TURBULENT HOLOMORPHIC FOLIATIONS ON COMPACT COMPLEX TORI AND TRANSVERSELY HOLOMORPHIC CARTAN GEOMETRY

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Abstract. We define a class of nonsingular holomorphic foliations on compact complex tori which generalizes (in higher codimension) the turbulent foliations of codimension one constructed by Ghys in [Gh]. For those smooth turbulent foliations we prove that all transversely holomorphic Cartan geometries are flat. We also establish a uniqueness result for the transversely holomorphic Cartan geometries.

1. Introduction

This article studies transversely holomorphic Cartan geometries on nonsingular holomorphic foliations on compact complex tori (i.e., compact complex manifolds which are biholomorphic to a quotient $\mathbb{C}^d/\Lambda$, with Λ being a normal lattice in $\mathbb{C}^d$, with $d \geq 2$).

A first motivation of our study comes from the following classification result for nonsingular codimension one holomorphic foliations on compact complex tori, proved by Ghys in [Gh]: Any nonsingular *codimension one* holomorphic foliation on a compact complex torus is either *linear* (meaning it is defined as the kernel of a global nonzero holomorphic one-form on the torus) or it is *turbulent*.

A codimension one holomorphic foliation on a compact complex torus A is *turbulent* in Ghys' sense if it is defined by the kernel of a closed *meromorphic* one-form constructed as follows. Assume that there exists a holomorphic submersion $\pi : A \longrightarrow T$, over an elliptic curve T and consider the closed meromorphic one-form $\eta = (\pi^*\omega) + \beta$, where ω is a meromorphic one-form on T and β is a (translation invariant) holomorphic one-form on the torus A that does not vanish on the fibers of the fibration π. The kernel of η defines (away from the polar divisor of η) a holomorphic codimension one foliation which extends to a nonsingular holomorphic foliation on the torus A (see [Gh]). Notice that in the case where the holomorphic one-form β vanishes on the fibers of π, the turbulent foliation degenerates to the fibration π.

The dynamics of the turbulent foliation was described in [Gh] (see also [Br2]): the inverse image through the fibration π of the polar divisor of the meromorphic one-form ω is a finite union of compact leafs and all other leafs are noncompact and accumulate on every compact leaf.

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Codimension one linear foliations on compact complex tori — being defined by a holomorphic one-form β (which is automatically closed and translation invariant) — are necessarily translation invariant and admit a transversely complex translation structure. The local (transverse) charts of the corresponding transversely complex translation structure are given by the local primitives of β (seen as local submersions $f_i : U_i \longrightarrow \mathbb{C}$ that locally define the foliation): the transition maps between two such local charts is necessarily given by the post-composition with a translation. More precisely, for each connected component of a nonempty intersection $U_i \cap U_j$ there exists a constant c_{ij} such that $f_i = f_j + c_{ij}$.

Recall that Lie's classification of dimension one homogeneous spaces implies that the possible transverse structures for a codimension one foliation are exactly of the following three types: translation structures, affine structures and projective structures. Codimension one holomorphic foliations which are transversely complex affine or transversely complex projective were studied by several authors (see, [Se, Sc, LP]).

In this context it was proved in [BD3] that there exists codimension one turbulent foliations on compact complex tori which do not admit any transversely complex projective structure. More precisely, the main result in [BD3] asserts that on the product of two elliptic curves, a generic smooth (nonsingular) turbulent foliation does not admit any transversely complex projective structure [BD3]. Indeed, for a polar part of the above meromorphic one-form ω of degree ≥ 8, generic turbulent foliations do not admit any transversely complex projective structure [BD3]. The proof of [BD3] employed a dimension counting argument, based on a proposition proving that smooth turbulent codimension one foliations admit at most one transversely complex projective structure. To the best of our knowledge these were the first known examples of nonsingular codimension one holomorphic foliations on a complex projective manifold admitting no transversely complex projective structures. Recall also that all smooth fibrations (submersions) over a Riemann surface admit a nonsingular transversely projective structure which is a consequence of the uniformization theorem for Riemann surfaces applied to the base of the fibration (which happens to be here a global transversal of the foliation).

It should be emphasized that those results in [BD3] deal with *regular* (meaning *nonsingular*) complex projective structures. Indeed, codimension one turbulent foliations on complex tori, being defined by global closed meromorphic one-forms η, admit a *singular* transversely complex translation structure, which in turn induces a singular transversely complex projective structure in the sense of [LPT]. Moreover, a consequence of Brunella's classification of nonsingular holomorphic foliations on compact complex surfaces, [Br1], says that all those foliations admit a singular transversely complex projective structure.

A recent result of Fazoli, Melo and Pereira, [FMP, Theorem 4.4], classifies all smooth (nonsingular) turbulent foliations on Kähler surfaces which do admit a regular (nonsingular) complex projective structure.

Our article here deals with smooth (nonsingular) holomorphic turbulent foliations, with arbitrary codimension, on compact complex tori (this notion is defined in Section 2). Our main result is that for such a smooth turbulent foliation, all transversely holomorphic Cartan geometries are *flat* (see Theorem 4.1). Recall that this means that the foliation is transversely modelled on a complex homogeneous space G/H, with G a complex Lie group and H a closed subgroup in G. It is equivalent to say that in an adapted holomorphic foliated atlas, the foliation is locally given by the level sets of local submersions to the complex homogeneous space G/H which differ on the overlaps of two open sets in the atlas by the post-composition with an element in G acting on G/H. The one dimensional homogeneous space $\mathrm{P}^1(\mathbb{C})$ acted on by $G = \mathrm{PSL}(2, \mathbb{C})$ (and with H being a maximal parabolic subgroup in G) corresponds to transversely complex projective structures (for foliations of codimension one). It should be emphasized that the codimension one is particular since all Cartan geometry are flat in dimension one and, consequently, all transversely Cartan geometries are automatically flat for codimension one foliations. In general, this is not the case anymore in higher codimension.

Given a holomorphic principal H–bundle E_H over a compact complex torus A equipped with a flat partial connection in the direction of a smooth holomorphic turbulent foliation $\mathcal{F}$, we also prove in Theorem 4.1 that for any given homogeneous model space G/H, the foliated torus $(A, \mathcal{F})$ admits at most one compatible transversely holomorphic G/H-structure. This generalizes to turbulent foliations — of arbitrary codimension and to any transversely holomorphic Cartan geometry — the uniqueness result proved in [BD3] for codimension one foliations and transversely complex projective structures.

Consider, for example, on a compact complex torus A a codimension one smooth holomorphic turbulent foliation $\mathcal{F}$ which admits a transversely complex projective structure (for tori of complex dimension two, those foliations were completely characterized in [FMP, Theorem 4.4]). Then the product $A \times A$ inherits a codimension two holomorphic foliation given by $\mathcal{F} \oplus \mathcal{F}$ which is transversely modelled on $\mathrm{P}^1(\mathbb{C}) \times \mathrm{P}^1(\mathbb{C})$. As a particular case of our Theorem 4.1 this transversely $\mathrm{P}^1(\mathbb{C}) \times \mathrm{P}^1(\mathbb{C})$-structure is unique (up to the choice of a natural principal bundle endowed with a flat partial connection along the foliation). Moreover, any transversely holomorphic Cartan geometry with infinitesimal model $G = \mathrm{PSL}(2, \mathbb{C}) \times \mathrm{PSL}(2, \mathbb{C})$ and $H = B \times B$, where B is the maximal parabolic subgroup in $\mathrm{PSL}(2, \mathbb{C})$, is flat and defines a transversely $\mathrm{P}^1(\mathbb{C}) \times \mathrm{P}^1(\mathbb{C})$-structure.

Moreover, Theorem 4.1 stands true also for branched Cartan geometries. Therefore one is allowed to apply Theorem 4.1 to transversely Cartan geometries having mild singularities (of *branched type*): the notion of *branched Cartan geometry* was introduced and studied in [BD] (see also the survey [BD2]).

A natural question left unsolved is to classify all smooth turbulent foliations on compact complex tori that admit a transversely holomorphic (or branched) Cartan geometry. Theorem 4.1 proves that the transversely Cartan geometry must be flat.

Let us mention some related works studying turbulent foliations. A more general notion of a possible singular turbulent foliation was defined and studied in [Br2], where Brunella classified all (possibly singular) codimension one foliations on compact complex tori (see also [PS] for a study of the space of all possible singular turbulent foliations having a fixed tangency divisor with respect to a given elliptic fibration). A general study of smooth holomorphic foliations on compact homogeneous Kähler manifolds was carried out in [LBP].

The organization of this article is as follows. Section 2 introduces a new class of subbundles (of arbitrary rank) in the holomorphic tangent bundle TA of a compact complex torus A. Those, so-called, *generating* subbundles are in some sense the opposite of the translation invariant subbundles. In particular, the corank one integrable generating subbundles in TA coincide with the tangent spaces of Ghys turbulent foliations. We prove vanishing properties for generating subbundles (see Lemma 2.1) which will be applied in the sequel to the tangent spaces of smooth turbulent foliations. Section 3 defines a smooth turbulent foliation on a compact complex torus as determined by an integrable generating subbudle in TA. We present the framework of transversely Cartan geometry and transversely branched Cartan geometry as introduced and developed in [BD, BD2]. Section 4 contains the proof of the main result, Theorem 4.1.

2. Generating distributions on a torus

Let A be a compact complex torus of complex dimension d. Denote by TA the holomorphic tangent bundle of A. Consider the trivial holomorphic vector bundle $A\times H^0(A,\, TA) \longrightarrow A$ with fiber $H^0(A,\, TA)$. Let

$$\Phi\ :\ A\times H^0(A,\, TA)\ \longrightarrow\ TA \tag{2.1}$$

be the evaluation map that sends any $(x,\, s)\ \in\ A\times H^0(A,\, TA)$ to $s(x)\ \in\ T_xA$. This homomorphism Φ is an isomorphism, because TA is holomorphically trivial.

Take a holomorphic subbundle

$$\mathcal{F}\ \subset\ TA \tag{2.2}$$

of rank r with $1\,\leq\, r\,\leq\, d-1$. So $\Phi^{-1}(\mathcal{F})\ \subset\ A\times H^0(A,\, TA)$ is a holomorphic subbundle, where Φ is the isomorphism in (2.1). For each $x\,\in\, A$, consider the subspace

$$\Phi^{-1}(\mathcal{F})_x\ \subset\ (A\times H^0(A,\, TA))_x\ =\ H^0(A,\, TA).$$

Let

$$G(\mathcal{F})\ :=\ \text{Span}\,\{\Phi^{-1}(\mathcal{F})_x\}_{x\in A}\ \subset\ H^0(A,\, TA) \tag{2.3}$$

be the subspace of $H^0(A,\, TA)$ generated by its subspaces $\{\Phi^{-1}(\mathcal{F})_x\}_{x\in A}$.

The subbundle $\mathcal{F}$ in (2.2) will be called *generating* if $G(\mathcal{F})\ =\ H^0(A,\, TA)$, where $G(\mathcal{F})$ is constructed in (2.3).

For a subbundle $\mathcal{F}\ \subset\ TA$ as in (2.2), let

$$\mathcal{N}\ :=\ (TA)/\mathcal{F} \tag{2.4}$$

be the quotient bundle. Let

$$q\ :\ TA\ \longrightarrow\ (TA)/\mathcal{F}\ =:\ \mathcal{N} \tag{2.5}$$

be the quotient map.

Fix a Kähler form ω_A on A in order to define the notion of *degree* for torsionfree coherent analytic sheaves on A. So, for a torsionfree coherent analytic sheaf E on A,

$$\text{degree}(E)\ :=\ \int_A c_1(E)\wedge\omega_A^{d-1}\ \in\ \mathbb{R}.$$

For a holomorphic vector bundle V on A, let

$$0 \,=\, V_0 \,\subset\, V_1 \,\subset\, \cdots \,\subset\, V_{\ell-1} \,\subset\, V_\ell \,=\, V$$

be the Harder–Narasimhan filtration of V (see [HL, p. 16, Theorem 1.3.4] for Harder–Narasimhan filtration). Then

$$\mu_{\max}(V) \,:=\, \frac{\text{degree}(V_1)}{\text{rank}(V_1)}, \quad \mu_{\min}(V) \,:=\, \frac{\text{degree}(V/V_{\ell-1})}{\text{rank}(V/V_{\ell-1})}. \tag{2.6}$$

Lemma 2.1. *Let $\mathcal{F}$ as in* (2.2) *be a generating subbundle. Let V be a holomorphic vector bundle on A with $\mu_{\max}(V) \leq 0$ (see* (2.6)*). Then*

$$H^0(A,\, V \otimes \bigwedge\nolimits^j \mathcal{N}^*) \,=\, 0$$

for all $j \geq 1$, where $\mathcal{N}$ is constructed in (2.4).

Proof. We will first prove that

$$\mu_{\min}(\mathcal{N}) \,>\, 0 \tag{2.7}$$

(see (2.6)). Let

$$0 \,=\, \mathcal{N}_0 \,\subset\, \mathcal{N}_1 \,\subset\, \cdots \,\subset\, \mathcal{N}_{m-1} \,\subset\, \mathcal{N}_m \,=\, \mathcal{N}$$

be the Harder–Narasimhan filtration of $\mathcal{N}$. Since $\mathcal{N}$ is a quotient of TA (see (2.5)), and $\mathcal{N}/\mathcal{N}_{m-1}$ is a quotient of $\mathcal{N}$, it follows that $\mathcal{N}/\mathcal{N}_{m-1}$ is a quotient of TA. As TA is semistable of degree zero, its quotient $\mathcal{N}/\mathcal{N}_{m-1}$ has the following property:

$$\mu_{\min}(\mathcal{N}) \,=\, \frac{\text{degree}(\mathcal{N}/\mathcal{N}_{m-1})}{\text{rank}(\mathcal{N}/\mathcal{N}_{m-1})} \,\geq\, 0. \tag{2.8}$$

In view of (2.8), to prove (2.7) by contradiction, assume the following:

$$\text{degree}(\mathcal{N}/\mathcal{N}_{m-1}) \,=\, 0. \tag{2.9}$$

If the rank of $\mathcal{N}/\mathcal{N}_{m-1}$ is b, choose a subspace $W \,\subset\, H^0(A,\, TA)$ of dimension b such that the composition of maps

$$A \times W \,\xrightarrow{\sigma}\, TA \,\longrightarrow\, \mathcal{N}/\mathcal{N}_{m-1} \tag{2.10}$$

is surjective over some point x_0 of A; the homomorphism σ in (2.10) is the evaluation map that sends any $(x,\, w) \,\in\, A \times W$ to $w(x) \,\in\, T_xA$ (see (2.1)). Consequently, the composition of maps in (2.10) is surjective over a Zariski open subset U of A containing x_0. Let

$$\varphi \,:\, \mathcal{W} \,:=\, A \times W \,\longrightarrow\, \mathcal{N}/\mathcal{N}_{m-1} \tag{2.11}$$

be the composition of maps in (2.10). It induces a homomorphism between the determinant line bundles

$$\det\varphi \,:\, \det\mathcal{W} \,=\, \bigwedge\nolimits^b \mathcal{W} \,\longrightarrow\, \det(\mathcal{N}/\mathcal{N}_{m-1})$$

(see [Ko, Ch. V, § 6] for the construction of the determinant line bundle). Since $\det\varphi$ does not vanish on the open subset U over which φ is surjective, it follows that

$$\text{degree}(\det(\mathcal{N}/\mathcal{N}_{m-1})) - \text{degree}(\det\mathcal{W}) \,=\, \text{degree}(\text{Divisor}(\det\varphi)).$$

Therefore, we have $\text{degree}(\text{Divisor}(\det\varphi)) \,=\, \text{degree}(\det(\mathcal{N}/\mathcal{N}_{m-1}))$, because $\mathcal{W}$ is a trivial vector bundle (see (2.11)). Hence from (2.9) we have $\text{degree}(\text{Divisor}(\det\varphi)) \,=\, 0$. But this implies that $\text{Divisor}(\det\varphi) \,=\, 0$ (note that $\text{Divisor}(\det\varphi)$ is an effective divisor). Since

Divisor$(\det \varphi) = 0$, it follows that φ is an isomorphism. Consequently, $\mathcal{N}/\mathcal{N}_{m-1}$ is a trivial vector bundle.

As noted before, $\mathcal{N}/\mathcal{N}_{m-1}$ is a quotient of TA. Let

$$\mathcal{K} \subset TA$$

be the kernel of the quotient map $TA \longrightarrow \mathcal{N}/\mathcal{N}_{m-1}$, so we have a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow TA \longrightarrow \mathcal{N}/\mathcal{N}_{m-1} \longrightarrow 0. \tag{2.12}$$

Note that we have

$$\mathcal{F} \subset \mathcal{K}, \tag{2.13}$$

because $\mathcal{N}/\mathcal{N}_{m-1}$ is a quotient of $(TA)/\mathcal{F} = \mathcal{N}$ (see (2.4)). Since φ is an isomorphism, we have

$$TA = \mathcal{K} \oplus \sigma(\mathcal{W})$$

(see (2.10) for σ). As TA is a trivial vector bundle, and $\mathcal{K}$ is a direct summand of it, we conclude that $\mathcal{K}$ is also a trivial vector bundle [At1, p. 315, Theorem 3].

Since $\mathcal{K}$ is trivial, it follows that $H^0(A, \mathcal{K})$ generates $\mathcal{K}$ using the evaluation map (see (2.1)). Hence from (2.13) it follows that $\mathcal{F}$ is contained in the subsheaf of TA generated by $H^0(A, \mathcal{K}) \subset H^0(A, TA)$. Since $\mathcal{F}$ is generating, this implies that $\mathcal{K} = TA$. Hence from (2.12) it follows that $\mathcal{N}/\mathcal{N}_{m-1} = 0$.

But this is a contradiction because $\mathcal{N}/\mathcal{N}_{m-1} \neq 0$. Since (2.9) led to this contradiction, we conclude that (2.7) holds.

From (2.7) it follows that

$$\mu_{\min}(\bigwedge^j \mathcal{N}) \geq j \cdot \mu_{\min}(\mathcal{N}) > 0$$

for all $1 \leq j \leq \operatorname{rank}(\mathcal{N}) = d - r$. Hence we have

$$\mu_{\max}(\bigwedge^j \mathcal{N}^*) = -\mu_{\min}(\bigwedge^j \mathcal{N}) < 0 \tag{2.14}$$

for all $1 \leq j \leq d - r$.

The given condition that $\mu_{\max}(V) \leq 0$ and (2.14) combine together to imply that

$$\mu_{\max}(V \otimes \bigwedge^j \mathcal{N}^*) = \mu_{\max}(V) + \mu_{\max}(\bigwedge^j \mathcal{N}^*) < 0$$

for all $1 \leq j \leq d - r$. From this it follows immediately that

$$H^0(A, V \otimes \bigwedge^j \mathcal{N}^*) = 0$$

for all $j \geq 1$. $\square$

3. Turbulent foliation and transversely branched Cartan geometry

Branched Cartan geometries on foliated manifolds were introduced in [BD]. In this section we recall their definition.

3.1. **Partial connection along a foliation.** Assume that $\mathcal{F}$ in (2.2) is closed under the operation of Lie bracket of vector fields, or in other words, $\mathcal{F}$ is a holomorphic foliation on A.

A nonsingular holomorphic foliation $\mathcal{F}$ on a compact complex torus A such that its holomorphic tangent bundle is a generating subbundle in TA is called a *smooth turbulent foliation.*

It should be emphasized that while the above notion of turbulent foliation coincides with that of Ghys for codimension one smooth foliations, it is less general that the definition adopted in [Br2, PS] where the authors work with possibly singular turbulent foliations.

Let H be a complex Lie group. Its Lie algebra will be denoted by $\mathfrak{h}$. Let

$$p : E_H \longrightarrow A \tag{3.1}$$

be a holomorphic principal H–bundle on A. Let

$$\mathrm{d}p : TE_H \longrightarrow p^*TA \tag{3.2}$$

be the differential of the projection map p in (3.1). The quotient

$$\mathrm{ad}(E_H) := \mathrm{kernel}(\mathrm{d}p)/H \longrightarrow A$$

is the adjoint vector bundle for E_H. The quotient

$$\mathrm{At}(E_H) := (TE_H)/H \longrightarrow A$$

is known as the Atiyah bundle for E_H [At2]. Consider the short exact sequence of holomorphic vector bundles on E_H

$$0 \longrightarrow \mathrm{kernel}(\mathrm{d}p) \longrightarrow \mathrm{T}E_H \xrightarrow{\mathrm{d}p} p^*TA \longrightarrow 0.$$

Taking its quotient by H, we get the following short exact sequence of vector bundles on A

$$0 \longrightarrow \mathrm{ad}(E_H) \xrightarrow{\iota''} \mathrm{At}(E_H) \xrightarrow{\widehat{\mathrm{d}p}} TA \longrightarrow 0, \tag{3.3}$$

where $\widehat{\mathrm{d}p}$ is constructed from $\mathrm{d}p$. Now define the subbundle

$$\mathrm{At}_{\mathcal{F}}(E_H) := (\widehat{\mathrm{d}p})^{-1}(\mathcal{F}) \subset \mathrm{At}(E_H). \tag{3.4}$$

So from (3.3) we get the short exact sequence

$$0 \longrightarrow \mathrm{ad}(E_H) \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H) \xrightarrow{\mathrm{d}'p} \mathcal{F} \longrightarrow 0\,, \tag{3.5}$$

where $\mathrm{d}'p$ is the restriction of $\widehat{\mathrm{d}p}$ in (3.3) to the subbundle $\mathrm{At}_{\mathcal{F}}(E_H)$.

A partial holomorphic connection on E_H in the direction of $\mathcal{F}$ is a holomorphic homomorphism

$$\theta : \mathcal{F} \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H)$$

such that $\mathrm{d}'p \circ \theta = \mathrm{Id}_{\mathcal{F}}$, where $\mathrm{d}'p$ is the homomorphism in (3.5). Giving such a homomorphism θ is equivalent to giving a homomorphism $\varpi : \mathrm{At}_{\mathcal{F}}(E_H) \longrightarrow \mathrm{ad}(E_H)$ such that the composition of maps

$$\mathrm{ad}(E_H) \hookrightarrow \mathrm{At}_{\mathcal{F}}(E_H) \xrightarrow{\varpi} \mathrm{ad}(E_H)$$

coincides with the identity map of $\mathrm{ad}(E_H)$, where the inclusion of $\mathrm{ad}(E_H)$ in $\mathrm{At}_{\mathcal{F}}(E_H)$ is the injective homomorphism in (3.5).

Given a partial connection $\theta : \mathcal{F} \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H)$, and any two locally defined holomorphic sections s_1 and s_2 of $\mathcal{F}$, consider the locally defined section $\varpi([\theta(s_1),\, \theta(s_2)])$ of $\mathrm{ad}(E_H)$. This defines an $\mathcal{O}_A$–linear homomorphism

$$\mathcal{K}(\theta) \in H^0(A,\, \mathrm{Hom}(\bigwedge^2 \mathcal{F},\, \mathrm{ad}(E_H))) \,=\, H^0(A,\, \mathrm{ad}(E_H) \otimes \bigwedge^2 \mathcal{F}^*),$$

which is called the *curvature* of the connection θ. The connection θ is called *flat* (or *integrable*) if $\mathcal{K}(\theta)$ vanishes identically.

The image of a flat connection θ in the subbundle $\mathrm{At}_{\mathcal{F}}(E_H)$ is a H-invariant foliation lifting $\mathcal{F}$.

Recall that a (flat partial) connection on the principal H-bundle E_H induces a canonical (flat partial) connection on any associated bundle obtained through a representation of the structural group H. In particular, it induces a (flat partial) connection on the adjoint bundle $\mathrm{ad}(E_H)$.

For any partial connection θ on E_H, let

$$\theta' \,:\, \mathcal{F} \,\longrightarrow\, \mathrm{At}(E_H) \tag{3.6}$$

be the homomorphism given by the composition of maps

$$\mathcal{F} \,\xrightarrow{\theta}\, \mathrm{At}_{\mathcal{F}}(E_H) \,\hookrightarrow\, \mathrm{At}(E_H).$$

Note that from (3.3) we have an exact sequence

$$0 \longrightarrow \mathrm{ad}(E_H) \xrightarrow{\iota'} \mathrm{At}(E_H)/\theta'(\mathcal{F}) \xrightarrow{\widehat{\mathrm{d}p}} TA/\mathcal{F} \,=\, \mathcal{N} \longrightarrow 0\,, \tag{3.7}$$

where ι' is given by ι'' in (3.3).

Consider the quotient $\mathcal{N}$ in (2.4). It is known that the normal bundle $\mathcal{N}$ to a holomorphic foliation $\mathcal{F}$ always admit a flat holomorphic partial connection $\nabla^{\mathcal{F}}$ in the direction of $\mathcal{F}$ called the *Bott connection*. For a local holomorphic section V of TA and a local holomorphic section N of $\mathcal{N}$, the Bott connection operates as $\nabla^{\mathcal{F}}_V N \,=\, q([V,\, \widetilde{N}])$, where $\widetilde{N}$ is a local holomorphic section of TA lifting N and q is the quotient map defined in (2.5): this does not depend on the chosen local lift $\widetilde{N}$ and defines a flat holomorphic partial connection in the direction of $\mathcal{F}$ (see, for instance, [BD2, Section 11.5]).

Lemma 3.1 ([BD, p. 40, Lemma 2.1])**.** *Let θ be a flat partial connection on E_H. Then θ produces a flat partial connection on $\mathrm{At}(E_H)/\theta'(\mathcal{F})$ that satisfies the condition that the homomorphisms in the exact sequence* (3.7) *are partial connection preserving (where the adjoint bundle ad(E_H) is endowed with the partial connection induced from (E_H, θ) and $\mathcal{N}$ is equipped with the Bott connection $\nabla^{\mathcal{F}}$).*

3.2. **Transversely branched Cartan geometry.** Let G be a connected complex Lie group and $H \,\subset\, G$ a complex Lie subgroup. The Lie algebra of G will be denoted by $\mathfrak{g}$. As in (3.1), E_H is a holomorphic principal H–bundle on A. Let

$$E_G \,=\, E_H \times^H G \,\longrightarrow\, A \tag{3.8}$$

be the holomorphic principal G–bundle on A obtained by extending the structure group of E_H using the inclusion of H in G. The inclusion of $\mathfrak{h}$ in $\mathfrak{g}$ produces a fiber-wise injective homomorphism of Lie algebra bundles

$$\iota \,:\, \mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G), \tag{3.9}$$

where $\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g}$ is the adjoint bundle for E_G. Let θ be a flat partial connection on E_H in the direction of $\mathcal{F}$. So θ induces flat partial connections on the associated bundles E_G, $\mathrm{ad}(E_H)$ and $\mathrm{ad}(E_G)$.

A *transversely branched holomorphic Cartan geometry* of type $(G,\, H)$ on the foliated torus $(A,\, \mathcal{F})$ is

- a holomorphic principal H–bundle E_H on A equipped with a flat partial connection θ in the direction of $\mathcal{F}$, and
- a holomorphic homomorphism

$$\beta \,:\, \mathrm{At}(E_H)/\theta'(\mathcal{F}) \longrightarrow \mathrm{ad}(E_G), \tag{3.10}$$

(see (3.6) for θ') such that the following three conditions hold:

1. β is partial connection preserving,
2. β is an isomorphism over a nonempty open subset of A, and
3. the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota'} & \mathrm{At}(E_H)/\theta'(\mathcal{F}) & \longrightarrow & \mathcal{N} & \longrightarrow & 0 \\ & & \| & & \Big\downarrow \beta & & \Big\downarrow \overline{\beta} & & \\ 0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota} & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{ad}(E_G)/\mathrm{ad}(E_H) & \longrightarrow & 0 \end{array} \tag{3.11}$$

where the top exact sequence is the one in (3.7), and ι is the homomorphism in (3.9).

Notice that condition (2) in the above definition implies that the codimension of the foliation $\mathcal{F}$ coincides with the dimension of the homogeneous model space G/H.

Let n be the complex dimension of $\mathfrak{g}$. Consider the homomorphism of n-th exterior products

$$\bigwedge^n \beta \,:\, \bigwedge^n (\mathrm{At}(E_H)/\theta'(\mathcal{F})) \longrightarrow \bigwedge^n \mathrm{ad}(E_G)$$

induced by β in (3.10). The divisor $\mathrm{Div}(\bigwedge^n \beta)$ is called the *branching divisor* for $((E_H,\, \theta),\, \beta)$. We call $((E_H,\, \theta),\, \beta)$ a *holomorphic Cartan geometry* if β is an isomorphism over the entire A.

3.3. Connection on E_G.

Let $((E_H,\, \theta),\, \beta)$ be a transversely branched Cartan geometry of type $(G,\, H)$ on the foliated manifold $(A,\, \mathcal{F})$. Consider the homomorphism

$$\mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G) \oplus \mathrm{At}(E_H)\,, \quad v \longmapsto (\iota(v),\, -\iota''(v)); \tag{3.12}$$

see (3.9) and (3.3) for ι and ι'' respectively. The corresponding quotient $(\mathrm{ad}(E_G)\oplus\mathrm{At}(E_H))/\mathrm{ad}(E_H)$ is identified with the Atiyah bundle $\mathrm{At}(E_G)$. Let

$$\beta' \,:\, \mathrm{At}(E_H) \longrightarrow \mathrm{ad}(E_G)$$

be the composition of homomorphisms

$$\mathrm{At}(E_H) \longrightarrow \mathrm{At}(E_H)/\theta'(\mathcal{F}) \xrightarrow{\beta} \mathrm{ad}(E_G),$$

where the first homomorphism is the quotient map, and θ' (respectively, β) is the one in (3.6) (respectively, (3.10)). The homomorphism

$$\mathrm{ad}(E_G) \oplus \mathrm{At}(E_H) \longrightarrow \mathrm{ad}(E_G)\,, \quad (v,\, w) \longmapsto v + \beta'(w) \tag{3.13}$$

vanishes on the image of $\mathrm{ad}(E_H)$ by the map in (3.12). Therefore, the homomorphism in (3.13) produces a homomorphism

$$\varphi \,:\, \mathrm{At}(E_G) \,=\, (\mathrm{ad}(E_G) \oplus \mathrm{At}(E_H))/\mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G)\,. \tag{3.14}$$

The composition

$$\mathrm{ad}(E_G) \hookrightarrow \mathrm{At}(E_G) \xrightarrow{\varphi} \mathrm{ad}(E_G)$$

clearly coincides with the identity map of $\mathrm{ad}(E_G)$. Consequently, φ defines a holomorphic connection on the principal G–bundle E_G [At2]. Let $\mathrm{Curv}(\varphi) \,\in\, H^0(A,\, \mathrm{ad}(E_G) \otimes \Omega^2_A)$ be the curvature of the connection φ on E_G.

Lemma 3.2 ([BD, p. 42, Lemma 3.1])**.** *The curvature* $\mathrm{Curv}(\varphi)$ *lies in the image of the homomorphism*

$$H^0(A,\, \mathrm{ad}(E_G) \otimes \bigwedge\nolimits^2 \mathcal{N}^*) \hookrightarrow H^0(A,\, \mathrm{ad}(E_G) \otimes \Omega^2_A)$$

given by the inclusion $q^* \,:\, \mathcal{N}^* \,\hookrightarrow\, \Omega^1_A$ *(the dual of the projection in* (2.5)*).*

The transversely branched Cartan geometry $((E_H,\, \theta),\, \beta)$ will be called *flat* (or *integrable*) if the curvature $\mathrm{Curv}(\varphi)$ vanishes identically.

Notice that for codimension one foliations, the normal bundle $\mathcal{N}$ has rank one and the above curvature tensor vanishes identically. Consequently, for codimension one foliations, any transversely branched holomorphic Cartan geometry is automatically flat.

A flat transversely holomorphic Cartan geometry with model $(G,\, H)$ induces on the foliated manifold $(A,\, \mathcal{F})$ a *transversely holomorphic G/H–structure.* This means that A admits an open cover by open sets U_i equipped with holomorphic submersions $f_i \,:\, U_i \,\longrightarrow\, G/H$ such that the restriction of the foliation $\mathcal{F}$ on U_i is given by the level sets of the map f_i. Moreover, for each connected component of the nonempty intersections $U_i \cap U_j$, there exists an element $g_{ij} \,\in\, G$ such that $f_i \,=\, g_{ij} \circ f_j$.

The above description holds also for flat *branched* holomorphic Cartan geometry, except that the local maps $f_i \,:\, U_i \,\longrightarrow\, G/H$ defining the foliation $\mathcal{F}$ in restriction to U_i are submersions at the generic point (the maps f_i are allowed to admit a branching divisor) (see [BD, BD2]).

4. Rigidity of branched Cartan geometries on torus

As before, G is a connected complex Lie group, and $H \,\subset\, G$ is a complex Lie subgroup and $(A,\, \mathcal{F})$ is a foliated compact complex torus. Let E_H be a holomorphic principal H–bundle on A equipped with a flat partial connection θ in the direction of $\mathcal{F}$.

Theorem 4.1. *Assume $\mathcal{F}$ is a smooth turbulent foliation on the compact complex torus A.*

1. *There can be at most one transversely branched holomorphic Cartan geometry of type $(G,\, H)$ on the foliated torus $(A, \mathcal{F})$ admitting (E_H, θ) as underlying H-principal bundle equipped with the flat partial connection θ along $\mathcal{F}$.*
2. *Any transversely branched holomorphic Cartan geometry of type $(G,\, H)$ on the foliated torus $(A, \mathcal{F})$ is flat.*

Proof. To prove (1), let β_1 and β_2 be two transversely branched Cartan geometries of type $(G,\, H)$ on $(E_H,\, \theta)$. Consider the principal G–bundle E_G in (3.8). Let φ_1 (respectively, φ_2) be the holomorphic connections on E_G produced by β_1 (respectively, β_2); see (3.14) for the constructions of φ_1 and φ_2. Since the space of holomorphic connections on E_G is an affine space for $H^0(A,\, \mathrm{ad}(E_G) \otimes \Omega^1_A)$, we have

$$\beta_1 - \beta_2 \ \in\ H^0(A,\, \mathrm{ad}(E_G) \otimes \Omega^1_A). \tag{4.1}$$

Let φ'_1 (respectively, φ'_2) be the partial connection on E_G given by φ_1 (respectively, φ_2). On the other hand, the partial connection θ on E_H induces a partial connection on E_G, because E_G is the principal G–bundle on A obtained by extending the structure group of E_H using the inclusion map $H \hookrightarrow G$. Let $\widetilde{\theta}$ be the partial connection on E_G induced by θ. Note that φ'_1 and φ'_2 coincide with $\widetilde{\theta}$. Consequently, from (4.1) we have

$$\beta_1 - \beta_2 \ \in\ H^0(A,\, \mathrm{ad}(E_G) \otimes \mathcal{N}^*); \tag{4.2}$$

note that the homomorphism

$$\mathrm{Id}_{\mathrm{ad}(E_G)} \otimes q^* \ :\ \mathrm{ad}(E_G) \otimes \mathcal{N}^* \ \longrightarrow\ \mathrm{ad}(E_G) \otimes \Omega^1_A,$$

where q^* is the dual of the homomorphism in (2.5), produces an injective homomorphism

$$H^0(A,\, \mathrm{ad}(E_G) \otimes \mathcal{N}^*) \ \longrightarrow\ H^0(A,\, \mathrm{ad}(E_G) \otimes \Omega^1_A).$$

Since $\mathrm{ad}(E_G)$ is a holomorphic vector bundle, admitting a holomorphic connection, on a compact complex torus, we know that $\mathrm{ad}(E_G)$ is semistable of degree zero [BG, p. 41, Theorem 4.1]. Hence from Lemma 2.1 we know that

$$H^0(A,\, \mathrm{ad}(E_G) \otimes \mathcal{N}^*) \ =\ 0.$$

Now from (4.2) it follows that $\beta_1 \ =\ \beta_1$. This proves (1).

To prove (2), recall from Lemma 3.2 that the curvature of a transversely branched Cartan geometry of type $(G,\, H)$ on $(E_H,\, \theta)$ is a holomorphic section of $\mathrm{ad}(E_G) \otimes \bigwedge^2 \mathcal{N}^*$. Again from Lemma 2.1 we know that

$$H^0(A,\, \mathrm{ad}(E_G) \otimes \bigwedge^2 \mathcal{N}^*) \ =\ 0.$$

Hence any transversely branched Cartan geometry of type $(G,\, H)$ on $(E_H,\, \theta)$ is flat. □

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