

# NOWHERE-ZERO 5-FLOW ON SIGNED LADDERS

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ABSTRACT. In 1983, Bouchet conjectured that every flow-admissible signed graph admits a nowhere-zero 6-flow. In this paper, we prove that Bouchet's conjecture holds for all signed ladders, circular and Möbius ladders. In fact, all signed ladders, circular and Möbius ladders admit a nowhere-zero 5-flow except for one case of signed circular ladders. Of course, the exception also has a nowhere-zero 6-flow.

## 1. INTRODUCTION

A *signed graph* is a graph with each edge labelled with a sign,  $+$  or  $-$ . An assignment of signs to each edge is a signature. An orientation of a signed graph is obtained by dividing each edge into two half-edges each of which receives its own direction. A positive edge has one half-edge directed from and the other half-edge directed to its end-vertex. Hence, a negative edge has both half-edges directed either towards or from their respective end-vertices.

Let  $v$  be a vertex of a signed graph  $G$ . *Vertex switching* at  $v$  changes the sign of each edge incident with  $v$  to its opposite. Let  $X \subseteq V$ . Switching a vertex set  $X$  means reversing the signs of all edges between  $X$  and its complement. Switching a set  $X$  has the same effect as switching all the vertices in  $X$ , one after another.

Two signed graphs  $G$  and  $G'$  with the same underlying graph but possibly different signatures on their edges are *switching equivalent*, if there is a series of switchings that transforms  $G$  to  $G'$ . Switching equivalence is an equivalence relation on the signatures of a fixed graph. If  $G'$  is isomorphic to a switching of  $G$ , we say that  $G$  and  $G'$  are *switching isomorphic*.

A signed graph is *balanced* if and only if it is switching equivalent to the signed graph with all-positive signature. And a signed graph is *anti-balanced* if it is switching equivalent to the signed graph with all-negative signature. In other words, a signed graph is balanced (anti-balanced) if and only if every circuit of the underlying graph contains an even (odd) number of negative edges, as vertex switching preserves the parity of the number of negative edges around a circuit.

Signed graphs were introduced by Harary [5] as a model for social networks. Also, signed graphs have diverse applications, a more recent one is in quantum computing [2]. We refer to [5, 9] for more information about signed graphs.

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In this paper, we use the symbols  $CL_n$  and  $ML_n$  to denote a signed circular ladder and a signed Möbius ladder of order  $2n$ , respectively. A *nowhere-zero  $k$ -flow* on a signed graph  $G$  is an assignment of an orientation and a value from  $\{\pm 1, \pm 2, \dots, \pm(k-1)\}$  to each edge in such a way that for each vertex of  $G$  the sum of incoming values equals the sum of outgoing values (Kirchhoff's law). We call such graphs flow-admissible. In 1983, Bouchet in [1] conjectured that every flow-admissible signed graph admits a nowhere-zero 6-flow. For example, Bouchet showed that there is a signature for the Petersen graph which admits no nowhere-zero 5-flow. Edita Máčajová found a case of  $CL_4$ , Fig. 10, using a computer search which has no nowhere-zero 5-flow but admits a nowhere-zero 6-flow. For further examples of signed graphs with this properties refer to [6].

However, signed circular ladders do not produce any more examples: in Sections 2 and 3 we show that all signed circular ladders apart from Fig. 10 and all signed Möbius ladders admit a nowhere-zero 5-flow.

## 2. NOWHERE-ZERO FLOW ON SIGNED CIRCULAR LADDER $CL_n$

**Theorem 2.1** ([8, Theorem 4.4]). *Let  $G$  be a flow-admissible signed cubic graph with two negative edges. If  $G$  is bipartite, then it has a nowhere-zero  $k$ -flow with  $k \leq 4$ .*

We have the following lemma due to König [7].

**Lemma 2.2.** *Every  $r$ -regular bipartite graph,  $r \geq 1$ , is 1-factorable.*

Therefore, we can decompose each cubic bipartite graph into 1-factors. Consider a signed graph  $G$  carrying a  $k$ -flow  $\phi$  and let  $P = e_1 e_2 \dots e_r$  be an  $u - v$  trail in  $G$ . By sending a value  $b \in \{\pm 1, \pm 2, \dots, \pm(k-1)\}$  from  $u$  to  $v$  along  $P$  we mean reversing the orientation of the edge  $e_1$  so that it leaves  $u$ , adding  $b$  to  $\phi(e_1)$ , and adding  $\pm b$  to  $\phi(e_i)$  for all other edges of  $P$  in such a way that Kirchhoff's law is fulfilled at each inner vertex of  $P$ .

**Theorem 2.3.** *Let  $G$  be a signed cubic bipartite graph and  $\{F_1, F_2, F_3\}$  be a 1-factorisation of  $G$ . Consider 2-factors  $F_1 \cup F_2$ ,  $F_1 \cup F_3$  and  $F_2 \cup F_3$ . If two of them are balanced, then  $G$  admits a nowhere-zero 4-flow.*

*Proof.* Without loss of generality, assume that  $F_1 \cup F_2$  and  $F_2 \cup F_3$  are balanced 2-factors. Nowhere-zero 4-flow is obtained by sending value 1 along each circuit of  $F_1 \cup F_2$  and value 2 along  $F_2 \cup F_3$ .  $\square$

**Remark 2.4.** Note that considering switching equivalence the maximum number of negative edges can occur in the subladder given in Fig. 1, is 3.

In fact, just in three cases three negative edges occur, and for other cases by switching at some vertices the number of negative edges declines. The possible cases are listed in Fig. 2, in which dashed lines denote negative edges.

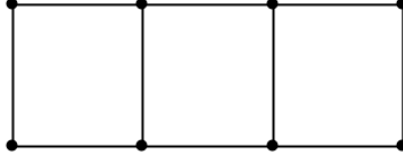


FIGURE 1.

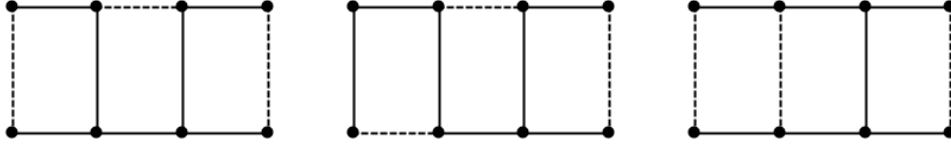
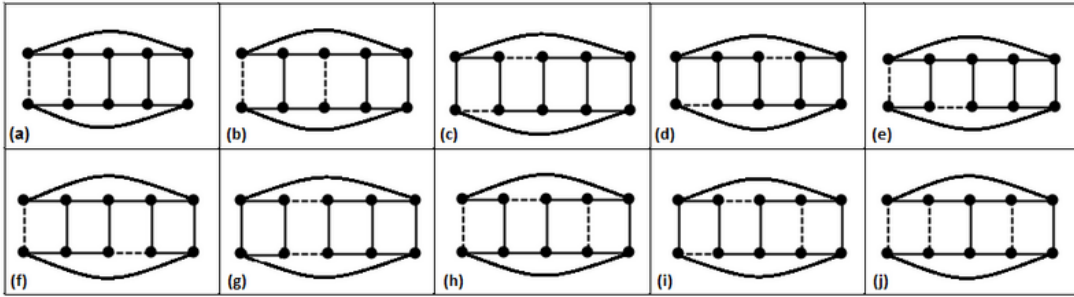


FIGURE 2.

**Remark 2.5.** One can check that in a signed circular ladder  $CL_n$ , up to switching equivalence there exist at most  $\lfloor \frac{n}{2} \rfloor + 1$  negative edges.

**Lemma 2.6.** *Signed circular ladders  $CL_5$  and  $CL_6$  with any signature have a nowhere-zero 5-flow.*

*Proof.* By Remark 2.5, a signed circular ladder  $CL_5$  has at most three negative edges. All types of the signed graph  $CL_5$  with two or three negative edges are listed in Fig. 3.

FIGURE 3.  $CL_5$  with two and three negative edges

It is not hard to check that all signed graphs  $CL_5$  with two negative edges given in Fig. 3, have a nowhere-zero 4-flow, and signed graphs  $CL_5$  with three negative edges, Graphs (h), (i), (j) in Fig. 3, admit a nowhere-zero 5-flow.

Now, we show that the signed graph  $CL_6$  with any signature (with at least two negative edges) has a nowhere-zero 5-flow. By Theorem 2.1, if the signed graph  $CL_6$  has two negative edges, it admits a nowhere-zero 4-flow. Moreover, all types of the signed graph  $CL_6$  with three and four negative edges are listed in Fig. 4. All of them

have a nowhere-zero 5-flow (Graphs (b), (f), (g), and (h) have nowhere-zero 4-flow, and Graphs (a), (c), (d), and (e) admit nowhere-zero 5-flow).

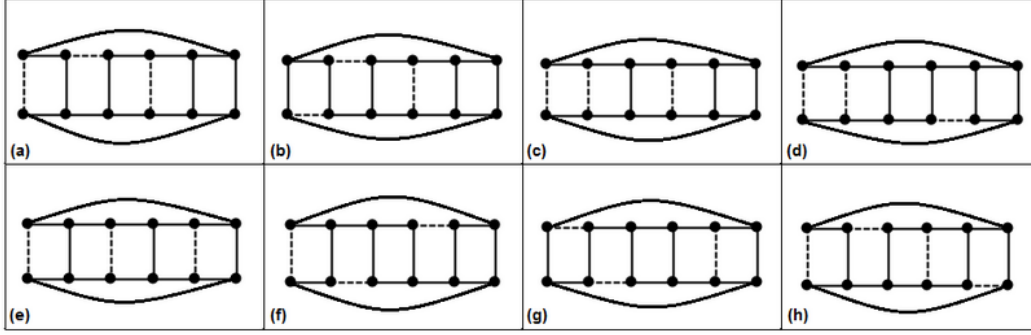


FIGURE 4.  $CL_6$  with three and four negative edges

By Remark 2.5, up to switching equivalence there is no  $CL_6$  with more than four negative edges. So, we conclude that the signed graph  $CL_6$  with two or three negative edges has nowhere-zero 5-flow.  $\square$

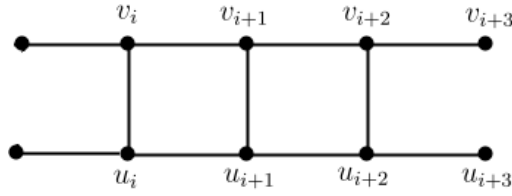


FIGURE 5. A subladder  $H$

**Lemma 2.7.** *Let  $(CL_n, \sigma)$  be a flow admissible signed circular ladder with  $n \geq 7$  having a positive square  $S$  (in Fig. 5,  $S = v_{i+1}v_{i+2}u_{i+2}u_{i+1}v_{i+1}$ .) If  $((CL_n \setminus S) \cup \{v_iv_{i+3}, u_iu_{i+3}\}, \sigma)$  with*

$$\sigma(v_iv_{i+3}) = \sigma(v_iv_{i+1})\sigma(v_{i+2}v_{i+3}) \quad \text{and} \quad \sigma(u_iu_{i+3}) = \sigma(u_iu_{i+1})\sigma(u_{i+2}u_{i+3})$$

*has a nowhere-zero 5-flow, then  $(CL_n, \sigma)$  has also a nowhere-zero 5-flow.*

*Proof.* Up to switching equivalence there are four cases for the edges of  $S$  and the edge set  $\{v_iv_{i+1}, v_{i+2}v_{i+3}, u_iu_{i+1}, u_{i+2}u_{i+3}\}$ :

- (i) If all edges of  $S \cup \{v_iv_{i+1}, v_{i+2}v_{i+3}, u_iu_{i+1}, u_{i+2}u_{i+3}\}$  are positive.
- (ii) If  $v_{i+1}u_{i+1}$  and  $v_{i+2}u_{i+2}$  are negative.
- (iii) If  $u_iu_{i+1}$  and  $v_{i+2}v_{i+3}$  are negative.
- (iv) If just an edge  $u_iu_{i+1}$  is negative.

Consider a nowhere-zero 5-flow on  $(CL_n \setminus S) \cup \{v_i v_{i+3}, u_i u_{i+3}\}$  with  $\sigma(v_i v_{i+3}) = \sigma(v_i v_{i+1})\sigma(v_{i+2} v_{i+3})$  and  $\sigma(u_i u_{i+3}) = \sigma(u_i u_{i+1})\sigma(u_{i+2} u_{i+3})$ . We show that  $(CL_n, \sigma)$  in each of the above cases has a nowhere-zero 5-flow. It is sufficient to prove the assertion for one of the cases, the rest cases are proved similarly. Let all edges of  $S \cup \{v_i v_{i+1}, v_{i+2} v_{i+3}, u_i u_{i+1}, u_{i+2} u_{i+3}\}$  are positive. Without loss of generality we can assume that  $v_i u_i$  is positive. Consider the left signed graph in Fig. 6, which  $a, b, b', c, c' \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ . It is not hard to check that one can find a value  $x \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$  such that  $x + c, x + c' \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ , see the right signed graph in Fig. 6.

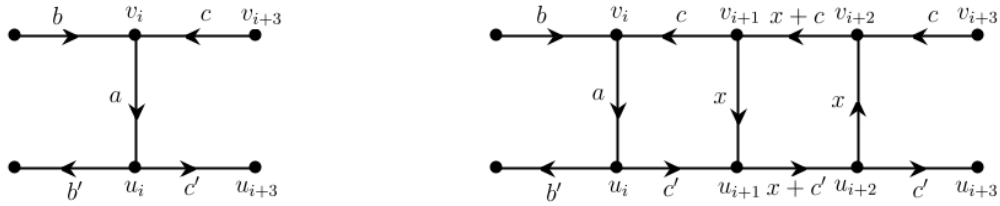


FIGURE 6. Constructing a nowhere-zero 5-flow on  $CL_n$

A note about other three cases, for example assume that  $u_i u_{i+1}$  and  $v_{i+2} v_{i+3}$  are negative. Delete  $S$  and consider a nowhere-zero 5-flow on the obtained signed circular ladder  $CL_{n-2}$ , see Fig. 7. Similar to the mentioned case, and using this note that if  $c = c' = 4$ , then we achieve a contradiction because  $(CL_n, \sigma)$  for  $n \geq 7$ , is flow admissible. (Since we assume that  $(CL_n, \sigma)$  for  $n \geq 7$ , is flow admissible, there is a positive integer  $\ell$  such that  $(CL_n, \sigma)$  admits a nowhere-zero  $\ell$ -flow. Considering the right subladder in Fig. 7, if  $c = c' = \ell - 1$ , we achieve a contradiction with flow admissibility of  $(CL_n, \sigma)$ . Now, we claim that  $(CL_n, \sigma)$  has a nowhere-zero 5-flow, so we can ignore this equality  $c = c' = 4$ ).

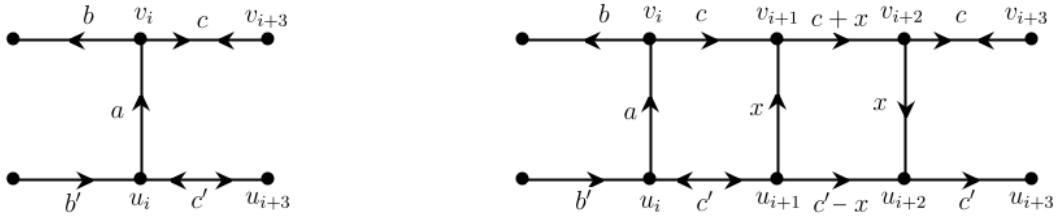


FIGURE 7. Constructing a nowhere-zero 5-flow on  $CL_n$

□

In the following theorem, we prove that the signed graph  $CL_n$  for  $n \geq 5$ , with any signature admits a nowhere-zero 5-flow.

**Theorem 2.8.** *Let  $n \geq 5$  be a positive integer. If a signed circular ladder  $CL_n$  is flow admissible, then it has a nowhere-zero 5-flow.*

*Proof.* If  $n = 5$  or  $6$ , then by Lemma 2.6, signed circular ladders  $CL_5$  and  $CL_6$  have nowhere-zero 5-flow. Assume that  $k \geq 3$  is a positive integer which shows the number of negative edges in  $CL_n$  for  $n \geq 7$ . It is not hard to check that if signed graph  $CL_{2k}$  has  $k$  negative edges (with any signature), it has at least a positive square except in one case which all negative edges occur on rungs alternately. This exceptional case admits a nowhere-zero 4-flow if  $k$  is even, see Theorem 2.3, and for odd  $k$ , one can find a pattern using Graphs (a) and (b) in Fig. 14, which shows that it has a nowhere-zero 5-flow. Also, if  $CL_{2k}$  has  $k + 1$  negative edges, there is just one signature with all negative squares; it is given in Fig. 8, three points among the rungs means there is a positive rung and then a negative rung, alternately. It has a nowhere-zero 5-flow, see Graphs (c) and (d) in Fig. 14. Note that if  $k$  is odd, then we can conclude that the exceptional case of  $CL_{2k}$  with  $k + 1$  negative edges, given in Fig. 8, has nowhere-zero 4-flow because it has two balanced 2-factors, see Theorem 2.3.

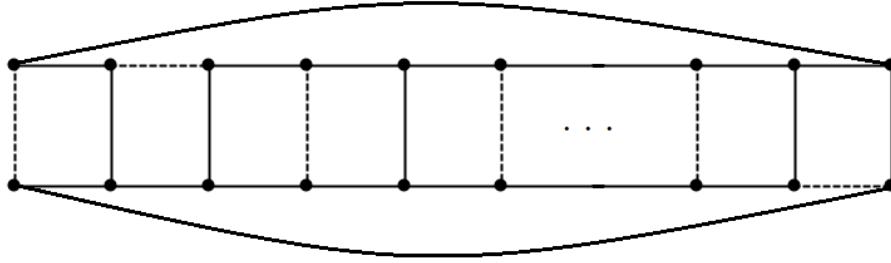
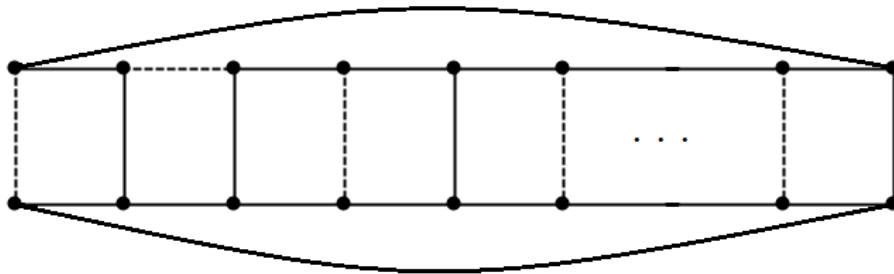


FIGURE 8. The exceptional case of  $CL_{2k}$  with  $k + 1$  negative edges

Similarly, signed graph  $CL_{2k+1}$  with  $k$  or  $k + 1$  negative edges, with any signature, has at least a positive square except in one case with  $k + 1$  negative edges given in Fig. 9. One can find a certain pattern to exist a nowhere-zero 5-flow on the exceptional case of  $CL_{2k+1}$  with  $k + 1$  negative edges, see Graphs (e), (f), (g), and (h) in Fig. 14. Now, ignore the three exceptional cases of signed graphs  $CL_{2k}$  and  $CL_{2k+1}$  with  $k$  and  $k + 1$  negative edges.

We claim that the signed graphs  $CL_{2k}$  and  $CL_{2k+1}$  with  $k$  and  $k + 1$  negative edges have nowhere-zero 5-flow. Note that  $k + 1$  is the maximum number of the negative edges can occur in  $CL_{2k}$  and  $CL_{2k+1}$ . We prove the claim by induction on  $k \geq 3$ . Let  $k = 3$ . We know that  $CL_6$  with three or four negative edges has nowhere-zero 5-flow. Also, we know that signed graph  $CL_7$  with three and four negative edges (except the exceptional case given in Fig. 9) has at least a positive square. So, by Lemma 2.7 and the existence of a nowhere-zero 5-flow on  $CL_5$ , we conclude that  $CL_7$  with three and four negative edges has nowhere-zero 5-flow. Now, assume that for each  $4 \leq i \leq k - 1$ ,  $CL_{2i}$  and  $CL_{2i+1}$  with  $i$  and  $i + 1$  negative edges have nowhere-zero 5-flow. This is the induction hypothesis. Consider  $i = k$ . Since  $CL_{2k}$  and  $CL_{2k+1}$  with  $k$  or  $k + 1$  negative edges have at least one positive square (without considering exceptional cases). Hence, by Lemma 2.7 and using the induction hypothesis, we



conclude that each signed graph  $CL_n$  for  $n \geq 5$ , with at least two negative edges has a nowhere-zero 5-flow.

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*Proof.* It is sufficient to check the signed circular ladders  $CL_4$  with two and three negative edges. If  $CL_4$  has two negative edges by Theorem 2.1, it has a nowhere-zero 4-flow. Also, there is just one signed circular ladder  $CL_4$  with three negative edges (up to switching equivalence). It is given in Fig. 10, and it has a nowhere-zero 6-flow. Note that it does not have a nowhere-zero  $k$ -flow for some positive integer  $k < 6$ .  $\square$

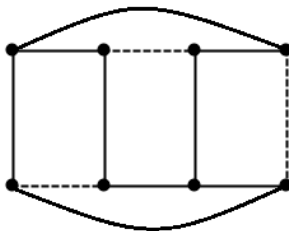


FIGURE 10.

### 3. NOWHERE-ZERO FLOW ON SIGNED MÖBIUS LADDER $ML_n$

**Remark 3.1.** Note that in a signed Möbius ladder  $ML_n$ , up to switching equivalence there exist at most  $\lfloor \frac{n+1}{2} \rfloor$  negative edges. Moreover, Möbius ladders  $ML_n$  for odd  $n$ , are bipartite.

So by Remark 3.1,  $ML_4$  and  $ML_5$  have at most 2 and 3 negative edges, respectively.

**Lemma 3.2.** *The signed Möbius ladders  $ML_4$  and  $ML_5$  admit a nowhere-zero 5-flow.*

*Proof.* All types of signed Möbius ladder  $ML_4$  with two negative edges (up to switching isomorphic) are listed in Fig. 11. It is not hard to find a nowhere-zero 4-flow on Graphs (a), (b), (c), and (e), and Graph (d) has a nowhere-zero 5-flow.

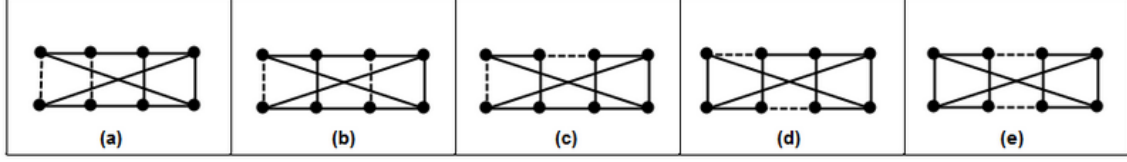


FIGURE 11. All types of  $ML_4$  with two negative edges

All types of signed Möbius ladder  $ML_5$  with two negative edges have a nowhere-zero 4-flow, see Theorem 2.1. Now, if signed Möbius ladder  $ML_5$  has three negative edges, up to switching isomorphic there are just two types, see Fig. 12. Graph (a) and Graph (b) given in Fig. 12, have nowhere-zero 5-flow and nowhere-zero 4-flow, respectively.  $\square$

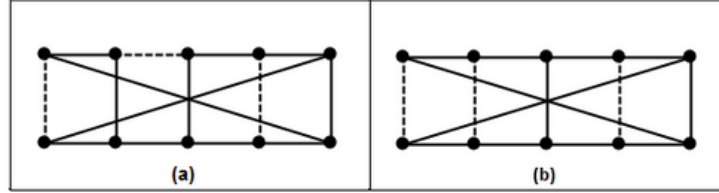


FIGURE 12.  $ML_5$  with three negative edges

We can state a lemma similar to Lemma 2.7 for signed Möbius ladders.

**Lemma 3.3.** *Let the signed Möbius ladder  $(ML_n, \sigma)$  for  $n \geq 6$ , be flow admissible and have a positive square,  $S$ . In Fig. 5, let  $S = v_{i+1}v_{i+2}u_{i+2}u_{i+1}v_{i+1}$ . If  $((ML_n \setminus S) \cup \{v_i v_{i+3}, u_i u_{i+3}\}, \sigma)$  with*

$$\sigma(v_i v_{i+3}) = \sigma(v_i v_{i+1})\sigma(v_{i+2} v_{i+3}) \quad \text{and} \quad \sigma(u_i u_{i+3}) = \sigma(u_i u_{i+1})\sigma(u_{i+2} u_{i+3})$$

*has a nowhere-zero 5-flow, then  $(ML_n, \sigma)$  has also a nowhere-zero 5-flow.*

*Proof.* Proceed as in the proof of Lemma 2.7.  $\square$

**Theorem 3.4.** *Let  $n \geq 4$  be a positive integer. If a signed Möbius ladder  $ML_n$  is flow admissible, then it has a nowhere-zero 5-flow.*

*Proof.* If  $n = 4$  or  $5$ , then by Lemma 3.2, signed Möbius ladders  $ML_4$  and  $ML_5$  with any signature have nowhere-zero 5-flow. Let  $k \geq 3$  is a positive integer which denotes the number of negative edges in  $ML_n$  for  $n \geq 6$ . One can check that if signed graph  $ML_{2k}$  has  $k - 1$  or  $k$  negative edges (with any signature), it has at least a positive

square except in one case which all negative edges occur on rungs alternately. This exceptional case admits a nowhere-zero 4-flow. Using nowhere-zero 4-flow on Graphs (a), (b), (c), and (d) in Fig. 15, we can obtain a pattern for the existence a nowhere-zero 4-flow on this exceptional case. Moreover, signed Möbius ladders  $ML_{2k+1}$  with  $k$  or  $k + 1$  negative edges, with any signature, have at least a positive square except in one case with  $k + 1$  negative edges given in Fig. 13. This exceptional case has nowhere-zero 5-flow, see Graphs (e), (f), (g), and (h) in Fig. 15. Now, ignore the three exceptional cases of signed graphs  $ML_{2k}$  with  $k - 1$  and  $k$  negative edges and  $ML_{2k+1}$  with  $k$  and  $k + 1$  negative edges.

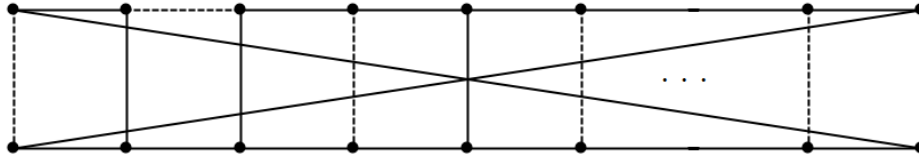
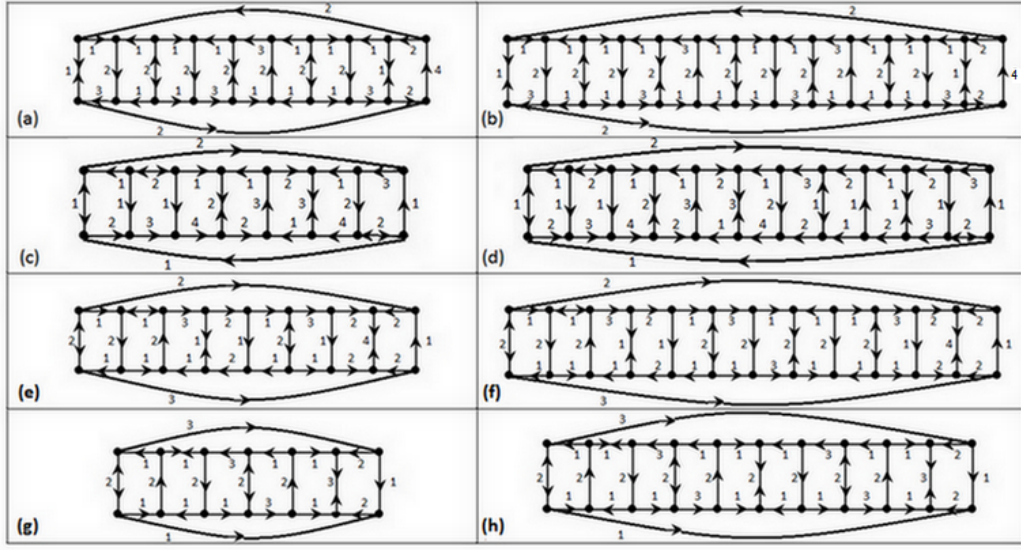
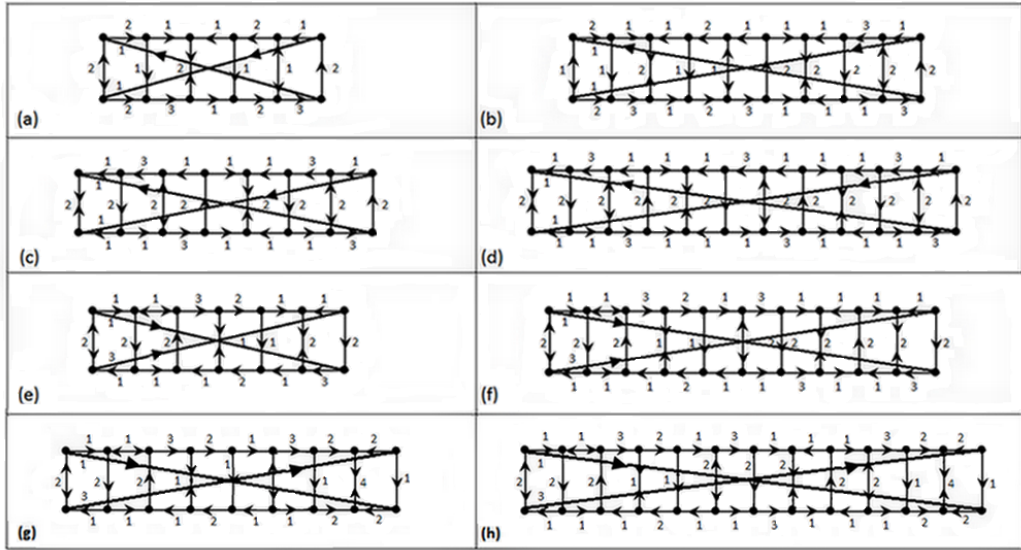


FIGURE 13. The exceptional case of  $ML_{2k+1}$  with  $k + 1$  negative edges

In order to prove the assertion, it is sufficient to show that signed graphs  $ML_{2k}$  with  $k$  negative edges and  $ML_{2k+1}$  with  $k + 1$  negative edges have nowhere-zero 5-flow. Note that  $k$  and  $k + 1$  are the maximum number of the negative edges can occur in  $ML_{2k}$  and  $ML_{2k+1}$ , respectively. To prove the claim use induction on  $k \geq 3$ . Let  $k = 3$ . One can check that  $ML_6$  with three negative edges has at least a positive square, (as it mentioned just in one case there is not a positive square). Also, signed graph  $ML_7$  with four negative edges (except the exceptional case given in Fig. 13) has at least a positive square. So, by Lemma 2.7 and the existence of a nowhere-zero 5-flow on  $ML_4$  and  $ML_5$ , we conclude that  $ML_6$  and  $ML_7$  with three and four negative edges, respectively have nowhere-zero 5-flow. Now, assume that for each  $4 \leq i \leq k - 1$ ,  $ML_{2i}$  and  $ML_{2i+1}$  with  $i$  and  $i + 1$  negative edges, respectively have nowhere-zero 5-flow. This is the induction hypothesis. Consider  $i = k$ . Since  $ML_{2k}$  and  $ML_{2k+1}$  with  $k$  and  $k + 1$  negative edges, respectively have at least one positive square (without considering exceptional cases). Hence, by Lemma 2.7 and using the induction hypothesis, we conclude that each signed graph  $ML_n$  for  $n \geq 4$ , with at least two negative edges has a nowhere-zero 5-flow.  $\square$

**Remark 3.5.** Note that the next natural class to look at are the generalized Petersen graphs, see [4]. The reason of considering this family of graphs is that the Petersen graph is exceptional among all generalized Petersen graphs by not admitting a nowhere-zero 4-flow. Since the proof by Castagna and Prins in [3] that shows that all other generalized Petersen graphs admit a nowhere-zero 4-flow (are 3-edge-colourable) is far from easy, and there is no other shorter proof, it is plausible, that establishing a nowhere-zero 5-flow for signed generalized Petersen graphs is not going to be easy.

FIGURE 14. Nowhere-zero 5-flow on the exceptional cases of  $CL_n$ FIGURE 15. Nowhere-zero 5-flow on the exceptional cases of  $ML_n$ 

## REFERENCES

- [1] A. BOUCHET, *Nowhere-zero integral flows on a bidirected graph*, J. Combin. Theory Ser. B 34 (1983), no. 3, 279–292.
- [2] J. BROWN, C. GODSIL, D. MALLORY, A. RAZ, C.TAMON, *Perfect state transfer on signed graphs*, Quantum Inf. Comput. 13 (2013), no. 5-6, 511–530.

- [3] F. CASTAGNA, G. PRINS, *Every generalized Petersen graph has a Tait coloring*, Pacific J. Math. 40 (1972), 53–58.
- [4] H. S. M. COXETER, *Self-dual configurations and regular graphs*, Bull. Amer. Math. Soc. 56 (1950), 413–455.
- [5] F. HARARY, *On the notion of balance of a signed graph*, Michigan Math. J. 2 (1953–1954) 143–146.
- [6] T. KAISER, R. LUKOĆKA, E. ROLLOVÁ, *Nowhere-zero flows in signed graphs: A survey*, Nowhere-zero flows in signed graphs: a survey. Selected topics in graph theory and its applications, 85–104, Lect. Notes Semin. Interdiscip. Mat., 14, Semin. Interdiscip. Mat. (S.I.M.), Potenza, 2017.
- [7] D. KÖNIG, *Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre*, (German) Math. Ann. 77 (1916), no. 4, 453–465.
- [8] E. ROLLOVÁ, M. SCHUBERT, E. STEFFEN, *Signed graphs with two negative edges*, Electron. J. Combin. 25 (2018), no. 2, Paper 2.40, 18 pp.
- [9] T. ZASLAVSKY, *Signed graphs*, Discrete Appl. Math. 4 (1982), no. 1, 47–74.

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