

CLIQUE NUMBER OF XOR-POWERS OF KNESER GRAPHS

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ABSTRACT. Let $f_\ell(n, k)$ denote the clique number of the xor-product of ℓ isomorphic Kneser graphs $KG(n, k)$. Alon and Lubetzky investigated the case of complete graphs as a coding theory problem and showed $f_\ell(n, 1) \leq \ell n + 1$. Imolay, Kocsis, and Schweitzer proved that $f_2(n, k) \leq \lfloor \frac{n}{k} \rfloor + c(k)$.

Here, the order of magnitude of $c(k)$ is determined to be $\Theta\left(k^{\binom{2k}{k}}\right)$. By explicit constructions and by an algebraic proof, it is shown that $\ell n - 2\ell - 1 \leq f_\ell(n, 1) \leq \ell n - \ell + 1$ (for all $n \geq 1$ and $\ell \geq 3$). Finally, it is proved that the order of magnitude of f lies between $\Omega\left(n^{\lfloor \log_2(\ell+1) \rfloor}\right)$ and $O\left(n^{\lfloor \frac{\ell+1}{2} \rfloor}\right)$ (as ℓ, k are given and $n \rightarrow \infty$).

We conjecture that the lower bound gives the correct exponent.

1. INTRODUCTION

1.1. Kneser graphs. A Kneser graph $G := KG(A, k)$ has a *base* set A , the vertex set of G consists of all subsets of k elements of A . We denote this as $V(G) := \binom{A}{k}$, and a pair $\{X, Y\}$ forms an edge of G when $X \cap Y = \emptyset$. We also use $KG(n, k)$ for a Kneser graph with an n -element base set. A complete subgraph in the Kneser graph corresponds to a family of mutually disjoint k -element sets in the base set A . So, the size of the largest clique $\omega(KG(n, k)) = \lfloor n/k \rfloor$.

The parameters of Kneser graphs are widely studied in combinatorics. Lovász [13] determined the chromatic number of Kneser graphs and Erdős, Ko and Rado [7] determined their independence number. Brešar and Valencia-Pabon [5] examined the independence number of Kneser graphs of different graph products. In this article, we study the clique number of the xor-products.

1.2. The xor-product. Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, their *xor-product* $G \cdot H$ is a graph with the vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') are connected in $G \cdot H$ if and only if either $gg' \in E(G)$ and $hh' \notin E(H)$ or $gg' \notin E(G)$ and $hh' \in E(H)$. The xor-product is not as well understood as other graph products, but there are a number of highly non-trivial results about it, e.g., by Alon and Lubetzky [2] and Thomason [15]. They were also motivated to compare it to the Shannon capacity of graphs, see Alon and Lubetzky [3], Lovász [14]. Let $f_\ell(n, k)$ denote the clique number of the xor-product of ℓ isomorphic Kneser graphs $KG(n, k)$.

Taking a clique $C \subset V(G)$ and a vertex $b \in V(H)$ the set $C \times \{b\}$ forms a clique in $G \cdot H$ so we obtain $\omega(G \cdot H) \geq \max\{\omega(G), \omega(H)\}$. Hence

$$\omega(KG(n, k) \cdot KG(n, k)) \geq \lfloor n/k \rfloor.$$

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Imolay, Kocsis, and Schweitzer [11] showed that the function $f_2(n, k) - \lfloor n/k \rfloor$ is bounded for any given k . Define

$$c(k) := \sup_{n \rightarrow \infty} \left\{ f_2(n, k) - \left\lfloor \frac{n}{k} \right\rfloor \right\}.$$

One of the objectives of this article is to determine the order of magnitude of $c(k)$ as $k \rightarrow \infty$.

Theorem 1.1. *For all $k \geq 1$ and $n \geq \frac{1}{2} \left(\binom{2k}{k} - 2 \right) k^2$,*

$$(1.1) \quad f_2(n, k) \geq \left\lfloor \frac{n}{k} \right\rfloor + \binom{2k}{k} \frac{k}{2} - k.$$

On the other hand, as $k \rightarrow \infty$ we have

$$(1.2) \quad c(k) \leq (1 + o(1))k \binom{2k}{k}.$$

The proof of Theorem 1.1 is presented in Section 2. It might be easier to determine $f_2(n, k)$ for large n . In [11] it was proved that $f_2(n, 2) = \lfloor n/2 \rfloor + 4$ for sufficiently large n . Let us define

$$c_\infty(k) := \limsup_{n \rightarrow \infty} \left\{ f(n, k) - \left\lfloor \frac{n}{k} \right\rfloor \right\}.$$

We have $c_\infty(1) = c(1) = 0$, $c_\infty(2) = 4$, and in general the true order of magnitudes

$$\binom{2k}{k} \frac{k}{2} - k \leq c_\infty(k) \leq c(k) \leq (1 + o(1))k \binom{2k}{k}.$$

We conjecture that here the equality holds for $c_\infty(k)$ for all $k \geq 1$ and the best construction is the one from Section 2.1.

Conjecture 1.2. *For all k , if n is large enough then*

$$f_2(n, k) = \left\lfloor \frac{n}{k} \right\rfloor + \binom{2k}{k} \frac{k}{2} - k.$$

Maybe more is true and $c_\infty(k) = c(k)$ for all k .

1.3. Multiproducts of the complete graphs. The rest of our article tackles the question of higher powers of Kneser graphs. We investigate $f_\ell(n, k)$ the clique number of the ℓ -th xor-power (the xor-product of ℓ isomorphic copies) of the Kneser graph $KG(n, k)$.

Even the case $k = 1$ is not trivial when $\ell \geq 3$. Note that $KG(n, 1)$ is the complete graph on n vertices. The function $f_\ell(n, 1)$ was considered by Alon and Lubetzky in [2], they proved an upper bound $f_\ell(n, 1) \leq \ell n + 1$. Here we give tighter bounds.

Theorem 1.3. *For all $\ell \geq 3$ and $n \geq 1$*

$$\ell n - 2\ell - 1 \leq f_\ell(n, 1) \leq \ell n - \ell + 1.$$

1.4. Higher powers of Kneser graphs. We give bounds for the magnitude of $f_\ell(n, k)$ for large n . In particular, we show that it is not necessarily linear in n .

Theorem 1.4. *We have*

$$(1.3) \quad f_\ell(n, k) \leq 2^{\lfloor \frac{\ell}{2} \rfloor} \cdot \left\lfloor \frac{\ell}{2} \right\rfloor! \cdot n^{\lfloor \frac{\ell+1}{2} \rfloor}.$$

On the other hand, if $k \geq \lfloor \log_2(\ell + 1) \rfloor$, then

$$(1.4) \quad f_\ell(n, k) \geq \left(\left\lfloor \frac{n}{k} \right\rfloor \right)^{\lfloor \log_2(\ell+1) \rfloor}.$$

This settles the exact magnitude for the cases $\ell \leq 4$.

Conjecture 1.5. For any fixed ℓ and k if k is large enough then

$$f_\ell(n, k) = \Theta(n^{\lfloor \log_2(\ell+1) \rfloor}).$$

1.5. Semi-intersecting families. The vertex set of a Kneser graph $KG(A, k)$ is a k -uniform hypergraph. Given ℓ Kneser graphs $KG(A_i, k)$ ($1 \leq i \leq \ell$) with pairwise disjoint n -element base sets $A_1, \dots, A_\ell$ the vertices of their xor-product are naturally correspond to those $k\ell$ -element subsets S of $A_1 \cup A_2 \cup \dots \cup A_\ell$ where $|S \cap A_i| = k$ for each i . The set pair $\{S, S'\}$ corresponds to an edge in the xor-product $KG(A_1, k) \cdots KG(A_\ell, k)$ if $S \cap S' \cap A_i = \emptyset$ in an odd number of cases for $1 \leq i \leq \ell$.

Definition 1.6. A family of sets $\mathcal{S}$ on the pairwise disjoint base sets $A_1 \cup A_2 \cup \dots \cup A_\ell$ is called an ℓ -semi-intersecting family with parameters n and k if

- $|A_1| = |A_2| = \dots = |A_\ell| = n$,
- $|S \cap A_1| = |S \cap A_2| = \dots = |S \cap A_\ell| = k$ for each $S \in \mathcal{S}$, and
- for distinct $S, T \in \mathcal{S}$, we have $S \cap T \cap A_i = \emptyset$ for an odd number of i 's, $1 \leq i \leq \ell$.

There is a one-to-one correspondence between ℓ -semi-intersecting families and cliques in $KG(n, k)^\ell$. Hence $f_\ell(n, k) = \omega(KG(n, k)^\ell)$ is the maximum size of an ℓ -semi-intersecting family with parameters n and k .

We prefer to work with this equivalent hypergraph reformulation. Similar questions in extremal combinatorics with two part set systems were studied extensively, see, e.g., [10] and [12].

2. DETERMINING THE ORDER OF MAGNITUDE OF $c(k)$

In this section we prove the bounds stated in Theorem 1.1. In this case $\ell = 2$, we use simply semi-intersecting instead of 2-semi-intersecting, and we denote the base sets A_1, A_2 by A and B .

2.1. Lower bound construction. Here we give a construction yielding (1.1).

Proof. For easier notation introduce $m := \frac{1}{2} \binom{2k}{k}$. Choose a subset of $K \subset A$ with $|K| = 2k$. Label the k -element subsets of K by

$$H_1, H_2, \dots, H_m, G_1, G_2, \dots, G_m$$

such that H_i and G_i are disjoint ($1 \leq i \leq m$). Let $L_2, L_3, \dots, L_m$ be pairwise disjoint k^2 -element subsets of B . This is possible as $n \geq \frac{1}{2} \left(\binom{2k}{k} - 2 \right) k^2$. Let us arrange the elements of L_i to a $k \times k$ rectangular point lattice. There are $n - (m-1)k^2$ elements of $B \setminus \bigcup L_i$, so we can select additional pairwise disjoint k -element subsets $F_1, F_2, \dots, F_d$ from them, where $d = \lfloor \frac{n}{k} \rfloor - (m-1)k$.

Define a semi-intersecting family $\mathcal{S}$ as follows. We let $S \subset A \cup B$ in $\mathcal{S}$ if one of the following holds.

- $S \cap A = H_i$ and $S \cap B$ corresponds to a row of the lattice in L_i ,
- $S \cap A = G_i$ and $S \cap B$ corresponds to a column of the lattice in L_i ,
- $S \cap A = H_1$ and $S \cap B = F_i$ for some $1 \leq i \leq d$.

This $\mathcal{S}$ is a semi-intersecting family with parameters n and k , because if $S, T \in \mathcal{S}$ with $S \neq T$, and they are disjoint in A then $\{S \cap A, T \cap A\} = \{H_i, G_i\}$ for some $2 \leq i \leq m$. On the other hand, this is the only case when S and T intersect in B , as they intersect only if $S \cap B$ is a row (or column) of some L_i and $T \cap B$ is a column (or row) of the same L_i .

It remains to count the members of $\mathcal{S}$. There are $(m-1)k$ sets from both the first and second bullet point, and d from the third one. Hence

$$|\mathcal{S}| = 2(m-1)k + d = \left(\binom{2k}{k} - 2 \right) k + \left\lfloor \frac{n}{k} \right\rfloor - \frac{\binom{2k}{k} - 2}{2} k = \left\lfloor \frac{n}{k} \right\rfloor + \binom{2k}{k} \frac{k}{2} - k.$$

□

2.2. Cross intersecting matchings. A set of hypergraphs $\mathcal{A}_1, \dots, \mathcal{A}_t$ is called a k -uniform *cross intersecting* matching of size t and of type $(d_1, \dots, d_t)$ if $t, k \geq 2$, $d_i \geq 2$ for $1 \leq i \leq t$, each $\mathcal{A}_i$ consists of d_i pairwise disjoint k -element sets and $X \cap Y \neq \emptyset$ for $X \in \mathcal{A}_i$, $Y \in \mathcal{A}_j$ whenever $1 \leq i \neq j \leq t$. Obviously, $d_i \leq k$ for every i . The classical set-pair theorem of Bollobás [4] implies that $t \leq \frac{1}{2} \binom{2k}{k}$ and here equality holds only if $d_1 = \dots = d_t = 2$ and each $\mathcal{A}_i$ consists of a complementary pair of k -sets of a base set L , $|L| = 2k$. There are many generalizations of Bollobás's theorem, see, e.g., Alon [1] where the exterior algebra method was introduced. Even the case of 2-independent d -partitions is highly non-trivial, i.e., when $\cup \mathcal{A}_i$ is the same dk -element set for all i . For this case Gargano, Körner, and Vaccaro [9] showed that for any fixed d one can have $t = \Omega(4^{k(1-o(1))})$ using their Sperner capacity method in information theory. Here we show an upper bound.

Lemma 2.1. *There exists a sequence $\gamma(2), \gamma(3), \dots$ with $\gamma(k) \rightarrow 0$ exponentially as $k \rightarrow \infty$, such that for every k -uniform cross intersecting matching of type $(d_1, \dots, d_t)$,*

$$(2.1) \quad \sum_i d_i(d_i - 1) \leq (1 + \gamma(k)) \binom{2k}{k}.$$

We conjecture that the true value of γ is zero, for all k .

Proof. Define $A := \cup_i (\cup \mathcal{A}_i)$, $n := |A|$. Given any ordering π of A and two non-empty subsets $X, X' \subset A$, we say that $X <_\pi X'$ if $\pi(x) < \pi(x')$ for all $x \in X$ and $x' \in X'$. We call π of *type i* if there are two sets $X, X' \in \mathcal{A}_i$ with $X <_\pi X'$. Every permutation π can have only at most one type. Indeed, if $X_i, X'_i \in \mathcal{A}_i$ with $X_i <_\pi X'_i$ and $X_j, X'_j \in \mathcal{A}_j$ with $X_j <_\pi X'_j$ then the cross intersection property implies that there are elements $u \in X_i \cap X'_j$ and $v \in X'_i \cap X_j$. From $X_i <_\pi X'_i$ we get $\pi(u) < \pi(v)$ and from $X_j <_\pi X'_j$ we get $\pi(v) < \pi(u)$, a contradiction.

Consider a uniform probability distribution on the $n!$ possible orderings of A . Let E_i denote the event that the random variable π is of type i . We have $\sum_i \Pr(E_i) \leq 1$ because for $i \neq j$ the events E_i and E_j cannot occur simultaneously. Fix i , our goal is to approximate $\Pr(E_i)$. From now on, $X_1, X_2, \dots, X_{d_i}$ denotes the members of $\mathcal{A}_i$. To simplify the presentation we leave out the index i from d_i in the following calculation.

Let $O_{\alpha, \beta}$ be the event that $X_\alpha < X_\beta$. There are $d(d-1)$ such events, because $1 \leq \alpha, \beta \leq d$ and $\alpha \neq \beta$. Since $E_i = \bigcup_{\alpha \neq \beta} O_{\alpha, \beta}$ the inclusion-exclusion principle yields

$$\Pr(E_i) \geq \sum_{\alpha \neq \beta} \Pr(O_{\alpha, \beta}) - \frac{1}{2} \sum_{\alpha_1 \neq \beta_1, \alpha_2 \neq \beta_2, (\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)} \Pr(O_{\alpha_1, \beta_1} \cap O_{\alpha_2, \beta_2}).$$

In the first sum, $\Pr(O_{\alpha, \beta}) = \frac{1}{\binom{2k}{k}}$, because this is the probability that π arranges the elements of $X_\alpha \cup X_\beta$ such that $X_\alpha < X_\beta$. Now we calculate $\Pr(O_{\alpha_1, \beta_1} \cap O_{\alpha_2, \beta_2})$ for all possible $\alpha_1, \beta_1, \alpha_2, \beta_2$. We categorize them into six groups.

— $\alpha_1 = \beta_2$ and $\alpha_2 = \beta_1$. In this case

$$\Pr(O_{\alpha_1, \beta_1} \cap O_{\alpha_2, \beta_2}) = 0.$$

— The numbers $\alpha_1, \beta_1, \alpha_2$ and β_2 are all distinct. In this case O_{α_1, β_1} and O_{α_2, β_2} are independent events, hence

$$\Pr(O_{\alpha_1, \beta_1} \cap O_{\alpha_2, \beta_2}) = \frac{1}{\binom{2k}{k} \binom{2k}{k}}.$$

Here there are $d(d-1)(d-2)(d-3)$ possibilities for $\alpha_1, \beta_1, \alpha_2, \beta_2$.

— $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 3$ and $\alpha_2 = \beta_1$. In this case $X_{\alpha_1} < X_{\beta_1} = X_{\alpha_2} < X_{\beta_2}$, therefore

$$\Pr(O_{\alpha_1, \beta_1} \cap O_{\alpha_2, \beta_2}) = \frac{1}{\binom{3k}{2k} \binom{2k}{k}}.$$

There are $d(d-1)(d-2)$ such possibilities.

— $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 3$ and $\alpha_1 = \beta_2$. This can be calculated in the same way as the previous case.

— $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 3$ and $\beta_1 = \beta_2$. This means that $X_{\alpha_1} < X_{\beta_1}$ and $X_{\alpha_2} < X_{\beta_1}$ are both true. Hence

$$\Pr(O_{\alpha_1, \beta_1} \cap O_{\alpha_2, \beta_2}) = \frac{1}{\binom{3k}{k}}.$$

As in the previous cases, there are $d(d-1)(d-2)$ possibilities for this.

— $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 3$ and $\alpha_1 = \alpha_2$. This is the same calculation as the previous case.

Combining the calculations above and using $2 \leq d \leq k$ we arrive at the following inequalities.

$$\begin{aligned} \Pr(E_i) &\geq \frac{d(d-1)}{\binom{2k}{k}} - \frac{1}{2} \frac{d(d-1)(d-2)(d-3)}{\binom{2k}{k} \binom{2k}{k}} - \frac{d(d-1)(d-2)}{\binom{3k}{2k} \binom{2k}{k}} - \frac{d(d-1)(d-2)}{\binom{3k}{k}} \\ &= \frac{d(d-1)}{\binom{2k}{k}} \left(1 - \frac{1}{2} \frac{(d-2)(d-3)}{\binom{2k}{k}} - \frac{(d-2)}{\binom{3k}{2k}} - \frac{(d-2)\binom{2k}{k}}{\binom{3k}{k}} \right) \\ &\geq \frac{d(d-1)}{\binom{2k}{k}} \left(1 - \frac{(k-2)(k-3)}{2\binom{2k}{k}} - \frac{(k-2)}{\binom{3k}{2k}} - \frac{(k-2)\binom{2k}{k}}{\binom{3k}{k}} \right) := \frac{d(d-1)}{\binom{2k}{k}} \cdot \frac{1}{1 + \gamma(k)}. \end{aligned}$$

Here $\gamma(2) = 0$, $\gamma(3) = 1/3$, $\gamma(4) < 0.44$, $\gamma(5) < 0.36$ and then it exponentially converges to 0 as $k \rightarrow \infty$.

Summing these lower bounds for all i we get (2.1). $\square$

2.3. Proof of the upper bound for $c(k)$. We prove (1.2) in the following form. For all $n, k \geq 2$,

$$f_2(n, k) \leq \left\lfloor \frac{n}{k} \right\rfloor + (1 + \gamma(k))k \binom{2k}{k},$$

where we define $\gamma(2) = \frac{1}{3}$ and $\gamma(k)$ comes from the proof of Lemma 2.1 for $k \geq 3$.

Consider a semi-intersecting family $\mathcal{S}$ with parameters n and k and base sets A and B . Since $(1 + \gamma(k))k \binom{2k}{k} \geq 2k^3$ we may suppose $|\mathcal{S}| > 2k^3$.

Lemma 2.2. *If $|\mathcal{S}| > 2k^3$ then either all degrees in A are at most k , or all degrees in B are at most k .*

Proof. Assume that there is an $a \in A$ with degree more than k . Let $S_1, S_2, \dots, S_{k+1} \in \mathcal{S}$ be some (distinct) sets containing a and define $X := \bigcup_{1 \leq i \leq k+1} (S_i \cap A)$. Note that $|X| \leq k^2$ as we take the union of $k+1$ sets with k elements, all containing a . The sets $B \cap S_i$ are pairwise disjoint for $1 \leq i \leq k+1$ so no $T \in \mathcal{S}$ can intersect each of them in B . Hence $T \cap X \neq \emptyset$ for all $T \in \mathcal{S}$.

We claim that the degree of each $y \in B$ is at most $|X|$, i.e., $\deg_{\mathcal{S}}(y) \leq k^2$. Indeed, a set $T \in \mathcal{S}$ with $y \in T$ contains a pair $\{x, y\}$ with $x \in X$ and every such pair can appear in at most one member of $\mathcal{S}$.

Similarly, $b \in B$ and $\deg_{\mathcal{S}}(b) > k$ imply $\deg_{\mathcal{S}}(x) \leq k^2$ for every $x \in A$.

Fix any member $T \in \mathcal{S}$. Since $\mathcal{S}$ is intersecting we obtain $|\mathcal{S}| \leq \sum_{z \in T} \deg_{\mathcal{S}}(z) \leq 2k \cdot k^2$, and we are done. $\square$

From now on, we may suppose that $\deg_{\mathcal{S}}(y) \leq k$ for all $y \in B$. Starting with $\mathcal{S}_0 := \mathcal{S}$ we define a series of families $\mathcal{S}_0 \supset \mathcal{S}_1 \supset \dots \supset \mathcal{S}_q$ as follows. If the families $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{i-1}$ have already been created, and the members of $\mathcal{S}_{i-1}$ were pairwise disjoint in B then we let $q := i - 1$ and $\mathcal{S}_q := \mathcal{S}_{i-1}$. Note that, $\mathcal{S}_q \leq \lfloor \frac{n}{k} \rfloor$.

Otherwise, define $\mathcal{S}_i$ as follows. Take a vertex $p_i \in B$ with maximum degree in $\mathcal{S}_{i-1}$, let $\mathcal{Z}_i \subset \mathcal{S}_{i-1}$ be the family of sets containing p_i , and let $d_i := |\mathcal{Z}_i|$. We have $2 \leq d_i \leq k$. Denote by $\mathcal{M}_i \subset \mathcal{S}_{i-1} \setminus \mathcal{Z}_i$ the family of sets which intersect at least one set of $\mathcal{Z}_i$ in B . Finally, let $\mathcal{S}_i := \mathcal{S}_{i-1} \setminus (\mathcal{Z}_i \cup \mathcal{M}_i)$.

Now we give an upper bound for $|\mathcal{S}_{i-1} \setminus \mathcal{S}_i|$. Since $p_i \in \cap \mathcal{Z}_i$ we have $|B \cap \bigcup_{Z \in \mathcal{Z}_i} Z| \leq 1 + (k-1)d_i$. An element of $B \cap (\bigcup \mathcal{Z}_i) \setminus \{p_i\}$ can meet at most $d_i - 1$ members of $\mathcal{M}_i$, so we get $|\mathcal{M}_i| \leq (k-1)d_i(d_i - 1)$. Hence $|\mathcal{S}_{i-1} \setminus \mathcal{S}_i| = |\mathcal{Z}_i \cup \mathcal{M}_i| \leq d_i + (k-1)d_i(d_i - 1)$ and

$$|\mathcal{S}| = |\mathcal{S}_q| + \sum_{1 \leq i \leq q} |\mathcal{S}_{i-1} \setminus \mathcal{S}_i| \leq \left\lfloor \frac{n}{k} \right\rfloor + \sum_{1 \leq i \leq q} (d_i + (k-1)d_i(d_i - 1)).$$

We get

$$|\mathcal{S}| - \left\lfloor \frac{n}{k} \right\rfloor \leq \sum_{1 \leq i \leq q} d_i + (k-1) \sum_{1 \leq i \leq q} d_i(d_i - 1) \leq k \sum_{1 \leq i \leq q} d_i(d_i - 1).$$

We need to bound $\sum_{1 \leq i \leq q} d_i(d_i - 1)$.

Observe that the sets in the family $\mathcal{Z}_i$ are pairwise disjoint in A as they have a common element in B . Define $\mathcal{A}_i$ as $\{Z \cap A : Z \in \mathcal{Z}_i\}$. If $S \in \mathcal{S}_i$ and $Z \in \mathcal{Z}_i$ then $S \cap Z \cap B = \emptyset$, as any set from $\mathcal{S}_{i-1}$ that intersects Z in B is in $\mathcal{Z}_i \cup \mathcal{M}_i$ by definition. Hence $S \cap Z \cap A \neq \emptyset$. In particular, any $Z \in \mathcal{Z}_i$ and $Z' \in \mathcal{Z}_j$ intersect in A if $i < j$, i.e., $X \cap X' \neq \emptyset$ if $X \in \mathcal{A}_i$, $X' \in \mathcal{A}_j$, and $i \neq j$.

If $q \geq 2$ then $\mathcal{A}_1, \dots, \mathcal{A}_q$ form a k -uniform cross intersecting matching and then Lemma 2.1 completes the proof. In case of $q \leq 1$ we have $|\mathcal{S}| \leq \left\lfloor \frac{n}{k} \right\rfloor + 2k^2(k-1)$ and we are done. $\square$

3. MULTIPRODUCTS OF THE COMPLETE GRAPHS

3.1. Algebraic upper bound for the product of complete graphs. Complete graphs are also Kneser graphs with $k = 1$. We prove the upper bound $f_{\ell}(n, 1) \leq n\ell - \ell + 1$ in Theorem 1.3 in the following stronger form. Suppose that $\ell \geq 2$, $n_1, \dots, n_{\ell}$ are positive integers and $A_1, \dots, A_{\ell}$ are disjoint sets of sizes $n_1, \dots, n_{\ell}$, $V := A_1 \cup A_2 \cup \dots \cup A_{\ell}$.

Theorem 3.1. *Let G be the xor-product of the complete graphs $K_{n_1}, \dots, K_{n_{\ell}}$. Then $\omega(G) \leq |V| - \ell + 1$.*

The vertices of G corresponds to ℓ -element sets T with $|T \cap A_i| = 1$ for each i . A clique in G corresponds to an ℓ -semi-intersecting family $\mathcal{S}$ of ℓ -element subsets of V , i.e., for distinct $S, T \in \mathcal{S}$ we have $|S \setminus T| = \ell - |S \cap T|$ is odd, so $(\ell + 1 + |S \cap T|)$ is even. Let $\mathbf{F}_2$ be the 2-element field. For every subset $X \subseteq V$ let $\hat{X} \in \mathbf{F}^V$ denote the characteristic vector X . Thus $\hat{\emptyset}$ is the $|V|$ dimensional zero-vector. Let $\mathcal{A}$ denote the family $\{A_1, \dots, A_{\ell}\}$.

Lemma 3.2. *Suppose that the cardinality $|\mathcal{S}|$ is odd and suppose that the vectors $\{\widehat{X} : X \in \mathcal{S} \cup \mathcal{A}\}$ have a non-trivial linear dependency, $\sum_{X \in \mathcal{S} \cup \mathcal{A}} \alpha(X) \widehat{X} = \widehat{\emptyset}$ for some $\alpha(X) \in \mathbf{F}_2$, not all coefficients are zero. Then this dependency is unique and $\alpha(X) = 1$ for each $X \in \mathcal{S} \cup \mathcal{A}$.*

Proof of Lemma 3.2. The scalar product $\langle \widehat{X}, \widehat{Y} \rangle = |X \cap Y|$. So for any $Y \subseteq V$,

$$0 = \langle \widehat{\emptyset}, \widehat{Y} \rangle = \sum_{X \in \mathcal{S} \cup \mathcal{A}} \alpha(X) |X \cap Y|.$$

Here every integer is taken modulo 2. Substituting to Y a single element $v \in A_i$, then a fixed member $A_i \in \mathcal{A}$, and finally a member $T \in \mathcal{S}$ we get

$$(3.1) \quad 0 = \sum_{S: v \in S \in \mathcal{S}} \alpha(S) + \alpha(A_i),$$

$$(3.2) \quad 0 = \left(\sum_{S \in \mathcal{S}} \alpha(S) \right) + \alpha(A_i) |A_i|,$$

$$(3.3) \quad 0 = \left(\sum_{S \in \mathcal{S}} |S \cap T| \alpha(S) \right) + \left(\sum_j \alpha(A_j) \right).$$

Add $(\ell + 1)$ times (3.2) to (3.3). For given T and A_i we get

$$0 = \left(\sum_{S \in \mathcal{S}} (\ell + 1 + |S \cap T|) \alpha(S) \right) + (\ell + 1) \alpha(A_i) |A_i| + \left(\sum_j \alpha(A_j) \right).$$

For distinct $S, T \in \mathcal{S}$ $(\ell + 1 + |S \cap T|)$ is even and for $S = T$ we have $\ell + 1 + |S \cap T| = 2\ell + 1 = 1$ (in $\mathbf{F}_2$). So the first term in the last displayed formula is exactly $\alpha(T)$. The second and the third terms are independent from T , so we obtain that all $\alpha(T)$ are equal.

If $\alpha(T) = 0$ for each $T \in \mathcal{S}$ then (3.1) gives that $\alpha(A_i) = 0$ for all i , a contradiction. Therefore each $\alpha(T) = 1$, so the first term in (3.2) is $|\mathcal{S}|$. By our assumptions this is odd, so $\alpha(A_i) |A_i|$ should be odd. In particular $\alpha(A_i) = 1$ for all i . $\square$

Proof of Theorem 3.1. If $|\mathcal{S}|$ is odd, then by Lemma 3.2 the vectors $\{\widehat{X} : X \in \mathcal{S} \cup \mathcal{A}\}$ are either linearly independent in $\mathbf{F}_2^V$ or has a unique linear dependency. So they generate a subspace of dimension at least $|\mathcal{S}| + |\mathcal{A}| - 1$. This is at most $|V|$ and we are done.

If $|\mathcal{S}|$ is even then we can assume that $|\mathcal{S}| \geq 2$. Take two distinct members $T_1, T_2 \in \mathcal{S}$. Then $|\mathcal{S} \setminus \{T_i\}|$ is odd (for $i = 1, 2$). If either of the set of vectors $\{\widehat{X} : X \in (\mathcal{S} \setminus \{T_i\}) \cup \mathcal{A}\}$ is independent, we get $|\mathcal{S}| + |\mathcal{A}| - 1 \leq |V|$ as desired. If both are dependent, then again by Lemma 3.2 they have unique linear dependencies, namely $\sum \{\widehat{X} : X \in (\mathcal{S} \setminus \{T_i\}) \cup \mathcal{A}\} = \widehat{\emptyset}$. Adding up these equations we get $\widehat{T}_1 + \widehat{T}_2 = \widehat{\emptyset}$. This contradiction completes the proof. $\square$

3.2. An explicit construction for the case of complete graphs. We prove the lower bound $f_\ell(n, 1) \geq \ell n - 2\ell - 1$ in Theorem 1.3 in the following stronger form. Suppose that $\ell \geq 3$, $A_1, \dots, A_\ell$ are disjoint sets of sizes $n_1, \dots, n_\ell$, $V := A_1 \cup A_2 \cup \dots \cup A_\ell$.

Theorem 3.3. *Let G be the xor-product of the complete graphs $K_{n_1}, \dots, K_{n_\ell}$. Suppose that $\ell \geq 3$ and $n_i \geq 2$ for each $i \in [\ell]$. Then $\omega(G) \geq |V| - 2\ell - 1$.*

Proof. We show a construction. For a given partition $A_1, \dots, A_\ell$ ($\ell \geq 3$) we call the family of sets $\mathcal{B} := \{B_1, \dots, B_\ell\}$ an ℓ -core if

- (i) each B_i is an $(\ell - 1)$ -set with $B_i \cap A_i = \emptyset$ but $|B_i \cap A_j| = 1$ for $i \neq j$ and
- (ii) $|B_i \cap B_j| \not\equiv \ell \pmod{2}$ for all $1 \leq i, j \leq \ell$.

This intersection condition can be reformulated as $|B_i \cap B_j| + \ell$ is always odd. Let $U(\mathcal{B})$ denote $\cup B_i$. A core $\mathcal{B}$ generates an ℓ -uniform family $\mathcal{S}(\mathcal{B})$ by enlarging each core set by extra elements as follows.

$$\mathcal{S}(\mathcal{B}) := \{B_i \cup \{x\} : i \in [\ell], x \in A_i \setminus U\}.$$

We claim that $\mathcal{S}(\mathcal{B})$ is an ℓ -semi-intersecting family with $k = 1$ of size $|V| - |U|$. Indeed, by definition each $S \in \mathcal{S}(\mathcal{B})$ intersects every A_i in exactly one element. Let $S, T \in \mathcal{S}(\mathcal{B})$ with $S \neq T$. Say $S = B_i \cup \{x\}$ and $T = B_j \cup \{y\}$. Then $S \cap T = B_i \cap B_j$. So $S \cap T \cap A_\alpha = \emptyset$ in $\ell - |B_i \cap B_j|$ cases of α . Here $\ell - |B_i \cap B_j|$ is odd by (ii), so $\mathcal{S}(\mathcal{B})$ is an ℓ -semi-intersecting family.

The next step in the proof of Theorem 3.3 is to find a core $\mathcal{B}$ with small $|U(\mathcal{B})|$. We need a couple of more definitions. The *type* of $\mathcal{B}$ is the multiset $\{|U \cap A_i| : i \in [\ell]\}$. The set $A_i \cap U$ is called the *i th class* of $\mathcal{B}$.

Lemma 3.4. *Suppose that $p, q \geq 3$, $\mathcal{B}'_p$ is a p -core of type $(x_1, \dots, x_p)$ and $\mathcal{B}''_q$ is a q -core of type $(y_1, \dots, y_q)$. Then there exists a $(p + q - 1)$ -core $\mathcal{B}$ of type $(x_1, \dots, x_{p-1}, x_p + y_1, y_2, \dots, y_q)$.*

Proof of Lemma 3.4. Suppose that $U(\mathcal{B}'_p)$ and $U(\mathcal{B}''_q)$ are disjoint. We have $|\mathcal{B}'_p| = p$, $|\mathcal{B}''_q| = q$. The classes of $\mathcal{B}'_p$ are denoted by $A'_1, \dots, A'_p$ ($|A'_i| = x_i$), its hyperedges are $B'_1, \dots, B'_p$. The classes of $\mathcal{B}''_q$ are denoted by $A''_1, \dots, A''_q$ ($|A''_j| = y_j$), its hyperedges are $B''_1, \dots, B''_q$. We define the core $\mathcal{B} = \mathcal{B}_{p+q-1}$ as follows. Its classes are $A_1, \dots, A_{p+q-1}$ where $A_i := A'_i$ for $1 \leq i \leq p-1$, $A_p := A'_p \cup A''_1$, and $A_j := A''_{j-p+1}$ for $p+1 \leq j \leq p+q-1$. The hyperedges $B_1, \dots, B_{p+q-1}$ of $\mathcal{B}_{p+q-1}$ are defined as unions of the form $B'_\alpha \cup B''_\beta$ as follows: $B_p := B'_p \cup B''_1$ and in general $B_i := B'_i \cup B''_1$ for $1 \leq i \leq p$ and $B_j := B'_p \cup B''_{j-p+1}$ for $p+1 \leq j \leq p+q-1$.

We claim that $\mathcal{B}$ is a $(p+q-1)$ -core. $B_\alpha \cap A_\alpha = \emptyset$ and $|B_\alpha \cap A_\beta| = 1$ for $\alpha \neq \beta$ follows from the definition of $\mathcal{B}$. Consider $|B_\alpha \cap B_\beta|$. We have to show that $|B_\alpha \cap B_\beta| + (p+q-1)$ is odd. Write B_α in the form $B'_e \cup B''_f$ and let $B_\beta = B'_g \cup B''_h$ where $B'_e, B'_g \in \mathcal{B}'$ and $B''_f, B''_h \in \mathcal{B}''$. We have $B_\alpha \cap B_\beta = (B'_e \cup B''_f) \cap (B'_g \cup B''_h)$ which is the disjoint union of $B'_e \cap B'_g$ and $B''_f \cap B''_h$. Since $|B'_e \cap B'_g| + p$ and $|B''_f \cap B''_h| + q$ are both odd their sum is even, so

$$|B_\alpha \cap B_\beta| + p + q - 1 = |B'_e \cap B'_g| + |B''_f \cap B''_h| + p + q - 1$$

is odd. So $\mathcal{B}$ is a $(p+q-1)$ -core, completing the proof of Lemma 3.4. $\square$

The procedure described in the proof of Lemma 3.4 will be referred to as the *fusion* of $\mathcal{B}_p$ and $\mathcal{B}_q$. Using this construction, we prove by induction that there exists an ℓ -core $\mathcal{B}$ with $|U(\mathcal{B})| \leq 2\ell + 1$ if $\max_i n_i \geq 3$. Note that we can assume that $\max_i n_i \geq 3$ as if each $n_i = 2$ then $|V| = 2\ell$, so the lower bound from Theorem 3.3 obviously holds. First, we define an ℓ -core for $\ell = 3, 4, 5$.

There is a 3-core $\mathcal{B}_3$ of type $(2, 2, 2)$ with three sets $B_\alpha := \{a_{\alpha-1, \alpha}, a_{\alpha+1, \alpha}\}$ (indices are taken modulo 3) where these a 's are six distinct vertices with $a_{i,j} \in A_i$ ($i, j \in \{1, 2, 3\}$, $i \neq j$).

There is a 4-core $\mathcal{B}_4$ of type $(2, 2, 2, 1)$ on 7 vertices $\{a_{1,2}, a_{1,3}, a_{2,1}, a_{2,3}, a_{3,1}, a_{3,2}, a_4\}$ where $\{a_{1,2}, a_{1,3}\} \subset A_1$, $\{a_{2,1}, a_{2,3}\} \subset A_2$, $\{a_{3,1}, a_{3,2}\} \subset A_3$, and $a_4 \in A_4$. The core sets are $B_1 := \{a_{2,1}, a_{3,1}, a_4\}$, $B_2 := \{a_{1,2}, a_{3,2}, a_4\}$, $B_3 := \{a_{1,3}, a_{2,3}, a_4\}$, and $B_4 := \{a_{1,3}, a_{2,1}, a_{3,2}\}$.

There is a 5-core $\mathcal{B}_5$ of type $(2, 2, 2, 1, 1)$ on 8 vertices $\{a_{1,2}, a_{1,3}, a_{2,1}, a_{2,3}, a_{3,1}, a_{3,2}, a_4, a_5\}$ where $\{a_{1,2}, a_{1,3}\} \subset A_1$, $\{a_{2,1}, a_{2,3}\} \subset A_2$, $\{a_{3,1}, a_{3,2}\} \subset A_3$, $a_4 \in A_4$, and $a_5 \in A_5$. The core sets are $B_1 := \{a_{2,1}, a_{3,1}, a_4, a_5\}$, $B_2 := \{a_{1,2}, a_{3,2}, a_4, a_5\}$, $B_3 := \{a_{1,3}, a_{2,3}, a_4, a_5\}$, $B_4 := \{a_{1,3}, a_{2,1}, a_{3,2}, a_5\}$, and $B_5 := \{a_{1,2}, a_{2,3}, a_{3,1}, a_4\}$.

Now we give the construction for any $\ell \geq 3$. Fusing m copies of $\mathcal{B}_5$ of type $(1, 2, 2, 2, 1)$ we obtain a $(4m + 1)$ -core $\mathcal{B}_{4m+1}$ of type $(1, 2, \dots, 2, 1)$ for each $m \geq 1$. The fusion of a $\mathcal{B}_4$ of type $(2, 2, 2, 1)$ and a $\mathcal{B}_{4m+1}$ defines a $(4m + 4)$ -core $\mathcal{B}_{4m+4}$ of type $(2, 2, \dots, 2, 1)$ for each $m \geq 0$. Fusing this with a $\mathcal{B}_4$ of type $(1, 2, 2, 2)$ yields a $(4m + 7)$ -core of type $(2, \dots, 2)$ (for each $m \geq 0$). Finally, fusing $\mathcal{B}_3$ of type $(2, 2, 2)$ and a $4m + 4$ -core of type $(1, 2, \dots, 2)$ ($m \geq 0$) one gets a $(4m + 6)$ -core of type $(2, 2, 3, 2, \dots, 2)$, which finishes the proof. $\square$

3.3. Remarks. Note that in the proof of Theorem 3.1 we have adapted a method of Deza, Frankl and Singhi [6]. Their ‘even town theorem’ can be applied to prove $f_\ell(n, 1) \leq \ell n - \ell + 1$ when ℓ and n are both even. If ℓ is even and n is odd one can still apply the even town theorem to get $\ell n + 1$, the same upper bound as in Alon and Lubetzky [2]. In the case ℓ is odd the bound $f_\ell(n, 1) \leq \ell n$ follows by a theorem of Frankl and Wilson [8].

Suppose that there exists a finite projective plane of order $\ell - 1$, i.e., an ℓ -uniform, ℓ -regular set system $\mathcal{L}$ of size $\ell^2 - \ell + 1$ such that $|L \cap L'| = 1$ for all pairwise intersections. Also suppose that ℓ is even. Take any vertex v , we have ℓ lines containing it, $L_1, \dots, L_\ell$. Set $A_i := L_i \setminus \{v\}$, and let $\mathcal{S} := \mathcal{P} \setminus \{L_1, \dots, L_\ell\}$. This $\mathcal{S}$ is an ℓ -semi-intersecting family of size $(\ell - 1)^2$ on $\ell \times (\ell - 1)$ vertices. Since such finite planes exist whenever $\ell - 1$ is a power of an odd prime, we got infinitely many cases when the lower bound is tight in Theorem 3.1. This example was also mentioned by Alon and Lubetzky [2, 3]. They were more interested from coding theory point of view, i.e., when n is fixed and $\ell \rightarrow \infty$.

Finding the exact value of $f_\ell(n, 1)$ is still open.

4. HIGHER POWERS, THE GENERAL CASE

In this section we study the order of magnitude of $f_\ell(n, k)$.

4.1. Proof of the upper bound (1.3) by induction on ℓ . Suppose that $\ell \geq 2$ and let $\mathcal{S}$ be an ℓ -semi-intersecting family with parameters n and k and base sets $A_1, \dots, A_\ell$. Take any $v \in A_i$. Define $\mathcal{S}[v] := \{S \setminus A_i : v \in S \in \mathcal{S}\}$. Then $\mathcal{S}[v]$ does not contain multiple hyperedges, it is an $(\ell - 1)$ -semi-intersecting family. Hence $|\mathcal{S}[v]| = \deg_{\mathcal{S}}(v) \leq f_{\ell-1}(n, k)$. Take this inequality for each $v \in A_i$ and suppose that $|\mathcal{S}|$ has maximum size. We get

$$(4.1) \quad f_\ell(n, k) = |\mathcal{S}| = \frac{1}{k} \sum_{v \in A_i} \deg_{\mathcal{S}}(v) \leq \frac{n}{k} f_{\ell-1}(n, k).$$

If ℓ is even then $\mathcal{S}$ is intersecting. Taking the degrees of any given $T \in \mathcal{S}$ we obtain

$$(4.2) \quad f_\ell(n, k) = |\mathcal{S}| \leq 1 + \sum_{v \in T} (\deg_{\mathcal{S}}(v) - 1) \leq 1 + k\ell (f_{\ell-1}(n, k) - 1) \leq k\ell f_{\ell-1}(n, k).$$

We have $f_1(n, k) \leq n/k$. Apply (4.2), we get $f_2(n, k) \leq 2kf_1(n, k) \leq 2n$. Then (4.1) gives $f_3(n, k) \leq (n/k)f_2(n, k) \leq 2n^2/k$. Apply again (4.2), we get $f_4(n, k) \leq 4kf_3(n, k) \leq 2 \cdot 4 \cdot n^2$. Continuing this way, we get for each even ℓ that $f_\ell(n, k) \leq 2 \cdot 4 \cdot \dots \cdot \ell \cdot n^{\ell/2}$ and $f_\ell(n, k) \leq 2 \cdot 4 \cdot \dots \cdot (\ell - 1) \cdot n^{(\ell+1)/2}/k$ when ℓ is odd. $\square$

4.2. Construction showing the lower bound (1.4). Note that $f_\ell(n, k)$ is monotonous in n and also increases monotonously in ℓ , since an ℓ -semi-intersecting family $\mathcal{S}$ can be extended to an $(\ell + 1)$ -semi-intersecting family by adding $A_{\ell+1}$ to the base sets and a fixed k -element $S_{\ell+1} \subset A_{\ell+1}$ to all $S \in \mathcal{S}$. So it is enough to prove the theorem for $\ell = 2^t - 1$ where $t \geq 2$ is an integer, and we also suppose that $k \geq t$. Let $m := \lfloor \frac{n}{k} \rfloor$.

Take ℓ disjoint sets $A_1, \dots, A_\ell$ of sizes $|A_\alpha| = mk$. We are going to define an ℓ -semi-intersecting family $\mathcal{S}$ of size m^t with parameters mk and k with these base sets. Let H be 0-1 matrix of size $\ell \times t$ with $2^t - 1$ pairwise distinct nonzero rows. Note that this matrix is unique up to a permutation of its rows. Let C be an $\ell \times t$ matrix with non-negative integer entries such that $C_{\alpha,\beta} = 0$ if and only if $H_{\alpha,\beta} = 0$ and the row sums are k , i.e., $\sum_{1 \leq \beta \leq t} C_{\alpha,\beta} = k$. This is possible, as $k \geq t$. Partition each A_α into subsets $A_{\alpha,\beta}^p$ where $1 \leq p \leq m$ and $|A_{\alpha,\beta}^p| = C_{\alpha,\beta}$. In particular, let $A_{\alpha,\beta}^p = \emptyset$ if $C_{\alpha,\beta} = 0$. Make another partition of $\cup A_\alpha$ into mt sets by joining some of the $A_{\alpha,\beta}^p$ as follows. For each β and p where $1 \leq \beta \leq t$ and $1 \leq p \leq m$ define

$$S_\beta^p := \bigcup_{1 \leq \alpha \leq \ell} A_{\alpha,\beta}^p.$$

For each of the m^t functions $\varphi : [t] \rightarrow [m]$ define $S_\varphi := \cup_{1 \leq \beta \leq t} S_\beta^{\varphi(\beta)}$. Finally, let $\mathcal{S} := \{S_\varphi : \varphi : [t] \rightarrow [m]\}$.

Let us prove that $\mathcal{S}$ is an ℓ -semi-intersecting family. Each $S_\varphi \in \mathcal{S}$ intersects every A_α in k elements, as

$$|S_\varphi \cap A_\alpha| = \left| \bigcup_{1 \leq \beta \leq t} (A_\alpha \cap S_\beta^{\varphi(\beta)}) \right| = \left| \bigcup_{1 \leq \beta \leq t} A_{\alpha,\beta}^{\varphi(\beta)} \right| = \sum_{1 \leq \beta \leq t} C_{\alpha,\beta} = k.$$

Let $D_\beta \subset [\ell]$ denote the set of indices of nonzero elements of the column β of H , i.e., $D_\beta := \{\alpha : H_{\alpha,\beta} = 1\}$. For any set $X \subset \cup A_\alpha$ let $\pi(X) \subset [\ell]$ denote its projection, $\pi(X) := \{\alpha \in [\ell] : A_\alpha \cap X \neq \emptyset\}$. We have $\pi(S_\beta^p) = D_\beta$ for all p . Even more, each such set has the type $(C_{1,\beta}, \dots, C_{\ell,\beta})$, i.e., $|S_\beta^p \cap A_\alpha|$ is exactly $C_{\alpha,\beta}$. Note that for any $Q \subseteq [t]$ we have $|\cup_{q \in Q} D_q| = 2^t - 2^{t-q}$, an even number except in the case $Q = [t]$.

Given two functions φ, σ we claim that $|\pi(S_\varphi \cap S_\sigma)|$ is even except in the case $\varphi = \sigma$. This implies that $\mathcal{S}$ is ℓ -semi-intersecting, as claimed. We have $S_\varphi \cap S_\sigma = \left(\cup S_\beta^{\varphi(\beta)} \right) \cap \left(\cup S_\beta^{\sigma(\beta)} \right)$. Since the sets S_β^p form a partition of $\cup A_\alpha$ we have that $S_\varphi \cap S_\sigma = \cup \{S_\beta^p : \varphi(\beta) = \sigma(\beta)\}$. Hence $\pi(S_\varphi \cap S_\sigma) = \cup \{D_\beta : \varphi(\beta) = \sigma(\beta)\}$. This set has even cardinality whenever $\varphi \neq \sigma$. So we find that S_φ and S_σ are disjoint in an odd number of base sets A_α which finishes the proof. $\square$

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