ON INTERTWINED POLYNOMIALS

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Abstract. Let A_1 and A_2 be polynomials of degree at least two over $\mathbb{C}$. We say that A_1 and A_2 are intertwined if the endomorphism (A_1, A_2) of $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by $(z_1, z_2) \mapsto (A_1(z_1), A_2(z_2))$ admits an irreducible periodic curve that is neither a vertical nor a horizontal line. We denote by $\mathrm{Inter}(A)$ the set of all polynomials B such that some iterate of B is intertwined with some iterate of A. In this paper, we prove a conjecture of Favre and Gauthier describing the structure of $\mathrm{Inter}(A)$. We also obtain a bound on the possible periods of periodic curves for endomorphisms (A_1, A_2) in terms of the sizes of the symmetry groups of the Julia sets of A_1 and A_2.

1. Introduction

In their seminal paper, Medvedev and Scanlon [19] described invariant curves of endomorphisms of $(\mathbb{CP}^1)^2$ of the form

$$(A_1, A_2)\colon (z_1, z_2) \mapsto (A_1(z_1), A_2(z_2)), \tag{1}$$

where A_1 and A_2 are polynomials, and showed that describing invariant curves for similar endomorphisms of $(\mathbb{CP}^1)^m$ with $m > 2$ reduces to the case $m = 2$. Since then, these results has been widely used in various contexts of complex and arithmetic dynamics (see, e.g., [2], [5], [9], [10], [11], [12], [13], [14], [20]).

From an algebraic perspective, the description of invariant—and more generally, periodic—curves for endomorphisms (1) reduces to a system of polynomial semiconjugacies involving iterates of A_1 and A_2. In [19], this system was analyzed using Ritt's theory of polynomial decompositions [31]. An alternative approach, combining Ritt's theory with results from [22], was proposed in [24]. In a sense, this approach is less technical and also allows one, to some extent, to complete the results of [19]. Nevertheless, the approaches of both [19] and [24] do not extend to the case where A_1 and A_2 are arbitrary rational functions. In this setting, a description of invariant and periodic curves for endomorphisms (1) was obtained in [30], based on a characterization of semiconjugacies between rational functions provided in a series of papers [23], [25], [26], [27], [28].

The goal of this paper remains the study of periodic curves for polynomial endomorphisms (1), but from a more general perspective. Here, we fix a polynomial A_1 and aim to describe all polynomials A_2 such that, for some *iterates* of A_1 and A_2, the corresponding endomorphism (1) admits an irreducible periodic curve. More precisely, we say that polynomials A_1 and A_2 of degree at least two are *intertwined* if the endomorphism (1) admits an irreducible periodic curve that is neither a vertical nor a horizontal line. Given a polynomial A, we denote by $\mathrm{Inter}(A)$ the collection of all polynomials B for which there exist $k, l \geq 1$ such that $A^{\circ k}$ and $B^{\circ l}$ are intertwined. We denote by $J(A)$ the Julia set of A, and by $\Sigma(A)$ the symmetry

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group of $J(A)$, consisting of all affine transformations $\mu = az + b$ that leave $J(A)$ invariant. Note that the group $\Sigma(A)$ is finite unless A is conjugate to a power.

In the above notation, our main result is the following statement, conjectured by Favre and Gauthier [10, p. 88].

Theorem 1.1. *For any $d \geq 2$, there exists a constant $r(d)$ such that for every polynomial A of degree d, there are polynomials $B_1, \dots, B_r$ with $r \leq r(d)$ satisfying the following property: if $B \in \mathrm{Inter}(A)$, then B is conjugate to $\mu \circ B_i^{\circ n}$ for some i, $1 \leq i \leq r$, some $\mu \in \Sigma(B_i)$, and some $n \geq 1$.*

As a by-product of our method, we obtain a bound on the period of an irreducible periodic curve for the endomorphisms (1) in the case when A_1 and A_2 are not special—that is, not conjugate to z^d or to $\pm T_d$—in terms of the sizes of $\Sigma(A_1)$ and $\Sigma(A_2)$. If such a curve exists, then the degrees of A_1 and A_2 coincide, and the existence of a bound in terms of this common degree d was shown in [12], where the method gives the bound $2d^4$. For a polynomial A of degree at least two that is not conjugate to a power, we set

$$N(A) = |\Sigma(A)|\, \varphi\big(|\Sigma(A)|\big),$$

where φ denotes Euler's totient function. In this notation, our result can be stated as follows.

Theorem 1.2. *Let A_1 and A_2 be non-special polynomials of degree $d > 2$, and C an irreducible (A_1, A_2)-periodic curve. Then its period divides* $\mathrm{LCM}(N(A_1), N(A_2))$.

For A_1 and A_2 of degree $d = 2$, the same method applies, but the conclusion must be replaced by the requirement that the period of C divides $2\,\mathrm{lcm}(N(A_1), N(A_2))$ (Corollary 5.3). Note that Theorem 1.1 yields the bound obtained in [12]. Moreover, since for $d = 2$ the groups $\Sigma(A_1)$ and $\Sigma(A_2)$ are never trivial, Theorem 1.1 implies the following corollary.

Corollary 1.3. *Let A_1 and A_2 be non-special polynomials of degree $d \geq 2$ such that the groups $\Sigma(A_1)$ and $\Sigma(A_2)$ are trivial. Then any (A_1, A_2)-periodic curve is invariant.*

The paper is organized as follows. In Section 2 we recall some classical results on the functional equation

$$A \circ C = D \circ B,$$

where A, B, C, D are polynomials, and then review basic results concerning its particular case, the functional equation

$$A \circ X = X \circ B,$$

which describes the semiconjugacy relation for polynomials.

The material of the next section is also known, but we provide complete proofs, as we believe the way these results are presented may be of independent interest. We begin by proving a particular case of the general result stated in [22]. This case characterizes polynomials A_1, A_2 and infinite compact sets $K_1, K_2 \subset \mathbb{C}$ satisfying

$$A_1^{-1}(K_1) = A_2^{-1}(K_2),$$

under the condition $\deg A_1 \mid \deg A_2$. From this result, we deduce a criterion for semiconjugacy of polynomials in terms of their Julia sets, first established in [24]. We also recover several results concerning the group $\Sigma(A)$ and polynomials sharing Julia sets, proved in [1], [3], [4], and [33].

In Section 4 we establish several results concerning the implications of the semiconjugacy relation for polynomials in the case when one of the polynomials involved is an lth iterate of some polynomial. In Section 5, building on these findings, we prove Theorem 1.2. In Section 6 we prove additional results that allow us to qualitatively describe semiconjugacy relations between iterates $A^{\circ l}$, $l \geq 1$, and $B^{\circ s}$, $s \geq 1$, assuming that one of A or B is fixed while the other, together with the integers s and l, may vary. Finally, we prove Theorem 1.1.

2. Polynomial semiconjugacies

2.1. **The theorems of Engstrom and Ritt.** In this section, we recall some results about polynomial solutions to the equation

$$A \circ C = D \circ B. \tag{2}$$

The first result, proved by Engstrom [7], roughly speaking, reduces the problem of finding solutions to (2) to the case

$$\mathrm{GCD}(\deg A, \deg D) = 1, \quad \mathrm{GCD}(\deg C, \deg B) = 1. \tag{3}$$

Theorem 2.1. *Let A, C, D, B be non-constant polynomials such that (2) holds. Then there exist polynomials $U, V, \widetilde{A}, \widetilde{C}, \widetilde{D}, \widetilde{B}$, where*

$$\deg U = \mathrm{GCD}(\deg A, \deg D), \quad \deg V = \mathrm{GCD}(\deg C, \deg B),$$

such that

$$A = U \circ \widetilde{A}, \quad D = U \circ \widetilde{D}, \quad C = \widetilde{C} \circ V, \quad B = \widetilde{B} \circ V,$$

and

$$\widetilde{A} \circ \widetilde{C} = \widetilde{D} \circ \widetilde{B}.$$

In particular, if $\deg C = \deg B$, then there exists a degree-one polynomial μ such that

$$A = D \circ \mu^{-1}, \quad C = \mu \circ B.$$

The second result, proved by Ritt [31], describes polynomial solutions to (2) that satisfy (3).

Theorem 2.2. *Let A, C, D, B be non-constant polynomials such that (2) and (3) hold. Then there exist polynomials $\sigma_1, \sigma_2, \mu, \nu$ of degree one such that, up to a possible replacement of A by D and of C by B, either*

$$A = \nu \circ z^s R^n(z) \circ \sigma_1^{-1}, \qquad C = \sigma_1 \circ z^n \circ \mu$$
$$D = \nu \circ z^n \circ \sigma_2^{-1}, \qquad B = \sigma_2 \circ z^s R(z^n) \circ \mu,$$

where R is a polynomial, $n \geq 1$, $s \geq 0$, and $\mathrm{GCD}(s, n) = 1$, or

$$A = \nu \circ T_m \circ \sigma_1^{-1}, \qquad C = \sigma_1 \circ T_n \circ \mu,$$
$$D = \nu \circ T_n \circ \sigma_2^{-1} \qquad B = \sigma_2 \circ T_m \circ \mu,$$

where T_n, T_m are the Chebyshev polynomials, $n, m \geq 1$, and $\mathrm{GCD}(n, m) = 1$.

Theorem 2.2 allows us to describe polynomial solutions of the equation

$$A \circ X = X \circ B \tag{4}$$

under the condition

$$\mathrm{GCD}(\deg X, \deg B) = 1. \tag{5}$$

We call such solutions of (4) *primitive*. Notice that Theorem 2.1, together with Lüroth's theorem, implies that a solution A, B, X of (4) is primitive if and only if

$$\mathbb{C}(X, B) = \mathbb{C}(z). \tag{6}$$

Below, we will use the conditions (5) and (6) interchangeably.

Specifically, Theorem 2.2 implies the following result (see [15]).

Theorem 2.3. *Let A, B, X be polynomials satisfying* (4) *and* (5). *Then there exist polynomials of degree one μ, ν such that either*

$$A = \nu \circ z^s R^n(z) \circ \nu^{-1}, \quad X = \nu \circ z^n \circ \mu, \quad B = \mu^{-1} \circ z^s R(z^n) \circ \mu, \tag{7}$$

where R is a polynomial, $n \geq 1$, $s \geq 0$, and $\mathrm{GCD}(s, n) = 1$, *or*

$$A = \nu \circ \pm T_m \circ \nu^{-1}, \quad X = \nu \circ T_n \circ \mu, \quad B = \mu^{-1} \circ \pm T_m \circ \mu,$$

where T_n, T_m are Chebyshev polynomials, $n, m \geq 1$, and $\mathrm{GCD}(n, m) = 1$.

Notice that Theorem 2.3 is a special case of the description of solutions to (4) in rational functions under condition (6), as obtained in [23]. For further details, we refer the reader to [23], [25], and [27]. Nevertheless, the conditions (5) and (6) are no longer equivalent for rational functions, since Theorem 2.1 does not hold in this broader setting.

The simplest counterexample to Theorem 2.1 in the rational case is given by the equality

$$\left(\frac{z^2-1}{z^2+1}\right) \circ \left(\frac{z^2-1}{z^2+1}\right) = -\frac{2z^2}{z^2-2} \circ \left(z + \frac{1}{z}\right).$$

Here all functions involved have degree two, yet the equality

$$\frac{z^2-1}{z^2+1} = \mu \circ \left(z + \frac{1}{z}\right)$$

for any Möbius transformation μ is impossible, because the two sides have different sets of critical points.

2.2. **Elementary transformations.** Let A and B be polynomials of degree at least two. Let us recall that if equality (4) is satisfied for some polynomial X of degree one, the polynomials A and B are called *conjugated*. If (4) is satisfied for some polynomial X of degree at least two, the polynomial B is called *semiconjugate* to A.

We will use the notation $A \leq B$ if, for polynomials A and B, there exists a non-constant polynomial X such that (4) holds, and the notation $A \underset{X}{\leq} B$ if A, B, and X satisfy (4). Notice that by writing equality (4) in the form of a commuting diagram:

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle X} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array}$$

we immediately see that $A \underset{X}{\leq} B$ implies $A^{\circ i} \underset{X}{\leq} B^{\circ i}$ for every $i \geq 2$.

In this section, we recall how an arbitrary solution of (4) can be reduced to a primitive one. Let A be a polynomial. A polynomial B is called an *elementary transformation* of A if there exist polynomials U and V such that

$$A = U \circ V \quad \text{and} \quad B = V \circ U.$$

Clearly, the identity

$$(U \circ V) \circ U = U \circ (V \circ U)$$

implies that $A \underset{U}{\leq} B$. Similarly, $B \underset{V}{\leq} A$.

We say that polynomials A and B are *equivalent*, and write $A \sim B$, if there exists a chain of elementary transformations connecting B and A. If

$$B \to B_1 \to B_2 \to \cdots \to B_s = A$$

is such a chain, and if U_i, V_i, $1 \leq i \leq s$, are the corresponding polynomials satisfying

$$B = V_1 \circ U_1, \quad B_i = U_i \circ V_i, \quad 1 \leq i \leq s,$$

and

$$U_i \circ V_i = V_{i+1} \circ U_{i+1}, \quad 1 \leq i \leq s-1,$$

then the composition

$$X = U_s \circ U_{s-1} \circ \cdots \circ U_1$$

makes the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle X} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutative. Thus, $A \sim B$ implies both $A \leq B$ and $B \leq A$.

Notice that, for any polynomial X of degree one, the equality

$$A = (A \circ X) \circ X^{-1}$$

implies

$$A \sim X^{-1} \circ A \circ X,$$

so that each equivalence class is a union of conjugacy classes. Moreover, the number of such classes is finite and can be bounded solely in terms of the degree of A (see [26], [28] for more details).

An arbitrary solution of equation (4) can be reduced to a primitive one by a sequence of elementary transformations. Specifically, if $\mathbb{C}(X, B) \neq \mathbb{C}(z)$, then the Lüroth theorem implies that there exists a polynomial W of degree greater than one such that

$$B = \widetilde{B} \circ W, \quad X = \widetilde{X} \circ W$$

for some polynomials $\widetilde{X}$ and $\widetilde{B}$ with $\mathbb{C}(\widetilde{X},\widetilde{B})=\mathbb{C}(z)$. Moreover, the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ {\scriptstyle W}\downarrow & & \downarrow{\scriptstyle W} \\ \mathbb{CP}^1 & \xrightarrow{W\circ\widetilde{B}} & \mathbb{CP}^1 \\ {\scriptstyle \widetilde{X}}\downarrow & & \downarrow{\scriptstyle \widetilde{X}} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutes. Hence, the triple $A,\widetilde{X},W\circ\widetilde{B}$ is another solution of (4). While this new solution may still be non-primitive, it satisfies $\deg\widetilde{X}<\deg X$. Therefore, by repeating this procedure, we eventually obtain a primitive solution. Thus, we conclude the following.

Lemma 2.4. *Let A and B be polynomials of degree at least two, and let X be a non-constant polynomial such that $A\underset{X}{\leq}B$. Then there exist polynomials X_0, B_0, and U such that:*

(a) *the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathbb{CP}^1 & \xrightarrow{B_0} & \mathbb{CP}^1 \\ {\scriptstyle X_0}\downarrow & & \downarrow{\scriptstyle X_0} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutes, and $X=X_0\circ U$;

(b) *the relation $B_0\sim B$ holds;*

(c) *the triple A,B_0,X_0 is a primitive solution of equation* (4). □

Notice that for arbitrary *rational* functions A, B, and X, Lemma 2.4 remains valid if one defines a primitive solution of (4) as a solution satisfying (6) and extends the definition of an elementary transformation by allowing U and V to be rational as well (for more details, see [26], [27]).

2.3. **Semiconjugacies between special polynomials.** We recall that a polynomial A of degree at least two is called *special* if it is conjugate to z^d or to $\pm T_d$. It is well known that special polynomials are characterized by their Julia sets: a circle if A is conjugate to z^d, and a segment if A is conjugate to $\pm T_d$. In particular, A is special if and only if some iterate of A is special.

The following result was established in [24]. Below, we give an alternative proof using Lemma 2.4 and Theorem 2.3.

Theorem 2.5. *Let A and B be polynomials of degree at least two such that $A\leq B$. Then A is conjugate to z^m if and only if B is conjugate to z^m. Similarly, A is conjugate to $\pm T_m$ if and only if B is conjugate to $\pm T_m$.*

Proof. It is well known, and follows easily from Theorem 2.1, that if $T_m=U_1\circ U_2$ for some polynomials U_1 and U_2 of degrees m_1 and m_2, respectively, then there

exists a polynomial of degree one α such that

$$U_1 = T_{m_1} \circ \alpha, \quad U_2 = \alpha^{-1} \circ T_{m_2}.$$

Similarly, the equality $z^m = U_1 \circ U_2$ implies that

$$U_1 = z^{m_1} \circ \alpha, \quad U_2 = \alpha^{-1} \circ z^{m_2}.$$

Therefore, a polynomial is conjugate to z^m or $\pm T_m$ if and only if all polynomials in its equivalence class share this property. Combining this with Lemma 2.4, we see that it suffices to prove the theorem under the assumption that A, B, X is a primitive solution of equation (4) with $\deg X \geq 2$.

Theorem 2.3 yields that, to prove the second part of the theorem, it is enough to show the following: if, in formulas (7) with non-constant R, the equality $B = \pm T_m$ holds, then A is conjugate to $\pm T_m$, and conversely, if $A = \pm T_m$, then B is conjugate to $\pm T_m$. It follows from the explicit formula

$$T_m = \frac{m}{2} \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k (m-k-1)!}{k!(m-2k)!} (2x)^{m-2k}$$

that T_m has nonzero coefficients c_m and c_{m-2} in degrees m and $m-2$, while the coefficient of x^{m-1} vanishes. Therefore, if $B = \pm T_m$ in (7), then $\mu(0) = 0$ and $n = 2$. Since $T_2 = (2z-1) \circ z^2$, this implies that $X = \gamma \circ T_2$ for some degree-one polynomial γ, and hence

$$X \circ B = \gamma \circ T_2 \circ \pm T_m = \gamma \circ T_m \circ T_2.$$

On the other hand,

$$A \circ X = A \circ \gamma \circ T_2.$$

Therefore,

$$\gamma \circ T_m = A \circ \gamma \quad \text{and} \quad A = \gamma \circ T_m \circ \gamma^{-1}.$$

Conversely, since T_m has only the critical values ± 1 and the multiplicity of T_m at any point of $\mathbb{C}$ is either one or two, the equality $A = \pm T_m$ in (7) implies that $n = 2$ and $\nu(0) = \pm 1$. In particular, $s \neq 0$ and m is odd because $\mathrm{GCD}(n, m) = 1$. Furthermore, since (7) gives $A(\nu(0)) = \nu(0)$ and $T_m(\pm 1) = (\pm 1)^m$, the oddness of m implies that the equality $A = -T_m$ cannot hold, and thus $A = T_m$. Now we have

$$A \circ X = T_m \circ \pm T_2 \circ \gamma = \pm T_2 \circ T_m \circ \gamma.$$

Since

$$X \circ B = \pm T_2 \circ \gamma \circ B,$$

it follows that

$$\pm T_2 \circ \gamma \circ B = \pm T_2 \circ T_m \circ \gamma,$$

which implies

$$\gamma \circ B = \pm T_m \circ \gamma \quad \text{and} \quad B = \gamma \circ (\pm T_m) \circ \gamma^{-1}.$$

By what has been proved above, to finish the proof of the theorem it is enough to show that in formulas (7), A is conjugate to z^m if and only if B is conjugate to z^m, which is straightforward. $\square$

3. Semiconjuagacies and Julia sets

3.1. **Polynomials sharing preimages of compact sets.** Theorem 3.3 below, providing a description of polynomial semicongugacies in terms of Julia sets, was obtained in [24] as a corollary of the results of [22], which describe polynomials A_1, A_2 and compact sets K_1, K_2 satisfying

$$A_1^{-1}(K_1) = A_2^{-1}(K_2), \tag{8}$$

in terms of solutions of the functional equation

$$B_1 \circ A_1 = B_2 \circ A_2, \tag{9}$$

where A_1, A_2, B_1, B_2 are polynomials.

It is easy to see that, for any polynomial solution of (9) and any compact set $K_3 \subset \mathbb{C}$, one obtains a solution of (8) by setting

$$K_1 = B_1^{-1}(K_3), \qquad K_2 = B_2^{-1}(K_3),$$

and the main result of [22] states, roughly speaking, that all solutions of (8) arise in this way whenever the set defined by (8) contains at least $\mathrm{LCM}(\deg A_1, \deg A_2)$ points.

For the sake of completeness, we provide a full proof of this result in the special case where $\deg A_1$ divides $\deg A_2$ (Theorem 3.2), which is sufficient to prove Theorem 3.3 and several other results from polynomial dynamics needed below. In the proof, we follow the ideas of [22], with appropriate simplifications arising from this special case. For simplicity, we assume that the compact sets involved are infinite. For generalizations of Theorem 3.2, we refer the reader to [22]. Note that these generalizations concern only polynomials; extensions to rational functions are not known.

We recall that, for a given compact set $K \subset \mathbb{C}$, a monic polynomial $P(z) \in \mathbb{C}[z]$ of degree $n > 0$ is called the nth polynomial of least deviation from zero if

$$\|P\|_K \leq \|Q\|_K$$

for any monic polynomial $Q(z) \in \mathbb{C}[z]$ of degree n, where

$$\|P\|_K := \max_{z \in K} |P(z)|.$$

It is well known that such a polynomial is unique whenever $\mathrm{card}\, K \geq n$. We denote it by $P_{n,K}$.

The following lemma was proved in [16] using the Kolmogorov criterion for polynomials of least deviation ([17]). For $n = 1$, it was also proved independently by a different method in the context of equation (8) in [21]. Below we provide a proof following the method of [22] (cf. Theorem 2.3 in [22]).

Lemma 3.1. *Let K be an infinite compact set in $\mathbb{C}$ and let X be a monic polynomial of degree $d \geq 1$. Then for any $n \geq 1$ the equality*

$$P_{n,K} \circ X = P_{nd, X^{-1}(K)}$$

holds.

Proof. For any polynomial Q, define its averaging by X by the formula

$$Q_X(z) = \frac{1}{d} \sum_{\substack{\zeta \in \mathbb{C}, \\ X(\zeta) = X(z)}} Q(\zeta),$$

where each root ζ of multiplicity k of

$$X(\zeta) - X(z) = 0 \tag{10}$$

is counted k times. Clearly, we have

$$\max_{z\in X^{-1}(K)} \Big|Q_X(z)\Big| \le \max_{z\in X^{-1}(K)} \frac{1}{d} \sum_{\substack{\zeta\in\mathbb{C},\\ X(\zeta)=X(z)}} \Big|Q(\zeta)\Big| \le \max_{z\in X^{-1}(K)} \Big|Q(z)\Big|. \tag{11}$$

Furthermore, for any polynomial A of degree less than d, the function $A_X(z)$ is constant. For $d = 1$, this is trivial, so assume $d \ge 2$. In this case, for the polynomials $A(z) = z^j$, $1 \le j \le d-1$, the claim follows from the Newton formulas, which express $A_X(z)$ in terms of the symmetric functions S_j, $1 \le j \le d-1$, of the roots ζ_i, $1 \le i \le d$, of (10). The general case then follows by linearity. Therefore, if Q is a polynomial of degree nd with X-adic decomposition

$$Q(z) = \sum_{i=0}^{n} A_i(z) X^i(z), \tag{12}$$

then

$$Q_X(z) = \sum_{i=0}^{n} a_i X^i(z), \tag{13}$$

where $a_i \in \mathbb{C}$, $1 \le i \le n$. Moreover, since $\deg Q = nd$, the degree of $A_n(z)$ in (12) is zero. Consequently, if Q is monic, then $A_n(z) = a_1 = 1$, and thus $Q_X(z)$ is monic since X is monic.

Equality (13) implies that, for any monic polynomial Q of degree dn,

$$\begin{aligned} \max_{z\in X^{-1}(K)} \Big|Q_X(z)\Big| = \max_{z\in K} \Big|\sum_{i=0}^{n} a_i z^i\Big| \ge \max_{z\in K} \Big|P_{n,K}(z)\Big| = \\ = \max_{z\in X^{-1}(K)} \Big|(P_{n,K}\circ X)(z)\Big|. \end{aligned} \tag{14}$$

It then follows from (11) and (14) that

$$\max_{z\in X^{-1}(K)} \Big|Q(z)\Big| \ge \max_{z\in X^{-1}(K)} \Big|(P_{n,K}\circ X)(z)\Big|.$$

Therefore, the polynomial $P_{n,K}\circ X$ is the ndth polynomial of least deviation from zero on $X^{-1}(K)$. □

Theorem 3.2. *Let A_1, A_2 be non-constant polynomials and K_1, K_2 infinite compact sets in $\mathbb{C}$ satisfying*

$$A_1^{-1}(K_1) \,=\, A_2^{-1}(K_2).$$

Assume that $\deg A_1 \mid \deg A_2$. *Then there exists a polynomial A such that*

$$A_2 = A\circ A_1 \quad \textit{and} \quad K_1 = A^{-1}(K_2).$$

Proof. Let a_1 and a_2 be the leading coefficients of the polynomials A_1 and A_2, n_1 and n_2 their degrees, and let

$$K = A_1^{-1}(K_1) \,=\, A_2^{-1}(K_2).$$

Further, let D be the disc of minimal radius containing K_2, and let $z_0 \in \mathbb{C}$ be its center.

Replacing K_2 and A_2 by $K_2 - z_0$ and $A_2 - z_0$, without loss of generality we may assume that $z_0 = 0$, which implies $P_{1,K_2} = z$. Moreover, in this case, for the set $\widehat{K_2} = a_2 z(K_2)$, we also have $P_{1,\widehat{K_2}} = z$. Since

$$\big((z/a_2) \circ A_2\big)^{-1}(\widehat{K_2}) = A_2^{-1}(K_2) = K,$$

Lemma 3.1 applied with $n = 1$ shows that the polynomial $(z/a_2) \circ A_2$ is the n_2-th polynomial of least deviation from zero on K.

On the other hand, setting $\widehat{K_1} = a_1 z(K_1)$, we have

$$K = (z/a_1 \circ A_1)^{-1}(\widehat{K_1}).$$

Thus, by Lemma 3.1, the polynomial

$$P_{n_2/n_1,\widehat{K_1}} \circ z/a_1 \circ A_1$$

is also the n_2-th polynomial of least deviation from zero on K. By the uniqueness of the polynomial of least deviation, we conclude that

$$P_{n_2/n_1,\widehat{K_1}} \circ z/a_1 \circ A_1 = z/a_2 \circ A_2,$$

which implies that the equality $A_2 = A \circ A_1$ holds for the polynomial

$$A = a_2 z \circ P_{n_2/n_1,\widehat{K_1}} \circ z/a_1.$$

Finally, from $A_2 = A \circ A_1$, we have

$$K = A_1^{-1}(K_1) = A_2^{-1}(K_2) = A_1^{-1}(A^{-1}(K_2)),$$

which implies that

$$A_1(K) = K_1 \quad \text{and} \quad A_1(K) = A^{-1}(K_2).$$

Thus, $K_1 = A^{-1}(K_2)$. $\square$

Theorem 3.3. *Let A and B be polynomials of degree at least two, and let X be a non-constant polynomial. Then the relation $A \underset{X}{\leq} B$ implies that*

$$X^{-1}(J(A)) = J(B). \tag{15}$$

Conversely, if for given B and X the condition

$$X^{-1}(K) = J(B) \tag{16}$$

holds for some compact set $K \subset \mathbb{C}$, then there exists a polynomial A such that

$$A \underset{X}{\leq} B \quad \textit{and} \quad J(A) = K.$$

Proof. The first part is known to hold for arbitrary rational functions A, B, and X (see e.g. [6], Lemma 5).

Assume now that (16) holds. Then

$$X^{-1}(K) = J(B) = (X \circ B)^{-1}(K).$$

Applying now Theorem 3.2 with $A_1 = X$ and $A_2 = X \circ B$, we conclude that there exists a polynomial A such that $A \underset{X}{\leq} B$. By the first part of the theorem, this implies that (15) holds. Finally, (15) combined with (16), gives $J(A) = K$. $\square$

3.2. **Polynomials with non-trivial $\Sigma(A)$ and polynomials sharing Julia sets.** Let

$$A(z) = a_0 z^d + a_1 z^{d-1} + \cdots + a_d$$

be a polynomial of degree $d \geq 2$. We say that A is *centered* if $a_1 = 0$. We recall that for a polynomial A, we defined $\Sigma(A)$ as the group of symmetries of $J(A)$; that is, the group of affine transformations $\mu(z) = az + b$ satisfying $\mu(J(A)) = J(A)$.

In this section, we recall two theorems concerning the Julia sets of polynomials. The first connects the nontriviality of $\Sigma(A)$ with the form of A, and the second provides a classification of polynomials that share a Julia set. These theorems follow from the results in [1], [3], [4], and [33], but they can also be obtained using Theorems 3.2 and 3.3. Since these alternative proofs may be of independent interest, we present them here. We denote by ε_n a primitive nth root of unity.

Theorem 3.4. *Let A be a polynomial of degree $d \geq 2$ not conjugate to z^d. Then $\Sigma(A)$ is a finite cyclic group. Moreover, if A is centered, then $\Sigma(A)$ is generated by the transformation $z \mapsto \varepsilon_n z$, where n is the largest integer such that A can be written in the form $A = z^s R(z^n)$, for some polynomial R and $s \geq 0$.*

Proof. Let us first show that $\Sigma(A)$ is a finite cyclic group. Let D be the disc of minimal radius containing $J(A)$, which we may assume, by conjugation, to be the unit disc, and let ν be an element of $\Sigma(A)$. The equality

$$\nu(J(A)) = J(A)$$

implies that $\nu(D) = D$, and hence $\nu(\partial D) = \partial D$. Thus, $\Sigma(A)$ is a subgroup of S^1 and is therefore either a finite cyclic group or dense in S^1. Moreover, since $J(A)$ is compact, in the latter case $J(A)$ would be a union of circles. For the compact set $K = z^d(J(A))$, we then have

$$(z^d)^{-1}(K) = A^{-1}(J(A)),$$

which, by Theorem 3.2, implies that A is a monomial, contradicting the assumption.

Let us now assume that A is centered. If A has the form $z^s R(z^n)$, then considering the semiconjugacy diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{z^s R(z^n)} & \mathbb{CP}^1 \\ {\scriptstyle z^n}\downarrow & & \downarrow{\scriptstyle z^n} \\ \mathbb{CP}^1 & \xrightarrow{z^s R^n(z)} & \mathbb{CP}^1 \end{array}$$

and applying Theorem 3.3, we obtain the equality

$$J(A) = J(z^s R(z^n)) = (z^n)^{-1}(J(z^s R^n(z))),$$

which shows that $J(A)$ is invariant under $z \mapsto \varepsilon_n z$. Thus, to prove the theorem, it suffices to show that if $\mu = az + b$ belongs to $\Sigma(A)$, then $\mu = \varepsilon_n z$ for some n, and A has the form $A = z^s R(z^n)$.

To prove this, observe that if $\mu \in \Sigma(A)$, then

$$A^{-1}(J(A)) = (A \circ \mu)^{-1}(J(A)).$$

Hence, by Theorem 3.2, the equality

$$A \circ \mu = \nu \circ A \tag{17}$$

holds for some degree-one polynomial $\nu = cz + d$. Moreover, ν belongs to $\Sigma(A)$, which implies that c is a root of unity, since $\Sigma(A)$ is finite. Since A is centered,

equality (17) implies that $b = 0$, and it is easy to see that if $b = 0$ and c is a root of unity, then a is also a root of unity. Furthermore, if a has order n, then A has the form $A = z^s R(z^n)$. □

Theorem 3.4 implies the following well-known fact, which we will often use without mentioning it explicitly.

Corollary 3.5. *Let A be a polynomial of degree at least two, and let $\mu \in \Sigma(A)$. Then $J(\mu \circ A) = J(A)$.*

Proof. Without loss of generality, we may assume that A is centered and not conjugate to a power, since in the latter case the corollary is obviously true.

Setting $n = |\Sigma(A)|$, we see that $A = z^s R(z^n)$ and $\mu = \varepsilon z$, where ε is an nth root of unity. Thus, $z^n \circ \mu = z^n$, implying that the relations

$$z^s R^n(z) \underset{z^n}{\leq} A \quad \text{and} \quad z^s R^n(z) \underset{z^n}{\leq} \mu \circ A$$

hold simultaneously. Hence, $J(\mu \circ A) = J(A)$ by the first part of Theorem 3.3. □

We recall that for a polynomial A of degree at least two that is not conjugate to a power, the natural number $N(A)$ is defined by

$$N(A) = |\Sigma(A)|\, \varphi(|\Sigma(A)|),$$

where φ denotes Euler's totient function.

Corollary 3.6. *Let A be a polynomial of degree at least two, not conjugate to a power, and let $\mu \in \Sigma(A)$. Assume that A and $\mu \circ A$ share an iterate. Then*

$$(\mu \circ A)^{\circ N(A)} = A^{\circ N(A)}. \tag{18}$$

Proof. As above, we may assume that $A = z^s R(z^n)$ and $\mu = \varepsilon z$, where ε is an nth root of unity. In this case, for any integer $N \geq 1$ we have

$$(\mu \circ A)^{\circ N} = \varepsilon^{\,1+s+s^2+\cdots+s^{N-1}}\, A^{\circ N},$$

implying that if A and $\mu \circ A$ share an iterate, then necessarily $\gcd(s, n) = 1$.

On the other hand, if the last condition holds, then the equalities

$$s^{\varphi(n)k} \equiv 1 \pmod{n}, \quad k \geq 0,$$

hold by Euler's theorem. Thus,

$$1 + s + s^2 + \cdots + s^{N(A)-1} = 1 + s + s^2 + \cdots + s^{n\varphi(n)-1} =$$
$$= (1 + s + s^2 + \cdots + s^{\varphi(n)-1}) \sum_{k=0}^{n-1} s^{\varphi(n)k} \equiv 0 \pmod{n},$$

implying (18). □

The following result provides a classification of polynomials sharing a Julia set.

Theorem 3.7. *Let $K \subset \mathbb{C}$ be a compact set that is the Julia set of a non-special polynomial of degree at least two. Then there exists a polynomial Q such that $J(Q) = K$, and any polynomial P with $J(P) = K$ has the form $P = \mu \circ Q^{\circ n}$ for some $\mu \in \Sigma(Q)$ and $n \geq 1$.*

Proof. Let Q be a non-special polynomial of minimal degree with $J(Q) = K$. We will show that for Q defined in this way, the conclusion of the theorem holds for any polynomial P with $P^{-1}(K) = K$. We begin by showing that the last equality implies that $\deg Q$ divides $\deg P$.

Since

$$(P \circ Q^{\circ i})^{-1}(K) = (Q^{\circ i} \circ P)^{-1}(K) = K, \quad i \geq 1,$$

it follows from Theorem 3.2 that for every $i \geq 1$ the equality

$$Q^{\circ i} \circ P = \mu_i \circ P \circ Q^{\circ i} \tag{19}$$

holds for some $\mu_i \in \Sigma(Q)$. Furthermore, since Q is not conjugate to a power, the group $\Sigma(Q)$ is finite, implying that there exist $i_2 > i_1$ such that $\mu_{i_1} = \mu_{i_2}$. Setting now $\mu = \mu_{i_1} = \mu_{i_2}$, we see that (19) yields

$$Q^{\circ(i_2 - i_1)} \circ \mu \circ P \circ Q^{\circ i_1} = \mu \circ P \circ Q^{\circ i_2},$$

which implies that $\mu \circ P$ commutes with $Q^{\circ(i_2 - i_1)}$.

Recall that by Ritt's theorem (see [32], and also [8], [29]), if rational functions A and B of degree at least two commute, then either they share an iterate, or they are both Lattès maps, or they are both conjugate to powers (with positive or negative exponents), or they are both conjugate to Chebyshev polynomials (with positive or negative signs). Since a polynomial cannot be a Lattès map, and Q is non-special, we conclude that

$$Q^{\circ l} = (\mu \circ P)^{\circ k}$$

for some $k, l \geq 1$, which implies

$$(\deg Q)^l = (\deg P)^k.$$

Since by assumption $\deg Q \leq \deg P$, we have $l \geq k$, which yields that $\deg Q$ divides $\deg P$.

Now the theorem follows easily from Theorem 3.2. Indeed, let P be an arbitrary polynomial with $P^{-1}(K) = K$. Then it follows from

$$P^{-1}(K) = Q^{-1}(K) = K$$

by Theorem 3.2 that $P = \widetilde{P} \circ Q$ for some polynomial $\widetilde{P}$ with $\widetilde{P}^{-1}(K) = K$. Applying Theorem 3.2 again to $\widetilde{P}$ and Q, and repeating this argument, we eventually obtain the representation

$$P = \mu \circ Q^{\circ n},$$

where $\mu \in \Sigma(Q)$ and $n \geq 1$. □

Corollary 3.8. *Let P and Q be polynomials of the same degree $d \geq 2$ such that $J(P) = J(Q)$. Then $Q = \mu \circ P$ for some $\mu \in J(P)$.*

Proof. For non-special polynomials P and Q, this follows from Theorem 3.7, and it is straightforward to verify that the same conclusion holds when one or both of them are special. Alternatively, the corollary is an immediate consequence of Theorem 3.2. □

4. Semiconjugacies involving iterates

In this section, we analyze the consequences of the condition $A \underset{X}{\leq} B$ for polynomials A and B in the case where one of them is an iterate of some polynomial.

We begin with the following result.

Theorem 4.1. *Let A and B be polynomials of degree at least two such that $A \underset{X}{\leq} B$ for some non-constant polynomial X. Then, for every polynomial $\widehat{B}$ with $J(\widehat{B}) = J(B)$, there exists a polynomial $\widehat{A}$ with $J(\widehat{A}) = J(A)$ such that $\widehat{A} \underset{X}{\leq} \widehat{B}$.*

Proof. By the first part of Theorem 3.3,

$$X^{-1}(J(A)) = J(B).$$

Since $J(\widehat{B}) = J(B)$, this implies by the second part of Theorem 3.3, that there exists a polynomial $\widehat{A}$ such that $\widehat{A} \underset{X}{\leq} \widehat{B}$ and $J(\widehat{A}) = J(A)$. □

Theorem 4.1 implies easily the following result.

Theorem 4.2. *Let A and B be polynomials of degree at least two, and X a non-constant polynomial such that $A \underset{X}{\leq} B$. Suppose that $B = \widehat{B}^{\circ l}$ for some polynomial $\widehat{B}$ and some $l \geq 2$. Then there exists a polynomial $\widehat{A}$ such that $\widehat{A} \underset{X}{\leq} \widehat{B}$ and $A = \widehat{A}^{\circ l}$.*

Proof. Since $J(\widehat{B}) = J(B)$, Theorem 4.1 implies that there exists a polynomial $\widehat{A}$ such that $\widehat{A} \underset{X}{\leq} \widehat{B}$. In turn, this relation gives $\widehat{A}^{\circ l} \underset{X}{\leq} \widehat{B}^{\circ l}$, which, together with $A \underset{X}{\leq} \widehat{B}^{\circ l}$, implies that $A = \widehat{A}^{\circ l}$. □

Corollary 4.3. *Let A and B be polynomials of degree at least two such that $A \sim B$. Then A is an lth iterate of some polynomial if and only if B is an lth iterate of some polynomial.*

Proof. Since the relation $A \sim B$ implies $A \leq B$, the corollary follows directly from Theorem 4.2. □

Lemma 4.4. *Let A be a polynomial of degree $d > 2$. Assume that the equality $A^{\circ l} = \mu \circ T_{d^l} \circ \nu$ holds for some $l \geq 2$ and some polynomials μ and ν of degree one. Then $A^{\circ l}$ is conjugate to $\pm T_{d^l}$.*

Proof. Clearly, without loss of generality, we may assume that $\nu = \mathrm{id}$, that is,

$$A^{\circ l} = \mu \circ T_{d^l}, \tag{20}$$

implying that

$$A = \mu \circ T_d \circ \alpha, \quad A^{\circ(l-1)} = \alpha^{-1} \circ T_{d^{l-1}} \tag{21}$$

for some degree one polynomial α.

It follows from (20) and (21) that

$$\mu \circ T_{d^l} = A^{\circ(l-1)} \circ A = \alpha^{-1} \circ T_{d^{l-1}} \circ \mu \circ T_d \circ \alpha,$$

so

$$T_{d^l} = (\mu^{-1} \circ \alpha^{-1} \circ T_{d^{l-1}}) \circ (\mu \circ T_d \circ \alpha),$$

which implies that

$$\mu^{-1} \circ \alpha^{-1} \circ T_{d^{l-1}} = T_{d^{l-1}} \circ \beta, \quad \mu \circ T_d \circ \alpha = \beta^{-1} \circ T_d \tag{22}$$

for some degree one polynomial β.

Now observe that the equality

$$T_m = \gamma \circ T_m \circ \delta, \quad m > 2, \tag{23}$$

for some degree one polynomials γ and δ implies that $\delta = \pm z$ and $\gamma = \pm z$. Indeed, writing $\delta = az + b$ and $\gamma = cz + d$, and arguing as in the proof of Theorem (2.5) with the same notation, we see that if (23) holds, then $b = 0$. Furthermore, since $m - 2 > 0$, we have

$$cc_m a^m = c_m \quad \text{and} \quad cc_{m-2}a^{m-2} = c_{m-2},$$

which implies $a = \pm 1$. Thus, $\delta = \pm z$, and consequently $\gamma = \pm z$. Applying this to equalities (22), we obtain

$$\alpha \circ \mu = \pm z, \quad \beta = \pm z, \quad \beta \circ \mu = \pm z, \quad \alpha = \pm z.$$

Hence $\mu = \pm z$, so by (20) we obtain $A^{\circ l} = \pm T_{d^l}$. □

The following lemma is well-known. For the sake of completeness, we provide a sketch of the proof.

Lemma 4.5. *Let A be a polynomial of degree $d \geq 2$, and let $n > 1$ be an integer. Then there exist at most two complex numbers α such that all but one of the multiplicities of A at the points of $A^{-1}(\alpha)$ are divisible by n. Furthermore, if there exist two such numbers, then $n = 2$ and there exist degree-one polynomials μ and ν such that $A = \mu \circ T_d \circ \nu$.*

Proof. It follows easily from the Riemann–Hurwitz formula that the preimage of a set $K \subset \mathbb{C}$ containing k elements under A contains at least $(k-1)d + 1$ points, with equality if and only if K contains all finite critical values of A. Applying this to the case where $K = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is the set of all complex numbers satisfying the condition of the theorem, we obtain

$$(k-1)d + 1 \leq k\left(\frac{d-1}{n} + 1\right), \tag{24}$$

which implies

$$k(d-1)\left(1 - \frac{1}{n}\right) \leq d - 1 \quad \text{and} \quad k\left(1 - \frac{1}{n}\right) \leq 1.$$

Thus, $k \leq 2$, and if $k = 2$, then $n = 2$ and the inequality in (24) becomes an equality. This implies that α_1 and α_2 are the only critical values of A, and that all multiplicities of A at the points in $A^{-1}\{\alpha_1, \alpha_2\}$ are two, except at two points whose multiplicities are one.

Finally, it is well known that in the last case $A = \mu \circ T_d \circ \nu$ for some degree-one polynomials μ and ν. An elegant way to see this is via the correspondence between equivalence classes of polynomial Belyi functions and plane trees, which is a special case of Grothendieck's theory of dessins d'enfants (see, e.g., [18]). Under this correspondence, the statement reduces to the fact that the only trees whose vertex valencies do not exceed two are "chains." □

The following result is an analogue of Theorem 4.2 with the roles of A and B reversed. Note that it is not symmetric to Theorem 4.2.

Theorem 4.6. *Let A and B be polynomials of degree at least two such that $A \leq B$. Suppose that $A = \widehat{A}^{\circ l}$ for some polynomial $\widehat{A}$ of degree greater than two and some $l \geq 2$. Then there exists a polynomial $\widehat{B}$ such that $B^{\circ e} = \widehat{B}^{\circ le}$ for some $e \geq 1$.*

Proof. In case A is a special, the claim follows from Theorem 2.5, taking into account that for even d the polynomials T_d and $-T_d$ are conjugate, while for d odd

their second iterates are equal. Thus, it suffices to consider the case where A, and hence B, are non-special.

Let X be a polynomial such that $A \underset{X}{\leq} B$. By Lemma 2.4, we can find polynomials B_0, U, and X_0 such that $B \sim B_0$, $X = X_0 \circ U$, the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathbb{CP}^1 & \xrightarrow{B_0} & \mathbb{CP}^1 \\ {\scriptstyle X_0}\downarrow & & \downarrow{\scriptstyle X_0} \\ \mathbb{CP}^1 & \xrightarrow{\widehat{A}^{\circ l}} & \mathbb{CP}^1 \end{array}$$

commutes, and $\widehat{A}^{\circ l}, B_0, X_0$ is a primitive solution of equation (4). Applying Theorem 2.3, without loss of generality, we may assume that

$$X_0 = z^n, \quad B_0 = z^s R(z^n), \quad \widehat{A}^{\circ l} = z^s R^n(z),$$

for some polynomial R, with $n \geq 1$, $s \geq 0$, and $\gcd(s, n) = 1$. Furthermore, we may assume that $n > 1$, since otherwise $\widehat{A}^{\circ l} \sim B$, and the statement of the theorem follows from Corollary 4.3.

Next, applying Theorem 2.1 to the equality

$$\widehat{A} \circ (\widehat{A}^{\circ (l-1)} \circ X_0) = X_0 \circ B_0,$$

we can find polynomials Y, B_1, B_2 such that $B_0 = B_1 \circ B_2$, and the diagram

$$\begin{array}{ccccc} \mathbb{CP}^1 & \xrightarrow{B_2} & \mathbb{CP}^1 & \xrightarrow{B_1} & \mathbb{CP}^1 \\ {\scriptstyle X_0}\downarrow & & {\scriptstyle Y}\downarrow & & {\scriptstyle X_0}\downarrow \\ \mathbb{CP}^1 & \xrightarrow{\widehat{A}^{\circ (l-1)}} & \mathbb{CP}^1 & \xrightarrow{\widehat{A}} & \mathbb{CP}^1 \end{array} \tag{25}$$

commutes. Moreover, since $n = \deg X_0$ is coprime to

$$\deg B_0 = \deg B = \deg A,$$

setting $d = \deg \widehat{A}$, we have

$$\deg B_2 = \gcd(\deg B_0, \deg (\widehat{A}^{\circ (l-1)} \circ X_0)) = \gcd(d^l, d^{l-1} \deg X_0) = d^{l-1} = \deg \widehat{A}^{\circ (l-1)}.$$

Thus,

$$\deg Y = \deg X_0 = n \quad \text{and} \quad \deg B_1 = \deg \widehat{A}.$$

The commutativity of (25) implies that the diagram

$$\begin{array}{ccccc} \mathbb{CP}^1 & \xrightarrow{B_1} & \mathbb{CP}^1 & \xrightarrow{B_2} & \mathbb{CP}^1 \\ {\scriptstyle Y}\downarrow & & {\scriptstyle X_0}\downarrow & & {\scriptstyle Y}\downarrow \\ \mathbb{CP}^1 & \xrightarrow{\widehat{A}} & \mathbb{CP}^1 & \xrightarrow{\widehat{A}^{\circ (l-1)}} & \mathbb{CP}^1 \end{array}$$

also commutes, which in turn yields the commutativity of

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B_2 \circ B_1} & \mathbb{CP}^1 \\ {\scriptstyle Y}\downarrow & & \downarrow{\scriptstyle Y} \\ \mathbb{CP}^1 & \xrightarrow{\widehat{A}^{\circ l}} & \mathbb{CP}^1. \end{array}$$

Since $\deg Y = n$, applying Theorem 2.3 again we conclude that there exist degree-one polynomials μ and ν such that

$$\widehat{A}^{\circ l} = \nu \circ z^{\widetilde{s}} \widetilde{R}^n(z) \circ \nu^{-1}, \quad Y = \nu \circ z^n \circ \mu, \quad B_2 \circ B_1 = \mu^{-1} \circ z^{\widetilde{s}} \widetilde{R}(z^n) \circ \mu,$$

for some polynomial $\widetilde{R}$, with $\widetilde{s} \geq 1$ and $\gcd(\widetilde{s}, n) = 1$. Furthermore, we have

$$\nu(z) = az, \quad a \in \mathbb{C}^*. \tag{26}$$

Indeed, if $\nu(0) \neq 0$, then from

$$\widehat{A}^{\circ l} = z^s R^n(z) = \nu \circ z^{\widetilde{s}} \widetilde{R}^n(z) \circ \nu^{-1},$$

it follows that the conditions of Lemma 4.5 are satisfied for $\alpha = \nu(0)$ and $\alpha = 0$, implying that

$$\widehat{A}^{\circ l} = \mu \circ T_m \circ \nu$$

for some degree-one polynomials μ and ν. Hence, $A = \widehat{A}^{\circ l}$ is special by Lemma 4.4, in contradiction with the assumption.

It follows from (26) that

$$Y = \nu \circ z^n \circ \mu = z^n \circ (\widehat{a} z) \circ \mu,$$

where $\widehat{a}^n = a$. Thus, the commutativity of the right square in (25) implies the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{\widehat{B}_0} & \mathbb{CP}^1 \\ {\scriptstyle z^n}\downarrow & & \downarrow{\scriptstyle z^n} \\ \mathbb{CP}^1 & \xrightarrow{\widehat{A}} & \mathbb{CP}^1 \end{array}$$

where

$$\widehat{B}_0 = B_1 \circ \mu^{-1} \circ (\widehat{a}^{-1} z).$$

Thus,

$$\widehat{A} \underset{z^n}{\leq} \widehat{B}_0, \quad \text{and hence} \quad \widehat{A}^{\circ l} \underset{z^n}{\leq} \widehat{B}_0^{\circ l}.$$

Combined with $\widehat{A}^{\circ l} \underset{z^n}{\leq} B_0$, this yields that

$$z^n \circ \widehat{B}_0^{\circ l} = z^n \circ B_0.$$

Therefore,

$$\widehat{B}_0^{\circ l} = \mu_0 \circ B_0, \tag{27}$$

where $\mu_0 = \varepsilon z$ with $\varepsilon^n = 1$.

Let us now observe that since B_0 has the form $B_0 = z^s R(z^n)$ with $\gcd(s, n) = 1$, an argument similar to that in Lemma 3.6 shows that for $e = n\varphi(n)$, the equality

$$(\mu_0 \circ B_0)^{\circ e} = B_0^{\circ e}$$

holds. This implies, via (27), that

$$\widehat{B}_0^{\circ le} = B_0^{\circ e}.$$

Finally, since any decomposition of B naturally induces a decomposition of $B^{\circ e}$, it follows from $B \sim B_0$ that $B^{\circ e} \sim B_0^{\circ e}$. Therefore,

$$B^{\circ e} \sim \widehat{B}_0^{\circ le},$$

implying by Corollary 4.3 that $B^{\circ e} = \widehat{B}^{\circ le}$ for some polynomial $\widehat{B}$. $\square$

Corollary 4.7. *Let A and B be polynomials of degree at least two such that $A \leq B$. Suppose that $A = \widehat{A}^{\circ l}$ for some polynomial $\widehat{A}$ and some $l \geq 2$. Then there exist a polynomial $\widehat{B}$ such that $B^{\circ 2e} = \widehat{B}^{\circ le}$ for some $e \geq 1$.*

Proof. Since $\widehat{A}^{\circ l} \underset{X}{\leq} B$ implies

$$(\widehat{A}^{\circ 2})^{\circ l} \underset{X}{\leq} B^{\circ 2},$$

applying Theorem 4.6 to the polynomial $\widehat{A}^{\circ 2}$ of degree at least four and the polynomial $B^{\circ 2}$, one concludes that

$$(B^{\circ 2})^{\circ e} = \widehat{B}^{\circ le}$$

for some $e \geq 1$. □

5. Proof of Theorem 1.2

The following result is an algebraic counterpart of Theorem 1.2.

Theorem 5.1. *Let A_1 and A_2 be polynomials of degree greater than two not conjugate to a power, and X_1 and X_2 non-constant polynomials satisfying*

$$A_1^{\circ k} \underset{X_1}{\leq} B, \quad A_2^{\circ k} \underset{X_2}{\leq} B$$

for some polynomial B and $k \geq 1$. Then there exists a polynomial $\widehat{B}$ and $\gamma_1 \in \Sigma(A_1)$, $\gamma_2 \in \Sigma(A_2)$ such that

$$\gamma_1 \circ A_1 \underset{X_1}{\leq} \widehat{B}, \quad \gamma_2 \circ A_2 \underset{X_2}{\leq} \widehat{B}.$$

Moreover, for $N = \mathrm{LCM}(N(A_1), N(A_2))$, we have

$$A_1^{\circ N} \underset{X_1}{\leq} \widehat{B}^{\circ N}, \quad A_2^{\circ N} \underset{X_2}{\leq} \widehat{B}^{\circ N}.$$

Proof. Applying Theorem 4.6 to either of the relations

$$A_1^{\circ k} \underset{X_1}{\leq} B, \quad A_2^{\circ k} \underset{X_2}{\leq} B,$$

we conclude that there exist an integer $e \geq 1$ and a polynomial $\widehat{B}$ such that $B^{\circ e} = \widehat{B}^{\circ ek}$ implying that

$$A_1^{\circ ek} \underset{X_1}{\leq} \widehat{B}^{\circ ek}, \quad A_2^{\circ ek} \underset{X_2}{\leq} \widehat{B}^{\circ ek}. \tag{28}$$

Therefore, by Theorem 4.1, there exist polynomials $\widehat{A}_1$ and $\widehat{A}_2$ such that

$$\widehat{A}_1 \underset{X_1}{\leq} \widehat{B}, \quad \widehat{A}_2 \underset{X_2}{\leq} \widehat{B}. \tag{29}$$

and

$$J(\widehat{A}_1) = J(A_1), \quad J(\widehat{A}_2) = J(A_2).$$

Moreover, since (28) and (29) imply that

$$\deg A_1 = \deg \widehat{A}_1, \quad \deg A_2 = \deg \widehat{A}_2,$$

it follows from Corollary 3.8 that there exist $\gamma_1 \in \Sigma(A_1)$ and $\gamma_2 \in \Sigma(A_2)$ such that

$$\widehat{A}_1 = \gamma_1 \circ A_1, \quad \widehat{A}_2 = \gamma_2 \circ A_2.$$

The relations

$$\gamma_1 \circ A_1 \underset{X_1}{\leq} \widehat{B}, \quad \gamma_2 \circ A_2 \underset{X_2}{\leq} \widehat{B} \tag{30}$$

yield the relations

$$(\gamma_1 \circ A_1)^{\circ ek} \underset{X_1}{\leq} \widehat{B}^{\circ ek}, \quad (\gamma_2 \circ A_2)^{\circ ek} \underset{X_2}{\leq} \widehat{B}^{\circ ek},$$

and together with (28), this shows that $\gamma_1 \circ A_1$ shares an iterate with A_1, and $\gamma_2 \circ A_2$ shares an iterate with A_2. Hence, by Corollary 3.6, for $N = \mathrm{LCM}\big(N(A_1), N(A_2)\big)$, we have

$$(\gamma_1 \circ A_1)^{\circ N} = A_1^{\circ N}, \quad (\gamma_2 \circ A_2)^{\circ N} = A_2^{\circ N},$$

implying by (30) that

$$A_1^{\circ N} \underset{X_1}{\leq} \widehat{B}^{\circ N}, \quad A_2^{\circ N} \underset{X_2}{\leq} \widehat{B}^{\circ N}. \qquad \square$$

Corollary 5.2. *Let A_1 and A_2 be polynomials of degree at least two not conjugate to a power, and X_1 and X_2 non-constant polynomials satisfying*

$$A_1^{\circ k} \underset{X_1}{\leq} B, \quad A_2^{\circ k} \underset{X_2}{\leq} B \tag{31}$$

for some polynomial B and $k \geq 1$. Then there exists a polynomial $\widehat{B}$ and $\gamma_1 \in \Sigma(A_1)$, $\gamma_2 \in \Sigma(A_2)$ such that

$$\gamma_1 \circ A_1^{\circ 2} \underset{X_1}{\leq} \widehat{B}, \quad \gamma_2 \circ A_2^{\circ 2} \underset{X_2}{\leq} \widehat{B}.$$

Furthermore, for $N = \mathrm{LCM}(N(A_1), N(A_2))$, we have

$$A_1^{\circ 2N} \underset{X_1}{\leq} \widehat{B}^{\circ N}, \quad A_2^{\circ 2N} \underset{X_2}{\leq} \widehat{B}^{\circ N}. \tag{32}$$

Proof. Since (31) implies

$$(A_1^{\circ 2})^{\circ k} \underset{X_1}{\leq} B^{\circ 2}, \quad (A_2^{\circ 2})^{\circ k} \underset{X_2}{\leq} B^{\circ 2},$$

where

$$\deg A_1^{\circ 2} = \deg A_2^{\circ 2} \geq 4,$$

the corollary follows from Theorem 5.1 applied to $A_1^{\circ 2}$, $A_2^{\circ 2}$, and $B^{\circ 2}$, taking into account that

$$\Sigma(A_1^{\circ 2}) = \Sigma(A_1), \quad \Sigma(A_2^{\circ 2}) = \Sigma(A_2). \qquad \square$$

Proof of Theorem 1.2. We recall that if A_1 and A_2 are non-special polynomials of degree at least two, then any irreducible (A_1, A_2)-invariant curve C that is neither a vertical nor a horizontal line has genus zero and admits a generically one-to-one parametrization by polynomials X_1 and X_2 such that the diagram

$$\begin{array}{ccc} (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{CP}^1)^2 \end{array} \tag{33}$$

commutes for some polynomial B (see Proposition 2.34 of [19] or Section 4.3 of [24]). Conversely, if diagram (33) commutes, then the curve C parametrized by $t \mapsto (X_1(t), X_2(t))$ is obviously invariant under (A_1, A_2).

Therefore, if C is a periodic curve, that is, if

$$(A_1, A_2)^{\circ k}(C) = C$$

for some $k \geq 1$, then there exist polynomials X_1, X_2, and B such that

$$A_1^{\circ k} \underset{X_1}{\leq} B, \quad A_2^{\circ k} \underset{X_2}{\leq} B. \tag{34}$$

By Theorem 5.1, this implies that

$$A_1^{\circ N} \underset{X_1}{\leq} \widehat{B}^{\circ N}, \quad A_2^{\circ N} \underset{X_2}{\leq} \widehat{B}^{\circ N}$$

for some polynomial $\widehat{B}$. Thus,

$$(A_1, A_2)^{\circ N}(C) = C. \qquad \square$$

Corollary 5.3. *Let A_1 and A_2 be non-special polynomials of degree $d \geq 2$, and C an irreducible (A_1, A_2)-periodic curve. Then its period divides* $2\mathrm{LCM}(N(A_1), N(A_2))$.

Proof. Applying to equalities (34) Corollary 5.2 instead of Theorem 5.1, we conclude that equalities (32) hold. $\square$

6. Semiconjugacies between iterates and proof of Theorem 1.1

The main results of this section are Theorems 6.3 and 6.4, which allow us to control polynomial solutions of

$$A^{\circ l} \underset{X}{\leq} B^{\circ s}, \qquad s, l \geq 1,$$

when one of the polynomials A or B is fixed, while the other, together with the integers s and l, may vary. Theorem 1.1 then follows easily from these results.

We begin with the following two theorems.

Theorem 6.1. *For any $d \geq 2$, there exists a constant $r(d)$ such that for every polynomial B of degree d, there are polynomials $A_1, A_2, \dots, A_r$ with $r \leq r(d)$ satisfying the following property: for a polynomial A, the relation $A \leq B$ implies that A is conjugate to one of $A_1, A_2, \dots, A_r$.*

Proof. If B is special, the claim follows from Theorem 2.5. For non-special B, Theorem 6.1 is a particular case of Theorem 1.1 in [28], which establishes the result in the broader context of rational functions. In the polynomial case, Theorem 6.1 also follows from Theorem 1.6 in [24]. $\square$

Theorem 6.2. *For any $d \geq 2$, there exists a constant $r(d)$ such that for every polynomial A of degree d, there are polynomials $B_1, B_2, \dots, B_r$ with $r \leq r(d)$ satisfying the following property: for a polynomial B, the relation $A \leq B$ implies that B is conjugate to one of $B_1, B_2, \dots, B_r$.*

Proof. It follows from Lemma 2.4 that to prove the theorem it suffices to establish the following two statements. First, for any $d \geq 2$, there exists a constant $r_1(d)$ such that for every polynomial A of degree d, there are polynomials $C_1, C_2, \dots, C_r$ with $r \leq r_1(d)$ such that if polynomials A, B_0, and X_0 form a primitive solution of (4), then B_0 is conjugate to one of $C_1, C_2, \dots, C_r$. Second, for any $d \geq 2$, there exists a constant $r_2(d)$ such that for every polynomial C of degree d, there are polynomials $B_1, B_2, \dots, B_r$ with $r \leq r_2(d)$ such that $C \sim B$ implies that B is conjugate to one of $B_1, B_2, \dots, B_r$.

The second statement follows from Theorem 6.1, since $C \sim B$ implies $B \leq C$. If A is special, the first statement follows from Theorem 2.5, while for non-special A it can be deduced from Theorem 2.3 as follows. If A is non-special and A, B_0, X_0 form a primitive solution of (4), then by Theorem 2.3 there exist degree-one polynomials μ and ν such that

$$A = \nu \circ z^s R^n(z) \circ \nu^{-1}, \quad X_0 = \nu \circ z^n \circ \mu, \quad B_0 = \mu^{-1} \circ z^s R(z^n) \circ \mu, \tag{35}$$

where R is a polynomial, $n \geq 1$, $s \geq 0$, and $\gcd(s, n) = 1$. Furthermore, $n < d$ implies that n can take only finitely many values among different solutions. In addition, if $n = 1$, then A and B_0 are conjugate. On the other hand, if $n > 1$, then Lemma 4.5 shows that $\nu(0)$ can take at most two distinct values in $\mathbb{C}$ among different solutions. Thus, it suffices to prove the first statement for fixed $n > 1$ and $\nu(0)$.

Let us assume that (35) is some primitive solution of (4), and let A, $\widetilde{X}_0$, $\widetilde{B}_0$ be another primitive solution satisfying the above conditions. Then

$$\widetilde{X}_0 = \widetilde{\nu} \circ z^n \circ \widetilde{\mu}$$

for some degree-one polynomials $\widetilde{\mu}, \widetilde{\nu}$. Therefore, since $\widetilde{\nu}(0) = \nu(0)$ by assumption,

$$\widetilde{X}_0 = X_0 \circ \delta$$

for some degree-one polynomial δ. Thus,

$$A \circ \widetilde{X}_0 = A \circ X_0 \circ \delta = X_0 \circ B_0 \circ \delta.$$

On the other hand,

$$A \circ \widetilde{X}_0 = \widetilde{X}_0 \circ \widetilde{B}_0 = X_0 \circ \delta \circ \widetilde{B}_0.$$

Hence,

$$X_0 \circ \delta \circ \widetilde{B}_0 = X_0 \circ B_0 \circ \delta. \tag{36}$$

Let μ_i, $1 \leq i \leq n$, be all Möbius transformations satisfying

$$X_0 \circ \mu = X_0.$$

Then (36) implies that

$$\delta \circ \widetilde{B}_0 = \mu_i \circ B_0 \circ \delta$$

for some i, $1 \leq i \leq n$. Consequently,

$$\widehat{B}_0 = \delta^{-1} \circ \mu_i \circ B_0 \circ \delta$$

for some i, $1 \leq i \leq n$. Thus, if polynomials A, B_0, and X_0 form a primitive solution of (4) with fixed $n > 1$ and $\nu(0)$, then, up to conjugacy, there are at most n possibilities for B_0. Since $n < d$, the theorem follows. □

Theorem 6.3. *For any $d \geq 2$, there exists a constant $r(d)$ such that for every non-special polynomial R of degree d, there are polynomials $B_1, \ldots, B_r$ of degree at most d with $r \leq r(d)$ satisfying the following property: for a polynomial B, the relation $B \leq \widehat{R}$, where $\widehat{R}$ is a polynomial with $J(\widehat{R}) = J(R)$, implies that B is conjugate to $\mu \circ B_i^{\circ n}$ for some i, $1 \leq i \leq r$, some $\mu \in \Sigma(B_i)$, and some $n \geq 1$.*

Proof. By Theorem 4.1, the relation $B \leq \widehat{R}$ implies that there exists a polynomial $\widehat{B}$ such that

$$\widehat{B} \leq R \qquad \text{and} \qquad J(\widehat{B}) = J(B).$$

On the other hand, by Theorem 6.1, there exist $r(d)$ and polynomials $\widehat{B}_1, \widehat{B}_2, \dots, \widehat{B}_r$ of degree d with $r \le r(d)$ such that $\widehat{B} \le R$ implies that $\widehat{B}$ is conjugate to one of $\widehat{B}_1, \widehat{B}_2, \dots, \widehat{B}_r$. Thus, the relation $B \le \widehat{R}$ implies that

$$J(B) = J\Big(\alpha \circ \widehat{B}_i \circ \alpha^{-1}\Big)$$

for some i, $1 \le i \le r$, and some polynomial α of degree one, or, equivalently, that

$$J\big(\alpha^{-1} \circ B \circ \alpha\big) = J(\widehat{B}_i). \tag{37}$$

Since R is non-special, Theorem 2.5 implies that B and $\widehat{B}_1, \widehat{B}_2, \dots, \widehat{B}_r$ are also non-special. Therefore, by Theorem 3.7, there are polynomials $B_1, B_2, \dots, B_r$ of degree at most d such that the equality

$$J(P) = J(\widehat{B}_i), \qquad 1 \le i \le r,$$

for a polynomial P implies that

$$P = \delta \circ B_i^{\circ s}, \qquad 1 \le i \le r,$$

for some $\delta \in \Sigma(B_i)$ and some $s \ge 1$. By (37), this completes the proof. □

Theorem 6.4. *For any $d \ge 2$, there exists a constant $r(d)$ such that for every polynomial A of degree d, there are polynomials $R_1, R_2, \dots, R_r$ of degree at most d^2 with $r \le r(d)$ satisfying the following property: for any polynomial R, the relation $A^{\circ l} \le R$, where $l \ge 1$, implies that R is conjugate to $\mu \circ R_i^{\circ n}$ for some i, $1 \le i \le r$, some $\mu \in \Sigma(R_i)$, and some $n \ge 1$.*

Proof. If A is special, Theorem 2.5 implies that the conclusion of the theorem holds for $r = 1$ and $R_1 = A$. Thus, we may assume that A is not special.

By Corollary 4.7, the relation $A^{\circ l} \le R$, implies there exist an integer $e \ge 1$ and a polynomial T such that

$$R^{\circ 2e} = T^{\circ el}. \tag{38}$$

Hence,

$$A^{\circ 2le} \le T^{\circ el},$$

implying by Theorem 4.1, that there exists a polynomial $\widehat{A}$ with $J(\widehat{A}) = J(A)$ such that $\widehat{A} \le T$. Furthermore, since $\deg \widehat{A} = \deg A^{\circ 2}$, it follows from Corollary 3.8 that

$$\widehat{A} = \gamma \circ A^{\circ 2}$$

for some $\gamma \in \Sigma(A)$.

Since Theorem 3.4 yields an explicit bound $|\Sigma(A)| \le d$, it follows from Theorem 6.2 that there exist $r(d)$ and polynomials $T_1, T_2, \dots, T_r$ of degree d^2 with $r \le r(d)$ such that

$$\gamma \circ A^{\circ 2} \le T$$

implies that T is conjugate to one of $T_1, T_2, \dots, T_r$. Since (38) implies $J(R) = J(T)$, we conclude that if R satisfies $A^{\circ l} \le R$, then

$$J(R) = J\big(\alpha \circ T_i \circ \alpha^{-1}\big)$$

for some i, $1 \le i \le r$, and some polynomial α of degree one, or, equivalently, that

$$J\big(\alpha^{-1} \circ R \circ \alpha\big) = J(T_i). \tag{39}$$

Since A is not special, Theorem 2.5 implies that the polynomials R, T, and $T_1, T_2, \dots, T_r$ are not special either. Therefore, by Theorem 3.7, there exist polynomials $R_1, R_2, \dots, R_r$ of degree at most d^2 such that the equality

$$J(P) = J(T_i), \quad 1 \le i \le r,$$

for a polynomial P implies that

$$P = \delta \circ R_i^{\circ s}$$

for some $\delta \in \Sigma(R_i)$ and some $s \ge 1$. By (39), this completes the proof. □

Proof of Theorem 1.1. The condition $B \in \mathrm{Inter}(A)$ means that

$$A^{\circ k} \le R, \quad B^{\circ l} \le R \tag{40}$$

for some polynomial R and integers $k, l \ge 1$. If A is special, Theorem 2.5 implies that the theorem holds with $r = 1$, where B_1 may be any polynomial that shares its Julia set with A, and whose degree d_0 is the minimal natural number such that $d = d_0^k$ for some $k \ge 1$. Thus, it is enough to prove the theorem for non-special A.

By Theorem 6.4, there exist polynomials $R_1, R_2, \dots, R_r$ of degree at most d^2 with $r \le r_1(d)$, where $r_1(d)$ depends only on d, such that the first relation in (40) implies that R is conjugate to $\delta \circ R_i^{\circ n}$ for some i, $1 \le i \le r$, some $\delta \in \Sigma(R_i)$, and some $n \ge 1$. On the other hand, Theorem 6.3 yields that for each fixed i, $1 \le i \le r$, there exist polynomials $B_1, \dots, B_{r'}$ with $r' \le r_2(d)$, where $r_2(d)$ depends only on d, such that the relation

$$B^{\circ l} \le \delta \circ R_i^{\circ n},$$

where $\delta \in \Sigma(R_i)$ and $n \ge 1$, implies that B is conjugate to $\mu \circ B_j^{\circ n}$ for some j, $1 \le j \le r'$, some $\mu \in \Sigma(B_j)$, and some $n \ge 1$.

Since, for any degree-one polynomial γ and any polynomial U, the set of polynomials U satisfying $U \le V$ coincides with the set of polynomials U satisfying $U \le \gamma \circ V \circ \gamma^{-1}$, the theorem follows. □

Notice that the definition of the group $\Sigma(A)$ for a polynomial A given in [10] is formally different from ours. However, these definitions coincide whenever A is not conjugate to z^d (see Definition 3.1 and Proposition 3.9 in [10]). In case A is conjugate to z^d, according to [10], the group $\Sigma(A)$ is defined as the group of all roots of unity. Obviously, Theorem 1.1 remains valid under this interpretation of $\Sigma(A)$ as well.

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