

On cuts of small chromatic number in sparse graphs

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Abstract

For a given integer k , let ℓ_k denote the supremum ℓ such that every sufficiently large graph G with average degree less than 2ℓ admits a separator $X \subseteq V(G)$ for which $\chi(G[X]) < k$. Motivated by the values of ℓ_1 , ℓ_2 and ℓ_3 , a natural conjecture suggests that $\ell_k = k$ for all k . We prove that this conjecture fails dramatically: asymptotically, the trivial lower bound $\ell_k \geq \frac{k}{2}$ is tight. More precisely, we prove that for every $\varepsilon > 0$ and all sufficiently large k , we have $\ell_k \leq (1 + \varepsilon)\frac{k}{2}$.

1 Introduction

For a given integer k , we define ℓ_k as the supremum ℓ such that every sufficiently large¹ graph G with average degree less than 2ℓ contains a set $X \subseteq V(G)$ with the properties that $G \setminus X$ is disconnected and $\chi(G[X]) < k$.

Observe first that any graph with average degree less than k and order at least $k+1$ contains a vertex of degree at most $k-1$, whose neighbourhood is thus a separator of chromatic number less than k . Conversely, for any $n \geq k$, one can construct an n -vertex graph with no such set X by taking a clique on k vertices and joining it completely to an independent set of size $n-k$, see [Figure 1](#) for an illustration. This graph has exactly $kn - \frac{k(k+1)}{2}$ edges and hence average degree less than $2k$, yet every separator must include the entire clique, which has chromatic number k . This shows that ℓ_k is well-defined and satisfies

$$\frac{k}{2} \leq \ell_k \leq k.$$

Having established general bounds, let us now consider the small cases. It is folklore that $\ell_1 = 1$: indeed, whenever $n \geq 2$, every n -vertex graph with fewer than $n-1$ edges is disconnected, while connected graphs with average degree 2 certainly exist (e.g. cycles).

The case $k = 2$ was resolved by Chen and Yu [[CY02](#)], confirming a conjecture of Caro with a very elegant inductive proof.

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¹As a function of k and ℓ .

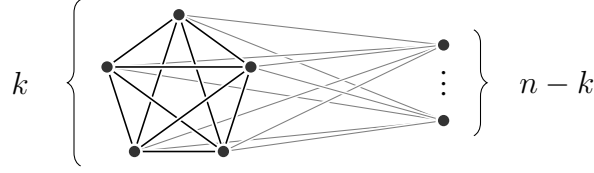


Figure 1: An n -vertex graph of average degree less than $2k$ in which every separator has chromatic number at least k .

Theorem 1.1 ([CY02]). *Any graph on n vertices with fewer than $2n - 3$ edges admits a stable cut, while some graphs with exactly $2n - 3$ edges do not. In particular, $\ell_2 = 2$.*

The extremal graphs with $2n - 3$ edges and no stable cut were first characterized by Le and Pfender [LP13], although their proof contained a gap later filled by Rauch and Rautenbach [RR24].

The next case, $k = 3$, was investigated by Bogdanov, Neustroeva, Sokolov, Volostnov, Russkin, and Voronov [BNS⁺25], who formulated the following conjecture.

Conjecture 1.2 ([BNS⁺25]). *Any n -vertex graph with fewer than $3n - 6$ edges admits a bipartite cut, while some graphs with $3n - 6$ edges do not. In particular, $\ell_3 = 3$.*

The extremal examples here are provided by 3-trees. In fact, an even stronger conjecture predates this one, namely that under the same conditions one can always find a cut inducing a forest [CRR25]. Partial progress was obtained in the same paper by Chernyshev, Rauch and Rautenbach, who proved that every n -vertex graph with fewer than $\frac{11}{5}n - \frac{18}{5}$ edges admits a forest cut. This bound was subsequently improved to $\frac{9}{4}n - \frac{15}{4}$ by Botler, Couto, Fernandes, de Figueiredo, Gómez, dos Santos and Sato [BCF⁺25], and then to $\frac{19}{8}n - \frac{28}{8}$ by Bogdanov et al. [BNS⁺25]. In the same work, the authors also established a bound of $\frac{80}{31}n - \frac{134}{31}$ for bipartite cuts. Related results were later obtained by Cheng, Tang and Zhan [CTZ26].

Taken together, these cases naturally suggest a bold generalization:

Conjecture 1.3. *For every integer k and every graph G on at least k vertices, if*

$$|E(G)| < k|V(G)| - \frac{k(k+1)}{2},$$

then G admits a cut X with $\chi(G[X]) < k$. In particular, $\ell_k = k$.

The main purpose of this note is to show that **Conjecture 1.3** is in fact far from correct.

Theorem 1.4. *For any $\varepsilon > 0$ and all sufficiently large k , we have*

$$\ell_k \leq (1 + \varepsilon) \frac{k}{2}.$$

While this does not fully determine ℓ_k , it provides an essentially sharp asymptotic estimate when combined with the lower bound:

$$\frac{k}{2} \leq \ell_k \leq (1 + o(1)) \frac{k}{2}$$

Thus we arrive at the following conclusion.

Theorem 1.5. *As k grows large, we have $\ell_k \sim \frac{k}{2}$.*

In fact, we prove a stronger statement:

Theorem 1.6. *For every integer k , there exist arbitrarily large graphs with average degree $(1 + o(1))k$ in which every separator contains a clique of size k .*

This construction is interesting in its own right, and appeared in [BRRS25] where it was used to establish lower bounds on the smallest maximum degree of a cut (instead of its chromatic number). Since the chromatic number is always at most the degeneracy plus one, **Theorem 1.4** also rules out the strengthening of **Conjecture 1.3** where $\chi(G[X]) < k$ is replaced by the requirement that $G[X]$ be $(k - 2)$ -degenerate. This would have tied in neatly with the already studied cases:

- the only -1 -degenerate graph is the empty one,
- a 0 -degenerate graph is stable,
- a 1 -degenerate graph is a forest.

Thus, for $k = 1, 2$, this is equivalent to the standard definition using chromatic number, while with $k = 3$ we retrieve the well-studied notion of forest-cuts. We note that the condition $\chi(G[X]) < k$ seems easier to work with when attempting to obtain positive results, as it is compatible with identifying a stable set into a single vertex².

2 Proofs

In a bipartite graph $G = (A \cup B, E)$, a *bi-hole of size k* is a pair (A', B') with $\min\{|A'|, |B'|\} = k$, $A' \subseteq A$, $B' \subseteq B$, such that there is no edge between A' and B' . In some sense, the size of a largest bi-hole in a bipartite graph corresponds to the “bipartite independence number” of that graph. Axenovich, Sereni, Snyder, and Weber [ASSW21] studied the following question: what is the largest integer $f(n, \Delta)$ such that every $n \times n$ bipartite graph $G = (A \cup B, E)$ with $\deg(a) \leq \Delta$ for every vertex $a \in A$ contains a bi-hole of size $f(n, \Delta)$? They proved that the asymptotic behaviour of the function $f(n, \Delta)$ is $\Theta\left(\frac{\ln \Delta}{\Delta} \cdot n\right)$. We make use of the following upper bound.

Theorem 2.1 ([ASSW21]). *Let $\Delta \geq 27$ be an integer and $n \geq \frac{\Delta}{\ln \Delta}$. Then, there exists an $n \times n$ bipartite graph $G = (A \cup B, E)$ with $\deg(a) \leq \Delta$ for every vertex $a \in A$, which contains no bi-hole of size at least $8 \cdot \frac{\ln \Delta}{\Delta} \cdot n$.*

Such a graph can be obtained with high probability from a random bipartite graph $G(2n, 2n, \Delta/(4n))$ by restricting one part to n vertices of degree at most Δ and the other part to any set of n vertices.

We now prove **Theorem 1.4**, which we restate for convenience.

Theorem 1.4. *For any $\varepsilon > 0$ and all sufficiently large k , we have*

$$\ell_k \leq (1 + \varepsilon) \frac{k}{2}.$$

Proof. Fix $\varepsilon > 0$, set $\eta := \varepsilon/2$ and let $\Delta \geq 27$ be large enough so that $1 - 8 \cdot \frac{\ln \Delta}{\Delta} \geq \frac{1}{1+\eta}$. Let k be an integer large enough so that $\eta k \geq 2\Delta$ and $(1 + \eta)k \geq \frac{\Delta}{\ln \Delta}$. Set $\ell := (1 + \varepsilon) \frac{k}{2}$. To show that $\ell_k \leq \ell$, it suffices to prove that there exist arbitrarily large graphs G with average degree less than 2ℓ and where every separator $X \subseteq V(G)$ of G satisfies $\chi(G[X]) \geq k$.

²For any smallest graph G with $\ell|V(G)| - |E(G)| > c$ and no cut of chromatic number less than k , we obtain that every subset X of vertices is either a clique or satisfies $\ell|X| - |E(G[X])| > \ell t - \frac{t(t-1)}{2}$, where $t = \chi(G[X])$.

Set $\alpha := \lceil (1+\eta)k \rceil \geq \frac{\Delta}{\ln \Delta}$, and let $\beta \geq 2$ be an integer. By [Theorem 2.1](#), there exists an $\alpha \times \alpha$ bipartite graph $H = (A \cup B, E)$ with $\deg(a) \leq \Delta$ for every vertex $a \in A$, which contains no bi-hole of size at least $8 \cdot \frac{\ln \Delta}{\Delta} \cdot \alpha$. Consider the graph G whose vertex set is the union of β pairwise disjoint sets $A_1, \dots, A_\beta$ of α vertices each, and whose edges are exactly such that:

- for every $i \in [1, \beta]$, the graph $G[A_i]$ is a clique, and
- for every $i \in [1, \beta - 1]$, the semi-induced subgraph $G[A_i, A_{i+1}]$ is isomorphic to H , with A_i mapped to the part A of H and A_{i+1} to the part B of H .

Claim 2.2. *Every separator $X \subseteq V(G)$ of G satisfies $\chi(G[X]) \geq k$.*

Proof. Consider a set $X \subseteq V(G)$ such that $G \setminus X$ is disconnected. Since each $G[A_i]$ is a clique, there exists an integer $i \in [1, \beta - 1]$ such that there is no edge in G between $A_i \setminus X$ and $A_{i+1} \setminus X$. By construction of G , this means that $(A_i \setminus X, A_{i+1} \setminus X)$ is a bi-hole in $G[A_i, A_{i+1}] \cong H$. Therefore, by definition of H , we have

$$\min\{|A_i \setminus X|, |A_{i+1} \setminus X|\} \leq 8 \cdot \frac{\ln \Delta}{\Delta} \cdot \alpha.$$

Thus, we have

$$\max\{|A_i \cap X|, |A_{i+1} \cap X|\} \geq \alpha \left(1 - 8 \cdot \frac{\ln \Delta}{\Delta}\right).$$

Since $G[A_i]$ and $G[A_{i+1}]$ are cliques, we deduce

$$\chi(G[X]) \geq \alpha \left(1 - 8 \cdot \frac{\ln \Delta}{\Delta}\right) \geq (1 + \eta)k \cdot \frac{1}{1 + \eta} \geq k. \quad \square$$

Claim 2.3. *G has average degree less than 2ℓ .*

Proof. In H , every vertex $a \in A$ satisfies $\deg(a) \leq \Delta$, so H has at most $\alpha\Delta$ edges. Therefore,

$$|E(G)| \leq \beta \left(\frac{\alpha(\alpha - 1)}{2} + \alpha\Delta \right).$$

Moreover $|V(G)| = \beta\alpha$. Thus, the average degree of G is

$$\frac{2|E(G)|}{|V(G)|} \leq \alpha - 1 + 2\Delta < (1 + 2\eta)k = (1 + \varepsilon)k = 2\ell. \quad \square$$

Since G can be made arbitrarily large by choosing appropriately the value of β , the two claims conclude the proof. $\square$

Remark 2.4. In the above proof, we can take $\Delta = \Theta\left(\frac{1}{\varepsilon} \ln \frac{1}{\varepsilon}\right)$ and $k = \Theta\left(\frac{1}{\varepsilon^2} \ln \frac{1}{\varepsilon}\right)$.

3 Conclusion

We disproved [Conjecture 1.3](#) in a strong form, but only for very large k . It would be interesting to establish the smallest k for which [Conjecture 1.3](#) strays from the truth, especially if it turns out to be already at $k = 3$. It also seems reasonable to believe that the stronger form, requiring a cut to be not only $(k - 1)$ -colourable but in fact $(k - 2)$ -degenerate, would break down earlier, maybe indeed for $k = 3$.

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