

Two conjectures on vertex-disjoint rainbow triangles

Xu Liu*, Bo Ning[†], Yuting Tian[‡]

Abstract

In 1963, Dirac proved that every n -vertex graph has k vertex-disjoint triangles if $n \geq 3k$ and minimum degree $\delta(G) \geq \frac{n+k}{2}$. The base case $n = 3k$ can be reduced to the Corrádi-Hajnal Theorem. Towards a rainbow version of Dirac's Theorem, Hu, Li, and Yang conjectured that for all positive integers n and k with $n \geq 3k$, every edge-colored graph G of order n with $\delta^c(G) \geq \frac{n+k}{2}$ contains k vertex-disjoint rainbow triangles. In another direction, Wu et al. conjectured an exact formula for anti-Ramsey number $ar(n, kC_3)$, generalizing the earlier work of Erdős, Sós and Simonovits. The conjecture of Hu, Li, and Yang was confirmed for the cases $k = 1$ and $k = 2$. However, Lo and Williams disproved the conjecture when $n \leq \frac{17k}{5}$. It is therefore natural to ask whether the conjecture holds for $n = \Omega(k)$.

In this paper, we confirm this by showing that the Hu-Li-Yang conjecture holds when $n \geq 42.5k + 48$. We disprove the conjecture of Wu et al. and propose a modified conjecture. This conjecture is motivated by previous works due to Allen, Böttcher, Hladký, and Piguet on Turán number of vertex-disjoint triangles.

1. Introduction

An edge-coloring of a graph G is a mapping $f : E(G) \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. An edge-colored subgraph H of G is called rainbow if the edges of H are assigned pairwise different colors. Given a positive integer n and a graph H , the *anti-Ramsey number* $ar(n, H)$ is the maximum number of colors in an edge-coloring of K_n such that it contains no rainbow H as a subgraph. For a vertex $v \in V(G)$, the *color degree* of v , denoted by $d^c(v)$, is defined to be the number of different colors which are assigned to all edges incident to v . The *minimum color degree*, denoted by $\delta^c(G)$, is defined to be $\min\{d^c(v) : v \in V(G)\}$. The *color number* of G , denoted by $c(G)$, is defined to be the size of $\{c(e) : e \in E(G)\}$. The *Turán number* $ex(n, H)$ is the maximum number of edges of a graph on n vertices that contains no subgraph isomorphic to H .

*College of Computer Science, Nankai University, Tianjin 300350, P.R. China.

[†]Corresponding author. College of Computer Science, Nankai University, Tianjin 300350, P.R. China. E-mail: bo.ning@nankai.edu.cn (B. Ning). Partially supported by the National Nature Science Foundation of China (No. 12371350) and Fundamental Research Funds for the Central Universities, Nankai University (No. 63243151).

[‡]College of Cryptology and Cyber Science, Nankai University, Tianjin 300350, P.R. China.

The main purpose of this paper is to study the following two conjectures on vertex-disjoint rainbow triangles. In fact, we disprove the first conjecture and confirm the second conjecture when $n = \Omega(k)$.

Conjecture 1 ([19]).

$$ar(n, kC_3) = \max \left\{ \binom{3k-1}{2} + n - 3k + 1, \left\lfloor \frac{(n-k+2)^2}{4} \right\rfloor + (k-2)(n-k+2) + \binom{k-2}{2} + 1 \right\}$$

for all $n \geq 3k$.

Conjecture 2 ([10]). *For all positive integers n and m with $n \geq 3k$, every edge-colored graph G of order n with $\delta^c(G) \geq (n+k)/2$ contains m vertex-disjoint rainbow triangles.*

The first conjecture is about the anti-Ramsey number of vertex-disjoint triangles. The concept of anti-Ramsey number of a graph H was introduced by Erdős, Sós, and Simonovits [8]. Although it is named anti-Ramsey numbers, it indeed is closely related Turán problems than Ramsey Theory. For example, for general graphs H , we have $\min\{ex(n, H-e) : e \in E(H)\} + 1 \leq ar(n, H) \leq ex(n, H)$. Erdős et al. [8] once conjectured that $ar(n, C_k) = (\frac{k-2}{2} + \frac{1}{k-1})n + O(1)$ and proved that $ar(n, C_3) = n - 1$. After several efforts (see [2, 11]), this problem was finally solved in [18]. Since the paper [8], there have been many works that study anti-Ramsey numbers of graphs. We refer the reader to the survey [9] and the results therein.

We define kC_3 as the vertex-disjoint union of k triangles. Yuan and Zhang [20] determined the exact values of $ar(n, kC_3)$ when n is sufficiently large using Simonovits' method. Wu et al. [19] improved the result in [20] by proving $ar(n, kC_3) = \lfloor \frac{(n-k+1)^2}{4} \rfloor + (k-2)(n-k+2) + \binom{k-2}{2} + 1$ for all $n \geq 2k^2 - k + 2$. Furthermore, for the base case, they [19] proved $ar(3k, kC_3) = \binom{3k-1}{2} + 1$, and $ar(n, kC_3) \leq \frac{(n-k+2)^2}{4} + (k-2)(n-k+2) + \binom{k-2}{2} + (k-1)^2 - \frac{n-3k}{2} + 1$ for all $3k \leq n \leq 2k^2 - k + 2$. Based on these results, they proposed Conjecture 1. Only when we are writing this manuscript, we find that Lu, Luo, and Ma [17] have very recently proved that for any two integers $n, k \geq 2$, and $n \geq 15k + 27$, we have $ar(n, kC_3) = \lfloor \frac{(n-k+1)^2}{4} \rfloor + (k-2)(n-k+2) + \binom{k-2}{2} + 1$. However, as shown in Section 2, we shall show that Conjecture 1 is false. Very interestingly for us, the construction is motivated by the work of Allen, Böttcher, Hladký, and Piguet [1] on Turán number of vertex-disjoint triangles.

Our first contribution to this paper is as follows.

Theorem 1. *Conjecture 1 is false.*

Remark 1. For all integers $n \geq 3k$ where $0.24n \lesssim k \lesssim 0.3n$, there exist two edge-colorings of K_n : $G_2(n, k)$ and $G_3(n, k)$ (for details, see the next section and Fig. 1), which contain no rainbow kC_3 , we can see that $c(G_2(n, k))$ and $c(G_3(n, k))$ are both larger than the conjectured values of $ar(n, kC_3)$, thus disproving Conjecture 1.

The second conjecture has many motivations. A well-known fact is that a minimum degree of at least $\frac{n+1}{2}$ ensures that every graph on n vertices contains a triangle. It is natural to ask an edge-colored version of this observation. In 2007 (and formally in 2012), Li and Wang [14] conjectured that every edge-colored graph has a rainbow triangle if $\delta^c(G) \geq \frac{n+1}{2}$. This conjecture was confirmed by H. Li [12] and independently in [13].

Theorem 2 ([12]). *Let G be an edge-colored graph on $n \geq 3$ vertices. If $\delta^c(G) \geq \frac{n+1}{2}$ then G contains a rainbow triangle.*

Theorem 3 ([13]). *Let G be an edge-colored graph on $n \geq 5$ vertices. If $\delta^c(G) \geq \frac{n}{2}$, then G contains a rainbow triangle, unless G is a properly colored $K_{\frac{n}{2}, \frac{n}{2}}$ where n is even.*

There are several extensions; for example, Czygrinow, Molla, Nagle, and Oursler [7] proved that H. Li's condition in Theorem 2 ensures a rainbow ℓ -cycle C_ℓ whenever $n > 432\ell$, which is sharp for a fixed odd integer $\ell \geq 3$ when n is sufficiently large. For more related results, see [3, 15, 4].

In 2020, Hu, H. Li, and Yang [10] proved that every edge-colored graph on $n \geq 20$ vertices has two vertex-disjoint rainbow triangles. This result was slightly improved to $n \geq 6$ in [5]. So, Conjecture 2 is true for $k = 1, 2$. Dirac [7] proved that every graph G on $n \geq 3k$ vertices has k vertex-disjoint triangles if $\delta(G) \geq \frac{n+k}{2}$. The base case $n = 3k$ is equivalent to the famous Corrádi-Hajnal Theorem [6] which states that every n -vertex graph with $n = 3k$ and minimum degree $\delta(G) \geq 2k$ has a triangle-factor. So, if it were true, Conjecture 2 can be seen as a rainbow version of Dirac's theorem. However, Lo and Williams [16] give a construction showing that this conjecture is false when $n \leq \frac{17k}{5}$. Thus, it is natural to ask whether Conjecture 2 holds for $n = \Omega(k)$. Our second result confirms this.

Theorem 4. *Conjecture 2 is true for $n \geq 42.5k + 48$.*

Remark 2. When we study $ar(n, kC_3)$, we need kC_3 to be rainbow, that is, the edges of all these vertex-disjoint triangles are assigned pairwise different colors. However, when we consider Conjecture 2, “vertex-disjoint rainbow triangles” means that each vertex-disjoint triangle is rainbow, but two different rainbow triangles can have the same color.

2. The disproof of Conjecture 1

We first introduce four classes of extremal graphs.

Definition 2.1. Let n and k be non-negative integers with $n \geq 3k$. We define four edge-colorings of K_n as follows.

- $G_1(n, k)$: Let $V(G_1(n, k)) = X \cup Y_1 \cup Y_2$, where $|X| = k - 2$, $|Y_1| = \lfloor \frac{n-k+2}{2} \rfloor$, and $|Y_2| = \lceil \frac{n-k+2}{2} \rceil$. We color all edges in $K_{|X|} \vee K_{|Y_1|, |Y_2|}$ with distinct colors, and color $E(K_n) \setminus E(K_{|X|} \vee K_{|Y_1|, |Y_2|})$ with another new color, depicted in Figure 1, so

$$c(G_1(n, k)) = \binom{k-2}{2} + (k-2)(n-k+2) + \left\lfloor \frac{(n-k+2)^2}{4} \right\rfloor + 1.$$

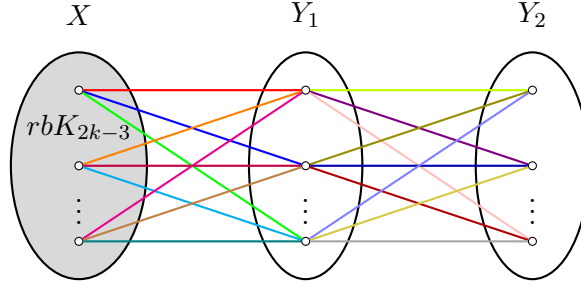


Figure 2: $G_2(n, k)$

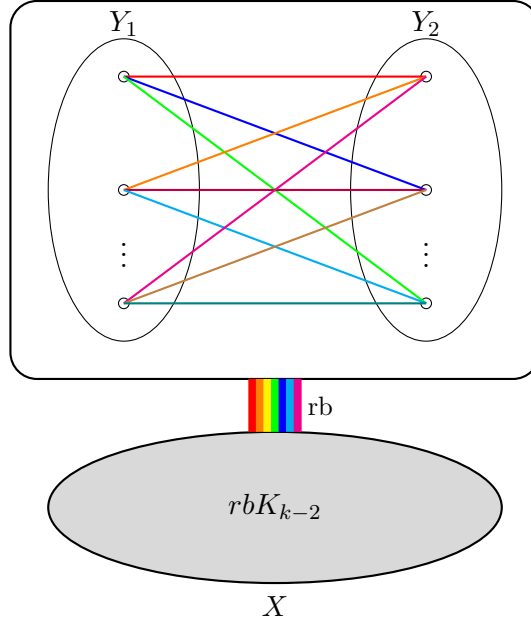


Figure 1: $G_1(n, k)$

- $G_2(n, k)$: The second-class of extremal graphs is defined only for $k < \frac{n+7}{4}$. Let $V_2(G_2(n, k)) = X \cup Y_1 \cup Y_2$ with $|X| = 2k - 3$, $|Y_1| = \lfloor \frac{n}{2} \rfloor$, and $Y_2 = \lceil \frac{n}{2} - 2k + 3 \rceil$ (or $|Y_1| = \lceil \frac{n}{2} \rceil$, and $Y_2 = \lfloor \frac{n}{2} - 2k + 3 \rfloor$). We color all edges in $|Y_1|K_1 \vee (K_{|X|} \cup |Y_2|K_1)$ with distinct colors, and color $E(K_n) \setminus E(|Y_1|K_1 \vee (K_{|X|} \cup |Y_2|K_1))$ with another new color, depicted in Figure 2, so

$$c(G_2(n, k)) = \binom{2k-3}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

- $G_3(n, k)$: Let $V(G_3(n, k)) = X \cup Y$ with $|X| = 2k - 3$ and $|Y| = n - 2k + 3$. We color all edges in $|Y|K_1 \vee K_{|X|}$ with distinct colors, and color $E(K_n) \setminus E(|Y|K_1 \vee K_{|X|})$ with

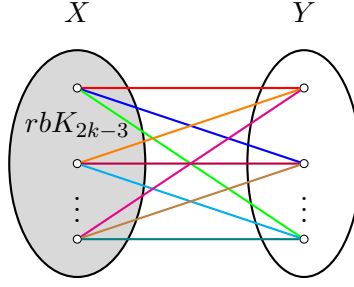


Figure 3: $G_3(n, k)$

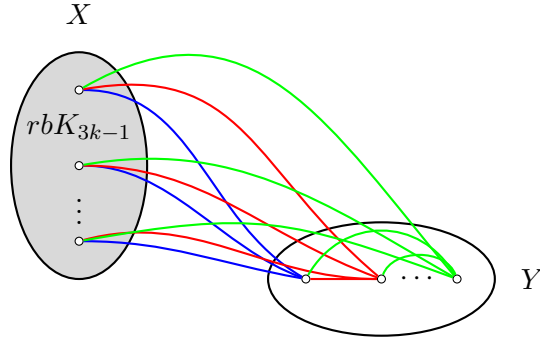


Figure 4: $G_4(n, k)$

another new color, depicted in Figure 3, so

$$c(G_3(n, k)) = \binom{2k-3}{2} + (n-2k+3)(2k-3) + 1.$$

- $G_4(n, k)$: Let $V(G_4(n, k)) = X \cup Y$ with $|X| = 3k-1$ and $|Y| = n-3k+1$. We color all edges in $K_{|X|}$ with distinct colors. Only one color is added for each vertex in Y added, depicted in Figure 4, so

$$c(G_4(n, k)) = \binom{3k-1}{2} + n-3k+1.$$

Proof of Theorem 1. For three sets A, B, C , a triangle uvw is said to be of type ABC if $u \in A$, $v \in B$, and $w \in C$.

For $G_1(n, k)$, any triangles with all edges in $E(K_{|X|} \vee K_{|Y_1|, |Y_2|})$ are of type XXX , XXY_1 , XXY_2 , or XY_1Y_2 , so every triangle has at least one vertex in X . There are rainbow $(k-2)C_3$ in $(V(K_n), E(K_{|X|} \vee K_{|Y_1|, |Y_2|}))$ because each edge is colored distinctly. In $(V(K_n), E(K_n) \setminus E(K_{|X|} \vee K_{|Y_1|, |Y_2|}))$, one rainbow C_3 can be added to rainbow $(k-2)C_3$ to form rainbow $(k-1)C_3$ in K_n . This is because all edges in $E(K_n) \setminus E(K_{|X|} \vee K_{|Y_1|, |Y_2|})$ are colored with the left one new color.

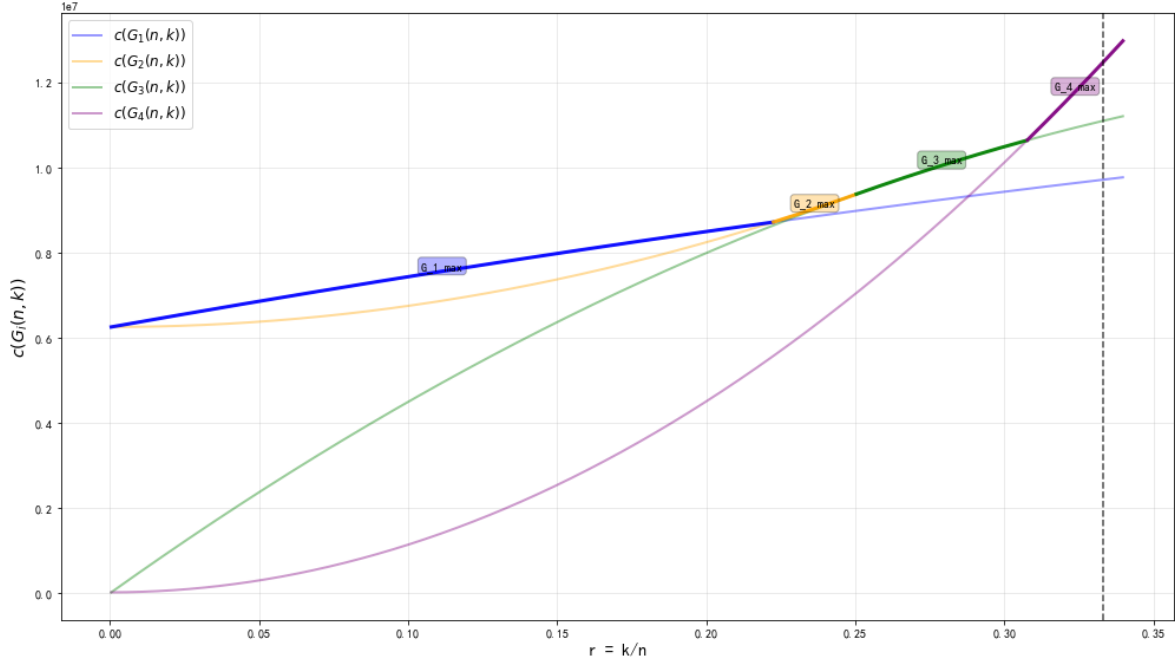


Figure 5: $c(G)$ of the graphs $G_i(n, k)$ where n ranges $3k$ to infinity.

For $G_2(n, k)$, all triangles with all edges in $E(|Y_1|K_1 \vee (K_{|X|} \cup |Y_2|K_1))$ are of type XXX , or XXY_1 . Thus, every triangle intersects X at least twice, and $(V(K_n), E(|Y_1|K_1 \vee (K_{|X|} \cup |Y_2|K_1)))$ contains rainbow $(k-2)C_3$ because each edge is colored distinctly. In $(V(K_n), E(K_n) \setminus E(|Y_1|K_1 \vee (K_{|X|} \cup |Y_2|K_1)))$, one rainbow C_3 can be added to rainbow $(k-2)C_3$'s to form rainbow $(k-1)C_3$ in K_n for the same reason.

Similarly, $G_3(n, k)$ has rainbow $(k-1)C_3$ in it. The reason why $G_4(n, k)$ has rainbow $(k-1)C_3$ but no rainbow kC_3 has already been given in Section 3 of [19].

When n range from $3k$ to $\frac{-13k^2+25k-8}{8-4k} \approx 3.25k$, $c(G_4(n, k))$ is the most; from about $\frac{-13k^2+25k-8}{8-4k} \approx 3.25k$ to about $4k-6$, $c(G_3(n, k))$ has the maximum value. Then $c(G_3(n, k))$ is larger than $c(G_2(n, k))$ and attains the maximum value until $n = \frac{9k^2-6k}{2k-4} \approx 4.5k$. Finally, $c(G_1(n, k))$ attains the maximum value from $\frac{9k^2-6k}{2k-4} \approx 4.5k$ to infinity. The values of the four functions $c(G_i(n, k))$ are potted in Figure 5, and the thresholds are listed in Table 1.

We can see that $c(G_2(n, k))$ and $c(G_3(n, k))$ are greater than the values in Conjecture 1 when n is approximately within the range of about $\frac{-13k^2+25k-8}{8-4k}$ to about $\frac{9k^2-6k}{2k-4}$, thus completing the proof of Theorem 1. $\square$

Motivated by the extremal graphs for vertex-disjoint triangles and Theorem 6 in [1], we dare to pose the following conjecture.

Conjecture 3. *There exists n_0 such that for each $n > n_0$ and each k , $n \geq 3k$, we have $ar(n, kC_3) = \max_{j \in [4]} c(G_j(n, k))$.*

Table 1: Transitions between $G_i(n, k)$.

graph	the range of n
$G_4(n, k)$	$3k \leq n \leq \frac{-13k^2 + 25k - 8}{8 - 4k} \approx 3.25k$
$G_3(n, k)$	$\frac{-13k^2 + 25k - 8}{8 - 4k} \approx 3.25k \leq n \leq 4k - 6$
$G_2(n, k)$	$4k - 6 \leq n \leq \frac{9k^2 - 6k}{2k - 4} \approx 4.5k$
$G_1(n, k)$	$n \geq \frac{9k^2 - 6k}{2k - 4} \approx 4.5k$

3. Proof of Theorem 5

Theorem 5. *Let n, k be two positive integers. Let G be an edge-colored graph of order n , where $n \geq 42.5k + 48$. If $\delta^c(G) \geq (n+k)/2$ with $\delta^c(G) \geq (n+k)/2$, then G contains k vertex-disjoint rainbow triangles.*

Lemma 1 ([15]). *Let G be an edge-colored graph on n vertices and $e(G)$ edges, $rt(G)$ the number of rainbow C_3 of G . Suppose that $\delta^c(G) \geq \frac{n+1}{2}$ and $e(G)$ is minimal subject to $\delta^c(G)$, then*

$$rt(G) \geq \frac{e(G)(2\delta^c(G) - n)}{3} \geq \frac{\delta^c(G)(2\delta^c(G) - n)n}{6}.$$

Lemma 2. *Let G be an edge-colored graph on n vertices and $e(G)$ edges, $rt(G)$ be the number of rainbow C_3 of G . Suppose that $\delta^c(G) \geq \frac{n+k}{2}$. Then*

$$rt(G) \geq \frac{kn(n+k)}{12}.$$

Proof. We remove edges of G as much as possible so that the resulting spanning subgraph G' satisfies $\delta^c(G') \geq \frac{n+k}{2}$. By Lemma 1, $rt(G') \geq \frac{\delta^c(G')(2\delta^c(G') - n)n}{6} \geq \frac{kn(n+k)}{12}$. Observe that $rt(G) \geq rt(G')$. This proves the lemma. $\square$

Proof of Theorem 5. We prove the theorem by induction on k . When $k = 1$ and $k = 2$, it is reduced to Theorem 2 and Hu-Li-Yang's theorem, respectively. Now assume $k \geq 3$ and suppose that the theorem holds for the case $k - 1$. We assume that G contains $k - 1$ vertex-disjoint rainbow triangles, but no k vertex-disjoint rainbow triangles. We denote by V_0 the vertex set of such $k - 1$ vertex-disjoint rainbow triangles.

Now we define some notation used below. Set $V_0 = \bigcup_{i=1}^{k-1} T_i$, where $T_i = \{u_{i,1}, u_{i,2}, u_{i,3}\}$ is the vertex set of the i -th rainbow triangle. Let $V_1 = V(G) \setminus V_0$. For $v \in V(G)$, $e \in E(G)$, let $rt(v)$ (resp. $rt(e)$) denote the number of rainbow triangles that contain v (resp. e). Similarly, for $v \in V_0$, $e \in E(G[V_0])$, let $rt_1(v)$ be the number of rainbow C_3 's which contain v and two vertices in V_1 , and $rt_2(e)$ the number of rainbow C_3 's which contain e and one vertex in V_1 .

For $s \in \mathbb{N}^+$ and $v \in V(G)$, let $RF_s(v)$ be an edge-colored friendship graph that consists of s triangles with exactly one common vertex v , in which each triangle is rainbow.

Claim 1. Let $v \in V_0$ and $s \in \mathbb{N}^+$.

1. If $rt(v) \geq (s-1)(n-1) + 1$, then G contains an $RF_s(v)$.
2. If $rt_1(v) \geq (s-1)(n-3k+2)$, then G contains an $RF_s(v)$ with $RF_s(v) \cap V_0 = \{v\}$.

Proof of Claim 1. Denote by $E_v = \{uw : u, w \in N_G(v), uvw \text{ is a rainbow triangle}\}$. So, for each $e \in E_v$, there is a rainbow triangle containing both v and e . If $rt(v) \geq (s-1)(n-1) + 1$, then $|E_v| \geq (s-1)(n-1) + 1$. Consider the subgraph $G[E_v]$. We have $v(G[E_v]) \leq d_G(v) \leq n-1$. Recall that the Erdős-Gallai Theorem on paths states that any graph on n vertices and m edges contains a path of length at least $\frac{2m}{n}$. Thus, there is a path of length at least $\frac{2((s-1)(n-1)+1)}{n-1} > 2s-2$, and therefore of length at least $2s-1$, in $G[E_v]$. Since any path with t edges contains a matching of size $\lceil t/2 \rceil$, the graph $G[E_v]$ admits a matching of size at least s . This matching together with v yields $RF_s(v)$.

Similarly, if $rt_1(v) \geq (s-1)(n-3k+2)$, then $G_v^1 = (V_1, E_v)$ contains a matching of size s . These s matching edges together with v produce $RF_s(v)$ intersecting V_0 only in v . $\square$

By induction, G has the following property.

Claim 2. (friendship subgraph) For any integer $s \geq 3k-2$ and each $v \in V(G)$, G is $RF_s(v)$ -free. Moreover, we have $rt(v) \leq (3k-3)(n-1)$ and $rt_1(v) \leq (3k-3)(n-3k+2)$.

Proof of Claim 2. Assume that there exists a vertex $v \in V(G)$ and an integer $s \geq 3k-2$ such that $RF_s(v) \subseteq G$. Let $G_v = G - v$. Observe that removing the vertex v decreases the color degree of any other vertex by at most 1, hence $\delta^c(G_v) \geq \delta^c(G) - 1 \geq \frac{n+k}{2} - 1 = \frac{(n-1)+(k-1)}{2}$. By the induction hypothesis, G_v contains $k-1$ vertex-disjoint rainbow triangles. Denote them by T_v^j ($j = 1, \dots, k-1$), which contains $3k-3$ vertices. Since $RF_s(v)$ consists of s rainbow triangles containing v and $s \geq 3k-2$, at least one of these rainbow triangles must be vertex-disjoint from each T_v^j ($j \in [k-1]$). So, there are k vertex-disjoint rainbow triangles in G , a contradiction. The other conclusion follows from Claim 1. $\square$

It is easy to see that each rainbow triangle must contain at least one vertex of V_0 . Since $\binom{3k-3}{3} < \frac{kn(n+k)}{12}$ when $n^2 \geq 54k^2$, there exists a rainbow C_3 that contains at least one vertex of V_1 .

For $t \in [k-1]$ and $\{u_1, \dots, u_{k-1}\} \subseteq V_0$, similar to the definition of $RF_s(v)$, we denote by $RF_{2,1}(t; u_1, \dots, u_{k-1})$ a graph which is the vertex-disjoint union of $RF_2(u_i)$ (the so called hourglass) ($i \in [t]$) and $RF_1(u_j)$ (the rainbow triangle) ($j \in [k-1] \setminus [t]$), as illustrated in Figure 6. Let t be the maximum integer in $[k-1]$ such that G contains a $RF_{2,1}(t; u_1, \dots, u_{k-1})$. Then, we relabel all vertices such that $V_0 = \bigcup_{i=1}^{k-1} T_i = \bigcup_{i=1}^{k-1} \{v_{i,1}, v_{i,2}, v_{i,3}\}$, where $T_1, \dots, T_{k-1}$ are $k-1$ vertex-disjoint rainbow C_3 , $v_{i,1}$ is the common vertex of two triangles of T_i , and $v_{i,4}, v_{i,5}$ are the remaining two vertices in $V(RF_2(v_i)) - V(T_i)$ ($i \in [t]$).

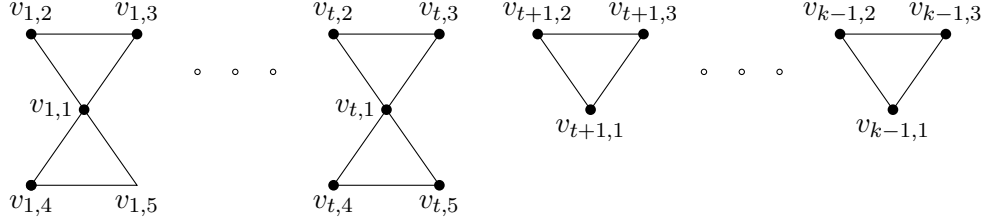


Figure 6: $RF_{2,1}(t; v_{1,1}, \dots, v_{k-1,1})$

Our main aim is to count $rt(G)$. The main idea is to show that $rt(G) < \frac{kn(n+k)}{12}$, which will contradict Lemma 2. To do so, we divide all rainbow triangles into three classes: the first class, denoted by RT_1 , consists of those containing exactly two vertices from V_1 ; the second class, denoted by RT_2 , consists of those containing at most one vertex of V_1 , hence each must contain an edge of $G[V_0]$, and the third class, denoted by RT_3 , consists of those entirely contained in V_0 .

We use

$$E_1 = \bigcup_{i \in [t], j \in [k-1] \setminus \{i\}} E[v_{i,1}, V(T_j)]$$

to denote the subset of edges in $G[V_0]$, in which one end-point of each edge is a center vertex from the first k triangles and the other end-point is from some triangle other than the one contains the center vertex.

We first obtain an upper bound of $|RT_1|$.

Claim 3. For any $i \in [k-1]$, let $u_1, u_2 \in V(T_i)$. If $rt_1(u_1) > n - 3k + 2$ then $rt_1(u_2) \leq 4$. Moreover, if $1 \leq rt_1(u_1) \leq n - 3k + 2$ then $rt_1(u_2) \leq 2(n - 3k + 2)$.

Proof of Claim 3. As $rt_1(u_1) > n - 3k + 2$, by Claim 1, G contains a $RF_2(u_1)$ with the unique vertex u_1 in V_0 . Suppose $V(RF_2(u_1)) = \{u_1, u_1^1, u_1^2, u_1^3, u_1^4\}$, where both $G[\{u_1, u_1^1, u_1^2\}]$ and $G[\{u_1, u_1^3, u_1^4\}]$ are rainbow C_3 's. Then, for each rainbow C_3 , named T_{u_2} , which contains u_2 and two vertices in V_1 , we infer $V(T_{u_2}) \cap \{u_1^1, u_1^2\} \neq \emptyset$ and $V(T_{u_2}) \cap \{u_1^3, u_1^4\} \neq \emptyset$; otherwise if $V(T_{u_2}) \cap \{u_1^1, u_1^2\} = \emptyset$, then $T_{iu_2}, G[\{u_1, u_1^1, u_1^2\}], T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_{k-1}$ are k vertex-disjoint rainbow C_3 's, a contradiction. Thus, $rt_1(u_2) \leq \binom{2}{1} \cdot \binom{2}{1} = 4$.

Assume that $rt_1(u_1) \leq n - 3k + 2$. By Claim 1 ($s = 3$), if $rt_1(u_2) > 2(n - 3k + 2)$, then G contains an $RF(u_2)$ in which u_2 is the unique vertex of V_0 . Suppose $e_0 = x_1x_2 \in E(G[V_1])$ together with u_1 forms a rainbow C_3 . Denote by $e_1, e_2, e_3 \in E(RF_3^3(u_2)) \cap E(G[V_1])$ the disjoint edges in which each edge together with u_2 forms a rainbow C_3 . Then at least one of them contains no x_1 and x_2 . Suppose such an edge is $e_1 = y_1y_2$. Then, $G[\{u_1, x_1, x_2\}], G[\{u_2, y_1, y_2\}], T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_{k-1}$ are k vertex-disjoint rainbow C_3 's. Thus, $rt_1(u_2) \leq 2(n - 3k + 2)$. $\square$

For $i \in [t]$, we estimate the number of rainbow C_3 's containing $v_{i,1}$ and some vertices in V_1 . Denote by $E_{1,i} = \{e \in E_1 : e \text{ is incident to } v_{i,1}\}$. It is easy to see $\bigcup_{i=1}^t E_{1,i} = E_1$. For a rainbow triangle, denote by $T(v_{i,1})$, which contains $v_{i,1}$ and some vertices in V_1 (maybe one or

two), if it contains two vertices in V_1 , then it contributes to $rt_1(v_{i,1})$; if it contains two vertices in V_0 , then it must contain an edge $e \in E_{1,i}$.

We now try to bound $rt_1(v_{i,1})$ by dealing with three cases: When $1 \leq rt_1(v_{i,1}) \leq 4$, by Claim 3, we have the following. If $n - 3k + 2 < rt_1(v_{i,2}) \leq 2(n - 3k + 2)$, then $rt_1(v_{i,3}) \leq 4$, and hence $rt_1(v_{i,2}) + rt_1(v_{i,3}) \leq 2(n - 3k + 2) + 4$. Otherwise, if $rt_1(v_{i,2}) \leq n - 3k + 2$, then $rt_1(v_{i,2}) + rt_1(v_{i,3}) \leq 2(n - 3k + 2)$. Then, by Claim 2, we obtain

$$\begin{aligned} \sum_{j=1}^3 rt_1(v_{i,j}) + \sum_{e \in E_{1,i}} rt_2(e) &\leq rt_1(v_{i,2}) + rt_1(v_{i,3}) + rt(v_{i,1}) \\ &\leq 3(n - 3k + 2) + (3k - 3)(n - 1) \\ &= 3kn - 12k + 9. \end{aligned}$$

When $4 < rt_1(v_{i,1}) \leq n - 3k + 2$, by Claim 3, we have $rt_1(v_{i,2}) \leq n - 3k + 2$ and $rt_1(v_{i,3}) \leq n - 3k + 2$. Here we note that if $rt_1(v_{i,2}) > n - 3k + 2$ or $rt_1(v_{i,3}) > n - 3k + 2$, then $rt_1(v_{i,1}) \leq 4$, which is a contradiction.

By Claim 2, we have

$$\begin{aligned} \sum_{j=1}^3 rt_1(v_{i,j}) + \sum_{e \in E_{1,i}} rt_2(e) &\leq rt_1(v_{i,2}) + rt_1(v_{i,3}) + rt(v_{i,1}) \\ &\leq (n - 3k + 2) + (n - 3k + 2) + (3k - 3)(n - 1) \\ &= (3k - 1)(n - 1) - 6k + 6 \\ &= (3k - 1)n - 9k + 7. \end{aligned}$$

When $rt_1(v_{i,1}) > n - 3k + 2$, by Claim 3, $rt_1(v_{i,2}) \leq 4$ and $rt_1(v_{i,3}) \leq 4$. Then, by Claim 2, we have

$$\begin{aligned} \sum_{j=1}^3 rt_1(v_{i,j}) + \sum_{e \in E_{1,i}} rt_2(e) &\leq rt_1(v_{i,2}) + rt_1(v_{i,3}) + rt(v_{i,1}) \\ &\leq 4 + 4 + (3k - 3)(n - 1) \\ &\leq (3k - 3)(n - 1) + 8 \\ &= (3k - 3)n - 3k + 11. \end{aligned}$$

By the above three inequalities, we have

Claim 4. $\sum_{i=1}^t \sum_{j=1}^3 rt_1(v_{i,j}) + \sum_{e \in E_1} rt_2(e) \leq t(3kn - 12k + 9)$.

Proof of Claim 4. For each $i \in [t]$, since $n \geq 42.5k + 48$, we know

$$\sum_{j=1}^3 rt_1(v_{i,j}) + \sum_{e \in E_{1,i}} rt_2(e) \leq 3kn - 12k + 9,$$

and so

$$\begin{aligned} \sum_{i=1}^t \sum_{j=1}^3 rt_1(v_{i,j}) + \sum_{e \in E_1} rt_2(e) &\leq \sum_{i=1}^t \left(\sum_{j=1}^3 rt_1(v_{i,j}) + \sum_{e \in E_{1,i}} rt_2(e) \right) \\ &\leq t(3kn - 12k + 9). \end{aligned}$$

□

Now we estimate $rt_1(v_{i,j})$ for $i \geq t+1$ and $j \in [3]$.

Claim 5. For any $i \in [k-1] \setminus [t]$, we have

$$rt_1(v_{i,1}) + rt_1(v_{i,2}) + rt_1(v_{i,3}) \leq \max\{2t(n-3k+2) + 8, 3(n-3k+2)\},$$

and

$$\sum_{i=t+1}^{k-1} \sum_{j=1}^3 rt_1(v_{i,j}) \leq \max\{(k-1-t)(2t(n-3k+2) + 8), 3(k-1-t)(n-3k+2)\}.$$

Proof of Claim 5. For each $j \in [3]$, let $T(v_{i,j})$ be a rainbow C_3 containing $v_{i,j}$ and two vertices of V_1 . We claim that

$$V(T(v_{i,j})) \cap \left(\bigcup_{s=1}^t \{v_{s,4}, v_{s,5}\} \right) \neq \emptyset;$$

Otherwise, we obtain a $RF_{2,1}(t+1; v_{1,1}, \dots, v_{t+1,j}, \dots)$, contradicting the maximality of t . There are at most $2t$ vertices in the set $\bigcup_{s=1}^t \{v_{s,4}, v_{s,5}\}$. The maximum number of rainbow C_3 's which contain some vertices in $\bigcup_{s=1}^t \{v_{s,4}, v_{s,5}\}$ is at most $2t(n-3k+2)$. Thus, $rt_1(v_{i,j}) < 2t(n-3k+2)$.

If there exists some $j \in [3]$ (assume $j = 1$) such that $rt_1(v_{i,1}) > n-3k+2$, then by Claim 3, we have

$$\sum_{j=1}^3 rt_1(v_{i,j}) \leq 2t(n-3k+2) + 4 + 4;$$

if $rt_1(v_{i,j}) < n-3k+2$ for each $j \in [3]$, then $\sum_{j=1}^3 rt_1(v_{i,j}) < 3(n-3k+2)$. Thus,

$$\sum_{i=t+1}^{k-1} \sum_{j=1}^3 rt_1(v_{i,j}) \leq \max\{(k-1-t)(2t(n-3k+2) + 8), (k-1-t)3(n-3k+2)\}.$$

□

Next, we obtain an upper bound of $|RT_2|$. For each $e \in E(G[V_0])$ and $v \in V_1$, we denote by $RT(e, v)$ the rainbow C_3 that contains e and v . To estimate $|RT_2|$, we divide the edges in $G[V_0]$ into five classes.

Recall

$$E_1 = \bigcup_{i \in [t], j \in [k-1] \setminus \{i\}} E[v_{i,1}, V(T_j)].$$

We use

$$E_2 = \bigcup_{i \in [t], j \in [k-1] \setminus [t]} E[V(T_i) - v_{i,1}, V(T_j)]$$

to denote the edges joining one non center-vertex from the first t rainbow triangles and one vertex from last $k - t - 1$ triangles;

$$E_3 = \bigcup_{t+1 \leq i < j \leq k-1} E[V(T_i), V(T_j)]$$

to denote all edges between any two triangles from the last $k - t - 1$ vertex disjoint rainbow triangles;

$$E_4 = \bigcup_{i=1}^{k-1} E(T_i)$$

to denote the edges within all these $k - 1$ vertex disjoint rainbow triangles; and

$$E_5 = \bigcup_{1 \leq i < j \leq t} E[V(T_i) - v_{i,1}, V(T_j) - v_{j,1}]$$

to denote the edges between all non-center vertices of the first t vertex-disjoint rainbow triangles. Obviously, E_i 's are pairwise disjoint, and $E(G[V_0]) = \bigcup_{i=1}^5 E_i$.

Claim 6. $\sum_{e \in E_3} rt_2(e) \leq 6 \binom{k-1-t}{2} (n - 3k + 3)$.

Proof of Claim 6. For each $t_1, t_2 \in [k-1]$ ($t_1 < t_2$), denote by e_j ($j = 1, 2, 3$) three disjoint edges which have end-vertices between $V(T_{t_1})$ and $V(T_{t_2})$. If two of the three edges, say e_1 and e_2 , satisfy that $rt_2(e_1) \geq 3$ and $rt_2(e_2) \geq 2$, suppose that G contains $RT(e_1, x_i)$ and $RT(e_2, y_j)$ with $i = 1, 2, 3$ and $j = 1, 2$, then $rt_2(e_3) = 0$. Otherwise, assume G contains $RT(e_3, z)$. Since $rt_2(e_1) \geq 3$ and $rt_2(e_2) \geq 2$, we may choose x_{i_0}, y_{j_0}, z to be distinct. Hence, $RT(e_1, x_{i_0}), RT(e_2, y_{j_0}), RT(e_3, z)$ together with $T_1, \dots, T_{t_1-1}, T_{t_1+1}, \dots, T_{t_2-1}, T_{t_2+1}, \dots, T_{k-1}$ form k vertex-disjoint rainbow C_3 's, a contradiction. Thus,

$$\sum_{j=1}^3 rt_2(e_j) \leq \max\{2(n - 3k + 3), 3 \cdot 2\} = 2(n - 3k + 3).$$

The nine edges in $G[V(T_{t_1}), V(T_{t_2})]$ can be partitioned into three sets, in which each consists of three pairwise disjoint edges. Denote by $E_3^{i,j}$ the edges in E_3 whose end vertices lie in T_i and T_j . Then for $t+1 \leq i < j \leq k-1$, we have $\sum_{e \in E_3^{i,j}} rt_2(e) \leq 3 \cdot 2(n - 3k + 3)$, thus

$$\begin{aligned} \sum_{e \in E_3} rt_2(e) &= \sum_{t+1 \leq i < j \leq k-1} \sum_{e \in E_3^{i,j}} rt_2(e) \\ &\leq \binom{k-1-t}{2} \cdot 6(n - 3k + 3). \end{aligned}$$

□

Claim 7. $\sum_{e \in E_2} rt_2(e) \leq t(k-1-t)(3(n - 3k + 3) + 9)$.

Proof of Claim 7. For $e \in E_2$, let $t_1 \in [t]$ and $t_2 \in [k-1] \setminus [t]$. Denote by $e_1 = u_{1,1}u_{1,2}$ and $e_2 = u_{2,1}u_{2,2}$ two disjoint edges between $T_{t_1} - v_{t_1,1}$ and T_{t_2} . Then, either $rt_2(e_1) \leq 3$ or $rt_2(e_2) \leq 3$; otherwise, suppose $rt_2(e_1) \geq 4$ and $rt_2(e_2) \geq 4$, denote by $N_{2,1}(e_j)$ ($j = 1, 2$) the set of vertices in V_1 in which each together with e_j forms a rainbow C_3 . Then, we can choose $w_1 \in N_{2,1}(e_1) \setminus \{v_{t_1,4}, v_{t_1,5}\}$ and $w_2 \in N_{2,1}(e_2) \setminus \{v_{t_1,4}, v_{t_1,5}, w_1\}$, and obtain three disjoint rainbow C_3 's: $RT(e_1, w_1), RT(e_2, w_2), G[\{v_{t_1,1}, v_{t_1,4}, v_{t_1,5}\}]$, which are disjoint with $T_1, \dots, T_{t_1-1}, T_{t_1+1}, \dots, T_{k-1}$, a contradiction. Thus,

$$rt_2(e_1) + rt_2(e_2) \leq (n - 3k + 3) + 3,$$

and $\sum_{e \in E[T_{t_1}-v_{t_1,1}, T_{t_2}]} rt_2(e) \leq 3((n - 3k + 3) + 3)$. Since $t_1 \in [t]$, $t_2 \in [k-1] \setminus [t]$, we finish the proof. $\square$

Claim 8. $\sum_{e \in E_5} rt_2(e) \leq 16 \cdot \binom{t}{2}$.

Proof of Claim 8. Let $1 \leq t_1 < t_2 \leq t$. For each edge $e = u_1u_2$ with $u_i \in V(T_{t_i}) - \{v_{t_i,1}\}$, if $rt_2(e) \geq 5$, then there exists $u \in V_1 - \{v_{t_1,4}, v_{t_1,5}, v_{t_2,4}, v_{t_2,5}\}$. Replacing T_{t_1} and T_{t_2} by three rainbow triangles $G[\{v_{t_1,1}, v_{t_1,4}, v_{t_1,5}\}], G[\{v_{t_2,1}, v_{t_2,4}, v_{t_2,5}\}], RT(e, u)$, we obtain k vertex-disjoint rainbow triangles. Thus, $rt_{2,1}(u_1u_2) \leq 4$. It follows

$$\sum_{\substack{u_1 \in V(T_{t_1}) - v_{t_1,1} \\ u_2 \in V(T_{t_2}) - v_{t_2,1}}} rt_2(u_1u_2) \leq \binom{2}{1} \binom{2}{1} \cdot 4.$$

Since there are $\binom{t}{2}$ such pairs (t_1, t_2) , summing over all of them yield

$$\sum_{e \in E_5} rt_2(e) \leq 16 \binom{t}{2}.$$

$\square$

We can now count $rt(G)$ as follows.

$$\begin{aligned} rt(G) &\leq |RT_1| + |RT_2| + |RT_3| \\ &\leq \sum_{i=1}^{k-1} \sum_{j=1}^3 rt_1(v_{i,j}) + \sum_{e \in E(G[V_0])} rt_2(e) + \binom{3k-3}{3} \\ &\leq \left\{ \sum_{i=1}^t \sum_{j=1}^3 rt_1(v_{i,j}) + \sum_{i=t+1}^{k-1} \sum_{j=1}^3 rt_1(v_{i,j}) \right\} \\ &\quad + \left\{ \sum_{e \in E_1} rt_2(e) + \sum_{e \in E_2} rt_2(e) + \sum_{e \in E_3} rt_2(e) + \sum_{e \in E_4} rt_2(e) + \sum_{e \in E_5} rt_2(e) \right\} \\ &\quad + \binom{3k-3}{3} := A_1. \end{aligned}$$

By Claim 4, 7, 6, 8 and Claim 5, we have an upper bound for the blue, purple, red, green and orange term, respectively. There also holds $\sum_{e \in E_4} rt_2(e) \leq \sum_{i=1}^t e(T_i)(n - 3k + 3) \leq (3k - 3)(n - 3k + 3)$. Thus, we have

$$A_1 \leq \max\{(k-1-t)(2t(n-3k+2)+8), 3(k-1-t)(n-3k+2)\}$$

$$\begin{aligned}
& + t(3kn - 12k + 9) + 3t(k - 1 - t)((n - 3k + 3) + 3) + 6 \binom{k-1-t}{2} (n - 3k + 3) \\
& + (3k - 3)(n - 3k + 3) + 16 \binom{t}{2} + \binom{3k-3}{3} := A_2.
\end{aligned}$$

For $a, b \in \mathbb{N}$, obviously $\binom{a}{b} \leq \frac{a^b}{b!}$.

If $t \geq 2$, then by $(k - 1 - t) \leq (k - t)$, $k \geq 3$ and $t \leq k$, we have

$$\begin{aligned}
A_2 & \leq t^2(-2n + 6k - 5) + t(2kn + 3k^2 - 8k + 1) \\
& + (3k^2n + (3k - 3)n - \frac{9}{2}k^3 - \frac{27}{2}k^2 + \frac{79}{2}k - \frac{27}{2}) \\
& \leq -2nt^2 + 2knt + k^2(6k - 5) + k(3k^2 - 8k + 1) + \\
& \quad \{(3k^2 + 3k - 3)n - \frac{9}{2}k^3 - \frac{27}{2}k^2 + \frac{79}{2}k\} \\
& = -2nt^2 + 2knt + (3k^2 + 3k - 3)n + (6 + 3 - \frac{9}{2})k^3 - (5 + 8 + \frac{27}{2})k^2 + (1 + \frac{79}{2})k \\
& \leq -2nt^2 + 2knt + (3k^2 + 3k - 3)n + \frac{9}{2}k^3 - 13k^2 - \frac{27}{2}k^2 + \frac{27}{2}k^2 \text{ (as } k \geq 3) \\
& = -2nt^2 + 2knt + (3k^2 + 3k - 3)n + \frac{9}{2}k^3 - 13k^2 := f(t) = B_1.
\end{aligned}$$

We claim $f(t)$ is maximal when $t = \frac{k}{2}$, and so,

$$B_1 \leq \frac{7}{2}k^2n + 3kn + \frac{9}{2}k^3 - 13k^2 := g(k) = B_2.$$

Set

$$n_0 = \frac{41k^2 + 36k + \sqrt{(41k^2 + 36k)^2 + 4k(54k^3 - 156k^2)}}{2k}.$$

By computing, we have

$$n \geq \frac{85}{2}k + 48 > \frac{41k^2 + 36k + (44k^2 + 36k)}{2k} \geq n_0.$$

By Lemma 2, we have

$$\begin{aligned}
12 \cdot (rt(G) - B_2) & \geq kn^2 + k^2n - \{(42k^2 + 36k)n + 54k^3 - 156k^2\} \\
& = kn^2 + (-41k^2 - 36k)n - 54k^3 + 156k^2 \\
& \geq kn^2 + (-41k^2 - 36k)n - 54k^3 + 156k^2 \text{ (as } n > n_0) \\
& > kn_0^2 + (-41k^2 - 36k)n_0 - 54k^3 + 156k^2 \\
& = 0,
\end{aligned}$$

which implies $rt(G) > rt(G)$, a contradiction.

If $t \leq 1$, then

$$A_2 \leq -t^2 + (-3n + 9k^2 - 3k + 3)t + 3k^2n + (6k - 3)n - \frac{9}{2}k^3 - \frac{45}{2}k^2 + \frac{75}{2}k - \frac{27}{2}$$

$$\begin{aligned}
&\leq -t^2 + (-3n + 3)t + 9k^2 - 3k + (3k^2 + 6k)n - \frac{9}{2}k^3 - \frac{45}{2}k^2 + \frac{75}{2}k \text{ (as } t \leq 1) \\
&\leq -t^2 + (-3n + 3)t + 9k^2 + (3k^2 + 6k)n - \frac{9}{2}k^3 - \frac{45}{2}k^2 + \frac{23}{2}k^2 \text{ (as } k \geq 3) \\
&= -t^2 + (-3n + 3)t + (3k^2 + 6k)n - \frac{9}{2}k^3 - 2k^2 := h(t) = C_1
\end{aligned}$$

Then, $h(t)$ is maximal when $t = 0$,

$$C_1 \leq (3k^2 + 6k)n - \frac{9}{2}k^3 - 2k^2 := C_2$$

Set

$$n_1 = \frac{35k^2 + 72k + \sqrt{(35k^2 + 72k)^2 - 4k(54k^3 + 24k^2)}}{2k}.$$

By computing, we have

$$n \geq \frac{85}{2}k + 48 > \frac{35k^2 + 72k + (32k^2 + 73k)}{2k} > n_1.$$

By Lemma 2, we have

$$\begin{aligned}
12(rt(G) - C_2) &\geq kn^2 + k^2n - (36k^2 + 72k)n + 54k^3 + 24k^2 \\
&> kn_1^2 - (35k^2 + 72k)n_1 + 54k^3 + 24k^2 \\
&= 0,
\end{aligned}$$

a contradiction. Thus, G contains a rainbow C_3 vertex-disjoint with each T_i ($i \in [k - 1]$). The proof is complete. $\square$

References

- [1] P. Allen, J. Böttcher, J. Hladký, D. Piguet, A density Corrádi-Hajnal theorem, *Canad. J. Math.* **67** (2015), no. 4, 721–758. [2](#), [6](#)
- [2] N. Alon, On a conjecture of Erdős, Simonovits, and Sós concerning anti-Ramsey theorems, *J. Graph Theory* **7** (1983), no. 1, 91–94. [2](#)
- [3] R. Čada, A. Kaneko, Z. Ryjáček, K. Yoshimoto, Rainbow cycles in edge-colored graphs. *Discrete Math.* **339** (2016), no. 4, 1387–1392. [3](#)
- [4] X. Chen, B. Ning, Rainbow triangles sharing one common vertex or edge, *Electron. J. Combin.* **32** (2025), no. 3, P3.30. [3](#)
- [5] X. Chen, X. Li, B. Ning, Note on rainbow triangles in edge-colored graphs, *Graphs Combin.* **38** (2022), no. 3, Paper No. 69, 13 pp. [3](#)
- [6] K. Corrádi, A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 423–439. [3](#)

- [7] G.A. Dirac, On the maximal number of independent triangles in graphs, *Abh. Math. Sem. Univ. Hamburg* **26** (1963), 78–82. [3](#)
- [8] P. Erdős, M. Simonovits, V.T. Sós (1965), Anti-Ramsey theorems. In *Infinite and Finite Sets* (Keszthely, Hungary, 1973), Vol. 10 of Coll. Math. Soc. J. Bolyai, pp. 657–665. [2](#)
- [9] S. Fujita, C. Magnant, K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs Combin.* **26** (2010), no. 1, 1–30. [2](#)
- [10] J. Hu, H. Li, D. Yang, Vertex-disjoint rainbow triangles in edge-colored graphs, *Discrete Math.* **343** (2020), no. 12, 112117, 5 pp. [2](#), [3](#)
- [11] T. Jiang, D.B. West, On the Erdős-Simonovits-Sós conjecture about the anti-Ramsey number of a cycle. Special issue on Ramsey theory, *Combin. Probab. Comput.* **12** (2003), no. 5-6, 585–598. [2](#)
- [12] H. Li, Rainbow C_3 's and C_4 's in edge-colored graphs, *Discrete Math.* **313** (2013), no. 19, 1893–1896. [2](#), [3](#)
- [13] B. Li, B. Ning, C. Xu, S. Zhang, Rainbow triangles in edge-colored graphs. *European J. Combin.* **36** (2014), 453–459. [2](#), [3](#)
- [14] H. Li, G. Wang, Color degree and heterochromatic cycles in edge-colored graphs, *European J. Combin.* **33** (2012), no. 8, 1958–1964. [2](#)
- [15] X. Li, B. Ning, Y. Shi, S. Zhang, Counting rainbow triangles in edge-colored graphs, *J. Graph Theory* 107 (2024), no. 4, 742–758. [3](#), [7](#)
- [16] A. Lo, E. Williams, Towards an edge-coloured Corrádi-Hajnal theorem, arXiv:2408.10651. [3](#)
- [17] H. Lu, X. Luo, X. Ma, New bounds on the Anti-Ramsey number of independent triangles, arXiv:2506.07115v1. 8 Jun 2025. [2](#)
- [18] J.J. Montellano-Ballesteros, V. Neuman-Lara, An anti-Ramsey theorem on cycles, *Graphs Combin.* **21** (2005) 343–354. [2](#)
- [19] F. Wu, S. Zhang, B. Li, J. Xiao, Anti-Ramsey numbers for vertex-disjoint triangles, *Discrete Math.* **346** (2023), 113123. [2](#), [6](#)
- [20] L. Yuan, X. Zhang, Anti-Ramsey numbers of graphs with some decomposition family sequences, arXiv:1903.10319. [2](#)