d'$, say the ℓ -th component ($\ell \in \{1, 2, 3\}$), is not 0. The vector $v = d - \varepsilon \text{sign}((\nabla Q(d'))_\ell) e_{q-4+\ell}$ satisfies the conditions in Lemma 4.2 for $0 < \varepsilon \ll 1$. Indeed, $v_1 = \dots = v_{q-4} = 0$, $v_{q-4+j} = d_{q-4+j} < 0$ for $j \neq \ell$, $v_{q-4+\ell} = d_{q-4+\ell} - \varepsilon \text{sign}((\nabla Q(d'))_\ell) < 0$ for $0 < \varepsilon \ll 1$, and

$$v' \cdot \nabla Q(d') = {}^t d' \text{Hess}(Q) d' - \varepsilon \text{sign}((\nabla Q(d'))_\ell) (\nabla Q(d'))_\ell < 0.$$

By Lemma 4.2, d is contained in $C^+(g)$. □

Proposition 4.37. *ACQ holds for g if and only if $\delta_1 = \delta_2 = \delta_3 = -1$, $\epsilon_q = \dots = \epsilon_n = -1$ and there exist distinct indices $i, j, k \in \{1, 2, 3\}$ such that either*

$$1. \alpha_{ij} \leq 0, \alpha_{ik} \leq 0, \alpha_{jk} < 2, \quad \text{or}$$

$$2. 0 < \alpha_{ij} < 2, 0 < \alpha_{ik} < 2, \alpha_{jk} + \frac{\alpha_{ij} \alpha_{ik}}{2} < 2 \sqrt{\left(1 - \frac{\alpha_{ij}^2}{4}\right) \left(1 - \frac{\alpha_{ik}^2}{4}\right)},$$

where we regard that α_{ij} is attached to the unordered pair $\{i, j\}$ (that is, we assume $\alpha_{ji} = \alpha_{ij}$).

Proof. Note that, if $\delta_1 = \delta_2 = \delta_3 = -1$ and $\epsilon_q = \dots = \epsilon_n = -1$ hold, ACQ holds if and only if

$$Q'(d_{q-3}, d_{q-2}, d_{q-1}) = -\sum_{j=1}^3 d_{q-4+j}^2 + \sum_{1 \leq i < j \leq 3} \alpha_{ij} d_{q-4+i} d_{q-4+j} \leq 0$$

holds for all $d_{q-3}, d_{q-2}, d_{q-1} \leq 0$.

Proof of “if” part: First of all, we show that ACQ holds for the case 1. Without loss of generality, we can assume $\alpha_{12}, \alpha_{13} \leq 0$ and $\alpha_{23} < 2$. Take any $d_{q-3}, d_{q-2}, d_{q-1} \leq 0$. Then,

$$\begin{aligned} Q' &= (-d_{q-3}^2 + \alpha_{12}d_{q-3}d_{q-2} + \alpha_{13}d_{q-3}d_{q-1}) + (-d_{q-2}^2 - d_{q-1}^2 + \alpha_{23}d_{q-2}d_{q-1}) \\ &\leq -d_{q-2}^2 - d_{q-1}^2 + \alpha_{23}d_{q-2}d_{q-1} = -(d_{q-2} - d_{q-1})^2 + (\alpha_{23} - 2)d_{q-2}d_{q-1} \leq 0 \end{aligned}$$

holds. Therefore, ACQ holds in the case 1.

Next, we show ACQ holds in the case 2. Without loss of generality, we can assume the inequalities in the case 2. hold for $i = 1, j = 2, k = 3$. Take any $d_{q-3}, d_{q-2}, d_{q-1} \leq 0$. Then,

$$\begin{aligned} Q' &= -\left(d_{q-3} - \frac{\alpha_{12}}{2}d_{q-2} - \frac{\alpha_{13}}{2}d_{q-1}\right)^2 \\ &\quad - \left(1 - \frac{\alpha_{12}^2}{4}\right)d_{q-2}^2 - \left(1 - \frac{\alpha_{13}^2}{4}\right)d_{q-1}^2 + \left(\alpha_{23} + \frac{\alpha_{12}\alpha_{13}}{2}\right)d_{q-2}d_{q-1} \quad (4.B) \end{aligned}$$

holds. Since the second line of the equation is non-positive for all $d_{q-2}, d_{q-1} \leq 0$ by the condition, ACQ holds.

Proof of “only if” part: The condition $\delta_1 = \delta_2 = \delta_3 = -1$ is necessary for ACQ to hold since if one of $\delta_1, \delta_2, \delta_3$ is 1, say $\delta_1 = 1$, then, $e_{q-3} \in L^+(g)$ whereas $Q(e_{q-3}) > 0$ and thus $e_{q-3} \notin C^+(g)$. The condition $\epsilon_q = \dots = \epsilon_n = -1$ is also necessary for ACQ to hold since if one of $\epsilon_q, \dots, \epsilon_n$ is 1, say $\epsilon_n = 1$, then, $e_n \in L^+(g)$ whereas $Q(e_n) > 0$ and thus $e_n \notin C^+(g)$. Therefore, in what follows, we assume $\delta_1 = \delta_2 = \delta_3 = -1$ and $\epsilon_q = \dots = \epsilon_n = -1$.

We show the contraposition of the claim. Suppose that the conditions 1. and 2. do not hold for any distinct $i, j, k \in \{1, 2, 3\}$. First, we show that ACQ does not hold if two of $\alpha_{12}, \alpha_{13}, \alpha_{23}$ is non-positive. By symmetry, we can assume $\alpha_{12}, \alpha_{13} \leq 0$. Since the condition 1. does not hold for $i = 1, j = 2, k = 3$, α_{23} is greater than 2. In this case, $Q'(0, -1, -1) = \alpha_{23} - 2 > 0$ holds and thus ACQ does not hold. In what follows, we assume more than one of $\alpha_{12}, \alpha_{13}, \alpha_{23}$ is positive. By symmetry, we can assume $\alpha_{12}, \alpha_{13} > 0$ without loss of generality. Then, Eq. (4.B), as a function of d_{q-3} , attains the maximum

$$Q'' = -\left(1 - \frac{\alpha_{12}^2}{4}\right)d_{q-2}^2 - \left(1 - \frac{\alpha_{13}^2}{4}\right)d_{q-1}^2 + \left(\alpha_{23} + \frac{\alpha_{12}\alpha_{13}}{2}\right)d_{q-2}d_{q-1}.$$

at $d_{q-3} = \frac{\alpha_{12}}{2}d_{q-2} + \frac{\alpha_{13}}{2}d_{q-1} \leq 0$. By noting that

$$\alpha_{12} \neq 2, \alpha_{13} \neq 2, \alpha_{23} + \frac{\alpha_{12}\alpha_{13}}{2} \neq 2\sqrt{\left(1 - \frac{\alpha_{12}^2}{4}\right)\left(1 - \frac{\alpha_{13}^2}{4}\right)}$$

holds, $Q'' \leq 0$ holds for all $d_{q-2}, d_{q-1} \leq 0$ if and only if the condition 2. (for $i = 1, j = 2, k = 3$) hold (cf. the proof of Proposition 4.25). $\square$

Proposition 4.38. *GCQ holds for g if and only if one of $\epsilon_q, \dots, \epsilon_n$ is -1 .*

Proof. It is easy to check that $L^+(g)^\circ$ is equal to

$$\{w \in \mathbb{R}^n \mid w_{q-3} \geq 0, w_{q-2} \geq 0, w_{q-1} \geq 0, w_q = \dots = w_n = 0\}.$$

Proof of “if” part: Without loss of generality, we can assume $\epsilon_q = -1$. Take any $w \in C^+(g)^\circ$. Since $\pm e_q \in C^+(g)$, $w_q = 0$ holds. Since $\pm e_q \pm e_j \in C^+(g)$ holds for $j \in$

$\{q+1, \dots, n\}$, $w_j = 0$ holds. Since $-e_j \pm e_q \in C^+(g)$ holds for $j \in \{q-3, q-2, q-1\}$, $w_j \geq 0$ holds for $j \in \{q-3, q-2, q-1\}$. This proves $w \in L^+(g)^\circ$.

Proof of “only if” part: We show the contraposition of the statement, that is, if all of

$\epsilon_q, \dots, \epsilon_n$ are 1, GCQ is violated. Let $\lambda_1, \lambda_2, \lambda_3 \neq 0$ be the eigenvalues of $\begin{pmatrix} \delta_1 & \alpha_{12}/2 & \alpha_{13}/2 \\ \alpha_{12}/2 & \delta_2 & \alpha_{23}/2 \\ \alpha_{13}/2 & \alpha_{23}/2 & \delta_3 \end{pmatrix}$.

We claim that $w = R(e_{q-1} + e_{q-2} + e_{q-3}) + e_q$ is contained in $C^+(g)^\circ$ for $R = \sqrt{\max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}}$. Take arbitrary $d \in C^+(g)$. Then,

$$\begin{aligned} \sum_{j=q}^n d_j^2 &\leq - \left(\sum_{j=1}^3 \delta_j d_{q-4+j}^2 + \sum_{1 \leq i < j \leq 3} \alpha_{ij} d_{q-4+i} d_{q-4+j} \right) \\ &\leq \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} \|(d_{q-3}, d_{q-2}, d_{q-1})\|^2 \end{aligned}$$

holds. Therefore, we obtain the following inequality:

$$\begin{aligned} w \cdot d &= R(d_{q-3} + d_{q-2} + d_{q-1}) + d_q \\ &\leq -R(|d_{q-1}| + |d_{q-2}| + |d_{q-3}|) + |d_q| \quad (\because d_{q-1}, d_{q-2}, d_{q-3} \leq 0) \\ &\leq -R\|(d_{q-1}, d_{q-2}, d_{q-3})\| + |d_q| \quad (\because \|(d_{q-1}, d_{q-2}, d_{q-3})\| \leq |d_{q-1}| + |d_{q-2}| + |d_{q-3}|) \\ &\leq -R \sqrt{\frac{\sum_{j=q}^n d_j^2}{\max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}}} + |d_q| \leq 0. \end{aligned}$$

This proves $w \in C^+(g)^\circ$. Since w is not contained in $L^+(g)^\circ$, GCQ is violated. $\square$

4.2.3 ACQ and GCQ in Table 3

Proposition 4.39. *ACQ does not hold for any germ in Table 3.*

Proof. Let $(g, h) = (x_1, \dots, x_q, h)$ be a germ in Table 3. It is easy to check that the linearized cone $L^+(g, h)$ is equal to $\{d \in \mathbb{R}^n \mid d_1, \dots, d_q \leq 0\}$. By Lemma 4.1, the tangent cone $C^+(g, h)$ is contained in $\overline{C}(q, Q) = \{d \in \mathbb{R}^n \mid d_1, \dots, d_q \leq 0, Q(d) = 0\}$, where Q is the quadratic part of h . One can easily check that e_n is contained in $L^+(g, h)$ but not in $\overline{C}(q, Q)$, in particular $C^+(g, h) \subset \overline{C}(q, Q) \subsetneq L^+(g, h)$. $\square$

The germ of type $(1, k)$ ($k \geq 3$) in Table 3

Let $(g, h) = (x_1, x_1^k + \sum_{j=2}^n \epsilon_j x_j^2)$, where $\epsilon_j \in \{1, -1\}$, $Q = \sum_{j=2}^n \epsilon_j x_j^2$, and $R = x_1^k$.

Proposition 4.40. *The tangent cone $C^+(g, h)$ is equal to $\{0\}$ if all the ϵ_j 's are $(-1)^k$, and $C^+(g, h)$ is equal to $\overline{C}(1, Q)$ otherwise.*

Proof. In the former case, it is easy to see that $M(g, h)$ is equal to $\{0\}$, and thus its tangent cone is also $\{0\}$. In the latter case, we can assume that $\epsilon_2 = (-1)^{k+1}$ without loss of generality. Let $d \in \overline{C}(1, Q)$. Since $R_k = x_1^k$ and $R_r = 0$ for $r \neq k$, one can deduce from Lemma 4.1 that d is contained in $C^+(g, h)$ if $d_1 = 0$. Suppose that $d_1 < 0$. If $(d_2, \dots, d_n) \neq (0, \dots, 0)$, the vector $v = (-1)^{k+1} \nabla Q(d)$ satisfies the conditions in Lemma 4.1. Indeed,

$$v \cdot \nabla Q(d) = (-1)^{k+1} \|(0, \dots, 2\epsilon_j d_j, \dots)\| = (-1)^{k+1} \sum_{j=2}^n d_j^2,$$

whose sign is opposite to $R_k(d) = d_1^k$. Thus, d is contained in $C^+(g, h)$ by Lemma 4.1. If $d_2 = \dots = d_n = 0$, $\nabla Q(d) = 0$ and the vector $v = e_2$ satisfies the conditions in Lemma 4.1. Indeed, ${}^t v \text{Hess}(Q)v = 2\epsilon_2 = 2(-1)^{k+1}$, whose sign is opposite to $R_k(d)$. Again, we can deduce from Lemma 4.1 that d is contained in $C^+(g, h)$. $\square$

Proposition 4.41. *ACQ does not hold for (g, h) .*

Proof. It is easy to check that $L^+(g, h)$ is equal to $\{d \in \mathbb{R}^n \mid d_1 \leq 0\}$, which contains $\overline{C}(1, Q) = \{d \in \mathbb{R}^n \mid d_1 \leq 0, \sum_{j=2}^n \epsilon_j d_j^2 = 0\}$ as a proper subset. Thus, $C^+(g, h)$ is not equal to $L^+(g, h)$ since $C^+(g, h) \subset \overline{C}(1, Q)$. $\square$

Proposition 4.42. *GCQ holds for (g, h) if and only if $\epsilon_j \epsilon_l = -1$ for some $j, l \in \{2, \dots, n\}$.*

Proof. It is easy to check that $L^+(g, h)^\circ$ is equal to $\{w \in \mathbb{R}^n \mid w_1 \geq 0, w_2 = \dots = w_n = 0\}$. **Proof of “if” part:** Take $w \in C^+(g, h)^\circ$. For any $j \in \{2, \dots, n\}$, we can take $l \in \{2, \dots, n\}$ so that $\epsilon_j \epsilon_l = -1$ by the assumption. Since $\pm e_j \pm e_l \in C^+(g, h)$, the following inequality holds for any pair of signs:

$$0 \leq w \cdot (\pm e_j \pm e_l) = \pm w_j \pm w_l.$$

We can thus deduce $w_j = 0$, and $C^+(g, h)^\circ \subset L^+(g, h)^\circ$. The opposite inclusion also holds since $C^+(g, h) \subset L^+(g, h)$.

Proof of “only if” part: If $\epsilon_j = (-1)^k$ for any $j \geq 0$, $C^+(g, h)$ is equal to $\{0\}$ by Proposition 4.40, and its polar is $\mathbb{R}^n \neq L^+(g, h)^\circ$. If $\epsilon_j = (-1)^{k+1}$ for any $j \geq 0$, $C^+(g, h)$ is equal to $\overline{C}(1, Q) = \{d \in \mathbb{R}^n \mid d_1 \leq 0, d_2 = \dots = d_n = 0\}$ by Proposition 4.40, and its polar is $\{w \in \mathbb{R}^n \mid w_1 \geq 0\} \neq L^+(g, h)^\circ$. $\square$

The germ of type (2) in Table 3

Let $(g, h) = (x_1, x_2^3 + x_1^2 + \sum_{j=3}^n \epsilon_j x_j^2)$, where $\epsilon_j \in \{1, -1\}$, $Q = x_1^2 + \sum_{j=3}^n \epsilon_j x_j^2$, and $R = x_2^3$.

Proposition 4.43. *The tangent cone $C^+(g, h)$ is equal to $\overline{C}(1, Q) \setminus \{(0, d, 0, \dots, 0) \in \mathbb{R}^n \mid d > 0\}$ if $\epsilon_3 = \dots = \epsilon_n = 1$, and $C^+(g, h)$ is equal to $\overline{C}(1, Q)$ otherwise.*

Proof. The tangent cone $C^+(g, h)$ is contained in $\overline{C}(1, Q)$ by Lemma 4.1. Let $d \in \overline{C}(1, Q)$. Since $R_3 = R = x_2^3$, d is contained in $\underline{C}(1, Q, R) \subset C^+(g, h)$ if $d_2 = 0$. If $d_2 \neq 0$ and d_j is not 0 for $j \geq 3$, the vector $v = -d_2 \epsilon_j d_j e_j$ satisfies the conditions in Lemma 4.1 for d . Indeed, $R_3(d) = d_2^3$ and $v \cdot \nabla Q(d) = -2d_2 \epsilon_j^2 d_j^2$, in particular the signs of these values are mutually opposite.

In what follows, we assume $d_2 \neq 0$ and $d_3 = \dots = d_n = 0$. Since $Q(d) = d_1^2$, d_1 is equal to 0 and $\nabla Q(d) = 0$. If $d_2 < 0$, the vector $v = -e_1$ satisfies the conditions in Lemma 4.1 for d since $v_1 = -1 < 0$, $R_3(d) = d_2^3 < 0$, $v \cdot \nabla Q(d) = 0$, and ${}^t v \text{Hess}(Q)v = 2 > 0$. If $d_2 > 0$ and some ϵ_j is -1 for $j \geq 3$, the vector $v = e_j$ satisfies the conditions in Lemma 4.1 for d since $R_3(d) = d_2^3 > 0$, $v \cdot \nabla Q(d) = 0$, and ${}^t v \text{Hess}(Q)v = -2 < 0$.

So far, we have shown that $\overline{C}(1, Q) \setminus \{(0, d, 0, \dots, 0) \in \mathbb{R}^n \mid d > 0\}$ is contained in $C^+(g, h)$, and $C^+(g, h) = \overline{C}(1, Q)$ if $\epsilon_j = -1$ for some $j \geq 3$. Lastly, we will show that $(0, d, 0, \dots, 0)$ ($d > 0$) is not contained in $C^+(g, h)$ when $\epsilon_3 = \dots = \epsilon_n = 1$. Let $x \in M(g, h)$. Since $h(x) = 0$, x_2^3 is equal to $-x_1^2 - \sum_{j=3}^n x_j^2 \leq 0$, and thus $x_2 \leq 0$. Thus, d_2 is less than or equal to 0 for any $d \in C^+(g, h)$. $\square$

Proposition 4.44. *GCQ holds for (g, h) if and only if $\epsilon_j \epsilon_l = -1$ for some $j, l \geq 3$.*

Proof. It is easy to check that $L^+(g, h)^\circ$ is equal to $\{w \in \mathbb{R}^n \mid w_1 \geq 0, w_2 = \dots = w_n = 0\}$.

Proof of “if” part: When $\epsilon_j \epsilon_l = -1$ for some $j, l \geq 3$, one can show that $C^+(g, h)^\circ = L^+(g, h)^\circ$ in the same way as in the proof of Proposition 4.42.

Proof of “only if” part: If $\epsilon_j = 1$ for any $j \geq 3$, $C^+(g, h)$ is equal to $\overline{C}(1, Q) \setminus \{(0, d, 0, \dots, 0) \in \mathbb{R}^n \mid d > 0\} = \{d \in \mathbb{R}^n \mid d_2 \leq 0, d_1 = d_3 = \dots = d_n = 0\}$ by Proposition 4.43, and its polar is $\{w \in \mathbb{R}^n \mid w_2 \geq 0\} \neq L^+(g, h)^\circ$. If $\epsilon_j = -1$ for any $j \geq 3$, $(1, 0, 1, 0, \dots, 0)$ is contained in $C^+(g, h)^\circ$. Indeed, any $d \in C^+(g, h) = \overline{C}(1, Q)$ satisfies $d_1^2 = \sum_{j=3}^n d_j^2 \geq d_3^2$, and thus $(1, 0, 1, 0, \dots, 0) \cdot d = d_1 + d_3 \leq 0$ since $d_1 \leq -|d_3|$. Hence $C^+(g, h)^\circ \neq L^+(g, h)^\circ$ in this case. $\square$

The germ of type $(3, k)$ in Table 3

Let $(g, h) = (x_1, x_2^k + \epsilon_1 x_1 x_2 + \sum_{j=3}^n \epsilon_j x_j^2)$, where $\epsilon_j \in \{1, -1\}$, $Q = \epsilon_1 x_1 x_2 + \sum_{j=3}^n \epsilon_j x_j^2$, and $R = x_2^k$.

Proposition 4.45. *If k is even and $\epsilon_3 = \dots = \epsilon_n = 1$, $C^+(g, h)$ is equal to $\overline{C}(1, Q) \setminus \{(0, d, 0, \dots, 0) \in \mathbb{R}^n \mid \epsilon_1 d < 0\}$. If k is odd, $\epsilon_1 = -1$, and $\epsilon_3 = \dots = \epsilon_n = \delta$ for some $\delta \in \{1, -1\}$, $C^+(g, h)$ is equal to $\overline{C}(1, Q) \setminus \{(0, d, 0, \dots, 0) \in \mathbb{R}^n \mid \delta d > 0\}$. In the other cases, $C^+(g, h)$ is equal to $\overline{C}(1, Q)$.*

Proof. The tangent cone $C^+(g, h)$ is contained in $\overline{C}(1, Q)$ by Lemma 4.1. Let $d \in \overline{C}(1, Q)$. Since $R_k = R = x_2^k$ and $R_j = 0$ for $j \neq k$, d is contained in $\underline{C}(1, Q, R) \subset C^+(g, h)$ if $d_2 = 0$. If $d_2 \neq 0$ and d_j is not 0 for $j \geq 3$, the vector $v = -\epsilon_j d_2^k d_j e_j$ satisfies the conditions in Lemma 4.1 for d . Indeed, $R_3(d) = d_2^k$ and $v \cdot \nabla Q(d) = -2d_2^k \epsilon_j^2 d_j^2$, in particular the signs of these values are mutually opposite.

In what follows, we assume $d_2 \neq 0$ and $d_3 = \dots = d_n = 0$. Since $Q(d) = \epsilon_1 d_1 d_2$, d_1 is equal to 0 and $\nabla Q(d) = (\epsilon_1 d_2, 0, \dots, 0)$. If k is even and $\epsilon_1 d_2 > 0$, the vector $v = -e_1$ satisfies the conditions in Lemma 4.1 for d since $v_1 = -1 < 0$, $R_3(d) = d_2^k > 0$, and $v \cdot \nabla Q(d) = -\epsilon_1 d_2 < 0$. If k is even and $\epsilon_j = -1$ for some $j \geq 3$, the vector $v = e_j$ satisfies the conditions in Lemma 4.1 for d since $v_1 = 0$, $R_3(d) = d_2^k > 0$, $v \cdot \nabla Q(d) = 0$, and ${}^t v \text{Hess}(Q)v = -2 < 0$. If k is odd and $\epsilon_1 = 1$, the vector $v = -e_1$ satisfies the conditions in Lemma 4.1 for d since $v_1 = -1 < 0$, and $v \cdot \nabla Q(d) = -d_2$, whose sign is opposite to $R_3(d) = d_2^k$. If k is odd, $\epsilon_j d_2 < 0$ for some $j \geq 3$, the vector $v = e_j$ satisfies the conditions in Lemma 4.1 for d since $v_1 = 0$, $v \cdot \nabla Q(d) = 0$, and ${}^t v \text{Hess}(Q)v = 2\epsilon_j$, whose sign is opposite to $R_3(d) = d_2^k$.

So far, we have shown that

- $C^+(g, h) = \overline{C}(1, Q)$ if one of the following conditions holds:
 - k is even and $\epsilon_j = -1$ for some $j \geq 3$,
 - k is odd and $\epsilon_1 = 1$,
 - k is odd and $\epsilon_j \epsilon_l = -1$ for some $j, l \geq 3$,
- $\overline{C}(1, Q) \setminus \{(0, d, 0, \dots, 0) \in \mathbb{R}^n \mid \epsilon_1 d < 0\}$ is contained in $C^+(g, h)$ when k is even and $\epsilon_3 = \dots = \epsilon_n = 1$,
- $\overline{C}(1, Q) \setminus \{(0, d, 0, \dots, 0) \in \mathbb{R}^n \mid \delta d > 0\}$ is contained in $C^+(g, h)$ when k is odd, $\epsilon_1 = -1$, and $\epsilon_3 = \dots = \epsilon_n = \delta$ for some $\delta \in \{1, -1\}$.

We will first show that $(0, d, 0, \dots, 0)$ ($\epsilon_1 d < 0$) is not contained in $C^+(g, h)$ when k is even and $\epsilon_3 = \dots = \epsilon_n = 1$. Let $x \in M(g, h)$. Since $h(x) = 0$, $\epsilon_1 x_1 x_2$ is equal to $-x_2^k - \sum_{j=3}^n x_j^2 \leq 0$. Since $g(x) = x_1 \leq 0$, $\epsilon_1 x_2 \geq 0$. Thus, $\epsilon_1 d_2$ is larger than or equal to 0 for any $d \in C^+(g, h)$.

We will next show that $(0, d, 0, \dots, 0)$ ($\delta d > 0$) is not contained in $C^+(g, h)$ when k is odd, $\epsilon_1 = -1$, and $\epsilon_3 = \dots = \epsilon_n = \delta$. Suppose that there exists a sequence $\{x^{(m)}\}_{m=1}^\infty$ of points in $M(g, h)$ and $t_m > 0$ such that $\lim_{m \rightarrow \infty} x^{(m)} = 0$ and $\lim_{m \rightarrow \infty} t_m x^{(m)} = (0, d, 0, \dots, 0)$. Since $h(x^{(m)}) = 0$, $x_1^{(m)} x_2^{(m)}$ is equal to $(x_2^{(m)})^k + \delta \sum_{j=3}^n (x_j^{(m)})^2$. By the assumption, $\delta x_2^{(m)}$ is larger than 0 for $m \gg 0$, and thus the following inequality holds:

$$x_1^{(m)} \cdot (\delta x_2^{(m)}) = \delta \left((x_2^{(m)})^k + \delta \sum_{j=3}^n (x_j^{(m)})^2 \right) > 0.$$

However, $x_1^{(m)} \cdot (\delta x_2^{(m)})$ never be larger than 0 since $x_1^{(m)} = g(x^{(m)}) \leq 0$. $\square$

Proposition 4.46. *GCQ holds for (g, h) if and only if one of the following holds:*

1. k is even and one of $\epsilon_3, \dots, \epsilon_n$ is -1 ,
2. k is odd and $(\epsilon_1 = 1 \text{ or } \{\epsilon_3, \dots, \epsilon_n\} = \{-1, 1\})$.

Proof. It is easy to check that $L^+(g, h)^\circ$ is equal to $\{w \in \mathbb{R}^n | w_1 \geq 0, w_2 = \dots = w_n = 0\}$.

Proof of “if” part: By the previous proposition, the conditions implies that $C^+(g, h)$ is equal to $\overline{C}(1, Q)$. Since $C^+(g, h) \subset L^+(g, h)$ always holds, it is enough to show $\overline{C}(1, Q)^\circ \subset L^+(g, h)^\circ$.

Take any $w \in \overline{C}(1, Q)^\circ$. Since $(d_1, 0, \dots, 0) \in \overline{C}(1, Q)$ holds for $d_1 \leq 0$, $w_1 \geq 0$ holds. Since $(0, d_2, 0, \dots, 0) \in \overline{C}(1, Q)$ holds for $d_2 \in \mathbb{R}$, $w_2 = 0$ holds. Since $-e_1 + \epsilon_1 \epsilon_j e_2 \pm e_j \in \overline{C}(1, Q)$ holds for all $j \in \{3, \dots, n\}$, the following inequality holds:

$$\begin{aligned} 0 &\geq w \cdot (-e_1 + \epsilon_1 \epsilon_j e_2 \pm e_j) = -w_1 \pm w_j \\ &\Leftrightarrow \pm w_j \leq w_1 \leq 0. \end{aligned}$$

We thus obtain $w_j = 0$ for all $j \in \{3, \dots, n\}$. This proves $w \in L^+(g, h)^\circ$.

Proof of “only if” part: We show the contraposition of the statement. If k is even and $\epsilon_3 = \dots = \epsilon_n = 1$, the previous proposition implies that $C^+(g, h)$ is equal to $\overline{C}(1, Q) \setminus \{(0, d, 0, \dots, 0) \in \mathbb{R}^n | \epsilon_1 d < 0\}$. We claim that $w = (0, w_2, 0, \dots, 0)$ for $\epsilon_1 w_2 \leq 0$ is contained in $C^+(g, h)^\circ$. Take any $d \in C^+(g, h)$. Then, $d_1 \leq 0$ and $\epsilon_1 d_1 d_2 + \sum_{j=3}^n d_j^2 = 0$ holds. The latter implies that $\epsilon_1 d_1 d_2 \leq 0$. If $d_1 < 0$, $\epsilon_1 d_2 \geq 0$ holds and thus $w \cdot d = w_2 d_2 \leq 0$ holds. If $d_1 = 0$, $d_3 = \dots = d_n = 0$ holds and thus $\epsilon_1 d_2 \geq 0$. Therefore, $w \cdot d = w_2 d_2 \leq 0$ holds. This proves the claim and thus GCQ is violated.

If k is odd and $\epsilon_1 = -1$, and $\epsilon_3 = \dots = \epsilon_n = \delta$ for some $\delta \in \{1, -1\}$, the previous proposition implies that $C^+(g, h)$ is equal to $\overline{C}(1, Q) \setminus \{(0, d, 0, \dots, 0) \in \mathbb{R}^n | \delta d > 0\}$. We claim that $w = (0, w_2, 0, \dots, 0)$ for $\delta w_2 \geq 0$ is contained in $C^+(g, h)^\circ$. Take any $d \in C^+(g, h)$. Then, $d_1 \leq 0$ and $-d_1 d_2 + \delta \sum_{j=3}^n d_j^2 = 0$ holds. The latter implies that $\delta d_1 d_2 \geq 0$. If $d_1 < 0$, $\delta d_2 \leq 0$ holds and thus $w \cdot d = w_2 d_2 \leq 0$ holds. If $d_1 = 0$, $d_3 = \dots = d_n = 0$ holds and thus $\delta d_2 \leq 0$. Therefore, $w \cdot d = w_2 d_2 \leq 0$ holds. This proves the claim and thus GCQ is violated. $\square$

The germ of type (4) in Table 3

Let $(g, h) = (x_1, x_2, \delta_1 x_1^2 + \delta_2 x_2^2 + \alpha x_1 x_2 + \sum_{j=3}^n \epsilon_j x_j^2)$, where $\delta_j, \epsilon_j \in \{1, -1\}$, $\alpha \in \mathbb{R}$, $4\delta_1\delta_2 - \alpha^2 \neq 0$, and $Q = \delta_1 x_1^2 + \delta_2 x_2^2 + \alpha x_1 x_2 + \sum_{j=3}^n \epsilon_j x_j^2$. In this case, $C^+(g, h) = \overline{C}(2, Q)$ holds.

Proposition 4.47. *GCQ holds for (g, h) if and only if $\{\epsilon_3, \dots, \epsilon_n\} = \{1, -1\}$ holds.*

Proof. It is easy to check that $L^+(g, h)^\circ$ is equal to $\{w \in \mathbb{R}^n | w_1 \geq 0, w_2 \geq 0, w_3 = \dots = w_n = 0\}$.

Proof of “if” part: Without loss of generality, we can assume $\epsilon_3 = 1$ and $\epsilon_4 = -1$. Take any $w \in C^+(g, h)^\circ$. For each $j \in \{3, \dots, n\}$, either $\pm e_j \pm e_3$ or $\pm e_j \pm e_4$ is contained in $C^+(g, h)$. This proves $w_j = 0$ for all $j \in \{3, \dots, n\}$. Since either $-e_j \pm e_3 \in C^+(g, h)$ or $-e_j \pm e_4 \in C^+(g, h)$ holds for $j \in \{1, 2\}$, $w_1 \geq 0$ and $w_2 \geq 0$ hold. This proves $C^+(g, h)^\circ \subset L^+(g, h)^\circ$ and thus GCQ holds in this case.

Proof of “only if” part: We assume $\epsilon_3 = \dots = \epsilon_n = \delta$ for some $\delta \in \{1, -1\}$. We take $P \in O(2)$ so that the following equality holds:

$$\delta_1 x_1^2 + \delta_2 x_2^2 + \alpha x_1 x_2 = (x_1, x_2)^t P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \neq 0$ is the eigenvalues of $\begin{pmatrix} \delta_1 & \alpha/2 \\ \alpha/2 & \delta_2 \end{pmatrix}$. The following then holds for $d \in C^+(g, h)$:

$$\begin{aligned} \delta_1 d_1^2 + \delta_2 d_2^2 + \alpha d_1 d_2 &= -\delta \sum_{j=3}^n d_j^2 \\ \Rightarrow \left| (d_1, d_2)^t P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \right| &= \left| \sum_{j=3}^n d_j^2 \right| \\ \Rightarrow \max\{|\lambda_1|, |\lambda_2|\} \|(d_1, d_2)\|^2 &\geq \left| \sum_{j=3}^n d_j^2 \right|. \end{aligned}$$

Let $R > \sqrt{\max\{|\lambda_1|, |\lambda_2|\}}$. For any $d \in C^+(g, h)$, the inner product $(R(e_1 + e_2) + e_3) \cdot d$ is estimated as follows:

$$\begin{aligned} &(R(e_1 + e_2) + e_3) \cdot d \\ &= R(d_1 + d_2) + d_3 \\ &\leq -R(|d_1| + |d_2|) + |d_3| && (\because d_1, d_2 \leq 0) \\ &\leq -R\|(d_1, d_2)\| + |d_3| && \left(\because \|(d_1, d_2)\| = \sqrt{d_1^2 + d_2^2} \leq |d_1| + |d_2| \right) \\ &\leq -R \sqrt{\frac{\left| \sum_{j=3}^n d_j^2 \right|}{\max\{|\lambda_1|, |\lambda_2|\}}} + |d_3| \\ &\leq \left(-\frac{R}{\sqrt{\max\{|\lambda_1|, |\lambda_2|\}}} + 1 \right) |d_3| \leq 0. \end{aligned}$$

Thus, $R(e_1 + e_2) + e_3$ is contained in $C^+(g, h)^\circ$. However, it is not in $L^+(g, h)^\circ$, and thus GCQ is violated. $\square$

The germ of type (5) in Table 3

Let $(g, h) = (x_1, x_2, x_1^3 + \epsilon_1 x_2^2 + \epsilon_2 x_1 x_2 + \sum_{j=3}^n \epsilon_j x_j^2)$, where $\epsilon_j \in \{1, -1\}$, $Q = \epsilon_1 x_2^2 + \epsilon_2 x_1 x_2 + \sum_{j=3}^n \epsilon_j x_j^2$, and $R = x_1^3$.

Proposition 4.48. *The tangent cone $C^+(g, h)$ is equal to $\overline{C}(2, Q) \setminus \{(d, 0, \dots, 0) \in \mathbb{R}^n \mid d < 0\}$ if $\epsilon_2 = \dots = \epsilon_n = -1$, and $C^+(g, h)$ is equal to $\overline{C}(2, Q)$ otherwise.*

Proof. The tangent cone $C^+(g, h)$ is contained in $\overline{C}(2, Q)$ by Lemma 4.1. Let $d \in \overline{C}(2, Q)$. Since $R_3 = R = x_1^3$ and $R_r = 0$ for $r \neq 3$, d is contained in $\underline{C}(2, Q, R) \subset C^+(g, h)$ if $d_1 = 0$. If $d_1 < 0$ and some d_j is not 0 for $j \geq 3$, the vector $v = \epsilon_j d_j e_j$ satisfies the conditions in Lemma 4.1 for d . Indeed, $R_3(d) = d_1^3 < 0$ and $v \cdot \nabla Q(d) = 2\epsilon_j^2 d_j^2 > 0$. Hence d is contained in $C^+(g, h)$.

In what follows, we assume $d_1 < 0$ and $d_3 = \dots = d_n = 0$. Since $Q(d) = \epsilon_1 d_2^2 + \epsilon_2 d_1 d_2$ is equal to 0, either $d_2 = 0$ or $\epsilon_2 d_1 = -\epsilon_1 d_2$. If $d_2 \neq 0$, $\nabla Q(d)$ is equal to

$$(\epsilon_2 d_2, 2\epsilon_1 d_2 + \epsilon_2 d_1, 0, \dots, 0) = (\epsilon_2 d_2, -\epsilon_2 d_1, 0, \dots, 0).$$

Since either the first or the second component is negative, we can take $v \in \mathbb{R}^n$ so that $v_1, v_2 \leq 0$ and $v \cdot \nabla Q(d) > 0$ (by putting $v = -e_1$ or $-e_2$). Hence d is contained in $C^+(g, h)$.

So far, we have shown that $\overline{C}(2, Q) \setminus \{(d, 0, \dots, 0) \in \mathbb{R}^n \mid d < 0\}$ is contained in $C^+(g, h)$. In what follows, we assume $d_2 = 0$. If $\epsilon_2 = 1$, the vector $v = -e_2$ satisfies the conditions in Lemma 4.1 for d since $v \cdot \nabla Q(d) = -\epsilon_2 d_1 = -d_1 > 0$. Hence d is contained in $C^+(g, h)$. If some ϵ_j is equal to 1 for $j \geq 3$, the vector $v = e_j$ satisfies the conditions in Lemma 4.1 since $v \cdot \nabla Q(d) = 0$ and ${}^t v \text{Hess}(Q)v = 2\epsilon_j = 2 > 0$. Lastly, we will show that $(d, 0, \dots, 0)$ ($d < 0$) is not contained in $C^+(g, h)$ when $\epsilon_2 = \dots = \epsilon_n = -1$. Let $x \in M(g, h)$. Since $x_1, x_2 \leq 0$, $\epsilon_1 x_2^2 - x_1 x_2 = -x_1^3 + \sum_{j \geq 3} x_j^2 \geq 0$. If $\epsilon_1 = -1$, $\epsilon_1 x_2^2 - x_1 x_2 = x_2(-x_1 - x_2) \leq 0$ since $x_1, x_2 \leq 0$. Thus, $x_2(-x_1 - x_2)$ and $-x_1^3 + \sum_{j \geq 3} x_j^2$ are both equal to 0, meaning that $x = 0$. Hence $M(g, h) = C^+(g, h) = \{0\}$. (Note that $\overline{C}(2, Q) = \{(d, 0, \dots, 0) \in \mathbb{R}^n \mid d \leq 0\}$ in this case.) If $\epsilon_1 = 1$, $x_2(x_1 - x_2) \leq 0$, and thus, $x_1 \geq x_2$. Thus, d_1 is not less than d_2 for any $d \in C^+(g, h)$, implying that $(d, 0, \dots, 0)$ is not in $C^+(g, h)$ for $d < 0$. $\square$

Proposition 4.49. *GCQ holds for (g, h) if and only if $\{\epsilon_3, \dots, \epsilon_n\} = \{1, -1\}$ holds.*

Proof. It is easy to check that $L^+(g, h)^\circ$ is equal to $\{w \in \mathbb{R}^n \mid w_1 \geq 0, w_2 \geq 0, w_3 = \dots = w_n = 0\}$.

Proof of “if” part: Under the assumption, $C^+(g, h) = \overline{C}(2, Q)$ holds by the previous proposition. In addition, we can assume $\epsilon_3 = 1$ and $\epsilon_4 = -1$ without loss of generality. Since either $\pm e_j \pm e_3$ or $\pm e_j \pm e_4$ is in $\overline{C}(Q, 2)$ for $j \in \{3, \dots, n\}$, $w_j = 0$ holds for all $j \in \{3, \dots, n\}$. Since either $-e_2 \pm e_3$ or $-e_2 \pm e_4$ is in $\overline{C}(2, Q)$, $w_2 \geq 0$ holds. Since $-e_1 \in \overline{C}(2, Q)$ holds, $w_1 \geq 0$ holds. This proves $w \in L^+(g, h)^\circ$.

Proof of “only if” part: We assume $\epsilon_3 = \dots = \epsilon_n$. Let $\lambda_1, \lambda_2 \neq 0$ be the eigenvalues of $\begin{pmatrix} 0 & \epsilon_2/2 \\ \epsilon_2/2 & \epsilon_1 \end{pmatrix}$. As in the proof of Proposition 4.48, we obtain the following inequality for $d \in C^+(g, h)$:

$$\max\{|\lambda_1|, |\lambda_2|\} \|(d_1, d_2)\|^2 \geq \left| \sum_{j=3}^n d_j^2 \right|.$$

We can thus show that $R(e_1 + e_2) + e_3$ is contained in $C^+(g, h)^\circ$ for $R > \sqrt{\max\{|\lambda_1|, |\lambda_2|\}}$ in the same way as in the proof of Proposition 4.48. $\square$

The germ of type (6) in Table 3

Let $(g, h) = (x_1, x_2, (x_1 + \epsilon_1 x_2)^2 + \epsilon_2 x_2^3 + \sum_{j=3}^n \epsilon_j x_j^2)$, where $\epsilon_j \in \{1, -1\}$, $Q = (x_1 + \epsilon_1 x_2)^2 + \sum_{j=3}^n \epsilon_j x_j^2$, and $R = \epsilon_2 x_2^3$.

Proposition 4.50. *The tangent cone $C^+(g, h)$ is equal to $\{0\}$ if $\epsilon_2 = -1$ and $\epsilon_3 = \dots = \epsilon_n = 1$, and $C^+(g, h)$ is equal to $\overline{C}(2, Q)$ otherwise.*

Proof. The tangent cone $C^+(g, h)$ is contained in $\overline{C}(2, Q)$ by Lemma 4.1. Let $d \in \overline{C}(2, Q)$. Since $R_3 = R = \epsilon_2 x_2^3$ and $R_j = 0$ for $j \neq 3$, d is contained in $\underline{C}(2, Q, R) \subset C^+(g, h)$ if $d_2 = 0$. If $d_2 \neq 0$ and d_j is not 0 for $j \geq 3$, the vector $v = -\epsilon_2 \epsilon_j d_2 d_j e_j$ satisfies the conditions in Lemma 4.1 for d . Indeed, $R_3(d) = d_2^3$ and $v \cdot \nabla Q(d) = -2d_2 \epsilon_j^2 d_j^2$, in particular the signs of these values are mutually opposite.

In what follows, we assume $d_2 \neq 0$ and $d_3 = \dots = d_n = 0$. Since $Q(d) = (d_1 + \epsilon_1 d_2)^2$, $d_1 + \epsilon_1 d_2 = 0$ and $\nabla Q(d) = 0$. If $\epsilon_2 = +1$, the vector $v = -e_1$ satisfies the conditions in Lemma 4.1 for d since $v_1 = -1 < 0$, $R_3(d) = \epsilon_2 d_2^3 < 0$ (note that $d_2 \leq 0$), and ${}^t v \text{Hess}(Q)v = 2 > 0$. If $\epsilon_2 \epsilon_j > 0$ for some $j \geq 3$, the vector $v = e_j$ satisfies the conditions in Lemma 4.1 for d since $v_1 = 0$, $R_3(d) = \epsilon_2 d_2^3$, and ${}^t v \text{Hess}(Q)v = 2\epsilon_j$.

So far, we have shown that $C^+(g, h)$ is equal to $\overline{C}(2, Q)$ unless $\epsilon_2 = -1$ and $\epsilon_3 = \dots = \epsilon_n = 1$. We can easily show that $M(g, h)$ is equal to $\{0\}$ if $\epsilon_2 = -1$ and $\epsilon_3 = \dots = \epsilon_n = 1$. Thus, the tangent cone $C^+(g, h)$ is also equal to $\{0\}$. $\square$

Proposition 4.51. *GCQ holds for (g, h) if and only if $\{\epsilon_3, \dots, \epsilon_n\} = \{1, -1\}$ holds.*

Proof. It is easy to check that $L^+(g, h)^\circ$ is equal to $\{w \in \mathbb{R}^n | w_1 \geq 0, w_2 \geq 0, w_3 = \dots = w_n = 0\}$.

Proof of “if” part: Under the assumption, $C^+(g, h) = \overline{C}(2, Q)$ holds. Without loss of generality, we can assume $\epsilon_3 = 1$ and $\epsilon_4 = -1$. Take any $w \in C^+(g, h)^\circ$. Since either $\pm e_j \pm e_3$ or $\pm e_j \pm e_4$ is contained in $\overline{C}(2, Q)$ for $j \in \{3, \dots, n\}$, $w_j = 0$ holds for all $j \in \{3, \dots, n\}$. Since either $-e_j \pm e_3$ or $-e_j \pm e_4$ is contained in $C^+(g, h)$ for $j \in \{1, 2\}$, $w_j \geq 0$ holds. This proves $w \in L^+(g, h)^\circ$.

Proof of “only if” part: We assume $\epsilon_3 = \dots = \epsilon_n$. The eigenvalues of $\begin{pmatrix} 1 & \epsilon_1 \\ \epsilon_1 & 1 \end{pmatrix}$ are 0 and 2. Thus, as in the proof of Proposition 4.48, we obtain $2\|(d_1, d_2)\|^2 \geq \left| \sum_{j=3}^n d_j^2 \right|$ for $d \in C^+(g, h)$. We can thus show that $2(e_1 + e_2) + e_3$ is contained in $C^+(g, h)^\circ$ in the same way as in the proof of Proposition 4.48. $\square$

The germ of type (7) in Table 3

Let $(g, h) = (x_1, x_2, x_3^3 + \epsilon_1 x_2 x_3 + \epsilon_2 x_3 x_1 + \epsilon_3 x_1 x_2 + \sum_{j=4}^n \epsilon_j x_j^2)$, where $\epsilon_j \in \{1, -1\}$, $Q = \epsilon_1 x_2 x_3 + \epsilon_2 x_3 x_1 + \epsilon_3 x_1 x_2 + \sum_{j=4}^n \epsilon_j x_j^2$, and $R = x_3^3$.

Proposition 4.52. *The tangent cone $C^+(g, h)$ is equal to $\overline{C}(2, Q) \setminus \{(0, 0, d, 0, \dots, 0) \in \mathbb{R}^n \mid \delta d > 0\}$ if $\epsilon_1 = \epsilon_2 = -1$ and $\epsilon_4 = \dots = \epsilon_n = \delta$ for some $\delta \in \{0, 1\}$, and $C^+(g, h) = \overline{C}(2, Q)$ otherwise.*

Proof. The tangent cone $C^+(g, h)$ is contained in $\overline{C}(2, Q)$ by Lemma 4.1. Let $d \in \overline{C}(2, Q)$. Since $R_3 = R = x_3^3$ and $R_j = 0$ for $j \neq 3$, d is contained in $\underline{C}(2, Q, R) \subset C^+(g, h)$ if $d_3 = 0$. In what follows, we assume $d_3 \neq 0$. If d_j is not 0 for $j \geq 4$, the vector $v = -\epsilon_j d_3 d_j e_j$ satisfies the conditions in Lemma 4.1 for d . Indeed, $R_3(d) = d_3^3$ and $v \cdot \nabla Q(d) = -2d_3 \epsilon_j^2 d_j^2$,

in particular the signs of these values are mutually opposite. If $\epsilon_1 d_2 + \epsilon_2 d_1$ is not 0, the vector $v = -d_3(\epsilon_1 d_2 + \epsilon_2 d_1)e_3$ satisfies the conditions in Lemma 4.1 for d . Indeed, $R_3(d) = d_3^3$ and $v \cdot \nabla Q(d) = -d_3(\epsilon_1 d_2 + \epsilon_2 d_1)^2$, in particular the signs of these values are mutually opposite.

In what follows, we assume $d_3 \neq 0$ and $\epsilon_1 d_2 + \epsilon_2 d_1 = d_4 = \dots = d_n = 0$. Since $Q(d) = \epsilon_3 d_1 d_2 = -\frac{\epsilon_1 \epsilon_3}{\epsilon_2} d_2^2$, $d_1 = d_2 = 0$ and $\nabla Q(d) = (\epsilon_2 d_3, \epsilon_1 d_3, 0, \dots, 0)$. If $\epsilon_i = 1$ for $i = 1$ or $i = 2$, the vector $v = -e_{i'}$ satisfies the conditions in Lemma 4.1 for d , where $\{i, i'\} = \{1, 2\}$. Indeed, $R_3(d) = d_3^3$, $v_i = 0$, $v_{i'} = -1 < 0$, and $v \cdot \nabla Q(d) = -d_3$. If $\epsilon_j d_3 < 0$ for some $j \geq 4$, the vector $v = e_j$ satisfies the conditions in Lemma 4.1 for d . Indeed, $v_1 = v_2 = 0$, $v \cdot \nabla Q(d) = 0$, and ${}^t v \text{Hess}(Q)v = 2\epsilon_j$, whose sign is opposite to $R_3(d) = d_3^3$.

So far, we have shown that

- $C^+(g, h)$ is equal to $\overline{C}(2, Q)$ unless $\epsilon_1 = \epsilon_2 = -1$ and $\epsilon_4 = \dots = \epsilon_n = \delta$ for some $\delta \in \{-1, 1\}$, and
- if $\epsilon_1 = \epsilon_2 = -1$ and $\epsilon_4 = \dots = \epsilon_n = \delta$ for some $\delta \in \{-1, 1\}$, the set

$$\overline{C}(2, Q) \setminus \{(0, 0, d, 0, \dots, 0) \in \mathbb{R}^n \mid \delta d > 0\}$$

is contained in $C^+(g, h)$.

In what follows, we show that $(0, 0, d, 0, \dots, 0)$ is not contained in $C^+(g, h)$ for $\delta d > 0$ when $\epsilon_1 = \epsilon_2 = -1$ and $\epsilon_4 = \dots = \epsilon_n = \delta$ for some $\delta \in \{-1, 1\}$. Let $x \in M(g, h)$. The following equality holds:

$$0 = h(x) = x_3(x_3^2 - x_1 - x_2) + \epsilon_3 x_1 x_2 + \delta \sum_{j=4}^n x_j^2.$$

If $\epsilon_3 x_1 x_2 + \delta \sum_{j=4}^n x_j^2$ is equal to 0, either x_3 or $x_3^2 - x_1 - x_2$ is also equal to 0. Since x_1 and x_2 are less than or equal to 0, x_3 is equal to 0 even in the latter case. If $\epsilon_3 x_1 x_2 + \delta \sum_{j=4}^n x_j^2 \neq 0$ and $\epsilon_3 = \delta$, the sign of $\epsilon_3 x_1 x_2 + \delta \sum_{j=4}^n x_j^2$ is same as that of δ . Since $x_3^2 - x_1 - x_2 > 0$, the sign of x_3 is opposite to that of δ . Thus, $(0, 0, d, 0, \dots, 0)$ is not contained in $C^+(g, h)$ for $\delta d > 0$ when $\epsilon_3 = \delta$. Suppose that ϵ_3 is equal to $-\delta$ and $\delta x_3 > -(x_1 + x_2)$. The following equality holds:

$$0 = \delta h(x) > -(x_1 + x_2)^3 + (x_1 + x_2)^2 - x_1 x_2 + \sum_{j=4}^n x_j^2.$$

However, the last value is larger than or equal to 0 since $(x_1 + x_2)^2 - x_1 x_2 \geq 0$. This contradiction implies that $\delta x_3 \leq -(x_1 + x_2)$, and thus $(0, 0, d, 0, \dots, 0)$ is not contained in $C^+(g, h)$ for $\delta d > 0$. $\square$

Proposition 4.53. *GCQ holds for (g, h) if and only if $\{\epsilon_4, \dots, \epsilon_n\} = \{1, -1\}$ or either ϵ_1 or ϵ_2 is 1.*

Proof. It is easy to check that $L^+(g, h)^\circ$ is equal to $\{w \in \mathbb{R}^n \mid w_1 \geq 0, w_2 \geq 0, w_3 = \dots = w_n = 0\}$.

Proof of “if” part: Under the assumption, $C^+(g, h) = \overline{C}(2, Q)$ holds. Take any $w \in C^+(g, h)^\circ$. Since $-e_1, -e_2, \pm e_3$ are contained in $\overline{C}(2, Q)$, $w_1, w_2 \geq 0$ and $w_3 = 0$ hold. For any $s > 0$, since $-\frac{e_2}{s} + \epsilon_1 \epsilon_j s e_3 \pm e_j$ is contained in $\overline{C}(2, Q)$, $-\frac{w_2}{s} \pm w_j \leq 0$ holds. Thus, $w_j = 0$ holds and w is contained in $L^+(g, h)^\circ$.

Proof of “only if” part: We assume $\epsilon_1 = \epsilon_2 = -1$ and $\epsilon_4 = \dots = \epsilon_n = \delta$. By Proposition 4.52, $C^+(g, h)$ is equal to $\overline{C}(2, Q) \setminus \{(0, 0, d, 0, \dots, 0) \in \mathbb{R}^n \mid \delta d > 0\}$. Let $d \in C^+(g, h)$.

If $d_1 = d_2 = 0$, $d_4, \dots, d_n$ are also equal to 0 since $Q(d) = \delta \sum_{j=4}^n d_j^2 = 0$, and $\delta d_3 \leq 0$. Otherwise, d_3 is equal to $\left(\epsilon_3 d_1 d_2 + \delta \sum_{j=4}^n d_j^2 \right) / (d_1 + d_2)$.

If $\epsilon_3 = \delta$, $w = \delta e_3$ is contained in $C^+(g, h)^\circ$. Indeed, for any $d \in C^+(g, h)$, $d \cdot w$ is equal to $\delta d_3 \leq 0$ if $d_1 = d_2 = 0$, and $\left(d_1 d_2 + \sum_{j=4}^n d_j^2 \right) / (d_1 + d_2) \leq 0$ otherwise. (Note that $d_1, d_2 \leq 0$.) If $\epsilon_3 = -\delta$, $w = e_1 + e_2 + \delta e_3$ is contained in $C^+(g, h)^\circ$. Indeed, for any $d \in C^+(g, h)$, $d \cdot w$ is equal to $\delta d_3 \leq 0$ if $d_1 = d_2 = 0$, and $\left((d_1 + d_2)^2 - d_1 d_2 + \sum_{j=4}^n d_j^2 \right) / (d_1 + d_2) \leq 0$ otherwise. In each case, the given w is not contained in $L^+(g, h)^\circ$. $\square$

The germ of type (8) in Table 3

Let $(g, h) = \left(x_1, x_2, x_3, \sum_{j=1}^3 \delta_j x_j^2 + \sum_{1 \leq i < j \leq 3} \alpha_{ij} x_i x_j + \epsilon_1 x_1 x_2 x_3 + \sum_{j=4}^n \epsilon_j x_j^2 \right)$, where $\delta_j, \epsilon_j \in \{1, -1\}$ and $\alpha_{ij} \in \mathbb{R}$ satisfies the condition $(**)$ in Table 2, $Q = \sum_{j=1}^3 \delta_j x_j^2 + \sum_{1 \leq i < j \leq 3} \alpha_{ij} x_i x_j + \sum_{j=4}^n \epsilon_j x_j^2$, and $R = \epsilon_1 x_1 x_2 x_3$.

Proposition 4.54. *The tangent cone $C^+(g, h)$ is equal to $\overline{C}(3, Q)$.*

Proof. The tangent cone $C^+(g, h)$ is contained in $\overline{C}(3, Q)$ by Lemma 4.1. Let $d \in \overline{C}(3, Q)$. Since $R_3 = R = \epsilon_1 x_1 x_2 x_3$ and $R_j = 0$ for $j \neq 3$, d is contained in $\underline{C}(3, Q, R) \subset C^+(g, h)$ if $d_1 d_2 d_3 = 0$. In what follows, we assume $d_1, d_2, d_3 < 0$. If d_j is not 0 for $j \geq 4$, the vector $v = \epsilon_1 \epsilon_j d_j e_j$ satisfies the conditions in Lemma 4.1 for d . Indeed, $R_3(d) = \epsilon_1 d_1 d_2 d_3$ and $v \cdot \nabla Q(d) = \epsilon_1 \epsilon_j^2 d_j^2$, in particular the signs of these values are mutually opposite. Suppose

that $d_4, \dots, d_n$ are all equal to 0. We put $d' = {}^t(d_1, d_2, d_3)$ and $A = \begin{pmatrix} 2\delta_1 & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & 2\delta_2 & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & 2\delta_3 \end{pmatrix}$, which

is a 3×3 -submatrix of $\text{Hess}(Q)$. By the direct calculation, we can deduce $\nabla Q(d) = Ad'$, which is not 0 since A is regular. Suppose that the ℓ -th component of Ad' , denoted by β , is not 0. Since d is contained in $\overline{C}(3, Q)$, $Q(d) = \frac{1}{2} {}^t d' A d'$ is equal to 0. For $0 < |\varepsilon| \ll 1$, the vector $v = d + \varepsilon e_\ell$ satisfies the conditions in Lemma 4.1. Indeed, $v_1, v_2, v_3 < 0$ since $|\varepsilon| \ll 1$ and $d_1, d_2, d_3 < 0$, and $v \cdot \nabla Q(d) = {}^t d' A d' + \varepsilon \beta = \varepsilon \beta$. We can make the sign of $\varepsilon \beta$ opposite to that of $R_3(d)$ by making an appropriate ε . $\square$

Proposition 4.55. *GCQ holds for (g, h) if and only if $\{\epsilon_4, \dots, \epsilon_n\} = \{1, -1\}$*

Proof. It is easy to check that $L^+(g, h)^\circ$ is equal to

$$\{w \in \mathbb{R}^n | w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, w_4 = \dots = w_n = 0\}.$$

Proof of “if” part: Without loss of generality, we can assume $\epsilon_4 = 1$ and $\epsilon_5 = -1$. Take any $w \in C^+(g, h)^\circ$. For each $j \in \{4, \dots, n\}$, either $\pm e_j \pm e_4$ or $\pm e_j \pm e_5$ is contained in $C^+(g, h)$. This proves $w_j = 0$ for all $j \in \{4, \dots, n\}$. Since either $-e_j \pm e_4 \in C^+(g, h)$ or $-e_j \pm e_5 \in C^+(g, h)$ holds for $j \in \{1, 2, 3\}$, $w_1 \geq 0$, $w_2 \geq 0$, and $w_3 \geq 0$ hold. This proves $C^+(g, h)^\circ \subset L^+(g, h)^\circ$ and thus GCQ holds in this case.

Proof of “only if” part: We assume $\epsilon_4 = \dots = \epsilon_n = \delta$ for some $\delta \in \{1, -1\}$. Let $\lambda_1, \lambda_2, \lambda_3 \neq 0$ be the eigenvalues of $\begin{pmatrix} \delta_1 & \alpha_{12}/2 & \alpha_{13}/2 \\ \alpha_{12}/2 & \delta_2 & \alpha_{23}/2 \\ \alpha_{13}/2 & \alpha_{23}/2 & \delta_3 \end{pmatrix}$. The following then holds for

$d \in C^+(g, h)$ (cf. the proof of Proposition 4.47):

$$\begin{aligned} \sum_{j=1}^3 \delta_j d_j^2 + \sum_{1 \leq i < j \leq 3} \alpha_{ij} d_i d_j &= -\delta \sum_{j=4}^n d_j^2 \\ \Rightarrow \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} \|(d_1, d_2, d_3)\|^2 &\geq \left| \sum_{j=4}^n d_j^2 \right|. \end{aligned}$$

Let $R = \sqrt{\max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}}$. For any $d \in C^+(g, h)$, the inner product $(R(e_1 + e_2 + e_3) + e_4) \cdot d$ is estimated as follows:

$$\begin{aligned} &(R(e_1 + e_2 + e_3) + e_4) \cdot d \\ &= R(d_1 + d_2 + d_3) + d_4 \\ &\leq -R(|d_1| + |d_2| + |d_3|) + |d_4| \quad (\because d_1, d_2, d_3 \leq 0) \\ &\leq -R \|(d_1, d_2, d_3)\| + |d_4| \quad \left(\because \|(d_1, d_2, d_3)\| = \sqrt{d_1^2 + d_2^2 + d_3^2} \leq |d_1| + |d_2| + |d_3| \right) \\ &\leq -R \sqrt{\frac{\left| \sum_{j=4}^n d_j^2 \right|}{\max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}}} + |d_4| \leq 0. \end{aligned}$$

Thus, $R(e_1 + e_2 + e_3) + e_4$ is contained in $C^+(g, h)^\circ$. However, it is not in $L^+(g, h)^\circ$, and thus GCQ is violated. $\square$

The results of this section are summarized in Theorem 4.2.

Theorem 4.2 (Generic CQ Classification). *Only the fully regular class (with $n = q + r$ and constraints locally equivalent to (g, h) with $g(x) = (x_1, \dots, x_q)$ and $h(x) = (x_{q+1}, \dots, x_{q+r})$) satisfies LICQ. All the other constraint classes (especially those in Tables 1–3) violate LICQ.*

MFCQ fails whenever a singular equality constraint is present, in particular all the classes in Tables 1 and 3 violate MFCQ. In classes consisting of only inequalities (Table 2), MFCQ holds if and only if the parameter l_1 in the normal form in the caption of Table 2 is positive.

The satisfaction or failure of ACQ and GCQ for the classes in Tables 1–3 are completely determined as summarized in Tables 4–6. (As explained in the captions, all the classes in Tables 1 and 3 violate ACQ.)

Type	Conditions for GCQ
$(1, k)$	$(k = 2 \text{ and one of } \epsilon_2, \dots, \epsilon_n \text{ is } -1)$ or $(k \geq 3 \text{ and } \{\epsilon_2, \dots, \epsilon_n\} = \{1, -1\})$
(2)	one of $\epsilon_4, \dots, \epsilon_n$ is -1

Table 4: Conditions for the classes in Table 1 to satisfy GCQ. Note that ACQ fail for all the classes in Table 1.

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Type	Conditions for ACQ	Conditions for GCQ
$(1, k)$	$(\epsilon_q = \dots = \epsilon_n = -1)$ $\vee (k \geq 3 \wedge \epsilon_{q+1} = \dots = \epsilon_n = -1)$	$(k = 2 \wedge (\epsilon_q, \dots, \epsilon_n) \neq (1, \dots, 1))$ $\vee ((\epsilon_{q+1}, \dots, \epsilon_n) \neq (1, \dots, 1))$
(2)	$\epsilon_{q+2} = \dots = \epsilon_n = -1$	$(\epsilon_{q+2}, \dots, \epsilon_n) \neq (1, \dots, 1)$
$(3, k)$	$(\epsilon_{q-1} = \dots = \epsilon_n = -1)$ $\vee (k \geq 3 \wedge \epsilon_q = \dots = \epsilon_n = -1)$	$(\epsilon_q, \dots, \epsilon_n) \neq (1, \dots, 1)$
$(4, k)$	$\perp$ (i.e., ACQ does not hold.)	$(\epsilon_{q+1}, \dots, \epsilon_n) \neq (1, \dots, 1) \vee \epsilon_q = (-1)^{k+1}$
(5)	$\epsilon_{q-1} = \epsilon_{q+1} = \dots = \epsilon_n = -1$	$(\epsilon_{q+1}, \dots, \epsilon_n) \neq (1, \dots, 1)$
(6)	$\delta_1 = \delta_2 = -1 \wedge \alpha < 2$ $\wedge \epsilon_q = \dots = \epsilon_n = -1$	$(\epsilon_q, \dots, \epsilon_n) \neq (1, \dots, 1)$
(7)	$\epsilon_{q-2} = \epsilon_q = \dots = \epsilon_n = -1$	$(\epsilon_q, \dots, \epsilon_n) \neq (1, \dots, 1)$
(8)	$\epsilon_{q-1} = \epsilon'_{q-1} = \epsilon_q = \dots = \epsilon_n = -1$	$(\epsilon_q, \dots, \epsilon_n) \neq (1, \dots, 1)$
(9)	$\perp$ (i.e., ACQ does not hold.)	$(\epsilon_{q+1}, \dots, \epsilon_n) \neq (1, \dots, 1) \vee \epsilon_{01} = 1 \vee \epsilon_{02} = 1$
(10)	(†)	$(\epsilon_q, \dots, \epsilon_n) \neq (1, \dots, 1)$

Table 5: Conditions for the classes in Table 2 *with* $l_1 = 0$ to satisfy ACQ or GCQ. Note that MFCQ holds if and only if $l_1 > 0$, and in this case ACQ and GCQ also hold. The condition (†) (for type (10)) is the following condition: $\delta_1 = \delta_2 = \delta_3 = -1$, $\epsilon_q = \dots = \epsilon_n = -1$ and there exist distinct indices $i, j, k \in \{1, 2, 3\}$ with

$$\left(\alpha_{ij} \leq 0 \wedge \alpha_{ik} \leq 0 \wedge \alpha_{jk} < 2 \right) \vee \left(0 < \alpha_{ij} < 2 \wedge 0 < \alpha_{ik} < 2 \wedge \alpha_{jk} + \frac{\alpha_{ij}\alpha_{ik}}{2} < 2\sqrt{\left(1 - \frac{\alpha_{ij}^2}{4}\right)\left(1 - \frac{\alpha_{ik}^2}{4}\right)} \right),$$

where we regard that α_{ij} is attached to the unordered pair $\{i, j\}$ (that is, we assume $\alpha_{ji} = \alpha_{ij}$).

Type	Conditions on which GCQ holds
$(1, k)$	$\{\epsilon_2, \dots, \epsilon_n\} = \{1, -1\}$
(2)	$\{\epsilon_3, \dots, \epsilon_n\} = \{1, -1\}$
$(3, k)$	$(k \text{ is even and one of } \epsilon_3, \dots, \epsilon_n \text{ is } -1) \text{ or } (k \text{ is odd and } (\epsilon_1 = 1 \text{ or } \{\epsilon_3, \dots, \epsilon_n\} = \{1, -1\}))$
(4)	$\{\epsilon_3, \dots, \epsilon_n\} = \{1, -1\}$
(5)	$\{\epsilon_3, \dots, \epsilon_n\} = \{1, -1\}$
(6)	$\{\epsilon_3, \dots, \epsilon_n\} = \{1, -1\}$
(7)	$\{\epsilon_4, \dots, \epsilon_n\} = \{1, -1\}$
(8)	$\{\epsilon_4, \dots, \epsilon_n\} = \{1, -1\}$

Table 6: Conditions for the classes in Table 3 to satisfy GCQ. Note that ACQ fails for all the classes in the table.

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A Genericity of counter examples to show the strictness of the hierarchy of the four classical constraint qualifications

To demonstrate the strictness of the hierarchy of the four classical constraint qualifications in Eq. (1.A), several counterexamples have been constructed in the literature [22, 24, 3, 2]. Here, we examine these examples through the lens of $\mathcal{K}[G]$ -equivalence and determine whether or not they are generic. As a measure of genericity, we compute the $\mathcal{K}[G]_e$ -codimension for each case. (For the computational details, see the Appendix of [12].)

Example A.1 (Peterson [22] (MFCQ but not LICQ)). Let $n = 1$, $q = 1$, and $r = 0$. Let $g(x) = (x, 2x)$. Then, LICQ is violated at the origin since both of the inequality constraints are active there and their gradients are linearly dependent. Contrastingly, MFCQ holds because one can take $d = (-1)$ as an MF-vector. In this case, $TK[G]_e(g) = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2x \end{pmatrix} \rangle_{\mathcal{E}_1}$ holds and thus $\frac{\mathcal{E}_1^2}{TK[G]_e(g)} \cong \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle_{\mathbb{R}}$ holds. Therefore, this constraint has $\mathcal{K}[G]_e$ -codimension 1.

Example A.2 (Wright [24] (MFCQ but not LICQ)). Let $n = 2$, $q = 2$, and $r = 0$. Let

$$g(x) = \left(\left(x_1 - \frac{1}{3} \right)^2 + x_2^2 - \frac{1}{9}, \left(x_1 - \frac{2}{3} \right)^2 + x_2^2 - \frac{4}{9} \right).$$

Then, LICQ is violated at the origin since both of the inequality constraints are active there but their gradients $(-2/3, 0)$ and $(-4/3, 0)$ are not linearly independent. However, there exists an MF-vector $d = (1, 0)$ and thus MFCQ holds at the origin. In this case, note that $g(x) = (-\frac{2}{3}x_1 + x_1^2 + x_2^2, -\frac{4}{3}x_1 + x_1^2 + x_2^2)$ holds. By the coordinate transformation $\phi: (x_1, x_2) \mapsto (X_1 = -\frac{2}{3}x_1 + x_1^2 + x_2^2, X_2 = x_2)$, we obtain

$$g \circ \phi^{-1}(X_1, X_2) = \left(X_1, X_1 - \frac{2}{9} \left(1 - \sqrt{1 + 9X_1 - 9X_2^2} \right) \right).$$

and its 2-jet at the origin is $j^2(g \circ \phi^{-1})(X) = (X_1, 2X_1 - X_2^2 + \frac{9}{4}X_1^2)$. This is $\mathcal{K}[G]^2$ -equivalent to $(X_1, X_1 - X_2^2)$, which coincides with the germ of type (1, 2) in Table 2 (with $\epsilon_2 = -1$). Since this normal form is 2- $\mathcal{K}[G]$ -determined, g itself is $\mathcal{K}[G]$ -equivalent to the normal form (1, 2)(with $\epsilon_2 = -1$) in Table 2. Therefore, g has $\mathcal{K}[G]_e$ -codimension 1.

Example A.3 (Peterson [22] (ACQ but not MFCQ)). Let $n = 2$, $q = 2$, and $r = 1$. Let $g(x) = (x_2 - x_1^2, -x_2 + x_1^2)$. This is $\mathcal{K}[G]$ -equivalent to $g'(x) = (x_1, -x_1)$. Since MFCQ and ACQ are invariant under the action of $\mathcal{K}[G]$, it is enough to consider g' . Then, MFCQ is violated because there is no MF-vector $d \in \mathbb{R}^2$ such that $dg'_{1,0}(d) = d_1 < 0$ and $dg'_{2,0}(d) = -d_1 < 0$ hold. ACQ holds because g' is linear in x_1, x_2 and thus the linearized cone coincides with the tangent cone. In this case, $TK[G]_e(g, h) = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \rangle_{\mathcal{E}_2}$ holds and thus

$\frac{\mathcal{E}_1^2}{TK[G]_e(g, h)} \supset \left\langle \begin{pmatrix} 0 \\ x_2^j \end{pmatrix} \middle| j \in \mathbb{N} \right\rangle_{\mathbb{R}}$ holds. This implies the $\mathcal{K}[G]_e$ -codimension of g is infinite.

Example A.4 (Andreani et al. [2] (ACQ but not MFCQ)). Let $n = 2$, $q = 2$, and $r = 0$. Let $g(x) = (-x_1, -x_1^2 - x_2^2)$. This is $\mathcal{K}[G]$ -equivalent to the germ of type $(3, 2)$ in Table 2, $\epsilon_1 = \epsilon_2 = -1$ of $\mathcal{K}[G]_e$ -codimension 2. In this case, $l_1 = 0$ and thus MFCQ is violated. In addition, the result in Table 5 implies that ACQ holds.

Example A.5 (Andreani and Silva [3] (GCQ but not ACQ)). Let $n = 2$, $q = 2$, and $r = 1$. Let $g(x) = (x_1, x_2)$ and $h(x) = x_1x_2$. Then, the feasible set-germ at the origin is $M(g, h) = \{(x_1, x_2) \in (\mathbb{R}^2, 0) \mid x_1 = 0, x_2 \leq 0 \text{ or } x_1 \leq 0, x_2 = 0\}$. In this case, it is easy to check that

$$L^+(g, h) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 \leq 0\}$$

and

$$C^+(g, h) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 = 0 \text{ or } x_1 = 0, x_2 \leq 0\}$$

hold. Therefore, ACQ does not hold. However, $L^+(g, h)^\circ = C^+(g, h)^\circ$ holds and thus

GCQ holds. In this case, $TK[G]_e(g, h) = \left\langle \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1x_2 \end{pmatrix} \right\rangle_{\mathcal{E}_2}$ holds

and thus $\frac{\mathcal{E}_1^2}{TK[G]_e(g, h)} \supset \left\langle \begin{pmatrix} 0 \\ 0 \\ x_1^{j_1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_2^{j_2} \end{pmatrix} \middle| j_1, j_2 \in \mathbb{N} \right\rangle_{\mathbb{R}}$ holds. This implies the $\mathcal{K}[G]_e$ -codimension of (g, h) is infinite.

Example A.6 (Peterson [22] (GCQ but not ACQ)). Let $n = 2$, $q = 1$, and $r = 0$. Let $g(x) = x_1^2x_2^2$. Then, the feasible set-germ at the origin is $M(g) = \{(x_1, x_2) \in (\mathbb{R}^2, 0) \mid x_1 = 0 \text{ or } x_2 = 0\}$. In this case, it is easy to check that $L^+(g) = \mathbb{R}^2$ and $C^+(g) = \{(x_1, x_2) \in (\mathbb{R}^2, 0) \mid x_1 = 0 \text{ or } x_2 = 0\}$ hold. Therefore, ACQ does not hold. However, $L^+(g, h)^\circ = C^+(g, h)^\circ$ holds and thus GCQ holds. In this case, $TK[G]_e(g) = \langle 2x_1x_2^2, 2x_1^2x_2, x_1^2x_2^2 \rangle_{\mathcal{E}_2} = \langle x_1x_2^2, x_1^2x_2 \rangle_{\mathcal{E}_2}$ and thus $\frac{\mathcal{E}_1^2}{TK[G]_e(g)} \supset \left\langle x_1^{j_1}, x_2^{j_2} \middle| j_1, j_2 \in \mathbb{N} \right\rangle_{\mathbb{R}}$ holds. This implies the $\mathcal{K}[G]_e$ -codimension of g is infinite.

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