

Hamiltonicity of Step-graphons

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Abstract

In this paper, we sample directed random graphs from (asymmetric) step-graphons and investigate the probability that the random graph has at least a Hamiltonian cycle (or a node-wise Hamiltonian decomposition). We show that for almost all step-graphons, the probability converges to either zero or one as the order of the random graph goes to infinity—we term it the zero-one law. We identify the key objects of the step-graphon that matter for the zero-one law, and establish a set of conditions that can decide whether the limiting value of the probability is zero or one.

1 Introduction

In this paper, a graphon W is a measurable function $W : [0, 1]^2 \rightarrow [0, 1]$. The graphon W is said to be *symmetric* if $W(s, t) = W(t, s)$ for almost all $(s, t) \in [0, 1]^2$. We do not require that W be symmetric. We treat graphon as a stochastic model and investigate its hamiltonicity. Specifically, we sample a *directed* graph $\vec{G}_n \sim W$ from a graphon W on n nodes via the following two-step procedure:

- S1.** Sample $t_1, \dots, t_n \sim \text{Uni}[0, 1]$ independently, where $\text{Uni}[0, 1]$ is the uniform distribution over the interval $[0, 1]$. We call t_i the *coordinate* of node v_i .
- S2.** For each pair of *distinct* nodes v_i and v_j , place independently a *directed* edge from v_i to v_j with probability $W(t_i, t_j)$ and a *directed* edge from v_j to v_i with probability $W(t_j, t_i)$.

A digraph $\vec{G}$ is said to have a node-wise *Hamiltonian decomposition* if it contains a subgraph $\vec{H}$, with the same node set as $\vec{G}$, such that $\vec{H}$ is a node-wise disjoint union of directed cycles of $\vec{G}$. If, further, $\vec{H}$ is a cycle (so it visits every node of $\vec{G}$), then $\vec{H}$ is said to be a *Hamiltonian cycle* of $\vec{G}$. We evaluate the probability that $\vec{G}_n \sim W$ has a Hamiltonian decomposition or cycle as $n \rightarrow \infty$. A precise problem formulation will be given shortly.

Our interest in Hamiltonian decomposition is rooted in structural system theory, which deals with the problem of understanding what type of network topology can sustain a desired system property. To elaborate, consider a network of n mobile agents whose

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communication topology is described by a digraph G , where the nodes represent the agents and the edges indicate the information flows. More specifically, a directed edge from v_j to v_i indicates that agent x_i can access the state information of agent x_j , so the dynamics of x_i are allowed to depend on the state of x_j . Said in other words, if the dynamics of the network system obey the differential equation $\dot{x}_i = f_i(x(t))$, for all $i = 1, \dots, n$, with each f_i a differentiable function, then $\partial f_i(x)/\partial x_j \neq 0$ only if there is an edge from v_j to v_i in G . We call such dynamics *compatible with G* . The central question of the structural system theory is then the following: Given the digraph G and given a desired system property (e.g., asymptotic stability with respect to the origin), is there a dynamical system $\dot{x} = f(x)$ such that it is compatible with G and satisfies the property? This framework can be extended to controlled system $\dot{x} = f(x, u)$, taking into account the constraint that for each control input u_j , there may be only few agents x_i under its direct influence, i.e., $\partial f_i(x)/\partial u_j \neq 0$.

It has been shown that existence of a Hamiltonian decomposition is essential for a network topology to sustain controllability [1] and stability [2], two of the fundamental properties of a dynamical control system. When a multi-agent system operates in an uncertain and/or adversarial environment, its network topology becomes a random object. We use a graphon W to represent the environment uncertainty and the random digraph $\vec{G}_n \sim W$ to represent the network topology, so the probability that an ordered pair of agents establishes an oriented communication link depends on their respective positions. The knowledge about how likely the network topology of a large-scale multi-agent system can have a Hamiltonian decomposition is critical for a network manager to understand whether the environment is in favor of or against them, to evaluate the risk-to-reward ratio, and to decide whether the system shall be deployed.

Problem formulation. We start by introducing the class of step-graphons:

Definition 1 (Step-graphon). *A graphon W is a **step-graphon** if there is a sequence $0 =: \sigma_0 < \sigma_1 < \dots < \sigma_m := 1$, for some $m \geq 1$, such that W is constant over $R_{ij} := [\sigma_{i-1}, \sigma_i) \times [\sigma_{j-1}, \sigma_j)$ for $1 \leq i, j \leq m$. We call $\sigma := (\sigma_0, \sigma_1, \dots, \sigma_m)$ a **partition** for W .*

We illustrate the definition of step-graphon in Figure 1a.

Definition 2 (H -property). *A graphon W has the **H -property** if*

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian decomposition}) = 1. \quad (1)$$

*A graphon W has the **strong H -property** if*

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian cycle}) = 1. \quad (2)$$

We show in this paper that the (strong) H -property is essentially a zero-one property for the class of step-graphons. Specifically, we show that for almost all step-graphons W , the limit on the left hand side of (1) or (2) is either zero or one. We present in the next section necessary and sufficient conditions for a graphon W to have the (strong) H -property.

The Main Theorem of this paper, which we present in Subsection 2.2, strengthens and generalizes the results of our earlier work [3, 4], in which we addressed only the

H -property for the class of *symmetric* step-graphons. Specifically, given a symmetric step-graphon W , we sample an undirected graph $G_n \sim W$ by placing an undirected edge between any two distinct nodes v_i and v_j with probability $W(t_i, t_j)$. We then obtain from G_n a directed graph $\vec{G}_n^s$ by replacing every undirected edge with a pair of oppositely oriented edges—we call such digraph *symmetric*. The step-graphon W is said to have the H -property if $\vec{G}_n^s \sim W$ has a Hamiltonian decomposition asymptotically almost surely (*a.a.s.*). We introduced in [3] a set of conditions that are necessary for W to have the H -property. Then, in [4], we showed that the same set of conditions is essentially sufficient. More specifically, we showed that if W satisfies the conditions, then *a.a.s.* $\vec{G}_n^s$ has a Hamiltonian decomposition which comprises mostly the 2-cycles. The main result of this paper, when specialized to symmetric step-graphons, implies that $\vec{G}_n^s \sim W$ has a Hamiltonian cycle *a.a.s.* The residual case where the probability that $\vec{G}_n^s \sim W$ has a Hamiltonian decomposition converges to neither zero nor one has been investigated in [5].

At the end of this section, we gather a few key notations and terminologies used throughout the paper.

Notation. In this paper, we consider both directed and undirected graphs. We will put an arrow on top of the letter (e.g., $\vec{G}$) to indicate that the graph it refers to is directed. All graphs considered in the paper do not have multiple edges, but can have self-loops. For a graph $\vec{G}$, let $V(\vec{G})$ and $E(\vec{G})$ be its node set and edge set, respectively. We use $v_i v_j$ to denote a directed edge from v_i to v_j , and use (v_i, v_j) to denote an undirected edge between v_i and v_j . A digraph $\vec{G}$ is said to be *strongly connected* if for any two distinct nodes v_i and v_j , there exist a path from v_i to v_j and a path from v_j to v_i .

Let $\mathbb{R}_{>0}$ (resp., $\mathbb{R}_{\geq 0}$) be the set of positive (resp., nonnegative) real numbers. Let $\mathbb{N}$ (resp., $\mathbb{N}_0$) be the set of positive (resp., nonnegative) integers.

Let $\mathbf{1}$ be the vector of all ones, and e_i be the i th column of the identity matrix. Their dimension will be clear in the context. The support of a vector x , denoted by $\text{supp}(x)$, is the set of indices i such that $x_i \neq 0$. Similarly, the support of a matrix $A = [a_{ij}]$, denoted by $\text{supp}(A)$, is the set of indices ij such that $a_{ij} \neq 0$. We will relate the support of a vector (resp., square matrix) to the node (resp., edge) set of a digraph. Specifically, for a vector $x \in \mathbb{R}^n$ and for a digraph $\vec{G}$ on n nodes, we can treat $\text{supp}(x)$ as a subset of $V(\vec{G})$ where $v_i \in \text{supp}(x)$ if and only if $x_i \neq 0$. Similarly, we treat $\text{supp}(A)$ as a subset of $E(\vec{G})$ where $v_i v_j \in E(\vec{G})$ if and only if $a_{ij} \neq 0$. For a subgraph $\vec{G}'$ of $\vec{G}$, we let $x|_{\vec{G}'}$ be the sub-vector of x obtained by deleting any entry x_i such that $v_i \notin \vec{G}'$.

2 Main Result

In this section, we identify the key objects associated with a step-graphon that matter for the (strong) H -property, and formulate a set of conditions about these objects for deciding whether a step-graphon has the (strong) H -property—this is the Main Theorem of the paper. We then illustrate and numerically validate the result. Toward the end, we provide a sketch of proof of the Main Theorem, highlighting the ideas and techniques that will be used to establish the result.

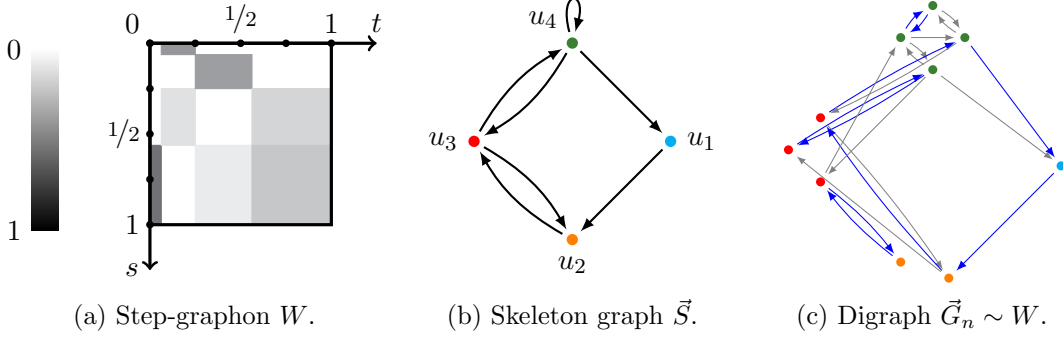


Figure 1: The step-graphon W in (a) has the partition sequence $\sigma = \frac{1}{16}(0, 1, 4, 9, 16)$. The value of W is shade coded, with black being 1 and white being 0. The digraph $\vec{S}$ in (b) is the skeleton graph associated with W with respect to the partition σ . The digraph $\vec{G}_n$, with $n = 10$, in (c) is sampled from W . It has a Hamiltonian decomposition, highlighted in blue, which comprises a 4-cycle and three 2-cycles.

2.1 Key objects

In this subsection, we introduce four key objects that are essential to deciding whether a step-graphon has the (strong) H -property. We start by introducing the definitions of concentration vector and of skeleton graph, which were introduced in [3, 4] for symmetric graphons but can be naturally extended to the general case here:

Definition 3 (Concentration vector). *Let W be a step-graphon with partition $(\sigma_0, \dots, \sigma_m)$. The associated **concentration vector** $x^* = (x_1^*, \dots, x_m^*)$ has entries defined as follows:*

$$x_i^* := \sigma_i - \sigma_{i-1}, \quad \text{for all } i = 1, \dots, m.$$

Next, we have

Definition 4 (Skeleton graph). *To a step-graphon W with a partition $\sigma = (\sigma_0, \dots, \sigma_m)$, we assign the digraph $\vec{S}$ on m nodes $\{u_1, \dots, u_m\}$, whose edge set $E(\vec{S})$ is defined as follows: There is a directed edge from u_i to u_j if and only if W is non-zero over R_{ij} . We call $\vec{S}$ the **skeleton graph** of W for the partition σ .*

Note that the concentration vector x^* and the skeleton graph $\vec{S}$ depend only on (and also, uniquely determine) the support of W .

The next two objects are derived from the skeleton graph $\vec{S}$, which are the node-cycle incidence matrix Z and the node-flow cone $\mathbf{X}$, i.e., the convex cone spanned by the column vectors of Z . We elaborate more on its name after the definition. These two objects will serve as the counterparts of the node-edge incidence matrix and of the edge-polytope that matter for the special case where we sample symmetric digraphs from symmetric graphons.

To this end, we label the cycles of $\vec{S}$ as $\vec{C}_1, \dots, \vec{C}_k$. A self-loop is a cycle of length 1.

Definition 5 (Node-cycle incidence vector/matrix). Let $\vec{C}_j$ be a cycle of the skeleton graph $\vec{S}$. The associated **node-cycle incidence vector** $z_j \in \mathbb{R}^m$ is given by

$$z_j := \sum_{u_i \in \vec{C}_j} e_i,$$

where we recall that $e_1, \dots, e_m$ is the standard basis of $\mathbb{R}^m$. Let

$$Z := [z_1 \ \cdots \ z_k] \in \mathbb{R}^{m \times k}$$

We call Z the **node-cycle incidence matrix** of $\vec{S}$.

In [3, 4], it was shown that the rank of the node-edge incidence matrix is a deciding factor for determining the H -property of a symmetric graphon. We will see soon the same holds for the case here. We define the *co-rank* of Z as

$$\text{co-rank}(Z) := m - \text{rank}(Z),$$

so Z has full row rank if and only if $\text{co-rank}(Z) = 0$. It is known [6] that the node-edge incidence matrix of an undirected graph has full row rank if and only if every connected component of the graph has an odd cycle. Similarly, the rank of the node-cycle incidence matrix can also be related to some relevant property of $\vec{S}$ (more precisely, the bipartite graph associated with $\vec{S}$). Since this graphical condition plays an important role in the analysis, we introduce it below.

We associate to the digraph $\vec{S}$ an undirected bipartite graph $B_{\vec{S}}$ with $2m$ nodes: The node set $V(B_{\vec{S}})$ is a disjoint union of two subsets

$$V'(B_{\vec{S}}) = \{u'_1, \dots, u'_m\} \quad \text{and} \quad V''(B_{\vec{S}}) = \{u''_1, \dots, u''_m\}.$$

The edge set $E(B_{\vec{S}})$ is such that

$$(u'_i, u''_j) \in E(B_{\vec{S}}) \iff u_i u_j \in E(\vec{S}).$$

The correspondence between $\vec{S}$ and $B_{\vec{S}}$ is illustrated in Figure 2.

Let $\vec{S}_1, \dots, \vec{S}_q$ be the *strongly connected components* (SCCs) of $\vec{S}$. We recall that they satisfy the following three (defining) conditions: (1) Every subgraph $\vec{S}_p$, for $p = 1, \dots, q$, is strongly connected; (2) The node sets $V(\vec{S}_1), \dots, V(\vec{S}_q)$ form a partition of $V(\vec{S})$; (3) If $\vec{S}'$ is any other strongly connected subgraph of $\vec{S}$, then $\vec{S}'$ is contained in some $\vec{S}_p$, for $p = 1, \dots, q$.

The following result can be obtained from [7, Corollary 5.6], which provides an explicit formula for the co-rank of Z :

Lemma 1. Let $B_{\vec{S}_p}$, for $p = 1, \dots, q$ be the bipartite graph associated with $\vec{S}_p$. Let τ_p be the number of connected components of $B_{\vec{S}_p}$. Then,

$$\text{co-rank}(Z) = \sum_{p=1}^q (\tau_p - 1).$$

In particular, $\text{co-rank}(Z) = 0$ if and only if every $B_{\vec{S}_p}$ is connected.

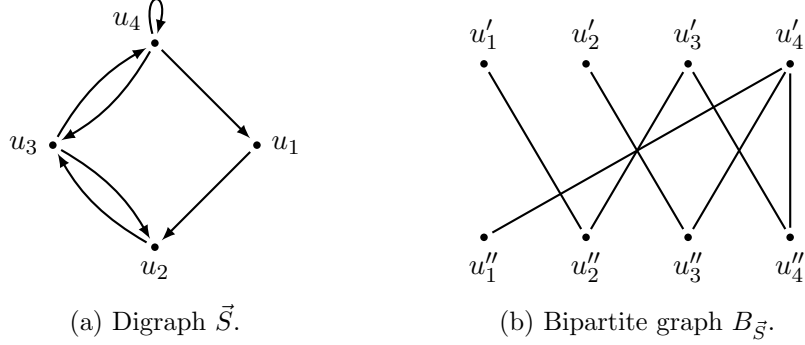


Figure 2: The bipartite graph $B_{\vec{S}}$ in (b) is associated with the digraph $\vec{S}$ in (a). Undirected edges $(u'_i, u''_j) \in E(B_{\vec{S}})$ one-to-one correspond to the directed edges $u_i u_j \in E(\vec{S})$.

To illustrate, consider the skeleton graph $\vec{S}$ in Figure 2a. To obtain the co-rank of Z , one way is to use the brute-force approach, i.e., we find all the cycles of $\vec{S}$, construct the node-cycle incidence matrix Z , and compute its co-rank. In this case, $\vec{S}$ has 4 cycles:

$$\vec{C}_1 = u_4 u_4, \quad \vec{C}_2 = u_3 u_4 u_3, \quad \vec{C}_3 = u_1 u_2 u_3 u_4 u_1, \quad \text{and} \quad \vec{C}_4 = u_2 u_3 u_2. \quad (3)$$

Thus,

$$Z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \text{so } \text{co-rank}(Z) = 0. \quad (4)$$

Another approach is to appeal to Lemma 1: In this case, $\vec{S}$ is strongly connected and the associated bipartite graph, shown in Figure 2b, is connected, so $\text{co-rank}(Z) = 0$.

Finally, we introduce the following object:

Definition 6 (Node-flow cone). *The node-flow cone X of $\vec{S}$ is the convex cone generated by the node-cycle incidence vectors $z_1, \dots, z_k$:*

$$\mathsf{X} := \left\{ \sum_{j=1}^k c_j z_j \mid c_j \geq 0 \right\}.$$

It is clear that $\dim \mathsf{X} = \text{rank}(Z)$.

The convex cone X has close relations with flows. To elaborate, recall that a *flow* f on $\vec{S}$ is a map $f : E(\vec{S}) \rightarrow \mathbb{R}_{\geq 0}$ satisfying the following balance condition at every node u_i :

$$\sum_{u_k : u_k u_i \in E(\vec{S})} f(u_k u_i) = \sum_{u_j : u_i u_j \in E(\vec{S})} f(u_i u_j) =: y_i(f). \quad (5)$$

It is not hard to see that X is the set of vectors $y(f) := (y_1(f), \dots, y_m(f))$ for all flows f .

2.2 Necessary and sufficient conditions for (strong) H -property

In this subsection, we state an essentially necessary and sufficient condition for a step-graphon W to have the (strong) H -property. Let σ be a partition for W , and x^* , S , Z , and $\mathbf{X}$ be the associated concentration vector, skeleton graph, node-cycle incidence matrix, and node-flow cone, respectively. We now introduce the following conditions:

Condition A: $\text{co-rank}(Z) = 0$.

Condition B: $x^* \in \text{int } \mathbf{X}$, where $\text{int } \mathbf{X}$ stands for the *relative interior* of $\mathbf{X}$.

Condition B': $x^* \in \mathbf{X}$.

Condition C: $\vec{S}$ is strongly connected.

These four conditions, though stated with respect to a specific σ , are in fact invariant under the choice of a partition. Precisely, we have

Proposition 1. *Let W be a step-graphon, and σ and σ' be two partitions for W . Let x^* , $\vec{S}$, Z , and $\mathbf{X}$ (resp., x'^* , $\vec{S}'$, Z' , and $\mathbf{X}'$) be the concentration vector, the skeleton graph, the node-cycle incidence matrix, and the node-flow cone of W for σ (resp., for σ'). Then, the following hold:*

1. *Suppose that both $\vec{S}$ and $\vec{S}'$ have at least two nodes; then, $\vec{S}$ is strongly connected if and only if $\vec{S}'$ is.*
2. *$\text{co-rank}(Z) = 0$ if and only if $\text{co-rank}(Z') = 0$.*
3. *$x^* \in \mathbf{X}$ if and only if $x'^* \in \mathbf{X}'$ ($x^* \in \text{int } \mathbf{X}$ if and only if $x'^* \in \text{int } \mathbf{X}'$).*

Note that for item 1, the hypothesis that both $\vec{S}$ and $\vec{S}'$ have at least two nodes is necessary, ruling out the special case where W is the zero function. To wit, if $W = 0$ and if the partition σ is chosen such that $\sigma = (0, 1)$, then the associated skeleton graph $\vec{S}$ comprises a single node u without self-loop. By default, $\vec{S}$ is strongly connected. However, any other partition σ' for W gives rise to a skeleton graph $\vec{S}'$ such that $\vec{S}'$ has multiple nodes but without any edge.

We provide a proof of the above Proposition 1 in Appendix A. With the result, we can now have the following definition:

Definition 7. *A step-graphon W is said to satisfy Condition $\star$, for $\star = A, B, B', C$, if there is a partition σ for W , with $|\sigma| \geq 2$, such that the associated objects (x^* , $\vec{S}$, Z , and $\mathbf{X}$) satisfy Condition $\star$.*

With the above definition, we can now state the main result of the paper:

Main Theorem. *Let W be a step-graphon. Then, the following hold:*

1. *If W does not satisfy Condition A or B', then*

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian decomposition}) = 0. \quad (6)$$

2. If W satisfies Conditions A and B, but not C, then

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian decomposition}) = 1, \quad (7)$$

and

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian cycle}) = 0. \quad (8)$$

3. If W satisfies Conditions A, B, and C, then

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian cycle}) = 1. \quad (9)$$

As mentioned earlier, the Main Theorem extends the results of [3, 4]. We substantiate our claim in Appendix B, where we specialize the Main Theorem to step-graphons with symmetric support.

2.3 Illustration and numerical validation

To illustrate the Main Theorem, we consider the four step-graphons in Figure 3. Over their respective support, W_a takes value 0.2 while W_b , W_c , and W_d take value 1. We let the partitions for the four step-graphons be

$$\begin{aligned} \sigma_a &= \frac{1}{16}(0, 1, 4, 9, 16), & \sigma_b &= \frac{1}{8}(0, 1, 3, 6, 8), \\ \sigma_c &= \frac{1}{20}(0, 5, 10, 16, 20), & \sigma_d &= \frac{1}{8}(0, 1, 3, 6, 8). \end{aligned}$$

The step-graphons in (a), (b), (c) share the same skeleton graph $\vec{S}$ as shown in (e), which is the same as the one in Figure 1b. The skeleton graph $\vec{S}'$ associated with the step-graphon in (d) is shown in (f), which can be obtained from $\vec{S}$ by removing the self-loop u_4u_4 .

The skeleton graph $\vec{S}$ has 4 cycles $\vec{C}_1, \dots, \vec{C}_4$ as shown in (3). The node-cycle incidence matrix Z has full row rank as argued in (4). The digraph $\vec{S}'$, being a subgraph of $\vec{S}$, has only three cycles $\vec{C}_2, \vec{C}_3, \vec{C}_4$. Its node-cycle incidence matrix Z' is given by

$$Z' := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{so } \text{co-rank}(Z') = 1. \quad (10)$$

We state without a proof that any three column vectors of Z form a facet-defining hyperplane of the cone $\mathbf{X}$. For each $i = 1, \dots, 4$, we let L_i be the subspace spanned by the z_j 's, for $j \neq i$. Let $g_i \in \mathbb{R}^4$ be the normal vector perpendicular to L_i of unit length such that $g_i^\top z_i > 0$. Then, it is not hard to obtain that

$$g_1 = \frac{1}{2}(-1, 1, -1, 1), \quad g_2 = \frac{1}{\sqrt{2}}(0, -1, 1, 0), \quad g_3 = (1, 0, 0, 0), \quad g_4 = \frac{1}{\sqrt{2}}(-1, 1, 0, 0).$$

Then, using the half-space representation, we can write

$$\mathbf{X} = \{y \in \mathbb{R}^4 \mid g_i^\top y \geq 0, \quad \text{for all } i = 1, \dots, 4\}.$$

We numerically validate the necessity and sufficiency of Conditions A , B (or B') for the step-graphon $W_\star$, for $\star = a, b, c, d$ to have the H -property. For each case (a)—(d) and for each $n = \{10, 50, 100, 500, 1000, 2000, 5000\}$, we sample 20,000 random graphs $\vec{G}_n \sim W$ and plot the empirical probability $p(n)$ that $\vec{G}_n$ has a Hamiltonian decomposition, i.e.,

$$p(n) := \frac{\text{number of } \vec{G}_n \sim W \text{ has a Hamiltonian decomposition}}{20,000}.$$

Case (a). The concentration vector is $x_a^* = \frac{1}{16}(1, 3, 5, 7)$. It belongs to the relative interior of $\mathbf{X}$ as we can express x_a^* as a positive combination of the z_j 's (the column vectors of Z given in (4)):

$$x_a^* = \frac{1}{4}z_1 + \frac{1}{8}z_2 + \frac{1}{16}z_3 + \frac{1}{8}z_4.$$

Also, the matrix Z has full row rank. Thus, W_a satisfies Conditions A and B . We see from the simulation that the empirical probability $p(n)$ converges to 1.

Case (b). The concentration vector is $x_b^* = \frac{1}{8}(1, 2, 3, 2)$. It belongs to the *boundary* of $\mathbf{X}$, i.e., $x_b^* \in \mathbf{X} - \text{int } \mathbf{X}$. To wit, we note that x_b^* is in the facet-defining hyperplane L_1 spanned by z_2 , z_3 , and z_4 :

$$x_b^* = \frac{1}{8}(z_2 + z_3 + z_4).$$

Thus, W_b satisfies Conditions A and B' , but not B . We see from the simulation that the empirical probability $p(n)$ converges to neither 1 nor 0. In fact, using the same arguments as in [5], we can show that the probability converges to 0.5, as demonstrated in the figure as well.

Case (c). The concentration vector is $x_c^* = \frac{1}{20}(5, 5, 6, 4)$. Since $g_1^\top x_c^* < 0$, x_c^* does not belong to $\mathbf{X}$. Thus, W_c satisfies Condition A but not B' . We see from the simulation that the empirical probability $p(n)$ converges to 0.

Case (d). The concentration vector is $x_d^* = \frac{1}{8}(1, 2, 3, 2)$, same as the one in Case (b). As argued above, we can write x_d^* as a positive combination of z_2 , z_3 , and z_4 , which are the three column vectors in Z' . Thus, $x_d^* \in \text{int } \mathbf{X}'$. Also, as shown in (10), Z' does not have full row rank. Thus, W_d satisfies Condition B but not A . We see from the simulation that the empirical probability $p(n)$ converges to 0.

2.4 Sketch of proof for the Main Theorem

We start by introducing two objects, which will be relevant to both necessity and sufficiency of Conditions A , B/B' (and C) for the (strong) H -property.

Definition 8 ($\vec{S}$ -partite graph). *Let $\vec{S}$ be an arbitrary digraph on m nodes, possibly with self-loops. A directed graph $\vec{G}$ is an $\vec{S}$ -partite graph if there exists a graph homomorphism $\pi : \vec{G} \rightarrow \vec{S}$. Further, $\vec{G}$ is a complete $\vec{S}$ -partite graph if*

$$v_i v_j \in E(\vec{G}) \iff \pi(v_i) \pi(v_j) \in E(\vec{S}).$$

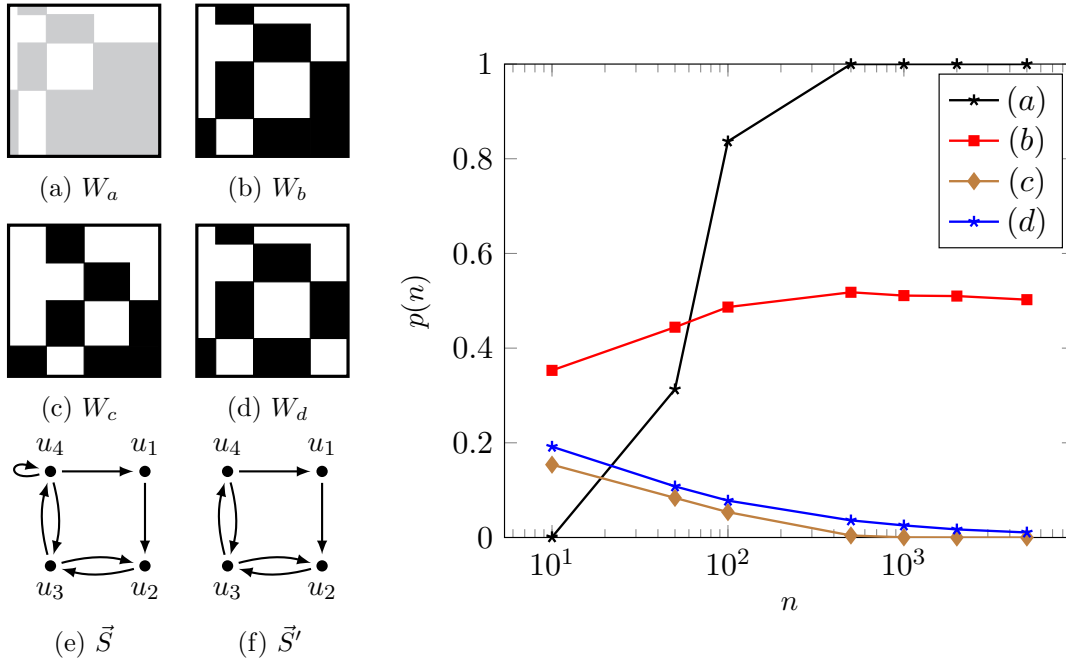


Figure 3: *Left:* Four step-graphons and the associated skeleton graphs, where $\vec{S}$ in (e) corresponds to W_a, W_b, W_c , and $\vec{S}'$ in (f) corresponds to W_d . *Right:* The empirical probability $p(n)$ that $\vec{G}_n \sim W_\star$ has a Hamiltonian decomposition, with 20,000 samples for each $n = 10, 50, 100, 500, 1000, 2000, 5000$.

For an $\vec{S}$ -partite graph $\vec{G}$, we let

$$y(\vec{G}) := (y_1, \dots, y_m) \quad \text{with } y_i := |\pi^{-1}(u_i)|, \quad \text{for all } i = 1, \dots, m.$$

Further, for a given vector $y \in \mathbb{N}_0^m$, we let $\vec{K}_y(\vec{S})$ (or simply $\vec{K}_y$) be the complete $\vec{S}$ -partite graph, with $y(\vec{K}_y) = y$.

The relevance of $\vec{S}$ -partite graphs is apparent. Any random graph $\vec{G}_n$ sampled from W is $\vec{S}$ -partite, where the homomorphism $\pi : \vec{G}_n \rightarrow \vec{S}$ is naturally the one that sends each node $v_j \in \vec{G}_n$, with coordinate $t_j \in [\sigma_{i-1}, \sigma_i)$, to u_i . It is also clear from the sampling procedure (more specifically, step *S1*) that $y(\vec{G}_n)$ is a multinomial random variable with n trials, m events, and x_i^* 's the event probabilities. Let

$$x(\vec{G}_n) := \frac{1}{n} y(\vec{G}_n). \tag{11}$$

We call $x(\vec{G}_n)$ the *empirical concentration vector*. It follows directly from the law of large numbers that

$$x(\vec{G}_n) \rightarrow x^* \text{ a.s.} \tag{12}$$

Next, for a square matrix $A = [a_{ij}] \in \mathbb{R}^{m \times m}$, let $\text{supp}(A)$ be the support of A , i.e., it is the set of indices ij such that $a_{ij} \neq 0$. One can also identify the index ij with a directed edge $u_i u_j$ of a digraph $\vec{S}$ on m nodes. We have the following definition:

Definition 9 (Edge-flow cone). *To the skeleton graph $\vec{S}$ on m nodes, we assign the set $\mathbf{A}$ of $m \times m$ nonnegative matrices A such that $\text{supp}(A) \subseteq E(\vec{S})$ and*

$$A^\top \mathbf{1} = A\mathbf{1}, \quad (13)$$

where $\mathbf{1}$ is the vector of all ones. We call $\mathbf{A}$ the **edge-flow cone**.

It is clear from the definition that $\mathbf{A}$ is a convex cone. Its relation to flows is as follows: Let $f : E(\vec{S}) \rightarrow \mathbb{R}_{\geq 0}$ be a flow, and let $A(f) = [a_{ij}(f)]$ be such that

$$a_{ij}(f) := \begin{cases} f(u_i u_j) & \text{if } u_i u_j \in E(\vec{S}), \\ 0 & \text{otherwise.} \end{cases}$$

The balance condition (5) of f guarantees that (13) is satisfied, so $A(f) \in \mathbf{A}$. It turns out that $\mathbf{A}$ is the set of $A(f)$ for all flows f on $\vec{S}$. In particular, the two sets $\mathbf{X}$ and $\mathbf{A}$ relate to each other in the following way:

$$\mathbf{X} = \mathbf{A}\mathbf{1} = \{A\mathbf{1} \mid A \in \mathbf{A}\}. \quad (14)$$

Our use of the edge-flow cone is through the Hamiltonian decompositions of $\vec{S}$ -partite graphs. Specifically, if $\vec{G}$ is an $\vec{S}$ -partite graph and if $\vec{G}$ has a Hamiltonian decomposition $\vec{H}$, then $\vec{H}$ induces an integer valued flow $f_{\vec{H}}$ on $\vec{S}$ in the way such that $f_{\vec{H}}(u_i u_j)$ records the total number of edges used in $\vec{H}$ from the nodes in $\pi^{-1}(u_i)$ to the nodes in $\pi^{-1}(u_j)$. Consider, for example, the digraph $\vec{G}_n$ in Figure 1c, with $n = 10$. It has a Hamiltonian decomposition $\vec{H}$ highlighted in blue. Then, the corresponding A -matrix is given by

$$A(f_{\vec{H}}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

It follows from the construction that

$$y(\vec{G}) = A(f_{\vec{H}})\mathbf{1} \in \mathbf{X}. \quad (15)$$

See Lemma 3 for a proof.

With the $\vec{S}$ -partite graphs and the edge-flow cone introduced above, we now sketch the proof of the Main Theorem:

2.4.1 On necessity of Conditions A , B' , and C

As argued above, if $\vec{G}_n \sim W$ has a Hamiltonian decomposition, then $y(\vec{G}_n) \in \mathbf{X}$. Thus, to establish the necessity of Conditions A and B' for the H -property (more precisely, to establish (6)), it suffices to show that

$$\neg A(\text{co-rank}(Z) > 0) \text{ or } \neg B'(x^* \notin \mathbf{X}) \implies y(\vec{G}_n) \notin \mathbf{X} \text{ a.a.s..}$$

The proof that

$$\neg B' \implies y(\vec{G}_n) \notin \mathbf{X} \text{ a.a.s.}$$

is straightforward, following directly from (12). The proof that

$$\neg A \implies y(\vec{G}_n) \notin \mathbf{X} \text{ a.a.s.}$$

uses the following arguments: Let Δ^{m-1} be the standard simplex in $\mathbb{R}^m$ and $\bar{\mathbf{X}} := \mathbf{X} \cap \Delta^{m-1}$. It is not hard to see that $y(G_n) \in \mathbf{X}$ if and only if $x(G_n) \in \bar{\mathbf{X}}$, where we recall that $x(G_n)$ is the empirical concentration vector (11). Note that if $\text{co-rank}(Z) \geq 1$, then $\dim \bar{\mathbf{X}} < \dim \Delta^{m-1} = m - 1$. Appealing to the central limit theorem, we have that the random variable $\omega(G_n) = \sqrt{n}(x(G_n) - x^*) + x^*$ converges in distribution to the Gaussian random variable ω^* whose support is known to be the entire hyperplane that contains Δ^{m-1} . As a consequence, it holds that $x(G_n) \notin \bar{\mathbf{X}}$ *a.a.s.*

The necessity of Condition *C* ($\vec{S}$ is strongly connected) for the strong *H*-property follows from the fact if $\vec{H}$ is a Hamiltonian cycle of $\vec{G}_n$, then $\pi(\vec{H})$ is a closed walk of $\vec{S}$. It is an immediate consequence of (12) that $\pi(\vec{H})$ visits every node of $\vec{S}$ *a.a.s.*, and hence, $\vec{S}$ must be strongly connected.

A complete proof of the necessity part will be presented in Section 3.

2.4.2 On sufficiency of Conditions *A*, *B*, and *C*

We introduce a subset $\mathbf{X}_0$ of $\mathbf{X}$, which comprises all integer-valued $y \in \mathbf{X}$ such that $\|y\|_1$ is sufficiently large and $y/\|y\|_1$ is sufficiently close to x^* . A precise definition of $\mathbf{X}_0$ will be given at the beginning of Section 5. The two conditions *A* and *B*, together with (12), guarantee that $y(G_n) \in \mathbf{X}_0$ *a.a.s.* The major task is then to show that

$$x(G_n) \in \mathbf{X}_0 \text{ (and } \vec{S} \text{ is strongly connected)} \implies G_n \text{ has a Hamiltonian decomposition (cycle) a.a.s.} \quad (16)$$

To accomplish the task, we take a two-step approach:

Step 1: We show that if $y \in \mathbf{X}_0$ (and if $\vec{S}$ is strongly connected), then the complete $\vec{S}$ -partite graph $\vec{K}_y$ has a Hamiltonian decomposition (cycle). The proof builds upon the following facts:

- 1.1. The first fact is a strengthened version [8] of the equality $\mathbf{X} = \mathbf{A}\mathbf{1}$, which states that if $y \in \mathbf{X}$ is *integer valued*, then there exists an *integer-valued* $A \in \mathbf{A}$ such that $A\mathbf{1} = y$.
- 1.2. We then express the matrix A , obtained from above, as an integer combination of the adjacency matrices A_j associated with the cycles $\vec{C}_j$ of $\vec{S}$, i.e., we write $A = \sum_{j=1}^k c_j A_j$ for $c_j \in \mathbb{N}_0$ (and $c_j \in \mathbb{N}$ in the case $y \in \mathbf{X}_0$). We show that $\vec{K}_y$ has a Hamiltonian decomposition, which contains c_j cycles that are isomorphic to $\vec{C}_j$ under the map $\pi : \vec{K}_y \rightarrow \vec{S}$.
- 1.3. If, further, $\vec{S}$ is strongly connected, then the cycles of the Hamiltonian decomposition exhibited above can be used to form a desired Hamiltonian cycle. The proof relies on the use of the induction hypothesis on the number of cycles in $\vec{S}$ and the (directed) ear decomposition of $\vec{S}$.

Complete arguments for this step will be presented in Section 5.

Step 2: Let $\vec{H}_y$ be the Hamiltonian decomposition (cycle) of $\vec{K}_y$ obtained in Step 1. We show that $\vec{G}_n$ contains $\vec{H}_{y(\vec{G}_n)}$ as a subgraph *a.a.s.*. Precisely, let $\psi_y : \vec{H}_y \rightarrow \vec{K}_y$ be the embedding. Composing ψ_y with π , we obtain the map $\pi \cdot \psi_y : \vec{H}_y \rightarrow \vec{S}$. We show that *a.a.s.* there exists an embedding $\phi : \vec{H}_{y(\vec{G}_n)} \rightarrow \vec{G}_n$ such that ϕ is compatible with $\psi_{y(\vec{G}_n)}$, i.e., $\pi \cdot \phi = \pi \cdot \psi_{y(\vec{G}_n)}$. The proof relies on the use of the *Blow-up Lemma* [9]. Roughly speaking, the lemma states that if an *undirected* graph H has its degree bounded above by a constant and if it can be embedded into a complete S -partite graph K_y , where S is an *undirected* graph *without* self-loop, then the graph H can be embedded into any S -partite graph G , with $y(G) = y$, as long as G satisfies some regularity condition. To enable its use, we take the following steps:

2.1 In Section 4, we show that if the step-graphon W has a nonzero diagonal block (i.e., $\vec{S}$ has a self-loop) and satisfies Condition $\star$, for $\star = A, B, C$, then there is a step-graphon W' such that $W' \leq W$ (i.e., $W'(s, t) \leq W(s, t)$ for all $(s, t) \in [0, 1]^2$), W' satisfies Condition $\star$ and, moreover, W' is “loop free”, i.e., the associated skeleton graph does not have any self-loop. This fact, combined with the monotonicity of the (strong) H -property, allow us to consider only the class of loop-free step-graphons for establishing the sufficiency of Conditions A , B , and C .

2.2 In Section 6, we introduce an auxiliary symmetric step-graphon W^s , which is derived from W , together with an auxiliary sampling procedure that allows us to draw undirected random graphs G_n from W^s . The graphon W^s and the sampling procedure are defined in a way such that the probability that $\vec{H}_{y(\vec{G}_n)}$ is embeddable into $\vec{G}_n$ is bounded above by the probability that $H_{y(G_n)}$ is embeddable into G_n , where $H_{y(G_n)}$ is the undirected counterpart of $\vec{H}_{y(\vec{G}_n)}$.

We then complete the proof by showing that *a.a.s.* the random graph G_n satisfies the aforementioned regularity condition. Thanks to the Blow-up lemma, $H_{y(G_n)}$ can be embedded into G_n *a.a.s.*

3 On Necessity of Conditions A , B' , and C

In this section, we establish (i) the necessity of Conditions A (i.e., $\text{co-rank } Z = 0$) and B' (i.e., $x^* \in X$) for the H -property, and (ii) the necessity of Condition C (i.e., $\vec{S}$ is strongly connected) for the strong H -property. The arguments for proving part (i) are similar to those used in [3], which dealt with symmetric step-graphons. For completeness of the presentation, we include the proofs of the relevant lemmas (but omit those for lemmas with exactly the same statements).

Recall that A is the edge-flow cone introduced in Definition 9. For each cycle $\vec{C}_j = u_{j_1} u_{j_2} \cdots u_{j_d} u_{j_1}$ of $\vec{S}$. We let A_j be the adjacency matrix associated with $\vec{C}_j$:

$$A_j := \sum_{i=1}^d e_{j_i} e_{j_{i+1}}^\top, \quad (17)$$

where j_{d+1} is identified with j_1 . It is clear that $A_j \in \mathbf{A}$ and that

$$z_j = A_j \mathbf{1}. \quad (18)$$

We have the following result:

Lemma 2. *The edge-flow cone $\mathbf{A}$ is generated by $A_1, \dots, A_k$:*

$$\mathbf{A} = \left\{ \sum_{j=1}^k c_j A_j \mid c_j \geq 0 \right\}. \quad (19)$$

In particular, we have that

$$\mathbf{X} = \mathbf{A} \mathbf{1}. \quad (20)$$

Proof. For any given $A = [a_{ij}] \in \mathbf{A}$, we show that there exist $c_1, \dots, c_k \geq 0$ such that $A = \sum_{j=1}^k c_j A_j$. Let A' be the diagonal matrix whose diagonal entries agree with those of A , and let $A'' := A - A'$. Note that if $a_{ii} > 0$, then it follows from Definition 9 that there is a self-loop on node u_i , whose corresponding A -matrix is $e_i e_i^\top$. It follows that

$$A' = \sum_{u_i u_i \in E(\vec{S})} a_{ii} e_i e_i^\top \in \mathbf{A}.$$

It remains to show that $A'' \in \mathbf{A}$. Let $\tilde{A}'' = [\tilde{a}_{ij}]$ be the weighted Laplacian defined as follows:

$$\tilde{a}_{ij} := \begin{cases} a_{ij} & \text{if } i \neq j \\ -\sum_{j=1, j \neq i}^n a_{ij} & \text{otherwise} \end{cases}$$

so $\tilde{A}$ has zero row-sum and zero column-sum. Similarly, let $\tilde{A}_j$ be the Laplacian matrix (with zero row- and column-sum) whose off-diagonal entries agree with those of A_j . It has been shown in [10, Proposition 3] that $\tilde{A}''$ is a nonnegative combination of $\tilde{A}_j$, which implies that A'' is a nonnegative combination of A_j . Finally, note that (20) is an immediate consequence of (18), (19), and the definition of $\mathbf{X}$ (Definition 6). $\square$

The next result establishes the necessity of $y(\vec{G}_n) \in \mathbf{X}$ for an $\vec{S}$ -partite graph to have a Hamiltonian decomposition:

Lemma 3. *Let $\vec{G}$ be an $\vec{S}$ -partite graph. If $\vec{G}$ has a Hamiltonian decomposition, then*

$$y(\vec{G}) \in \mathbf{X}.$$

Proof. Let $\pi : \vec{G} \rightarrow \vec{S}$ be the graph homomorphism, and $\vec{H}$ be a Hamiltonian decomposition of $\vec{G}$. Note that $\vec{H}$ is also an $\vec{S}$ -partite graph and $y(\vec{H}) = y(\vec{G})$. Given $1 \leq i, j \leq m$, let n_{ij} be the number of directed edges of $\vec{H}$ from nodes in $\pi^{-1}(u_i)$ to nodes in $\pi^{-1}(u_j)$. It is clear that for all $u_i \in V(\vec{S})$,

$$|\pi^{-1}(u_i)| = \sum_{j=1}^m n_{ij} = \sum_{j=1}^m n_{ji}. \quad (21)$$

Now, consider the matrix $A := [n_{ij}]_{1 \leq i, j \leq m}$. It is clear that $\text{supp}(A) \subseteq E(\vec{S})$. Also, by (21), we have that

$$A^\top \mathbf{1} = A\mathbf{1} = y(\vec{G}),$$

so $A \in \mathcal{A}$. By Lemma 2 and the fact that $A\mathbf{1} = y(\vec{G})$, we conclude that $y(\vec{G}) \in \mathcal{X}$. $\square$

With Lemma 3 above, we establish the necessity of Condition B' .

Proof of necessity of Condition B' for the H -property. We show that if $x^* \notin \mathcal{X}$, then (6) holds. Recall that for a random graph $\vec{G}_n \sim W$, $x(\vec{G}_n) = \frac{1}{n}y(\vec{G}_n)$ is the empirical concentration vector of $\vec{G}_n$, and it converges to x^* *a.a.s.*. Since $x^* \notin \mathcal{X}$ and since $\mathcal{X}$ is a closed subset of $\mathbb{R}^m$, it holds that $x(\vec{G}_n) \notin \mathcal{X}$ *a.a.s.*. By Lemma 3, if $x(\vec{G}_n) \notin \mathcal{X}$ (and hence, $y(\vec{G}_n) \notin \mathcal{X}$), then $\vec{G}_n$ cannot have a Hamiltonian decomposition. This completes the proof. $\square$

Next, given $\vec{G}_n \sim W$, we define

$$\omega(\vec{G}_n) := \sqrt{n}(x(\vec{G}_n) - x^*) + x^*. \quad (22)$$

The following result is known [3]:

Lemma 4. *The random variable $\omega(\vec{G}_n)$ converges in distribution to the Gaussian random variable $\omega^* \sim N(x^*, \Sigma)$, where $\text{Diag}(x^*)$ is the diagonal matrix whose i th entry is x_i^* and $\Sigma := \text{Diag}(x^*) - x^*x^{*\top}$. The rank of Σ is $(m-1)$ and its null space is spanned by $\mathbf{1}$.*

With Lemmas 3 and 4, we establish the necessity of Condition A :

Proof of necessity of Condition A for the H -property. We show that if

$$\text{co-rank}(Z) \geq 1, \quad (23)$$

then (6) holds. We may as well assume that Condition B' holds, i.e., $x^* \in \mathcal{X}$.

To proceed, we first normalize the node-cycle incidence vectors z_j so that their one-norm is 1:

$$\bar{z}_j := \frac{z_j}{\|z_j\|_1}, \quad \text{for all } j = 1, \dots, k.$$

Let $\bar{\mathcal{X}}$ be the convex hull generated by $\bar{z}_1, \dots, \bar{z}_k$:

$$\bar{\mathcal{X}} := \left\{ \sum_{j=1}^k c_j \bar{z}_j \mid c_j \geq 0 \text{ for all } j, \text{ and } \sum_{j=1}^k c_j = 1 \right\}.$$

Equivalently, $\bar{\mathcal{X}}$ is the set of all $x \in \mathcal{X}$ such that $\|x\|_1 = 1$. In particular, since $\|x^*\|_1 = 1$ and since $x^* \in \mathcal{X}$, we have that

$$x^* \in \bar{\mathcal{X}}. \quad (24)$$

Similarly, since $\|x(\vec{G}_n)\|_1 = 1$, we have that

$$x(\vec{G}_n) \in \mathcal{X} \iff x(\vec{G}_n) \in \bar{\mathcal{X}}.$$

Next, let L be the affine hyperplane in $\mathbb{R}^m$ spanned by $e_1, \dots, e_m$, which contains the standard simplex. Let L' be the affine space spanned by $\bar{z}_1, \dots, \bar{z}_k$, which is the affine space of least dimension that contains $\bar{X}$. By our hypothesis (23),

$$\dim L' \leq m - 2 < m - 1 = \dim L,$$

i.e., L' is a *proper* affine subspace of L .

We now establish a sequence of inequalities that bound from above the left hand side of (6). By Lemma 3, it is necessary that $x(\vec{G}_n) \in X$ for $\vec{G}_n$ to have a Hamiltonian decomposition, so

$$\begin{aligned} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian decomposition}) &\leq \mathbf{P}(x(\vec{G}_n) \in X) \\ &= \mathbf{P}(x(\vec{G}_n) \in \bar{X}) \leq \mathbf{P}(x(\vec{G}_n) \in L'). \end{aligned} \quad (25)$$

Then, by (22) and (24), we have that

$$x(\vec{G}_n) \in L' \iff \omega(\vec{G}_n) \in L',$$

so

$$\mathbf{P}(x(\vec{G}_n) \in L') = \mathbf{P}(\omega(\vec{G}_n) \in L'). \quad (26)$$

Combining (25) and (26), we have that

$$\mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian decomposition}) \leq \mathbf{P}(\omega(\vec{G}_n) \in L'). \quad (27)$$

Finally, we appeal to Lemma 4 to obtain that

$$\lim_{n \rightarrow \infty} \mathbf{P}(\omega(\vec{G}_n) \in L') = \lim_{n \rightarrow \infty} \mathbf{P}(\omega^* \in L') = 0,$$

where the last equality follows from the fact that L' is a proper affine subspace of L and the fact that the Gaussian random variable ω^* has the entire L as its support. $\square$

Finally, we establish the necessity of Condition C:

Proof of necessity of Condition C for the strong H-property. Let $\vec{G}$ be an $\vec{S}$ -partite graph such that $y(\vec{G}) \in \mathbb{N}^m$, so $\pi^{-1}(u_i)$ contains at least one node. If $\vec{G}$ has a Hamiltonian cycle $\vec{H}$, then $\pi(\vec{H})$ is a closed walk of $\vec{S}$ that visits every node at least once, which implies that $\vec{S}$ is strongly connected. In other words, we have just shown that

$\vec{S}$ is not strongly connected and $y(\vec{G}) \in \mathbb{N}^m \implies \vec{G}$ does not have a Hamiltonian cycle.

Now, let $\vec{G}_n \sim W$. Since $x(\vec{G}_n)$ converges to x^* *a.a.s.* and since all the entries x_i^* are positive, we have that

$$y_i(\vec{G}_n) = |\pi^{-1}(u_i)| = \Theta(n) \text{ a.a.s..}$$

The above arguments then imply that if $\vec{S}$ is not strongly connected, then *a.a.s.* $\vec{G}_n$ does not have a Hamiltonian cycle, i.e., (8) holds. $\square$

4 Pre-processing: Removal of self-loops

Let W be a step-graphon and $\sigma = (\sigma_0, \dots, \sigma_{m-1}, \sigma_*)$ be a partition for W , with $\sigma_0 = 0$ and $\sigma_* = 1$. Let $\vec{S}$ be the associated skeleton graph, and $\vec{S}_1, \dots, \vec{S}_q$ be the SCCs of $\vec{S}$. Suppose that the skeleton graph $\vec{S}$ associated with W has a self-loop, say, on node u_m . Without loss of generality, we assume that $u_m \in V(\vec{S}_q)$ and the partition σ is fine enough such that $\vec{S}_q$ has at least two nodes.

Surgery on the nonzero diagonal block of W : We introduce a new step-graphon W' as follows. First, let

$$\sigma_m := \frac{1}{2}(\sigma_{m-1} + 1). \quad (28)$$

Then, we set

$$W'(s, t) := \begin{cases} 0 & \text{if } (s, t) \in [\sigma_{m-1}, \sigma_m]^2 \cup [\sigma_m, 1]^2, \\ W(s, t) & \text{otherwise.} \end{cases} \quad (29)$$

In words, W' is obtained from W by first subdividing the block $R_{mm} = [\sigma_{m-1}, 1]^2$ into four sub-blocks:

$$\begin{aligned} R_{mm,11} &:= [\sigma_{m-1}, \sigma_m]^2, & R_{mm,12} &:= [\sigma_{m-1}, \sigma_m) \times [\sigma_m, 1], \\ R_{mm,21} &:= [\sigma_m, 1] \times [\sigma_{m-1}, \sigma_m), & R_{mm,22} &:= [\sigma_m, 1]^2. \end{aligned}$$

and then, setting the value of $W(s, t)$ to 0 if $(s, t) \in R_{mm,11} \cup R_{mm,22}$ while keeping $W(s, t)$ unchanged otherwise. See Figure 4 for illustration.

The goal of this section is to show that W' inherits any Condition $\star$, for $\star = A, B, C$, satisfied by W . Precisely, we have

Theorem 2. *Let W and W' be given as above. If W satisfies Condition $\star$, for $\star = A, B, C$, then so does W' .*

Let $\sigma' := (\sigma_0, \dots, \sigma_{m-1}, \sigma_m, \sigma_*)$. It is clear that σ' is a partition for W' . Let x'^* , $\vec{S}'$, Z' , and $\mathbf{X}'$ be the concentration vector, the skeleton graph, the node-cycle incidence matrix, and the node-flow cone of W' for σ' , respectively. With slight abuse of terminology, we say that $\vec{S}'$ is obtained from $\vec{S}$ by performing the surgery on node u_m . It is clear that $\vec{S}'$ has one less self-loop than $\vec{S}$ does.

If W' still has a nonzero diagonal block (equivalently, $\vec{S}'$ has a self-loop), then we perform the surgery again for W' (resp., $\vec{S}'$) on the corresponding block (resp. node). Iterating this procedure until we obtain a graphon which admits a partition such that its associated skeleton graph does not have any self-loop. We introduce the following definition:

Definition 10. *A step-graphon W is **loop free** if there is a (and hence, any) partition such that the associated skeleton graph does not have any self-loop.*

The following result is then a corollary of Theorem 2:

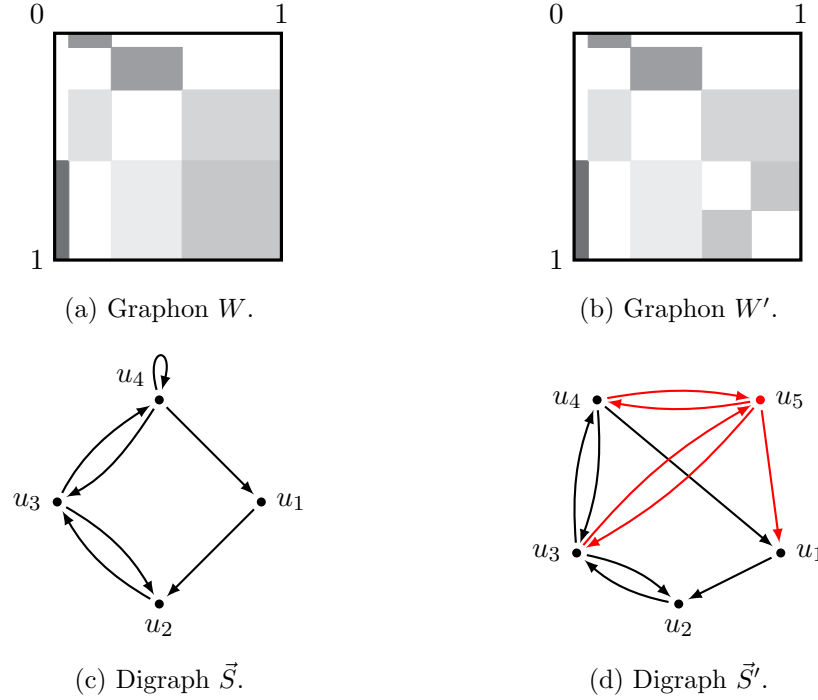


Figure 4: The step-graphon W' in (b) is obtained from W in (a) by first subdividing the right-bottom block into 2-by-2 sub-blocks and then setting the value of the two diagonal sub-blocks to zero. A partition sequence σ for W is $\sigma = \frac{1}{16}(0, 1, 4, 9, 16)$. The subdivision then gives rise to partition sequence $\sigma' = \frac{1}{16}(0, 1, 4, 9, 12.5, 16)$ for W' . The two digraphs $\vec{S}$ and $\vec{S}'$ shown in (c) and (d) are the skeleton graphs associated with W and W' , respectively. The digraph $\vec{S}'$ can be obtained from $\vec{S}$ by removing the self-loop u_4u_4 and by adding the node u_5 and the edges u_5u_1 , u_5u_3 , u_3u_5 , u_5u_4 , and u_4u_5 , which are highlighted in red—we call this procedure a surgery of $\vec{S}$ on node u_4 .

Corollary 3. *If a step-graphon W satisfies Condition $\star$, for $\star = A, B, C$, then there exists a loop-free step-graphon W' such that $W' \leq W$ and satisfies Condition $\star$.*

The remainder of the section is devoted to the proof of Theorem 2. We deal with the three conditions in the order of C , A , and B in three subsections.

4.1 Proof for Condition C

In this subsection, we show that

$$\vec{S} \text{ is strongly connected (with at least 2 nodes)} \implies \vec{S}' \text{ is strongly connected.}$$

First, note that by (29), the digraph $\vec{S}'$ can be obtained from $\vec{S}$ by first adding a new node u_{m+1} and the following set of new edges:

$$E_{m+1} := \{u_i u_{m+1} \mid u_i u_m \in E(\vec{S})\} \cup \{u_{m+1} u_j \mid u_m u_j \in E(\vec{S})\}, \quad (30)$$

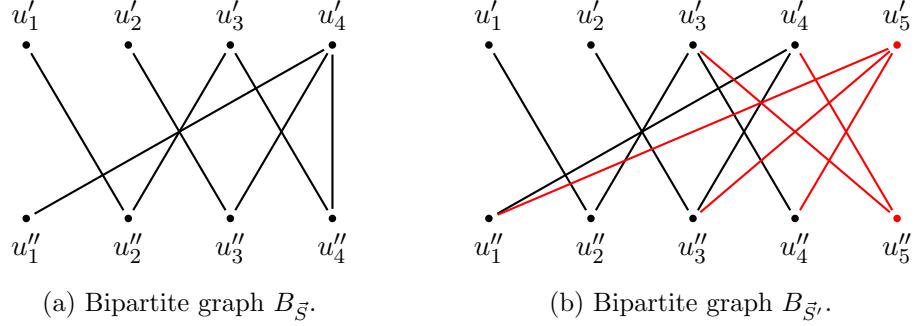


Figure 5: The bipartite graph $B_{\vec{S}}$ in (a) (resp., $B_{\vec{S}'}$ in (b)) is associated with $\vec{S}$ in 4c (resp., $\vec{S}'$ in 4d). The graph $B_{\vec{S}'}$ can be obtained from $B_{\vec{S}}$ by removing the edge (u'_4, u''_4) and by adding nodes u'_5 and u''_5 and edges (u'_5, u''_1) , (u'_5, u''_3) , (u'_3, u''_5) , (u'_5, u''_4) , and (u'_4, u''_5) , highlighted in red.

and then deleting the self-loop $u_m u_m$. Precisely,

$$V(\vec{S}') = V(\vec{S}) \cup \{u_{m+1}\} \quad \text{and} \quad E(\vec{S}') = E(\vec{S}) \cup E_{m+1} - \{u_m u_m\}. \quad (31)$$

We now show that for any two distinct nodes $u_i, u_j \in V(\vec{S}')$, there is a walk from u_i to u_j . Consider the following three cases:

Case 1: $u_i \neq u_{m+1}$ and $u_j \neq u_{m+1}$. Since $\vec{S}$ is strongly connected, there is a path $\vec{P}$ from u_i to u_j in $\vec{S}$. By (30) and (31), $\vec{P}$ is also a path of $\vec{S}'$.

Case 2: $u_i = u_{m+1}$. Let $u_{i_1} \cdots u_{i_\ell}$, with $u_{i_1} = u_m$ and $u_{i_\ell} = u_j$, be a walk of $\vec{S}$ from u_m to u_j . In the case $u_j = u_m$, the walk is closed—such a closed walk exists because $\vec{S}$ is strongly connected and has $m \geq 2$ nodes. Since $u_m u_{i_2} \in E(\vec{S})$, by (30) we have that $u_{m+1} u_{i_2} \in E(\vec{S}')$ and, hence, $u_{m+1} u_{i_2} \cdots u_{i_\ell}$ is a walk from u_{m+1} to u_j .

Case 3: $u_j = u_{m+1}$. Similarly, if $u_{j_1} \cdots u_{j_\ell}$ is a walk of $\vec{S}$, with $u_{j_1} = u_i$ and $u_{j_\ell} = u_m$, then $u_{j_1} \cdots u_{j_{\ell-1}} u_{m+1}$ is a walk of $\vec{S}'$ from u_i to u_{m+1} . $\square$

4.2 Proof for Condition A

In this subsection, we show that

$$\text{co-rank}(Z) = 0 \implies \text{co-rank}(Z') = 0.$$

Recall that $\vec{S}_1, \dots, \vec{S}_q$ are the SCCs of $\vec{S}$. Let $\vec{S}'_p := \vec{S}_p$, for $p = 1, \dots, q-1$, and $\vec{S}'_q$ be obtained from $\vec{S}_q$ by performing the surgery on the node u_m . Then, it should be clear from (30) and (31) that $\vec{S}'_1, \dots, \vec{S}'_q$ are the SCCs of $\vec{S}'$.

Also, recall that we use the notation $B_{\vec{S}}$ to denote the bipartite graph associated with $\vec{S}$. For $B_{\vec{S}'}$, it follows from (30) and (31) that

$$V'(B_{\vec{S}'}) = V'(B_{\vec{S}}) \cup \{u'_{m+1}\}, \quad V''(B_{\vec{S}'}) = V''(B_{\vec{S}}) \cup \{u''_{m+1}\},$$

and

$$E(B_{\vec{S}'}) = E(B_{\vec{S}}) \cup \{(u'_i, u''_{m+1}) \mid (u'_i, u''_m) \in E(B_{\vec{S}})\} \\ \cup \{(u'_{m+1}, u''_i) \mid (u'_m, u''_i) \in E(B_{\vec{S}})\} - \{(u'_m, u''_m)\}. \quad (32)$$

See Figure 4 for illustration.

Since $\text{co-rank}(Z) = 0$, it follows from Lemma 1 that every bipartite graph $B_{\vec{S}_p}$, for $p = 1, \dots, q$, is connected. Using the same lemma, we have that $\text{co-rank}(Z') = 0$ if and only if $B_{\vec{S}'_p}$ is connected for all $p = 1, \dots, q$. Since $\vec{S}'_p = \vec{S}_p$ for $p = 1, \dots, q-1$, it suffices to show that $B_{\vec{S}'_q}$ is connected. We establish this fact by proving the following lemma:

Lemma 5. *Let S' be obtained from $\vec{S}$ by performing the surgery on the node u_m . If $B_{\vec{S}}$ is connected, then so is $B_{\vec{S}'}$.*

Proof. We show that for any $u'_i \in V'(B_{\vec{S}'})$ and any $u''_j \in V''(B_{\vec{S}'})$, there is a path of $B_{\vec{S}'}$ from u'_i to u''_j (by reversing the order, we obtain a path from u''_j to u'_i). We consider the following four cases:

Case 1: $u'_i \neq u'_{m+1}$ and $u''_j \neq u''_{m+1}$. Let P be a path of $B_{\vec{S}}$ that connects u'_i and u''_j . If the path does not contain the edge (u'_m, u''_m) , then P is also a path of $B_{\vec{S}'}$. We thus assume that P contains (u'_m, u''_m) . Since $m \geq 2$ and since $B_{\vec{S}}$ is connected, at least one of the two nodes u'_m and u''_m has degree at least 2 within $B_{\vec{S}}$. Without loss of generality, we assume that $\deg(u'_m) \geq 2$ and that (u'_m, u''_ℓ) , with $u''_\ell \neq u''_m$, is an edge of $B_{\vec{S}}$. By (32), we have that (u'_m, u''_ℓ) , (u'_{m+1}, u''_ℓ) , and (u'_{m+1}, u''_m) are edges of $B_{\vec{S}'}$. Replacing the segment $u'_m u''_m$ in P with $u'_m u''_\ell u'_{m+1} u''_m$, we obtain a walk of $B_{\vec{S}'}$ that connects u'_i and u''_j .

Case 2: $u'_i \neq u'_{m+1}$ and $u''_j = u''_{m+1}$. Let P be a path of $B_{\vec{S}}$ from u'_i to u''_m . Replacing the last node u''_m of P with u''_{m+1} , we obtain a path of $B_{\vec{S}'}$ from u'_i to u''_{m+1} .

Case 3: $u'_i = u'_{m+1}$ and $u''_j \neq u''_{m+1}$. Similarly, let P be a path of $B_{\vec{S}}$ from u'_m to u''_j . Replacing the first node u'_m of P with u'_{m+1} , we obtain a path of $B_{\vec{S}'}$ from u'_{m+1} to u''_j .

Case 4: $u'_i = u'_{m+1}$ and $u''_j = u''_{m+1}$. By the same arguments in Case 1, we can assume without loss of generality that (u'_m, u''_ℓ) , with $u''_\ell \neq u''_m$, is an edge of $B_{\vec{S}}$. Then, $u'_{m+1} u''_\ell u'_m u''_{m+1}$ is a path from u'_{m+1} to u''_{m+1} . $\square$

4.3 Proof for Condition B

In this subsection, we show that

$$x^* \in \text{int } X \implies x'^* \in \text{int } X'.$$

We start by relating the cycles of $\vec{S}'$ to those of $\vec{S}$. Label the cycles of $\vec{S}$ in a way such that the first ℓ cycles $\vec{C}_1, \dots, \vec{C}_\ell$, for some $\ell \leq k$, contain the node u_m and that $\vec{C}_1 = u_m u_m$ is the self-loop.

The self-loop $\vec{C}_1$ induces the 2-cycle $\vec{C}'_1 := u_m u_{m+1} u_m$ of $\vec{S}'$. Each cycle $\vec{C}_p$, for $2 \leq p \leq \ell$, induces four different cycles of $\vec{S}'$ as follows: $\vec{C}'_{p,1} := \vec{C}_p$ and $\vec{C}'_{p,2}, \vec{C}'_{p,3}, \vec{C}'_{p,4}$ are obtained from $\vec{C}_p$ by substituting the node u_m with u_{m+1} , $u_m u_{m+1}$, $u_{m+1} u_m$, respectively. Thus, the set of cycles of $\vec{S}'$ is given by

$$\{\vec{C}'_1\} \cup \{\vec{C}'_{p,i} \mid 2 \leq p \leq \ell \text{ and } 1 \leq i \leq 4\} \cup \{\vec{C}'_q \mid \ell + 1 \leq q \leq k\}.$$

To illustrate, consider the digraph $\vec{S}$ in Figure 4c and the corresponding digraph $\vec{S}'$ in Figure 4d. The digraph $\vec{S}$ has 4 cycles as exhibited in (3). The first three cycles contain the node u_4 . The self-loop $\vec{C}_1$ induces the 2-cycle $\vec{C}'_1 = u_4 u_5 u_4$ in $\vec{S}'$. The cycle $\vec{C}_2$ induces four cycles of $\vec{S}'$:

$$\vec{C}_{2,1} = u_3 u_4 u_3, \quad \vec{C}_{2,2} = u_3 u_5 u_3, \quad \vec{C}_{2,3} = u_3 u_4 u_5 u_3, \quad \text{and} \quad \vec{C}_{2,4} = u_3 u_5 u_4 u_3.$$

Similarly, the four cycles of $\vec{S}'$ induced by $\vec{C}_3$ are

$$\vec{C}_{3,1} = u_1 u_2 u_3 u_4 u_1, \quad \vec{C}_{3,2} = u_1 u_2 u_3 u_5 u_1, \quad \vec{C}_{3,3} = u_1 u_2 u_3 u_4 u_5 u_1, \quad \text{and} \quad \vec{C}_{3,4} = u_1 u_2 u_3 u_5 u_4 u_1.$$

Thus, the digraph $\vec{S}'$ has ten cycles $\vec{C}'_1, \vec{C}_{2,1}, \dots, \vec{C}_{2,4}, \vec{C}_{3,1}, \dots, \vec{C}_{3,4}$, and $\vec{C}_4$.

Let $z'_1, z'_{p,i}$, and z'_q be the node-cycle incidence vectors of $\vec{S}'$ corresponding to $\vec{C}'_1, \vec{C}_{p,i}$, and $\vec{C}_q$, respectively. To relate these vectors to the z_j 's, we first augment each z_j by adding a zero entry at the end. Precisely, we define

$$\hat{z}_j := \begin{bmatrix} z_j \\ 0 \end{bmatrix} \in \mathbb{R}^{m+1}, \quad \text{for all } j = 1, \dots, k.$$

Then, we have that

$$\begin{cases} z'_1 = e_m + e_{m+1}, \\ z'_{p,1} = \hat{z}_p, \quad z'_{p,2} = \hat{z}_p - e_m + e_{m+1}, \quad z'_{p,3} = z'_{p,4} = \hat{z}_p + e_{m+1}, & \text{for } 2 \leq p \leq \ell, \\ z'_q = \hat{z}_q, & \text{for } \ell + 1 \leq q \leq k. \end{cases} \quad (33)$$

Note that

$$z'_{p,3} = z'_{p,4} = \frac{1}{2}(z'_1 + z'_{p,1} + z'_{p,2}).$$

which implies that $z'_{p,3}$ and $z'_{p,4}$ are *not* extremal generators of $\mathbf{X}'$.

It now suffices to show that x'^* can be expressed as a *positive* combination of $z'_1, z'_{p,1}$'s, $z'_{p,2}$'s, and z'_q 's. First, by (28), we have that

$$x'^* = (x_1^*, \dots, x_{m-1}^*, x_m^*/2, x_m^*/2).$$

Let $\hat{x}^* := (x^*; 0)$. Then, we can express x'^* as

$$x'^* = \hat{x}^* + \frac{x_m^*}{2}(e_{m+1} - e_m). \quad (34)$$

Since $x^* \in \text{int } \mathbf{X}$, there exist positive coefficients c_j 's such that $x^* = \sum_{j=1}^k c_j z_j$. It follows from the definitions of $\hat{z}_j$ and of $\hat{x}^*$ that

$$\hat{x}^* = \sum_{j=1}^k c_j \hat{z}_j. \quad (35)$$

Since $\vec{C}_1, \dots, \vec{C}_\ell$ are the cycles of $\vec{S}$ that contain u_m ,

$$x_m^* = \sum_{j=1}^{\ell} c_j. \quad (36)$$

We define positive coefficients as follows:

$$\begin{cases} c'_1 := c_1/2, \\ c'_{p,i} := c_p/2 & \text{for } 2 \leq p \leq \ell \text{ and } 1 \leq i \leq 2, \\ c'_q := c_q & \text{for } \ell + 1 \leq q \leq k. \end{cases} \quad (37)$$

Then, following (34), we have that

$$\begin{aligned} x'^* &= \hat{x}^* + \frac{x_m^*}{2}(e_{m+1} - e_m) \\ &= \sum_{j=1}^k c_j \hat{z}_j + \frac{x_m^*}{2}(e_{m+1} - e_m) \\ &= \sum_{q=\ell+1}^k c'_q \hat{z}_q + \sum_{p=2}^{\ell} \left[\sum_{i=1}^2 c'_{p,i} \right] \hat{z}_p + c_1 e_m + \frac{x_m^*}{2}(e_{m+1} - e_m) \\ &= \sum_{q=\ell+1}^k c'_q z'_q + \sum_{p=2}^{\ell} \sum_{i=1}^2 c'_{p,i} z'_{p,i} + c_1 e_m + \frac{1}{2} \left[x_m^* - \sum_{j=2}^{\ell} c_j \right] (e_{m+1} - e_m) \\ &= \sum_{q=\ell+1}^k c'_q z'_q + \sum_{p=2}^{\ell} \sum_{i=1}^2 c'_{p,i} z'_{p,i} + \frac{1}{2} c_1 (e_{m+1} + e_m) \\ &= \sum_{q=\ell+1}^k c'_q z'_q + \sum_{p=2}^{\ell} \sum_{i=1}^2 c'_{p,i} z'_{p,i} + c'_1 z'_1, \end{aligned}$$

where the second equality follows from (35), the third equality follows from (37), the fourth equality follows from (33) and (37), the fifth equality follows from (36), and the last equality follows from (33) and (37). This completes the proof. $\square$

5 Hamiltonicity of complete $\vec{S}$ -partite graphs

Recall that $\vec{K}_y$ is the complete $\vec{S}$ -partite graph, with $y_i = |\pi^{-1}(u_i)|$, for all $i = 1, \dots, m$. In this section, we investigate when $\vec{K}_y$ can have a Hamiltonian decomposition (cycle).

In the sequel, we assume that $\vec{S}$ does *not* have a self-loop and that Condition B ($x^* \in \text{int } X$) is satisfied. Let U be an open neighborhood of x^* in X . Then, there is a continuous function $\gamma : U \rightarrow \mathbb{R}_{>0}^k$ such that

$$x = Z\gamma(x) \quad \text{for all } x \in U. \quad (38)$$

Let

$$\gamma_0 := \frac{1}{2} \min_{j=1}^k \gamma_j(x^*).$$

Shrink U if necessary so that

$$\gamma_j(y) > \gamma_0, \quad \text{for all } j = 1, \dots, k \text{ and for all } y \in U. \quad (39)$$

We now introduce the following subset of X :

$$\mathsf{X}_0 := \{y \in \mathsf{X} \cap \mathbb{N}^m \mid \|y\|_1 \geq 1/\gamma_0 \text{ and } y/\|y\|_1 \in \mathsf{U}\}. \quad (40)$$

In words, X_0 collects all integer-valued $y \in \mathsf{X}$ such that $\|y\|_1$ is sufficiently large and $y/\|y\|_1$ is sufficiently close to x^* . The main result of this section is as follows:

Theorem 4. *The following two items hold:*

1. *For any integer-valued $y \in \mathsf{X}$, $\vec{K}_y$ has a Hamiltonian decomposition.*
2. *If $\vec{S}$ is strongly connected, then for any $y \in \mathsf{X}_0$, $\vec{K}_y$ has a Hamiltonian cycle.*

By Lemma 3, if $\vec{K}_y$ has a Hamiltonian decomposition, then $y \in \mathsf{X}$. Combining this fact with item 1 of the above theorem, we have that $y \in \mathsf{X} \cap \mathbb{N}_0^m$ is both necessary and sufficient for $\vec{K}_y$ to have a Hamiltonian decomposition.

We establish the two items of Theorem 4 in two subsections.

5.1 Proof of item 1

We start by decomposing $y \in \mathsf{X}$ into an integer combination of the node-cycle incidence vectors z_j . This is feasible as we show in the following lemma:

Lemma 6. *For any integer-valued $y \in \mathsf{X}$, there exist $c_1, \dots, c_k \in \mathbb{N}_0$ such that*

$$y = \sum_{j=1}^k c_j z_j. \quad (41)$$

Proof. Since y is integer valued, it is known [8, Theorem 1.2] that there exists an integer-valued $A \in \mathsf{A}$ such that

$$y = A\mathbf{1}. \quad (42)$$

We show that there exist $c_1, \dots, c_k \in \mathbb{N}_0$ such that

$$A = \sum_{j=1}^k c_j A_j. \quad (43)$$

Since $A \in \mathsf{A}$, there exist $r_1, \dots, r_k \in \mathbb{R}_{\geq 0}$ such that $A = \sum_{j=1}^k r_j A_j$. Since A is integer valued, it holds that if $r_j > 0$ for some $j = 1, \dots, k$, then $A' := (A - A_j)$ has nonnegative entries and is integer valued. We claim that $A' \in \mathsf{A}$. To wit, note that $\text{supp}(A') \subseteq \text{supp}(A)$ and $\text{supp}(A) \subseteq E(\vec{S})$, so $\text{supp}(A') \subseteq E(\vec{S})$. Also, note that

$$A'\mathbf{1} = A\mathbf{1} - A_j\mathbf{1} = A^\top \mathbf{1} - A_j^\top \mathbf{1} = A'^\top \mathbf{1}.$$

This establishes the claim. If $A' \neq 0$, then we can repeat the same arguments to find some $j' = 1, \dots, k$ such that $(A' - A_{j'}) \in \mathsf{A}$. This iteration will terminate in finite steps and we obtain (43). Now, using (42), (43), and the fact that $A_j\mathbf{1} = z_j$, we obtain (41). $\square$

Using the decomposition (41), we exhibit a desired Hamiltonian decomposition of $\vec{K}_y$ in the following lemma:

Lemma 7. *Let $c_1, \dots, c_k \in \mathbb{N}_0$ be given as in Lemma 6 so that (41) holds. Then, $\vec{K}_y$ has a Hamiltonian decomposition $\vec{H}$ such that $\vec{H}$ contains, for each $j = 1, \dots, k$, c_j cycles that are isomorphic to $\vec{C}_j$ under the map π .*

Proof. The proof will be carried out by induction on $c := \sum_{j=1}^k c_j$. For the base case $c = 0$, item 2 holds trivially. For the inductive step, we assume that item 2 holds for $(c - 1) \geq 0$ and prove for c .

Without loss of generality, we assume that $c_1 \geq 1$ and write $\vec{C}_1 = u_1 u_2 \dots u_{d_1} u_1$, where d_1 is the length of $\vec{C}_1$. Since $\vec{S}$ does not have a self-loop, $d_1 \geq 2$. It follows from (41) that for each $i = 1, \dots, d_1$, $y_i = |\pi^{-1}(u_i)| \geq 1$, so there is at least a node, say v_i , contained in $\pi^{-1}(u_i)$.

Because $\vec{K}_y$ is complete $\vec{S}$ -partite and because $\vec{C}_1$ is a cycle of $\vec{S}$, we have that $\vec{D}_1 := v_1 v_2 \dots v_{d_1} v_1$ is a cycle of $\vec{K}_y$. It is clear that $\vec{D}_1$ is isomorphic to $\vec{C}_1$ under the map π .

We now remove $\vec{D}_1$ from $\vec{K}_y$ and the edges incident to $\vec{D}_1$. Then, the resulting graph is the complete $\vec{S}$ -partite graph $\vec{K}_{y'}$, where

$$y' := y - z_1 = (c_1 - 1)z_1 + \sum_{j=2}^k c_j z_j.$$

By the induction hypothesis, $\vec{K}_{y'}$ has a Hamiltonian decomposition $\vec{H}'$ which contains $(c_1 - 1)$ cycles isomorphic to $\vec{C}_1$ and c_j cycles isomorphic to $\vec{C}_j$ for $j = 2, \dots, k$, under the map π . Taking the union of $\vec{H}'$ and the cycle $\vec{D}_1$, we obtain the desired Hamiltonian decomposition for $\vec{K}_y$. $\square$

5.2 Proof of item 2

Under the assumption that $\vec{S}$ is strongly connected and $y \in \mathbf{X}_0$, the two lemmas we established in the previous subsection can be strengthened.

We first have the following result, which is a strengthened version of Lemma 6:

Lemma 8. *For any $y \in \mathbf{X}_0$, there exist positive integers $c_1, \dots, c_k$ such that*

$$y = \sum_{j=1}^k c_j z_j. \tag{44}$$

Proof. For convenience, we let $n := \|y\|_1$. Since $y \in \mathbf{X}_0$, $y/n \in \mathbf{U}$. By (38), we can write

$$y = \sum_{j=1}^k n \gamma_j(x) z_j. \tag{45}$$

Now, let

$$c'_j := \lfloor n \gamma_j(x) \rfloor \quad \text{and} \quad r'_j := n \gamma_j(x) - c'_j, \quad \text{for all } j = 1, \dots, k.$$

By (39), (40), and the hypothesis that $y \in X_0$, we have that

$$c'_j \geq \lfloor n\gamma_0 \rfloor \geq 1, \quad \text{for all } j = 1, \dots, k.$$

If $r'_j = 0$ for all $j = 1, \dots, k$, then we set $c_j := c'_j$ and (44) holds. Otherwise, let

$$y' := \sum_{j=1}^k c'_j z_j \quad \text{and} \quad y'' := y - y' = \sum_{j=1}^k r'_j z_j. \quad (46)$$

It is clear that both y' and y'' are integer valued and belong to X . By Lemma 6, there exist $c''_1, \dots, c''_k \in \mathbb{N}_0$ such that

$$y'' = \sum_{j=1}^k c''_j z_j. \quad (47)$$

We then let $c_j := c'_j + c''_j$ for $j = 1, \dots, k$. It is clear that all the c_j 's are positive. Using (46) and (47), we conclude that (44) holds. $\square$

We now show that whenever $\vec{S}$ is strongly connected and y can be expressed as (44), with $c_1, \dots, c_k$ positive integers, the digraph $\vec{K}_y$ has a Hamiltonian cycle.

Lemma 9. *Suppose that $\vec{S}$ is strongly connected and that (44) holds for some positive integers $c_1, \dots, c_k$; then, $\vec{K}_y$ has a Hamiltonian cycle.*

Proof. The proof will be carried out by induction on k , the number of cycles in $\vec{S}$.

Base case $k = 1$. In this case, $\vec{S}$ is itself a cycle. We write $\vec{S} = u_1 u_2 \cdots u_m u_1$, for $m \geq 2$. By Lemma 7, there exists a Hamiltonian decomposition $\vec{H}$ of $\vec{K}_y$ which comprises c_1 cycles that are isomorphic to $\vec{S}$ under π . We label these cycles as $\vec{D}_1, \dots, \vec{D}_{c_1}$ and write

$$\vec{D}_j = v_{j,1} v_{j,2} \cdots v_{j,m} v_{j,1}, \quad \text{for all } j = 1, \dots, c_1,$$

where the nodes are labeled such that

$$\pi^{-1}(u_i) = \{v_{j,i} \mid j = 1, \dots, c_1\}, \quad \text{for all } i = 1, \dots, m.$$

Since $\vec{K}_y$ is complete $\vec{S}$ -partite, we have that $v_{i,m} v_{j,1}$ is an edge of $\vec{K}_y$ for any $1 \leq i, j \leq c_1$. It follows that

$$\vec{H} := v_{1,1} \cdots v_{1,m} v_{2,1} \cdots v_{2,m} v_{3,1} \cdots v_{c_1,m} v_{1,1}$$

is a Hamiltonian cycle of $\vec{K}_y$.

Inductive step. We assume that the lemma holds for any $k' \leq k - 1$ and prove for k . Since $\vec{S}$ is strongly connected, $\vec{S}$ admits an *ear decomposition*. See, e.g., [11, Chapter 7.2] and also Figure 6 for an illustration. In particular, $\vec{S}$ can be obtained by gluing an ear $\vec{P} = u_1 \cdots u_r$ to a strongly connected subgraph $\vec{S}'$, where the starting node u_1 and the ending node u_r of the ear are nodes of $\vec{S}'$ while the other nodes of the ear do not belong to $\vec{S}'$. Note that u_1 and u_r can be the same (in this case, $\vec{P}$ is a cycle).

Let k' be the number of cycles in $\vec{S}'$. We claim that $k' < k$, i.e., $\vec{S}$ contains more cycles than its subgraph $\vec{S}'$ does. To wit, if $u_1 = u_r$, then $\vec{P}$ is a cycle of $\vec{S}$ but not of $\vec{S}'$.

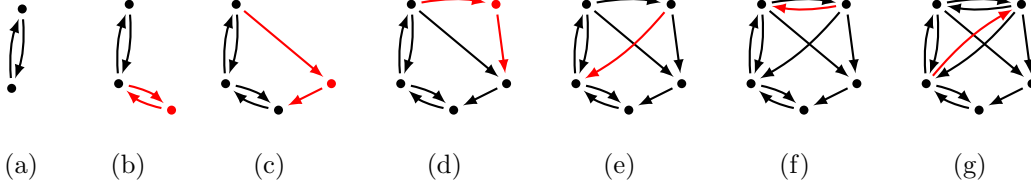


Figure 6: Starting with the 2-cycle in (a), we iteratively add ears, highlighted in red in each step, to obtain the strongly connected digraph in (g).

If $u_1 \neq u_r$, then we let $\vec{P}'$ be a path in $\vec{S}'$ from u_r to u_1 . Concatenating $\vec{P}'$ with $\vec{P}$, we obtain a cycle in $\vec{S}$, which is not in $\vec{S}'$. This establishes the claim.

Re-label the cycles of $\vec{S}$, if necessary, so that $\vec{C}_1, \dots, \vec{C}_{k'}$ are the cycles of $\vec{S}'$, and $\vec{C}_{k'+1}, \dots, \vec{C}_k$ are the cycles of $\vec{S}$ but not of $\vec{S}'$. Each $\vec{C}_j$, for $j = k' + 1, \dots, k$, must contain the ear $\vec{P}$. Let $y^{(j)} := c_j z_j$ for $j = 1, \dots, k$ and $y' := \sum_{j=1}^{k'} y^{(j)}$. Since all the c_j 's are positive, we have that $\text{supp}(y^{(j)}) = V(\vec{C}_j)$. Since $\vec{S}'$ is strongly connected, every node of $\vec{S}'$ is contained in some cycle $\vec{C}_j$, for $j = 1, \dots, k'$, and hence, $\text{supp}(y') = V(\vec{S}')$. We then truncate y' and the $y^{(j)}$'s by setting

$$\tilde{y}' := y'|_{\vec{S}'} \quad \text{and} \quad \tilde{y}^{(j)} := y^{(j)}|_{\vec{C}_j}, \quad \text{for all } j = k' + 1, \dots, k.$$

For ease of notation, let

$$\vec{K} := \vec{K}_y(\vec{S}), \quad \vec{K}' := \vec{K}_{\tilde{y}'}(\vec{S}'), \quad \text{and} \quad \vec{K}^{(j)} := \vec{K}_{\tilde{y}^{(j)}}(\vec{C}_j) \quad \text{for all } j = k' + 1, \dots, k.$$

Since $y = y' + \sum_{j=k'+1}^k y^{(j)}$, one can embed simultaneously $\vec{K}'$ and $\vec{K}^{(j)}$, for $j = k' + 1, \dots, k$, into $\vec{K}$. In other words, $\vec{K}$ contains these $(k - k' + 1)$ subgraphs whose node sets are pairwise disjoint.

Since $\vec{S}'$ is strongly connected and has k' cycles, for $k' < k$, and since $y' = \sum_{j=1}^{k'} c_j z_j$ with the c_j 's positive, we can appeal to the induction hypothesis to obtain a Hamiltonian cycle $\vec{H}'$ of $\vec{K}'$. Through the embedding of $\vec{K}'$ into $\vec{K}$, we treat $\vec{H}'$ as a cycle of $\vec{K}$. Because $\vec{S}'$ contains the node u_1 , there is a node v'_1 in $\vec{H}'$ such that $\pi(v'_1) = u_1$. We write $\vec{H}'$ explicitly as

$$\vec{H}' := v'_1 \cdots v'_{n'} v'_1, \tag{48}$$

where $n' := \|y'\|_1 = |V(\vec{K}')|$.

For each $j = k' + 1, \dots, k$, we use the same arguments as in the base case to obtain a Hamiltonian cycle $\vec{H}^{(j)}$ of $\vec{K}^{(j)}$. Similarly, we treat $\vec{H}^{(j)}$ as a cycle of $\vec{K}$. Because $\vec{H}^{(j)}$ contains the ear $\vec{P}$ and, hence, the node u_1 , there exists a node $v_{j,1}$ in $\vec{H}^{(j)}$ such that $\pi(v_{j,1}) = u_1$. We write $\vec{H}^{(j)}$ explicitly as

$$\vec{H}^{(j)} := v_{j,1} \cdots v_{j,n_j} v_{j,1}, \tag{49}$$

where $n_j := \|y^{(j)}\|_1 = |V(\vec{K}^{(j)})|$.

Since $V(\vec{K}')$ and the $V(\vec{K}^{(j)})$ form a partition of $V(\vec{K})$, their respective Hamiltonian cycles, namely, $\vec{H}'$ and the $\vec{H}^{(j)}$'s, form a Hamiltonian decomposition of $\vec{K}$. We will now

use these cycles to construct a Hamiltonian cycle of $\vec{K}$. Since $\vec{K}$ is complete $\vec{S}$ -partite and since the nodes v'_1 and $v_{j,1}$, for $j = k' + 1, \dots, k$, belong to $\pi^{-1}(u_1)$, we have that $v'_{n'}v_{k'+1,1}$, $v_{k,n_k}v'_1$, and $v_{j,n_j}v_{j+1,1}$, for $j = k' + 1, \dots, k - 1$, are edges of $\vec{K}$. Thus,

$$\vec{H} := v'_1 \cdots v'_{n'} v_{k'+1,1} \cdots v_{k'+1,n_{k'+1}} v_{k'+2,1} \cdots v_{k,n_k} v'_1$$

is a desired Hamiltonian cycle of $\vec{K}$. □

6 On Sufficiency of Conditions of A , B , and C

In this section, we show that if a step-graphon W satisfies Conditions A , B (and C), then W has the (strong) H -property.

The condition that a graph has a Hamiltonian decomposition (cycle) is monotone with respect to edge addition. Specifically, if $\vec{G}$ and $\vec{G}'$ are two graphs on the same node set, with $E(\vec{G}) \supseteq E(\vec{G}')$, then

$$\begin{aligned} G' \text{ has a Hamiltonian decomposition (cycle)} \\ \implies G \text{ has a Hamiltonian decomposition (cycle)}. \end{aligned}$$

This monotonicity is carried over to graphons. Specifically if W' and W are two graphons, with $W' \leq W$, then

$$\begin{aligned} \mathbf{P}(\vec{G}_n \sim W' \text{ has a Hamiltonian decomposition (cycle)}) \\ \leq \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian decomposition (cycle)}) \end{aligned}$$

which implies that

$$W' \text{ has the (strong) } H\text{-property} \implies W \text{ has the (strong) } H\text{-property}.$$

Thus, by Corollary 3, we can assume that W is loop free.

By Theorem 4, for any $y \in \mathbf{X}_0$, $\vec{K}_y$ has a Hamiltonian decomposition. If, further, $\vec{S}$ is strongly connected, then $\vec{K}_y$ has a Hamiltonian cycle. Denote the Hamiltonian decomposition (cycle) by $\vec{H}_y$. We show below that if W satisfies Conditions A and B , then $\vec{H}_y$ can be embedded into $\vec{G}_n \sim W$ *a.a.s.*. We make the statement precise below.

To this end, let ψ_y be the embedding (i.e., a one-to-one graph homomorphism) of $\vec{H}_y$ into $\vec{K}_y$:

$$\psi_y : \vec{H}_y \rightarrow \vec{K}_y. \tag{50}$$

Composing ψ_y with π , we obtain the graph homomorphism $\pi \cdot \psi_y : \vec{H}_y \rightarrow \vec{S}$, which assigns to each node of $\vec{H}_y$ a node of $\vec{S}$. We introduce the following definition:

Definition 11. Let $\vec{G}$ be an $\vec{S}$ -partite graph, with $y(\vec{G}) \in \mathbf{X}_0$. An embedding $\phi : \vec{H}_{y(\vec{G})} \rightarrow \vec{G}$, if exists, is **compatible with** $\psi_{y(\vec{G})}$ if

$$\pi \cdot \phi = \pi \cdot \psi_{y(\vec{G})}.$$

We denote by $\vec{\mathcal{H}}$ the set of all $\vec{S}$ -partite graphs $\vec{G}$ such that $y(\vec{G}) \in \mathbf{X}_0$ and that $\vec{G}$ admits an embedding $\phi : \vec{H}_{y(\vec{G})} \rightarrow \vec{G}$, compatible with $\psi_{y(\vec{G})}$. We now state the main result of the section:

Theorem 5. *Let W be a loop-free step graphon. If W satisfies Conditions A and B, then*

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \in \vec{\mathcal{H}}) = 1.$$

We take a two-step approach to establish the result: In Subsection 6.1, we associate to W a *symmetric* step-graphon W^s and use it to sample *undirected* random graph $G_n \sim W^s$. The two random graphs G_n and $\vec{G}_n$ relate to each other in the way that the probability of the event that $\vec{G}_n \in \vec{\mathcal{H}}$ is bounded from below by the probability of the event that $H_{y(G_n)}$ is embeddable into G_n , where H_y is the undirected counterpart of $\vec{H}_y$. This step allows for the use of the Blow-up Lemma, which we do in Subsection 6.2.

6.1 Reduction by symmetrization

Let $\sigma = (\sigma_0, \dots, \sigma_m)$ be a partition for W . Recall that p_{ij} is the value of W over $R_{ij} = [\sigma_{i-1}, \sigma_i) \times [\sigma_{j-1}, \sigma_j)$. We define a *symmetric* step-graphon W^s as follows:

$$W^s(s, t) := \begin{cases} \max\{p_{ij}, p_{ji}\} & \text{if } p_{ij}p_{ji} = 0 \\ p_{ij}p_{ji} & \text{otherwise,} \end{cases} \quad \text{for } (s, t) \in R_{ij} \text{ and for } 1 \leq i, j \leq m. \quad (51)$$

We use q_{ij} to denote the value of W^s over R_{ij} (and R_{ji}). See Figure 7 for illustration.

To the step-graphon W^s with partition σ , there corresponds the *undirected* graph S on m nodes defined as follows: We still use $u_1, \dots, u_m$ to denote the nodes of S . A pair (u_i, u_j) is an edge of S if $q_{ij} > 0$. It follows from (51) that (u_i, u_j) is an edge of S if and only if $\vec{S}$ contains either $u_i u_j$ or $u_j u_i$, or both. Since $\vec{S}$ does not have any self loop (as W is loop free), neither does S .

We use W^s to sample an *undirected* graph G_n on n nodes as follows: First, follow step S1 to obtain the coordinates t_i 's of the n nodes. Then,

S'2. For each pair of two distinct nodes v_i and v_j , place an *undirected* edge (v_i, v_j) with probability $W^s(t_i, t_j)$.

We next introduce the set of S -partite graphs:

Definition 12. *An undirected graph G is **S -partite** if there is a graph homomorphism $\pi : G \rightarrow S$. Further, G is **complete S -partite** if*

$$(v_i, v_j) \in E(G) \iff (\pi(v_i), \pi(v_j)) \in E(S).$$

Similarly, for a given S -partite graph G , we let $y(G) := (y_1(G), \dots, y_m(G))$, with $y_i := |\pi^{-1}(u_i)|$ for all $i = 1, \dots, m$. Given a vector $y \in \mathbb{N}_0^m$, we use K_y to denote the complete S -partite graph, with $y(K_y) = y$.

It is clear from the sampling procedure (more specifically, step S'2) that $G_n \sim W^s$ is S -partite, with the graph homomorphism π being the one that sends each node $v_j \in V(G_n)$ to u_i if $t_j \in [\sigma_{i-1}, \sigma_i)$.

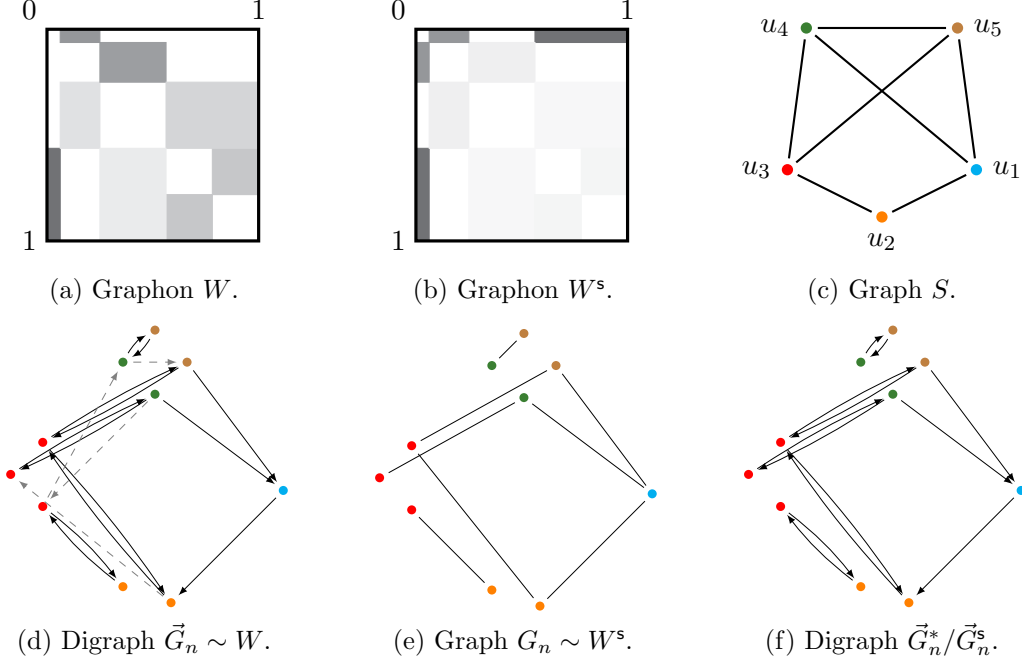


Figure 7: Given the step-graphon W in (a), we use the rule (51) to obtain the symmetric graphon W^s in (b). The undirected graph in (c) is the skeleton graph of W^s for the partition $\sigma = \frac{1}{16}(0, 1, 4, 9, 12.5, 16)$. The digraph $\vec{G}_n$ (same as the one in Figure 1c) is sampled from W . The undirected graph G_n in (e) is sampled from W^s , following steps $S1$ and $S'2$. Finally, consider the following two sampling procedures: One is to trim $\vec{G}_n \sim W$ by removing certain edges specified in step $S3$ —these edges are dashed and in gray. We denote by the resulting graph $\vec{G}_n^* \sim W^*$. The other is to perform step $S'3$ on G_n to transform it into the digraph $\vec{G}_n^s$. In this case, $\vec{G}_n^*$ and $\vec{G}_n^s$ are the same, given in (f). In Lemma 12, we argued that the two sampling procedures are equivalent in the sense that $\vec{G}_n^*$ and $\vec{G}_n^s$ have the same distribution.

Given the $\vec{S}$ -partite graph $\vec{H}_y$ introduced right above (50), we let H_y be the S -partite graph obtained from $\vec{H}_y$ by ignoring the orientations of its edges, i.e.,

$$V(H_y) = V(\vec{H}_y),$$

$$E(H_y) = \{(v_i, v_j) \mid \vec{H}_y \text{ contains at least one of the two edges } v_i v_j \text{ and } v_j v_i\}.$$

Note that if $\vec{H}_y$ is a cycle and if it has more than two nodes, then H_y is an (undirected) cycle. If $\vec{H}_y$ is a node-wise disjoint union of cycles, then H_y is a node-wise disjoint union of cycles and edges, where the edges correspond to the 2-cycles in $\vec{H}_y$.

For any $y \in \mathbf{X}_0$, the embedding ψ_y given in (50) induces the embedding of H_y into K_y , which sends the edges (v_i, v_j) of H_y to $(\psi_y(v_i), \psi_y(v_j))$. With slight abuse of notation, we still use $\psi_y : H_y \rightarrow K_y$ to denote the induced embedding.

Let $\mathcal{H}$ be the set of all S -partite graphs G such that $y(G) \in \mathbf{X}_0$ and that there exists an embedding $\phi : H_{y(G)} \rightarrow G$ which is compatible with $\psi_{y(G)}$, i.e., $\pi \cdot \phi = \pi \cdot \psi_{y(G)}$.



Figure 8: The graph G in (b) is obtained from $\vec{G}$ by ignoring the orientations of the edges (the two directed edges v_2v_3 and v_3v_2 are reduced to the same edge (v_2, v_3)). A Hamiltonian cycle (HC) $v_1v_2v_3v_4v_1$ of $\vec{G}$ induces an HC of G . However, the converse is in general not true. For example, $v_1v_3v_2v_4v_1$ is an HC of G , but does not induce an HC of $\vec{G}$ because v_1v_3 is not an edge of $\vec{G}$.

The main result of the subsection is the following:

Proposition 6. *For any $n \in \mathbb{N}$,*

$$\mathbf{P}(G_n \sim W^s \in \mathcal{H}) \leq \mathbf{P}(\vec{G}_n \in W \in \vec{\mathcal{H}}).$$

We establish below Proposition 6. Given an S -partite graph G , we perform the following operation on its edge set to obtain an $\vec{S}$ -partite digraph:

S'3. For each edge (v_i, v_j) of G , we consider the following three cases:

Case 1: If $u_iu_j \in \vec{S}$ and $u_ju_i \notin \vec{S}$, then replace (v_i, v_j) with v_iv_j ;

Case 2: If $u_ju_i \in \vec{S}$ and $u_iu_j \notin \vec{S}$, then replace (v_i, v_j) with v_jv_i ;

Case 3: If $u_iu_j, u_ju_i \in \vec{S}$, then replace (v_i, v_j) with two edges v_iv_j and v_jv_i .

We denote by $\vec{G}^s$ the resulting digraph.

Note that an embedding $\phi : H_y \rightarrow G$, with $y(G) = y$, does not necessarily induce an embedding $\phi : \vec{H}_y \rightarrow \vec{G}^s$; indeed, there may exist an edge v_iv_j of $\vec{H}_y$ such that $\phi(v_i)\phi(v_j)$ is not an edge of $\vec{G}^s$ (even though $(\phi(v_i), \phi(v_j))$ is an edge of G). See Figure 8 for illustration.

The following lemma shows that the induced embedding always exists if ϕ is compatible with ψ_y :

Lemma 10. *Let $G \in \mathcal{H}$ and $\phi : H_{y(G)} \rightarrow G$ be an embedding compatible with $\psi_{y(G)}$. Then, ϕ induces an embedding of $\vec{H}_{y(G)}$ to $\vec{G}^s$. In particular, we have that*

$$G \in \mathcal{H} \iff \vec{G}^s \in \vec{\mathcal{H}}.$$

Proof. Within the proof, we will simply write ψ by omitting its sub-index. We show that

$$v_iv_j \in E(\vec{H}_{y(G)}) \implies \phi(v_i)\phi(v_j) \in E(\vec{G}^s).$$

Since ϕ is compatible with ψ , we have that

$$u_i := \pi \cdot \psi(v_i) = \pi \cdot \phi(v_i) \quad \text{and} \quad u_j := \pi \cdot \psi(v_j) = \pi \cdot \phi(v_j).$$

Since $\pi \cdot \psi : \vec{H}_y \rightarrow \vec{S}$ is a graph homomorphism, $u_i u_j$ is an edge of $\vec{S}$. Also, since $\phi : H_y \rightarrow G$ is a graph homomorphism, $(\phi(v_i), \phi(v_j))$ is an edge of G . Then, by the operation given in the step $S'3$, we conclude that $\phi(v_i)\phi(v_j)$ is an edge of $\vec{G}^s$. $\square$

With slight abuse of notation, we denote by $\vec{G}_n^s \sim W^s$ the random digraph on n nodes obtained by following the steps $S1$, $S'2$, and $S'3$. An immediate consequence of Lemma 10 is then the following:

Lemma 11. *For any $n \in \mathbb{N}$,*

$$\mathbf{P}(G_n \sim W^s \in \mathcal{H}) = \mathbf{P}(\vec{G}_n^s \sim W^s \in \vec{\mathcal{H}}). \quad (52)$$

The following lemma relates the event $\vec{G}_n^s \sim W^s \in \mathcal{H}$ to the event $\vec{G}_n \sim W \in \mathcal{H}$, and completes the proof of Proposition 6.

Lemma 12. *For any $n \in \mathbb{N}$,*

$$\mathbf{P}(\vec{G}_n^s \sim W^s \in \vec{\mathcal{H}}) \leq \mathbf{P}(\vec{G}_n \sim W \in \vec{\mathcal{H}}).$$

Proof. Given an arbitrary $\vec{S}$ -partite graph $\vec{G}_n$, we let $\vec{G}_n^*$ be obtained by removing certain edges out of $\vec{G}_n$ as specified below:

S3. An edge $v_i v_j \in E(\vec{G}_n)$ will be removed if the following two conditions hold:

- 1: Both $\pi(v_i)\pi(v_j)$ and $\pi(v_j)\pi(v_i)$ are edges of $\vec{S}$.
- 2: $v_j v_i$ is not an edge of $\vec{G}_n$.

We denote by $\vec{G}_n^* \sim W^*$ the random digraph obtained by following the steps $S1$, $S2$, and $S3$. Let $u_i, u_j \in V(\vec{S})$ be such that $u_i u_j, u_j u_i \in E(\vec{S})$. It is clear that for two distinct nodes $v_i \in \pi^{-1}(u_i)$ and $v_j \in \pi^{-1}(u_j)$, the probability that $\vec{G}_n \sim W$ has both edges $v_i v_j$ and $v_j v_i$ is $p_{ij} p_{ji} = q_{ij}$. Thus, by step $S3$,

$$\mathbf{P}(v_i v_j \in \vec{G}_n^* \text{ and } v_j v_i \in \vec{G}_n^*) = q_{ij} \quad \text{and} \quad \mathbf{P}(v_i v_j \notin \vec{G}_n^* \text{ and } v_j v_i \notin \vec{G}_n^*) = 1 - q_{ij}.$$

This, in particular, implies that the two sampling procedures, namely, the one ($S1$ - $S'2$ - $S'3$) for sampling $\vec{G}_n^s \sim W^s$ and the other ($S1$ - $S2$ - $S3$) for sampling $\vec{G}_n^* \sim W^*$, are in fact equivalent to each other. It follows that

$$\mathbf{P}(\vec{G}_n^s \sim W^s \in \vec{\mathcal{H}}) = \mathbf{P}(\vec{G}_n^* \sim W^* \in \vec{\mathcal{H}}). \quad (53)$$

The condition that an $\vec{S}$ -partite graph belongs to $\vec{\mathcal{H}}$ is monotone with respect to edge addition. Since $\vec{G}_n^*$ is obtained from $\vec{G}_n$ by removing edges, $\vec{G}_n^* \in \vec{\mathcal{H}}$ implies $\vec{G}_n \in \vec{\mathcal{H}}$. Thus,

$$\mathbf{P}(\vec{G}_n^* \sim W^* \in \vec{\mathcal{H}}) \leq \mathbf{P}(\vec{G}_n \sim W \in \vec{\mathcal{H}}). \quad (54)$$

The lemma then follows from (53) and (54). $\square$

6.2 On the use of the Blow-up Lemma

Let U be the open neighborhood of x^* in X , which is introduced at the beginning of Section 5. Since $x^* \in \text{int } X$ and since $x(G_n)$ converges to x^* *a.a.s.*, it holds that $y(G_n) \in X_0$ *a.a.s.* We show below that

$$\lim_{n \rightarrow \infty} \mathbf{P}(G_n \sim W^s \in \mathcal{H} \mid y(G_n) \in X_0) = 1. \quad (55)$$

In words, we show that if $y(G_n) \in X_0$, then *a.a.s.* there exists an embedding $\phi : H_{y(G_n)} \rightarrow G_n$, compatible with $\psi_{y(G_n)} : H_{y(G_n)} \rightarrow K_{y(G_n)}$. Note that if (55) holds, then by Proposition 6, we have that

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \in \vec{H}) = 1,$$

i.e., Theorem 5 holds, which will then complete the proof of (7) and (9).

The proof of (55) relies on the use of the Blow-up lemma, which we recall below. Let G be an arbitrary undirected graph. For two disjoint subsets X and Y of $V(G)$, we let $e(X, Y)$ be the number of edges between X and Y . We need the following definition:

Definition 13 (Super-regular pair). *Let G be an undirected graph, and A, B be two disjoint subsets of $V(G)$. The pair (A, B) is (ϵ, δ) -super-regular if*

$$e(X, Y) > \delta|X||Y|, \quad \text{for any } X \subseteq A \text{ and } Y \subseteq B, \text{ with } |X| > \epsilon|A| \text{ and } |Y| > \epsilon|B|, \quad (56)$$

and, moreover,

$$e(a, B) > \delta|B| \quad \text{for any } a \in A, \quad \text{and} \quad e(b, A) > \delta|A| \quad \text{for any } b \in B. \quad (57)$$

We extend the above definition to the S -partite graphs:

Definition 14 (Super-regular S -partite graphs). *Let S be an undirected graph, without self-loops, on m nodes. An S -partite graph G , with $y(G) \in \mathbb{N}^m$, is (ϵ, δ) -super-regular if for any two distinct nodes $u_i, u_j \in V(S)$, $(\pi^{-1}(u_i), \pi^{-1}(u_j))$ is (ϵ, δ) -super-regular.*

For an arbitrary graph H , we let $\Delta(H)$ be the degree of H (i.e., the maximum of the degrees of its nodes). We reproduce below the Blow-up lemma [9]:

Lemma 13 (Blow-up Lemma). *Let S be an undirected graph, without self-loops, on m nodes. Then, given parameters $\delta > 0$ and $\Delta \in \mathbb{N}$, there exists an $\epsilon = \epsilon(\delta, \Delta, m) > 0$ such that for any $y \in \mathbb{N}^m$, the following holds: If H is an undirected graph with $\Delta(H) \leq \Delta$ and if there is an embedding $\psi : H \rightarrow K_y(S)$, then for any (ϵ, δ) -super-regular S -partite graph G , with $y(G) = y$, there is an embedding $\phi : H \rightarrow G$, compatible with ψ .*

We now return to the proof of (55). For any $y \in X_0$, we let H_y be given as in the previous subsection. As argued earlier, H_y is either a cycle or a node-wise disjoint union of cycles and possibly edges. Thus,

$$\Delta(H_y) \leq 2, \quad \text{for all } y \in X_0.$$

Also, it has been shown (as a consequence of Theorem 4) that there is an embedding $\psi_y : H_y \rightarrow K_y$ for all $y \in X_0$. Let

$$\delta := \frac{1}{2} \min\{q_{ij} \mid (u_i, u_j) \in E(S)\}. \quad (58)$$

and $\epsilon := \epsilon(\delta, 2, m) > 0$ be given as in the statement of Lemma 13. It remains to show that *a.a.s.* $G_n \sim W^s$ is (ϵ, δ) -super-regular.

Proposition 7. *For any $\epsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mathbf{P}(G_n \sim W^s \text{ is } (\epsilon, \delta)\text{-super-regular}) = 1.$$

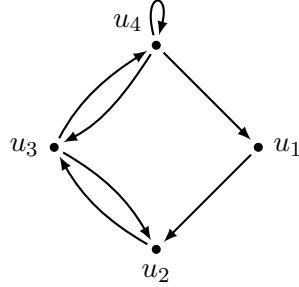
The proof of the proposition uses standard arguments in random graph theory. For completeness of presentation, we include it in Appendix C. $\square$

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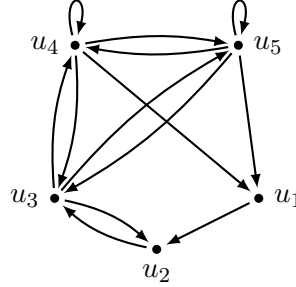
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A Proof of Proposition 1

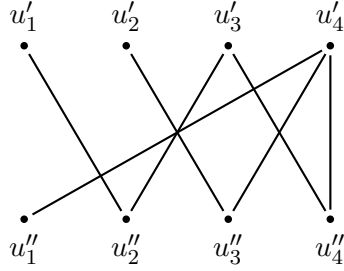
We say that σ' is a *refinement* of σ if σ' contains σ as a subsequence. Furthermore, σ' is a *one-step refinement* of σ if σ' contains one more element than σ does. It is clear that any refinement can be obtained by iterating one-step refinements. Note that for any two arbitrary partitions σ and σ' , there exists a partition σ'' as a refinement of both σ and σ' . The arguments above then imply that to establish Proposition 1, it suffices to prove for the case where σ' is a one-step refinement of σ .



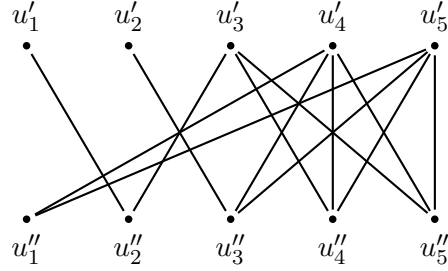
(a) Digraph $\vec{S}$.



(b) Digraph $\vec{S}'$.



(c) Bipartite graph $B_{\vec{S}}$.



(d) Bipartite graph $B_{\vec{S}'}$.

Figure 9: Two digraphs $\vec{S}$ and $\vec{S}'$ are skeleton graphs of W in Figure 1a for $\sigma = \frac{1}{16}(0, 1, 4, 9, 16)$ and $\sigma' = \frac{1}{16}(0, 1, 4, 9, 12.5, 16)$, where σ' is a one-step refinement of σ . The two bipartite graphs $B_{\vec{S}}$ and $B_{\vec{S}'}$ in (c) and (d) are associated with $\vec{S}$ and $\vec{S}'$, respectively. We show in Appendix A that $\vec{S}'$ is strongly connected if and only if $\vec{S}$ is, and that $B_{\vec{S}'}$ is connected if and only if $B_{\vec{S}}$ is (in this case, both $\vec{S}'$ and $\vec{S}$ are strongly connected, and both $B_{\vec{S}'}$ and $B_{\vec{S}}$ are connected).

Let $\sigma = (\sigma_0, \dots, \sigma_{m-1}, \sigma_*)$, with $\sigma_0 = 0$ and $\sigma_* = 1$. We assume, without loss of generality, that σ' is obtained from σ by inserting an element σ_m between σ_{m-1} and σ_* :

$$\sigma' = (\sigma_0, \dots, \sigma_{m-1}, \sigma_m, \sigma_*).$$

Then, the following hold for x'^* , $\vec{S}'$, and $B_{\vec{S}'}$:

1. Let $\hat{x}^* := (x^*; 0) \in \mathbb{R}^{m+1}$. Then,

$$x'^* = \hat{x}^* + (1 - \sigma_m)(e_{m+1} - e_m). \quad (59)$$

2. The skeleton graph $\vec{S}'$ can be obtained from $\vec{S}$ by adding the new node u_{m+1} , as a “copy” of u_m , and new edges incident to u_{m+1} , i.e.,

$$V(\vec{S}') = V(\vec{S}) \cup \{u_{m+1}\},$$

and

$$\begin{aligned} E(\vec{S}') = E(\vec{S}) \cup \{u_i u_{m+1} \mid u_i u_m \in E(\vec{S})\} \cup \{u_{m+1} u_j \mid u_m u_j \in E(\vec{S})\} \\ \cup \{u_{m+1} u_{m+1} \mid u_m u_m \in E(\vec{S})\}. \end{aligned} \quad (60)$$

With slight abuse of terminology, we call any such digraph $\vec{S}'$, obtained from $\vec{S}$ via the above operation, a *one-step refinement* on node u_m .

3. Correspondingly, the bipartite graph $B_{\vec{S}'}$ is given by

$$V'(B_{\vec{S}'}) = V'(B_{\vec{S}}) \cup \{u'_{m+1}\}, \quad V''(B_{\vec{S}'}) = V''(B_{\vec{S}}) \cup \{u''_{m+1}\}$$

and

$$\begin{aligned} E(B_{\vec{S}'}) = E(B_{\vec{S}}) \cup \{(u'_i, u''_{m+1}) \mid (u'_i, u''_m) \in E(B_{\vec{S}})\} \\ \cup \{(u'_{m+1}, u''_j) \mid (u'_m, u''_j) \in E(B_{\vec{S}})\} \\ \cup \{(u'_{m+1}, u''_{m+1}) \mid (u'_m, u''_m) \in E(B_{\vec{S}})\}. \end{aligned} \quad (61)$$

We now establish the three items of the proposition:

A.1 Proof of item 1

Consider the graph homomorphism $\theta : \vec{S}' \rightarrow \vec{S}$ defined by

$$\theta(u_i) := \begin{cases} u_i & \text{if } 1 \leq i \leq m \\ u_m & \text{if } i = m+1 \end{cases}$$

In words, θ identifies the node u_{m+1} with u_m . It follows directly from (60) that

$$u_i u_j \in E(\vec{S}') \iff \theta(u_i) \theta(u_j) \in E(\vec{S}).$$

Thus,

$$\vec{P}' = u_{i_1} \cdots u_{i_\ell} \text{ is a walk of } \vec{S}' \iff \theta(\vec{P}') := \theta(u_{i_1}) \cdots \theta(u_{i_\ell}) \text{ is a walk of } \vec{S}. \quad (62)$$

Proof that $\vec{S}'$ is strongly connected $\Rightarrow \vec{S}$ is strongly connected. Let u_i and u_j be two distinct nodes of $\vec{S}$. We pick nodes $u_{i'} \in \theta^{-1}(u_i)$ and $u_{j'} \in \theta^{-1}(u_j)$. Since $\vec{S}'$ is strongly connected, there is a path $\vec{P}'$ of $\vec{S}'$ from $u_{i'}$ to $u_{j'}$. By (62), we have that $\theta(\vec{P}')$ is a walk of $\vec{S}$ from u_i to u_j .

Proof that $\vec{S}$ is strongly connected $\Rightarrow \vec{S}'$ is strongly connected. Let $u_{i'}$ and $u_{j'}$ be two distinct nodes of $\vec{S}'$. We first consider the case where $u_i := \theta(u_{i'})$ and $u_j := \theta(u_{j'})$ are two

distinct nodes. In this case, there is a path $\vec{P} = u_{i_1} \cdots u_{i_\ell}$, with $u_{i_1} = u_i$ and $u_{i_\ell} = u_j$, of $\vec{S}$ from u_i to u_j . We pick nodes $u_{i'_j} \in \theta^{-1}(u_{i_j})$, for $j = 2, \dots, \ell - 1$. Then, by (62), $u_{i'} u_{i'_2} \cdots u_{i'_{\ell-1}} u_{j'}$ is a path of $\vec{S}'$ from $u_{i'}$ to $u_{j'}$. We now assume that $u_i = u_j$. Since $\vec{S}$ has at least 2 nodes ($m \geq 2$), there exists a node u_k of $\vec{S}$ such that $u_k \neq u_i$. Let $\vec{P}_1 = u_{i_1} \cdots u_{i_{\ell_1}}$ (resp., $\vec{P}_2 = u_{i_{\ell_1+1}} \cdots u_{i_\ell}$) be a path of $\vec{S}$ from u_i to u_k (resp., from u_k to u_i), where $u_{i_1} = u_{i_\ell} = u_i$ and $u_{i_{\ell_1}} = u_k$. Concatenating $\vec{P}_1$ and $\vec{P}_2$, we obtain a closed walk. Pick nodes $u_{i'_j} \in \theta^{-1}(u_{i_j})$, for $j = 2, \dots, \ell - 1$. Using again (62), we conclude that $u_{i'} u_{i'_2} \cdots u_{i'_{\ell-1}} u_{j'}$ is a walk of $\vec{S}'$ from $u_{i'}$ to $u_{j'}$. $\square$

A.2 Proof of item 2

Let $\vec{S}_1, \dots, \vec{S}_q$ be the SCCs of $\vec{S}$, and $\vec{S}'_1, \dots, \vec{S}'_{q'}$ be the SCCs of $\vec{S}'$. Without loss of generality, we assume that $u_m \in V(\vec{S}_q)$. By Lemma 1, it suffices to show that

$$B_{\vec{S}_1}, \dots, B_{\vec{S}_q} \text{ are connected} \iff B_{\vec{S}'_1}, \dots, B_{\vec{S}'_{q'}} \text{ are connected.} \quad (63)$$

If $\vec{S}_q$ comprises the single node u_m without self-loop, then $\vec{S}'$ has $(q + 1)$ SCCs $\vec{S}'_1, \dots, \vec{S}'_{q+1}$, where $\vec{S}'_p := \vec{S}_p$, for $p = 1, \dots, q$, and $\vec{S}'_{q+1}$ comprises the single node u_{m+1} without self-loop. It follows that the bipartite graph $B_{\vec{S}_q}$ has two nodes u'_m and u''_m , without the edge (u'_m, u''_m) , so $B_{\vec{S}_q}$ is disconnected. The same applies to $B_{\vec{S}'_{q+1}}$. Then, by Lemma 1, $\text{co-rank}(Z) \geq 1$ and $\text{co-rank}(Z') \geq 2$.

To prove item 2, we must assume either $\text{co-rank}(Z) = 0$ or $\text{co-rank}(Z') = 0$ and establish the other. By the above arguments, either $\vec{S}_q$ has at least two nodes, or, $\vec{S}_q$ comprises the single node u_m with self-loop. It follows that $\vec{S}'$ has q SCCs $\vec{S}'_1, \dots, \vec{S}'_q$, where $\vec{S}'_p := \vec{S}_p$ for all $p = 1, \dots, q - 1$, and $\vec{S}'_q$ is a one-step refinement of $\vec{S}_q$ on u_m . Thus, to prove 63, it now suffices to establish the following result:

Lemma 14. *Let $\vec{S}$ be an arbitrary digraph on m nodes, with $m \geq 1$, possibly with self-loops. Let $\vec{S}'$ be obtained from $\vec{S}$ by performing the one-step refinement on node u_m as described in (60). Then, $B_{\vec{S}}$ is connected if and only if $B_{\vec{S}'}$ is.*

Proof. The arguments are similar to those for proving item 1 of Proposition 1. With slight abuse of notation, we now let $\theta : B_{\vec{S}'} \rightarrow B_{\vec{S}}$ be the graph homomorphism defined as

$$\theta(u'_i) := \begin{cases} u'_i & \text{if } 1 \leq i \leq m \\ u''_m & \text{if } i = m + 1, \end{cases} \quad \text{and} \quad \theta(u''_i) := \begin{cases} u''_i & \text{if } 1 \leq i \leq m \\ u''_m & \text{if } i = m + 1. \end{cases}$$

It follows from (61) that

$$(u'_i, u''_j) \in E(B_{\vec{S}'}) \iff (\theta(u'_i), \theta(u''_j)) \in E(B_{\vec{S}}),$$

and hence,

$$u'_{i_1} u''_{j_1} \cdots u'_{i_\ell} u''_{j_\ell} \text{ is a walk of } B_{\vec{S}'} \iff \theta(u'_{i_1}) \theta(u''_{j_1}) \cdots \theta(u'_{i_\ell}) \theta(u''_{j_\ell}) \text{ is a walk of } B_{\vec{S}}. \quad (64)$$

The lemma is then an immediate consequence of (64). $\square$

A.3 Proof of item 3

We consider two cases: (1) u_m does not have a self-loop and (2) u_m has a self-loop.

A.3.1 Case 1: u_m does not have a self-loop

Let $\vec{C}_1, \dots, \vec{C}_\ell$, for $\ell \leq k$, be the cycles of $\vec{S}$ that contain u_m . Every such cycle $\vec{C}_p$, for $1 \leq p \leq \ell$, induces two different cycles in $\vec{S}'$: one is $\vec{C}_{p,1} := \vec{C}_p$ and the other is obtained by replacing the node u_m in $\vec{C}_p$ with u_{m+1} . We denote it by $C_{p,2}$. The set of cycles of $\vec{S}'$ is thus given by

$$\{\vec{C}_{p,i} \mid 1 \leq p \leq \ell \text{ and } 1 \leq i \leq 2\} \cup \{\vec{C}_q \mid \ell+1 \leq q \leq k\}.$$

Let $z'_{p,i}$ and z'_q be the node-cycle incidence vectors of $\vec{S}'$ corresponding to $\vec{C}_{p,i}$ and $\vec{C}_q$. Then,

$$z'_{p,1} = \hat{z}_p, \quad z'_{p,2} = \hat{z}_p - e_m + e_{m+1}, \quad \text{and} \quad z'_q = \hat{z}_q, \quad (65)$$

where we recall that $\hat{z}_j = (z_j; 0)$.

Proof that $x^ \in X \Rightarrow x'^* \in X'$ ($x^* \in \text{int } X \Rightarrow x'^* \in \text{int } X'$).* We write $x^* = \sum_{j=1}^k c_j z_j$ with $c_j \geq 0$. Since $\vec{C}_1, \dots, \vec{C}_\ell$ are the cycles that contain u_m , we have that

$$\sum_{p=1}^{\ell} c_p = x_m^* = (1 - \sigma_{m-1}). \quad (66)$$

Now, let

$$c'_{p,1} := \frac{\sigma_m - \sigma_{m-1}}{1 - \sigma_{m-1}} c_p, \quad c'_{p,2} := \frac{1 - \sigma_m}{1 - \sigma_{m-1}} c_p, \quad \text{and} \quad c'_q := c_q. \quad (67)$$

Note that $c'_{p,1} + c'_{p,2} = c_p$ for all $1 \leq p \leq \ell$. Then,

$$\begin{aligned} x'^* &= \hat{x}^* + (1 - \sigma_m)(e_{m+1} - e_m) \\ &= \sum_{j=1}^k c_j \hat{z}_j + (1 - \sigma_m)(e_{m+1} - e_m) \\ &= \sum_{p=1}^{\ell} \left[\sum_{i=1}^2 c'_{p,i} \right] \hat{z}_p + \sum_{q=\ell+1}^k c'_q \hat{z}_q + (1 - \sigma_m)(e_{m+1} - e_m) \\ &= \sum_{p=1}^{\ell} \sum_{i=1}^2 c'_{p,i} z'_{p,i} + \sum_{q=\ell+1}^k c'_q z'_q + \left[(1 - \sigma_m) - \sum_{p=1}^{\ell} c'_{p,2} \right] (e_{m+1} - e_m) \\ &= \sum_{p=1}^{\ell} \sum_{i=1}^2 c'_{p,i} z'_{p,i} + \sum_{q=\ell+1}^k c'_q z'_q + (1 - \sigma_m) \left[1 - \frac{1}{1 - \sigma_{m-1}} \sum_{p=1}^{\ell} c_p \right] (e_{m+1} - e_m) \\ &= \sum_{p=1}^{\ell} \sum_{i=1}^2 c'_{p,i} z'_{p,i} + \sum_{q=\ell+1}^k c'_q z'_q, \end{aligned} \quad (68)$$

where the first equality follows from (59), the fourth equality follows from (65), the fifth equality follows from (67), and the last equality follows from (66). By (68), $x'^* \in \mathbf{X}'$. If, further, the coefficients c_j 's are positive (which holds if $x^* \in \text{int } \mathbf{X}$), then by (67) the $c'_{p,i}$'s and the c'_q 's are positive as well and hence, $x'^* \in \text{int } \mathbf{X}'$.

Proof that $x'^ \in \mathbf{X}' \Rightarrow x^* \in \mathbf{X}$ ($x'^* \in \text{int } \mathbf{X}' \Rightarrow x^* \in \text{int } \mathbf{X}$).* We write

$$x'^* = \sum_{p=1}^{\ell} \sum_{i=1}^2 c'_{p,i} z'_{p,i} + \sum_{q=\ell+1}^k c'_q z'_q,$$

where the $c'_{p,i}$'s and the c'_q 's are nonnegative. For $p = 1, \dots, \ell$ and for $q = \ell + 1, \dots, k$, we define

$$c_p := c'_{p,1} + c'_{p,2} \quad \text{and} \quad c_q := c'_q. \quad (69)$$

Let $J \in \mathbb{R}^{m \times (m+1)}$ be defined as follows:

$$J := \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 1 \end{bmatrix}.$$

It follows from (65) and (59) that

$$z_p = J z'_{p,i}, \quad z_q = J z'_q, \quad \text{and} \quad x^* = J x'^*.$$

Thus,

$$\begin{aligned} x^* &= J x'^* \\ &= \sum_{p=1}^{\ell} \sum_{i=1}^2 c'_{p,i} J z'_{p,i} + \sum_{q=\ell+1}^k c'_q J z'_q \\ &= \sum_{p=1}^{\ell} \left[\sum_{i=1}^2 c'_{p,i} \right] z_p + \sum_{q=\ell+1}^k c'_q z_q \\ &= \sum_{j=1}^k c_j z_j, \end{aligned}$$

which shows that $x^* \in \mathbf{X}$. By (69), if the $c'_{p,i}$'s and the c'_q 's are positive, then so are the c_j 's, which implies that if $x'^* \in \text{int } \mathbf{X}'$, then $x^* \in \text{int } \mathbf{X}$. $\square$

A.3.2 Case 2: u_m has a self-loop

We again let $\vec{C}_1, \dots, \vec{C}_\ell$, for $\ell \leq k$, be the cycles of $\vec{S}$ that contain u_m , with $\vec{C}_1 = u_m u_m$ the self-loop. The self-loop $\vec{C}_1$ induces three cycles in $\vec{S}'$:

$$\vec{C}_{1,1} = u_m u_m, \quad \vec{C}_{1,2} = u_{m+1} u_{m+1}, \quad \text{and} \quad \vec{C}_{1,3} = u_m u_{m+1} u_m.$$

As argued at the beginning of Subsection 4.3, each cycle $\vec{C}_p$, for $2 \leq p \leq \ell$, induces four different cycles: $\vec{C}_{p,1} := \vec{C}_p$ and $\vec{C}_{p,2}, \vec{C}_{p,3}, \vec{C}_{p,4}$ are obtained by replacing $u_m \in \vec{C}_p$ with $u_{m+1}, u_m u_{m+1}, u_{m+1} u_m$, respectively.

Let $z'_{1,i}$, for $i = 1, \dots, 3$, be the node-cycle incidence vectors corresponding to $\vec{C}_{1,i}$, which are given by

$$z'_{1,1} = e_m, \quad z'_{1,2} = e_{m+1}, \quad \text{and} \quad z'_{1,3} = e_m + e_{m+1}.$$

Let $z'_{p,i}$, for $2 \leq p \leq \ell$ and $1 \leq i \leq 4$, be the node-cycle incidence vectors corresponding to $\vec{C}_{p,i}$, as given in (33). Note, in particular, that

$$\begin{aligned} z'_{1,3} &= z'_{1,1} + z'_{1,2}, \\ z'_{p,3} &= z'_{p,4} = z'_{p,1} + z'_{1,2} = z'_{p,2} + z'_{1,1}, \quad \text{for } p = 2, \dots, \ell, \end{aligned}$$

which implies that none of the vectors $z'_{1,3}$, $z'_{p,3}$, and $z'_{p,4}$ is an extremal generator of $\mathbf{X}'$, and can thus be suppressed in the nonnegative (positive) combination of x'^* . The same arguments used in the above case can be used to establish the current case. $\square$

B On graphons with symmetric support

Let W be a step-graphon with symmetric support, i.e., $W(s, t) \neq 0$ if and only if $W(t, s) \neq 0$. Let σ be a partition for W and $\vec{S}$ be the associated skeleton graph. Note that $\vec{S}$ is symmetric. Let S be the undirected graph obtained from $\vec{S}$ by ignoring the orientations of the self-loops and by replacing every pair of oppositely oriented edges $\{u_i u_j, u_j u_i\}$, for $u_i \neq u_j$, with the corresponding undirected edge (u_i, u_j) .

Definition 15. Let $f_1, \dots, f_\ell$ be the edges of S . To each f_j , we associate the **node-edge incidence vector** $z'_j := \sum_{u_i \in f_j} e_i$. The **node-edge incidence matrix** of S is given by

$$Z' := \begin{bmatrix} z'_1 & \cdots & z'_\ell \end{bmatrix}.$$

We further let $\mathbf{X}'$ be the convex cone spanned by $z'_1, \dots, z'_\ell$. We establish the following result:

Lemma 15. *It holds that $\mathbf{X}' = \mathbf{X}$.*

Proof. Note that each edge f_j of S corresponds to a cycle of $\vec{S}$; indeed, a self-loop (u_i, u_i) corresponds to $u_i u_i$ and an edge (u_i, u_j) between two distinct nodes corresponds to the 2-cycle $u_i u_j u_i$. Relabel the cycles of $\vec{S}$ such that the first ℓ cycles $\vec{C}_j$, for $j = 1, \dots, \ell$, correspond to the edges f_j of S . It is clear from the definition that the node-cycle incidence vector z_j of $\vec{S}$ coincides with the node-edge incidence vector z'_j of S . Thus, $\mathbf{X}' \subseteq \mathbf{X}$. It now suffices to show that for any cycle $\vec{C}_j$ of $\vec{S}$, with length greater than 2, the associated node-cycle incidence vector z_j can be expressed as a nonnegative combination of the z'_j 's. We write $\vec{C}_j = u_1 u_2 \cdots u_\ell u_1$ for $\ell > 2$. Then, $f_1 := (u_1, u_2), f_2 := (u_2, u_3), \dots, f_\ell := (u_\ell, u_1)$ are edges of S . It follows that $z_j = \frac{1}{2} \sum_{i=1}^{\ell} z'_i$. $\square$

An immediate consequence of Lemma 15 is that $\text{co-rank}(Z) = \text{co-rank}(Z')$. Also note that a symmetric digraph $\vec{S}$ is strongly connected if and only if S is connected. The following result is thus a corollary of the Main Theorem specializing to the class of graphons with symmetric support (which include the class of symmetric graphons).

Corollary 8. *Let W be a step-graphon with symmetric support. Let σ be a partition for W , and let x^* and S be the associated concentration vector and the undirected skeleton graph. Further, let Z' be the node-edge incidence matrix of S and $\mathbf{X}'$ be the convex cone spanned by the columns of Z' . Then, the following items hold:*

1. *If $\text{co-rank}(Z') > 0$ or if $x^* \notin \mathbf{X}'$, then*

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian decomposition}) = 0.$$

2. *If $\text{co-rank}(Z') = 0$ and $x^* \in \text{int } \mathbf{X}'$, and if S is not connected, then*

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian decomposition}) = 1,$$

and

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian cycle}) = 0.$$

3. *If $\text{co-rank}(Z') = 0$, $x^* \in \text{int } \mathbf{X}'$, and S is connected, then*

$$\lim_{n \rightarrow \infty} \mathbf{P}(\vec{G}_n \sim W \text{ has a Hamiltonian cycle}) = 1.$$

At the end of the section, we relate the above corollary to our earlier work [3, 4], which deal with the H -property for symmetric graphons. As mentioned in Section 1, the sampling procedure considered in [3, 4] is slightly different from the one considered in this paper. Specifically, in [3, 4], we first sample an undirected graph $G_n \sim W$, with W a symmetric graphon, and then obtain the symmetric digraph $\vec{G}_n^s$ from G_n by replacing each undirected edge with a pair of oppositely oriented edges, i.e., we follow steps $S1$, $S'2$, and $S'3$ (see Subsection 6.1), where W^s in step $S'2$ is replaced with W . We denote by $\vec{G}_n^s \sim W$ the symmetric digraph obtained in this way. For ease of presentation and to avoid any confusion, we say that the symmetric step-graphon W has the (strong) H^s -property if $\vec{G}_n^s \sim W$ has a Hamiltonian decomposition (cycle) *a.a.s.* We establish the following result:

Lemma 16. *A symmetric step-graphon W has the (strong) H^s -property if and only if it has the (strong) H -property.*

Proof. Let $\overline{W}$ be the saturation of W , i.e.,

$$\overline{W}(s, t) := \begin{cases} 1 & \text{if } W(s, t) \neq 0, \\ 0 & \text{if } W(s, t) = 0. \end{cases}$$

It is clear that $\overline{W}$ and W share the same support. By Corollary 8, $\overline{W}$ has the (strong) H -property if and only if W does. Similarly, it has been shown in [3, 4] that $\overline{W}$ has

the (strong) H^s -property if and only if W does. It thus remains to show that $\overline{W}$ has the (strong) H -property if and only if it has the (strong) H^s -property. But this follows from the fact that the two sampling procedures, $\vec{G}_n \sim \overline{W}$ and $\vec{G}_n^s \sim \overline{W}$, are equivalent with each other. To wit, since $\overline{W}$ takes value 1 over its support, the two digraphs are completely determined by their respective empirical concentration vectors. Furthermore, it follows from the two sampling procedures that if $x(\vec{G}_n) = x(\vec{G}_n^s)$, then $\vec{G}_n = \vec{G}_n^s$. We conclude the proof by pointing out that $x(\vec{G}_n)$ and $x(\vec{G}_n^s)$ are identically distributed. $\square$

C Proof of Proposition 7

The proof relies on the use of the Chernoff bound for Binomial random variable, which we recall below:

Lemma 17. *Suppose that $X \sim \text{Bin}(N, p)$; then, for any $r \in [0, 1]$,*

$$\mathbf{P}(X \leq (1 - r)Np) \leq \exp\left(-\frac{r^2}{2}Np\right).$$

Now, let $G_n \sim W^s$. Recall that $x(G_n) = y(G_n)/n$ is the empirical concentration vector, which converges to x^* a.a.s.. It follows that a.a.s.

$$x_i(G_n) > \frac{1}{2} \min\{x_i^* \mid i = 1, \dots, m\} =: \alpha, \quad \text{for all } i = 1, \dots, m. \quad (70)$$

For the remainder of the section, we assume that (70) holds. Let $\mathcal{E}_n(u_i, u_j)$ be the event that the pair $(\pi^{-1}(u_i), \pi^{-1}(u_j))$ is (ϵ, δ) -super-regular. We have the following result:

Lemma 18. *For any $(u_i, u_j) \in E(S)$, the event $\mathcal{E}_n(u_i, u_j)$ holds a.a.s..*

Proof. For convenience, let $A := \pi^{-1}(u_i)$ and $B := \pi^{-1}(u_j)$. We show below that (56) and (57) hold a.a.s..

Proof that (56) holds a.a.s.. For any given $X \subseteq A$ and $Y \subseteq B$, $e(X, Y)$ is a binomial $(|X||Y|, q_{ij})$ random variable. If

$$|X| > \epsilon|A| \quad \text{and} \quad |Y| > \epsilon|B|, \quad (71)$$

then

$$\begin{aligned} \mathbf{P}(e(X, Y) \leq \delta|X||Y|) &= \mathbf{P}\left(e(X, Y) \leq \left(1 - \frac{q_{ij} - \delta}{q_{ij}}\right) q_{ij}|X||Y|\right) \\ &\leq \exp\left(-\frac{(q_{ij} - \delta)^2|X||Y|}{2q_{ij}}\right) \\ &\leq \exp\left(-\frac{(q_{ij} - \delta)^2\epsilon^2\alpha^2}{2q_{ij}}n^2\right) \\ &\leq \exp\left(-\frac{q_{ij}\epsilon^2\alpha^2}{8}n^2\right), \end{aligned}$$

where the first inequality follows from Lemma 17, the second inequality follows from (70) and (71), and the last inequality follows from (58) and, hence, $\delta \leq q_{ij}/2$. The number of

pairs that satisfy (71) is bounded above by the total number of pairs $(X, Y) \in 2^A \times 2^B$, which is $2^{|A|+|B|} \leq 2^n$. It follows that

$$\mathbf{P}(\text{event (56) does not hold}) \leq 2^n \exp\left(-\frac{q_{ij}\epsilon^2\alpha^2}{8}n^2\right) \xrightarrow{n \rightarrow \infty} 0.$$

Proof that (57) holds a.a.s.. For any $a \in A$, $e(a, B)$ is a binomial $(|B|, q_{ij})$ random variable. Using the same arguments as above, we obtain that

$$\begin{aligned} \mathbf{P}(e(a, B) \leq \delta|B|) &= \mathbf{P}\left(e(a, B) \leq \left(1 - \frac{q_{ij} - \delta}{q_{ij}}\right)q_{ij}|B|\right) \\ &\leq \exp\left(-\frac{(q_{ij} - \delta)^2|B|}{2q_{ij}}\right) \leq \exp\left(-\frac{q_{ij}\alpha}{8}n\right). \end{aligned}$$

Similarly,

$$\mathbf{P}(e(b, A) \leq \delta|A|) \leq \exp\left(-\frac{q_{ij}\alpha}{8}n\right).$$

We conclude that

$$\mathbf{P}(\text{event (57) does not hold}) \leq (|A| + |B|) \exp\left(-\frac{q_{ij}\alpha}{8}n\right) \leq n \exp\left(-\frac{q_{ij}\alpha}{8}n\right) \xrightarrow{n \rightarrow \infty} 0.$$

This completes the proof. □

Proposition 7 is then an immediate consequence of Lemma 18; indeed,

$$\mathbf{P}(G_n \sim W^s \text{ is } \epsilon\text{-}\delta\text{-super-regular}) \geq 1 - \sum_{(u_i, u_j) \in E(S)} \mathbf{P}(\neg \mathcal{E}_n(u_i, u_j)) \xrightarrow{n \rightarrow \infty} 1.$$

This completes the proof. □