

# Triacontagonal proofs of the Bell-Kochen-Specker theorem

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**Abstract** Coxeter pointed out that a number of polytopes can be projected orthogonally into two dimensions in such a way that their vertices lie on a number of concentric regular triacontagons (or 30-gons). Among them are the 600-cell and 120-cell in four dimensions and Gosset's polytope  $4_{21}$  in eight dimensions. We show how these projections can be modified into Kochen-Specker diagrams from which parity proofs of the Bell-Kochen-Specker theorem are easily extracted. Our construction trivially yields parity proofs of fifteen bases for all three polytopes and also allows many other proofs of the same type to be constructed for two of them. The defining feature of these proofs is that they have a fifteen-fold symmetry about the center of the Kochen-Specker diagram and thus involve both rays and bases that are multiples of fifteen. Any proof of this type can be written as a word made up of an odd number of distinct letters, each representing an orbit of fifteen bases. Knowing a word makes it possible to write down all the characteristics of the associated proof without first having to recover its bases. A comparison is made with earlier approaches that have been used to obtain parity proofs in these polytopes, and two questions related to possible applications of these polytopes are raised.

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## 1 Introduction

Coxeter pointed out that a number of four and eight dimensional polytopes can be projected orthogonally into two dimensions in such a way that their vertices lie on a number of concentric regular triacontagons (or 30-gons). He dubbed this the triacontagonal projection and the frontispiece of his monograph *Regular Polytopes*[1] shows such a projection of the 600-cell. His book *Introduction to Geometry*[2] shows a similar projection of the 120-cell and his paper with Shephard[3] shows the projection of  $4_{21}$ <sup>1</sup>. All three projections have  $C_{30}$  point symmetry, which gives them a highly symmetrical appearance. Table 1 lists the features of these polytopes that are of interest in connection with proofs of the Bell-Kochen-Specker (BKS) theorem[5][6]<sup>2</sup>, namely, the system of rays and bases they give rise to. A ray is an undirected line through the center of the polytope that passes through a pair of antipodal vertices, either of which may be taken as a representative of the ray. The number of rays is half the number of vertices of the polytope. A basis is a set of four or eight mutually orthogonal rays (four for the 600-cell and 120-cell and eight for Gosset's polytope).

| Polytope                      | Dimension | Ray-Basis symbol     |
|-------------------------------|-----------|----------------------|
| 600-cell or $\{3, 3, 5\}$     | 4         | $60_5 - 75_4$        |
| 120-cell or $\{5, 3, 3\}$     | 4         | $300_9 - 675_4$      |
| Gosset's polytope or $4_{21}$ | 8         | $120_{135} - 2025_8$ |

**Table 1** The names and symbols of the three polytopes studied in this paper are listed in the first column, their dimension in the second and their ray-basis symbol in the third. The ray-basis symbol lists the number of rays to the left of the dash and the number of bases to the right, with the subscript on the left indicating the number of bases to which each ray belongs and that on the right the number of rays in each basis.

The purpose of this paper is to show how the triacontagonal projections of these polytopes can be modified into Kochen-Specker diagrams from which parity proofs of the BKS theorem are easily extracted. It is interesting that these projections, which arose out of purely geometrical considerations, should lead so directly to an important physical result.

The plan of this paper is as follows. The next section spells out the simple procedure that turns a triacontagonal projection into a Kochen-Specker diagram and points out the forms that the rays and bases of the polytope take in it, particularly their fifteen-fold symmetry about its center. The notion of a

<sup>1</sup> Interestingly, hand drawn versions of all three projections existed before the computer versions were made. According to Coxeter, the projection of the 600-cell was drawn by van Oss, that of the 120-cell by Chilton and, even earlier, by Wythoff and that of  $4_{21}$  by Peter McMullen.

<sup>2</sup> This theorem rules out the existence of deterministic noncontextual hidden variables theories as an alternative to quantum mechanics.

parity proof is introduced and its relationship to the basis table is explained. We do this to make it easier for readers with no knowledge of the BKS theorem to follow the chain of arguments leading from Coxeter's projections to parity proofs, the objects of interest in this paper. The three sections that follow discuss how parity proofs can be extracted for each of the polytopes. Since the construction is basically the same in all the cases, we describe it in detail only for the 120-cell and discuss only the individual peculiarities and results obtained in the other two cases. The final section compares the approach used here with others that have been used to find parity proofs in these polytopes and comments on its advantages and limitations relative to them. We end by raising two questions suggested by this work, to which we do not have answers.

Before proceeding, we say a few words about the field of contextuality studies in which this work is situated. The basic result that laid the foundations for the field was the discovery by Bell[5] and Kochen and Specker[6] of the incompatibility between deterministic noncontextual hidden variables theories and quantum mechanics. Bell arrived at this conclusion by a continuum argument, whereas Kochen and Specker established it using a finite set of rays in a three-dimensional Hilbert space. The latter approach has been greatly generalized over the years, and many examples of rays are now known in real and complex Hilbert spaces of all dimensions that prove the theorem (see [7] for many references to such work). The three polytopes studied in this paper are just one of the many settings in which such proofs arise. What makes them particularly interesting is that they each have well over a billion parity proofs in them, in addition to a vast number of other contextuality proofs that are not as easy to characterize.

The notion of noncontextuality has evolved considerably over the years as a result of questions about how effectively it can be tested through experiments of limited precision [8]-[11]. In a notable development, Spekkens et. al.[12] have provided an operationally meaningful definition of a noncontextual theory in terms of the predictions it makes for the most general types of preparations and measurements that can be carried out on a quantum system. The advantage of this approach is that it makes it possible to design experiments in which noncontextual theories predict results only within a restricted range of parameters, whereas quantum mechanics (and/or experiment) yield results outside that range. Two examples of processes for which this can be done are parity-oblivious multiplexing[13] and universal quantum computation via magic state distillation[14]. In both cases, the "quantum advantage" obtained by implementing the process with quantum resources can be related to the breakdown of noncontextuality (and hence the relevance of contextuality) as an explicative mechanism for it.

Kochen-Specker sets have also been shown to be useful for a variety of other tasks such as "all versus nothing" proofs[15], bipartite perfect quantum strategies[16], the quantum computational advantage of shallow circuits[17]

and random number generation[29]. For all these reasons, the study of quantum contextuality remains an active area of research.

## 2 From the triacontagonal projection to the Kochen-Specker diagram

The triacontagonal projections show the vertices of the polytopes as points and their edges as line segments joining the points. Two changes must be made to convert them into Kochen-Specker diagrams<sup>3</sup>: firstly, every diametrically opposite pair of points in the projection must be replaced by just one of them and, secondly, the line segments representing the edges of the polytope must be replaced by line segments joining orthogonal pairs of its rays. The first change is needed because diametrically opposite pairs of points in the projection correspond to antipodal pairs of vertices of the polytope, which define the same ray. This change can be effected by keeping only the alternate vertices of all the triacontagons, thereby converting them into pentadecagons (or 15-gons). The second change can be effected by joining all pairs of pentadecagon vertices that correspond to orthogonal rays (this is easily done by using the triacontagonal coordinates of the polytopes<sup>4</sup>).

The resulting Kochen-Specker diagram differs in a number of ways from the triacontagonal projection: it has only half the number of points, a very different pattern of lines and a fifteen-fold symmetry about its center (instead of the thirty-fold symmetry of the original). However its visual appearance is confusing and not particularly helpful, so we will replace it by a table of numbers, the so called “basis table”, that captures all the information in it. The basis table is just a list of all the bases formed by the rays of the polytope. Since all the polytopes we are considering are saturated (i.e., they have no orthogonalities between their rays other than those represented in their bases), the basis table captures all the orthogonalities in the Kochen-Specker diagram and serves as a convenient substitute for it.

The Kochen-Specker diagrams we will be dealing with, which are all derived from triacontagonal projections, have the special feature that both their rays *and their bases* have a fifteen-fold symmetry about the center of the diagram. The symmetry of the rays is evident, but that of the bases less so. However it follows from the fact that fifteen-fold rotations about the center of the Kochen-Specker diagram that send it into itself correspond to four-dimensional rotations of the polytope that send it into itself<sup>5</sup>. We will exploit

<sup>3</sup> The Kochen-Specker diagram of a system of rays is a graph whose vertices are the rays and whose edges join orthogonal pairs of rays.

<sup>4</sup> The triacontagonal coordinates of the 600-cell and 120-cell are given in [1] and those of Gosset’s polytope can be found in [20].

<sup>5</sup> This is most easily seen from the triacontagonal coordinates of the polytopes.

this symmetry by introducing a numbering scheme for the rays that greatly simplifies the specification of the bases. This turns out to be crucial because the bases in two of the polytopes number in the hundreds or thousands and a direct specification of them would be impractical.

Our numbering scheme is as follows. The rays of any polytope are numbered in counterclockwise order around its pentadecagons, starting from the outermost pentadecagon and proceeding inwards, with the lowest number on any pentadecagon being attached to the ray/vertex lying on, or just above, the positive x-axis. The fifteen-fold symmetry of the bases about the center of the pentadecagons then makes it necessary to specify only one basis in each orbit, because the others can be obtained by repeatedly adding one to each of its rays, provided that one goes back to the ray at the beginning of a pentadecagon when one overshoots the one at the end (a procedure we will refer to as “wraparound”). We term the representative basis we pick from each orbit the “generator” and label it by a lower case letter. The basis table can then be expressed as a “word” made up of a number of distinct letters, each representing a generator. The entire basis table can be easily reconstructed from the word representing it.

A parity proof of the BKS theorem<sup>6</sup> can be obtained from a basis table by picking an odd number of bases in such a way that each of the rays that occurs in them occurs in an even number of them.<sup>7</sup> The parity proofs of interest in this paper are those having a fifteen-fold symmetry about the center of the Kochen-Specker diagram and, as such, can be represented by “words” made up of an odd number of distinct letters representing generators (and their orbits of fifteen bases). How such proofs can be picked out in the three polytopes is a task that we will address in the next three sections.

We end with a word about terminology. Since the parity proofs of this paper have a fifteen-fold symmetry, they might be termed “pentadecagonal” proofs. However we have referred to them as “triacontagonal” proofs in the title of this paper to acknowledge their origin in the projections of the same name due to Coxeter.

### 3 The 600-cell

The 600-cell is a four-dimensional regular polytope with 120 vertices lying on the surface of a 3-sphere. Its vertices come in antipodal pairs, so it has 60 rays.

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<sup>6</sup> To get a better feeling for what a parity proof is, go to the website <https://quantumgames.wpi.edu/> and click on the tab “600-cell” to see many examples of such proofs in this polytope.

<sup>7</sup> The reason this proves the theorem is that it is impossible to assign a 0 or a 1 to the rays in a noncontextual fashion in such a way that each basis has exactly one ray assigned the value 1 in it.

The rays form 75 bases, with each ray occurring in five bases, so the ray-basis symbol of this polytope is  $60_5 - 75_4$ .

The triacontagonal projection of the 600-cell consists of 120 points representing its vertices and 720 line segments representing its edges. The Kochen-Specker diagram obtained from it has 60 points representing the rays and 450 line segments connecting orthogonal pairs of rays. The rays lie, in sets of fifteen, at the vertices of four concentric regular pentadecagons; see Table 2 for the numbering of the rays and their polar coordinates in the plane of the projection<sup>8</sup>.

| Pentadecagon | Circumradius | Angle | Rays  |
|--------------|--------------|-------|-------|
| A            | 1.0000       | 0°    | 1-15  |
| B            | 0.8135       | 6°    | 16-30 |
| C            | 0.6728       | 6°    | 31-45 |
| D            | 0.3383       | 0°    | 46-60 |

**Table 2** Pentadecagons of the 600-cell, labeled A through D in order of decreasing circumradius. The rays, which lie at the vertices of the pentadecagons, are numbered in counter-clockwise order around them, beginning from the outermost pentadecagon and proceeding inwards. The ray with the lowest number on each pentadecagon is the one lying on or just above the positive x-axis. The rays on A and D begin from the x-axis while those on B and C begin from 6° above the x-axis. The rays on any pentadecagon are spaced 24° apart.

The 75 bases of the 600-cell fall into five orbits of fifteen bases each. Table 3 gives one generator for each orbit, with its basis profile shown in the column to its left; the four letters of the profile indicate the pentadecagons to which the rays of the generator (as well as all the bases in its orbit) belong. The first two generators have unique profiles, whereas the last three have a common profile. However, the last three generators have orbits that are distinct and have no bases in common. The two polytopes we study later also have this feature, i.e., they have generators with the same profile that nevertheless have no bases in common.

Table 4 shows the orbits of fifteen bases arising from each of the generators *a* through *e* of Table 3. The 75 bases in Table 4 constitute the entire basis table of the 600-cell, which can be described by the word *abcde*. The bases associated with generators *a* and *b* each yield a  $30_2 - 15_4$  parity proof, since they involve an odd number of bases (fifteen) with each ray occurring twice over them. It is interesting that the triacontagonal projection yields these proofs simply as a consequence of the way the bases are generated with it.

<sup>8</sup> Expressions for the circumradii of the pentadecagons can be given in terms of radicals but they are not simple, particularly for the two polytopes studied later. Thus we content ourselves by giving numerical values for the circumradii in Tables 2, 5 and 10. The expressions for the circumradii can be found in the references mentioned in footnote 4.

| Basis profile | Generator                                                             |
|---------------|-----------------------------------------------------------------------|
| AADD          | $a = \{1, 5, 55, 56\}$                                                |
| BBCC          | $b = \{16, 18, 36, 43\}$                                              |
| ABCD          | $c = \{1, 19, 43, 49\}, d = \{1, 20, 41, 58\}, e = \{1, 27, 42, 46\}$ |

**Table 3** Generators of the 600-cell. The generators are shown in the second column, with each being labeled by a lower case letter. The first column shows the basis profiles of the generators, with each capital letter indicating the pentadecagon to which the corresponding ray of the generator belongs. The first two generators have unique profiles, whereas the last three all have a common profile. The orbits of these generators are shown in Table 4.

| A  | A        | D         | D         | B         | B         | C         | C         | A  | B         | C         | D         | A  | B         | C         | D         | A  | B         | C         | D         |
|----|----------|-----------|-----------|-----------|-----------|-----------|-----------|----|-----------|-----------|-----------|----|-----------|-----------|-----------|----|-----------|-----------|-----------|
| 1  | <b>5</b> | <b>55</b> | <b>56</b> | <b>16</b> | <b>18</b> | <b>36</b> | <b>43</b> | 1  | <b>19</b> | <b>43</b> | <b>49</b> | 1  | <b>20</b> | <b>41</b> | <b>58</b> | 1  | <b>27</b> | <b>42</b> | <b>46</b> |
| 2  | 6        | 56        | 57        | 17        | 19        | 37        | 44        | 2  | 20        | 44        | 50        | 2  | 21        | 42        | 59        | 2  | 28        | 43        | 47        |
| 3  | 7        | 57        | 58        | 18        | 20        | 38        | 45        | 3  | 21        | 45        | 51        | 3  | 22        | 43        | 60        | 3  | 29        | 44        | 48        |
| 4  | 8        | 58        | 59        | 19        | 21        | 39        | 31        | 4  | 22        | 31        | 52        | 4  | 23        | 44        | 46        | 4  | 30        | 45        | 49        |
| 5  | 9        | 59        | 60        | 20        | 22        | 40        | 32        | 5  | 23        | 32        | 53        | 5  | 24        | 45        | 47        | 5  | 16        | 31        | 50        |
| 6  | 10       | 60        | 46        | 21        | 23        | 41        | 33        | 6  | 24        | 33        | 54        | 6  | 25        | 31        | 48        | 6  | 17        | 32        | 51        |
| 7  | 11       | 46        | 47        | 22        | 24        | 42        | 34        | 7  | 25        | 34        | 55        | 7  | 26        | 32        | 49        | 7  | 18        | 33        | 52        |
| 8  | 12       | 47        | 48        | 23        | 25        | 43        | 35        | 8  | 26        | 35        | 56        | 8  | 27        | 33        | 50        | 8  | 19        | 34        | 53        |
| 9  | 13       | 48        | 49        | 24        | 26        | 44        | 36        | 9  | 27        | 36        | 57        | 9  | 28        | 34        | 51        | 9  | 20        | 35        | 54        |
| 10 | 14       | 49        | 50        | 25        | 27        | 45        | 37        | 10 | 28        | 37        | 58        | 10 | 29        | 35        | 52        | 10 | 21        | 36        | 55        |
| 11 | 15       | 50        | 51        | 26        | 28        | 31        | 38        | 11 | 29        | 38        | 59        | 11 | 30        | 36        | 53        | 11 | 22        | 37        | 56        |
| 12 | 1        | 51        | 52        | 27        | 29        | 32        | 39        | 12 | 30        | 39        | 60        | 12 | 16        | 37        | 54        | 12 | 23        | 38        | 57        |
| 13 | 2        | 52        | 53        | 28        | 30        | 33        | 40        | 13 | 16        | 40        | 46        | 13 | 17        | 38        | 55        | 13 | 24        | 39        | 58        |
| 14 | 3        | 53        | 54        | 29        | 16        | 34        | 41        | 14 | 17        | 41        | 47        | 14 | 18        | 39        | 56        | 14 | 25        | 40        | 59        |
| 15 | 4        | 54        | 55        | 30        | 17        | 35        | 42        | 15 | 18        | 42        | 48        | 15 | 19        | 40        | 57        | 15 | 26        | 41        | 60        |

**Table 4** Bases of the 600-cell. The generators  $a$  through  $e$ , shown in bold font, give rise to the orbits of bases shown below them. The 75 bases in this table make up the basis table of the 600-cell. Note that the ray numbers keep increasing by one as one moves down a column, except when the end of a pentadecagon is reached, when one goes back to the ray at the beginning of that pentadecagon ("wraparound"). The bases associated with the generators  $a$  and  $b$  (which have the profiles AADD and BBCC, respectively) give rise to  $30_2 - 15_4$  parity proofs involving thirty rays that each occur twice over fifteen bases.

Any fifteen-fold symmetric parity proof of the 600-cell must be made up of a combination of the columns in Table 4. Accordingly, it can be written as a "word" made up of an odd number of distinct letters chosen from  $a, b, c, d$  and  $e$ . An odd number of letters is needed to ensure that the total number of bases associated with the proof is odd. Additionally, the number of times each of the pentadecagons occurs over the letters of the word must be even to ensure that each ray occurs an even number of times over the bases. These two conditions restrict the allowed proofs to the words  $a, b, acd, ace, ade, bcd, bce$  and  $bde$ . Let us term a parity proof "minimal" if it cannot be reduced to a shorter proof by leaving out some of the letters in it. Then the only minimal proofs in the preceding list are  $a$  and  $b$  because all the other proofs can be reduced to them by dropping two of their letters.

Although the triacontagonal projection yields just two fifteen-fold symmetric parity proofs of the 600-cell, it has billions of other proofs of the BKS theorem among its bases, as discussed in [18] and [30].

#### 4 The 120-cell

The 120-cell is a four-dimensional regular polytope with 600 vertices lying on the surface of a 3-sphere. It has 300 rays that form 675 bases, with each ray occurring in nine bases, so its ray-basis symbol is  $300_9 - 675_4$ .

The triacontagonal projection arranges the rays at the vertices of twenty concentric regular pentadecagons, sixteen of which lie in pairs on eight different circles and four of which lie on circles of their own. Following Chilton[4], we label the pentadecagons by the letters A through L in order of decreasing circumradius, but add subscripts to eight of the letters to distinguish between pentadecagons lying on circles of the same radius; see Table 5 for the numbering of the rays and their coordinates in the plane of the projection.

The 675 bases of the 120-cell fall into forty five orbits of fifteen bases each. Table 6 lists one generator for each orbit, along with their basis profiles. Most profiles have only one orbit associated with them, but two of them have three each. Four of the generators, namely, those labeled  $j, q, r'$  and  $s'$ , directly give  $30_2 - 15_4$  parity proofs. This can be seen simply from their basis profiles, listed to their left, since each has the rays of two pentadecagons occurring twice over its fifteen bases. However there are many other parity proofs that can be constructed by combining the generators in Table 6, and we now discuss how that can be done.

We use a technique due to Lisonek, Raussendorf and Singh[23]. Consider the matrix  $M$  whose  $i, j$ -th element,  $M_{i,j}$ , is the number of times pentadecagon  $i$  of Table 5 occurs in the basis profile of generator  $j$  of Table 6 (we take  $i$  and  $j$  to be the order in which the pentadecagons or generators occur in Table 5 and 6). The matrix  $M$  obtained in this way is a  $20 \times 45$  matrix. A parity proof can be extracted from  $M$  by picking an odd number of its columns in such a way that if they are stacked next to each other, the sum of the numbers in each of the rows is even. The word describing the resulting parity proof has for its letters the generators corresponding to the columns of  $M$  that are picked. It remains only to explain how the right columns of  $M$  are to be picked.

The problem is solved if we find all 45-dimensional column vectors  $X$ , with elements of 0 or 1, for which the equation  $MX = 0 \pmod{2}$  is satisfied, for then the non-vanishing elements of  $X$  will indicate the columns of  $M$  to be picked. The solutions to the foregoing equation are all the vectors in the nullspace of  $M$  (by which we mean all vectors  $X$  such that  $MX = 0 \pmod{2}$ ). We used Maple to find that the nullspace has dimension 30 and obtained a set of linearly independent vectors in it. Taking all possible linear combinations of these vectors, with coefficients of 0 or 1, gives  $2^{30}$  solutions to the equation  $MX = 0 \pmod{2}$ . However, only half of these solutions (namely, those with an odd number of 1's as elements) give parity proofs. Thus the number of parity proofs is  $2^{29}$ , and each of them can be constructed as the union of the bases

| Pentadecagon   | Circumradius | Angle  | Rays    |
|----------------|--------------|--------|---------|
| A              | 1.0000       | 6°     | 1-15    |
| B <sub>1</sub> | 0.9515       | 1.76°  | 16-30   |
| B <sub>2</sub> | 0.9515       | 10.24° | 31-45   |
| C              | 0.9004       | 0°     | 46-60   |
| D <sub>1</sub> | 0.8673       | 3.13°  | 61-75   |
| D <sub>2</sub> | 0.8673       | 8.87°  | 76-90   |
| E <sub>1</sub> | 0.8110       | 2.07°  | 91-105  |
| E <sub>2</sub> | 0.8110       | 9.93°  | 106-120 |
| F <sub>1</sub> | 0.7741       | 4.24°  | 121-135 |
| F <sub>2</sub> | 0.7741       | 7.76°  | 136-150 |
| G <sub>1</sub> | 0.6402       | 4.24°  | 151-165 |
| G <sub>2</sub> | 0.6402       | 7.76°  | 166-180 |
| H <sub>1</sub> | 0.5927       | 4.58°  | 181-195 |
| H <sub>2</sub> | 0.5927       | 7.42°  | 196-210 |
| I <sub>1</sub> | 0.5067       | 3.31°  | 211-225 |
| I <sub>2</sub> | 0.5067       | 8.69°  | 226-240 |
| J              | 0.4452       | 0°     | 241-255 |
| K <sub>1</sub> | 0.3219       | 1.76°  | 256-270 |
| K <sub>2</sub> | 0.3219       | 10.24° | 271-285 |
| L              | 0.0947       | 6°     | 286-300 |

**Table 5** Pentadecagons of the 120-cell, labeled A through L in order of decreasing circumradius, with pairs having the same circumradius being distinguished by a subscript. The rays are numbered in counter-clockwise order around the pentadecagons, beginning from the outermost pentadecagon and proceeding inwards. The ray with the lowest number on each pentadecagon is the one lying on or just above the positive x-axis. The starting angles of rays on all pentadecagons are indicated in the third column. The rays on any pentadecagon are spaced 24° apart.

associated with the generators picked out by the non-vanishing elements of  $X$ .

The procedure just described can be translated into a simple algorithm for constructing the parity proofs. Let a “word” be a sequence of distinct letters representing the generators in Table 6. Then the 30 linearly independent vectors in the nullspace of  $M$ , which each pick out a set of generators, can be written in the form of the words shown in the Table 7. The words are listed in order of increasing length and the letters of any word are arranged in alphabetical order, with all primed letters coming after the unprimed ones (this convention eliminates the confusion caused by writing the same word in different forms).

The algorithm for obtaining parity proofs can now be stated: combine the words in Table 7 in all possible ways that lead to words of odd length. Each such word represents a parity proof.

To complete the description of the algorithm, we must explain how two or more words in Table 7 are to be combined. We give a procedure for combining two words which can be used repeatedly to combine any number of words from Table 7, even when one or both of the words do not belong to that table. Let

| Basis profile                                               | Generator                                                                                   | Basis profile                                               | Generator                     |
|-------------------------------------------------------------|---------------------------------------------------------------------------------------------|-------------------------------------------------------------|-------------------------------|
| AB <sub>1</sub> K <sub>2</sub> L                            | $a = \{1, 20, 274, 296\}$                                                                   | CD <sub>2</sub> H <sub>2</sub> K <sub>2</sub>               | $z = \{46, 79, 201, 280\}$    |
| AB <sub>2</sub> K <sub>1</sub> L                            | $b = \{1, 42, 268, 291\}$                                                                   | CD <sub>2</sub> I <sub>2</sub> J                            | $a' = \{46, 80, 235, 244\}$   |
| ACJL                                                        | $c = \{1, 50, 245, 286\}$                                                                   | CE <sub>1</sub> H <sub>2</sub> J                            | $b' = \{46, 94, 200, 247\}$   |
| AD <sub>1</sub> I <sub>1</sub> L                            | $d = \{1, 65, 222, 289\}$                                                                   | CE <sub>2</sub> H <sub>1</sub> J                            | $c' = \{46, 110, 184, 250\}$  |
| AD <sub>2</sub> I <sub>2</sub> L                            | $e = \{1, 87, 230, 298\}$                                                                   | CF <sub>1</sub> G <sub>2</sub> J                            | $d' = \{46, 125, 168, 246\}$  |
| AE <sub>1</sub> H <sub>2</sub> L                            | $f = \{1, 95, 207, 292\}$                                                                   | CF <sub>2</sub> G <sub>1</sub> J                            | $e' = \{46, 139, 156, 251\}$  |
| AE <sub>2</sub> H <sub>1</sub> L                            | $g = \{1, 117, 185, 295\}$                                                                  | D <sub>1</sub> D <sub>2</sub> I <sub>1</sub> I <sub>2</sub> | $f' = \{61, 80, 212, 229\}$   |
| AG <sub>1</sub> H <sub>1</sub> I <sub>1</sub>               | $h = \{1, 155, 192, 215\}$                                                                  | D <sub>1</sub> E <sub>1</sub> H <sub>2</sub> I <sub>1</sub> | $g' = \{61, 94, 209, 215\}$   |
| AG <sub>2</sub> H <sub>2</sub> I <sub>2</sub>               | $i = \{1, 177, 200, 237\}$                                                                  | D <sub>1</sub> E <sub>2</sub> F <sub>1</sub> L              | $h' = \{61, 118, 131, 298\}$  |
| B <sub>1</sub> B <sub>1</sub> K <sub>1</sub> K <sub>1</sub> | $j = \{16, 20, 265, 266\}$                                                                  | D <sub>1</sub> E <sub>2</sub> H <sub>1</sub> I <sub>1</sub> | $i' = \{61, 116, 193, 224\}$  |
| B <sub>1</sub> D <sub>1</sub> I <sub>2</sub> K <sub>2</sub> | $k = \{16, 65, 228, 275\}$                                                                  | D <sub>1</sub> F <sub>2</sub> G <sub>1</sub> I <sub>2</sub> | $j' = \{61, 140, 162, 234\}$  |
| B <sub>1</sub> E <sub>1</sub> H <sub>1</sub> K <sub>2</sub> | $l = \{16, 102, 193, 280\}$                                                                 | D <sub>2</sub> E <sub>1</sub> F <sub>2</sub> L              | $k' = \{76, 94, 141, 289\}$   |
| B <sub>1</sub> E <sub>2</sub> I <sub>1</sub> J              | $m = \{16, 117, 221, 246\}$                                                                 | D <sub>2</sub> E <sub>1</sub> H <sub>2</sub> I <sub>2</sub> | $l' = \{76, 96, 199, 228\}$   |
| B <sub>1</sub> F <sub>2</sub> G <sub>2</sub> K <sub>1</sub> | $n = \{16, 139, 178, 259\},$<br>$o = \{16, 140, 176, 268\},$<br>$p = \{16, 147, 177, 256\}$ | D <sub>2</sub> E <sub>2</sub> H <sub>1</sub> I <sub>2</sub> | $m' = \{76, 118, 183, 237\}$  |
| B <sub>2</sub> B <sub>2</sub> K <sub>2</sub> K <sub>2</sub> | $q = \{31, 35, 280, 281\}$                                                                  | D <sub>2</sub> F <sub>1</sub> G <sub>2</sub> I <sub>1</sub> | $n' = \{76, 132, 170, 218\}$  |
| B <sub>2</sub> D <sub>2</sub> I <sub>1</sub> K <sub>1</sub> | $r = \{31, 87, 224, 267\}$                                                                  | E <sub>1</sub> E <sub>2</sub> H <sub>1</sub> H <sub>2</sub> | $o' = \{91, 116, 187, 200\}$  |
| B <sub>2</sub> E <sub>1</sub> I <sub>2</sub> J              | $s = \{31, 95, 231, 244\}$                                                                  | E <sub>1</sub> F <sub>1</sub> G <sub>2</sub> H <sub>1</sub> | $p' = \{91, 124, 172, 192\}$  |
| B <sub>2</sub> E <sub>2</sub> H <sub>2</sub> K <sub>1</sub> | $t = \{31, 110, 199, 262\}$                                                                 | E <sub>2</sub> F <sub>2</sub> G <sub>1</sub> H <sub>2</sub> | $q' = \{106, 148, 160, 200\}$ |
| B <sub>2</sub> F <sub>1</sub> G <sub>1</sub> K <sub>2</sub> | $u = \{31, 125, 155, 271\},$<br>$v = \{31, 132, 156, 274\},$<br>$w = \{31, 133, 154, 283\}$ | F <sub>1</sub> F <sub>1</sub> G <sub>1</sub> G <sub>1</sub> | $r' = \{121, 123, 156, 163\}$ |
| CD <sub>1</sub> H <sub>1</sub> K <sub>1</sub>               | $x = \{46, 65, 183, 269\}$                                                                  | F <sub>2</sub> F <sub>2</sub> G <sub>2</sub> G <sub>2</sub> | $s' = \{136, 138, 171, 178\}$ |
| CD <sub>1</sub> I <sub>1</sub> J                            | $y = \{46, 64, 224, 253\}$                                                                  |                                                             |                               |

**Table 6** Generators of the 120-cell. The generators are shown in the second and fourth columns, along with their letter labels. The generators have the 41 basis profiles shown in the first and third columns, with the letters of any profile indicating the pentadecagons to which the corresponding rays of the generator(s) belong. Most of the profiles have just one generator associated with them, but two of them have three each. Each generator gives rise to an orbit of fifteen bases, and the union of all the orbits yields the 675 bases of the 120-cell.

|                 |                                                                       |
|-----------------|-----------------------------------------------------------------------|
| 1-letter words  | $j$                                                                   |
| 2-letter words  | $jq, jr', js', no, np, uv, uw$                                        |
| 4-letter words  | $cdjy, ceja', cfjb', cgjc', dejf', dfjg', dgji', efjl', egjm', fgjo'$ |
| 6-letter words  | $abdekr, abfglt, bcklsx$                                              |
| 8-letter words  | $ahklmsuh'$                                                           |
| 10-letter words | $abhijlmsnq', acdefgklmz, acfghimsud', bcfghimnse', fghijslmsup'$     |
| 12-letter words | $abfghijkmnsj', bdefgiklmnsk', defghijkmsun'$                         |

**Table 7** The 30 words representing a set of linearly independent vectors in the nullspace of the matrix  $M$ . The first word is of odd length, while all the others are of even length.

$U$  and  $V$  be the sets of letters corresponding to the two words that must be combined. Then the word obtained by combining them is represented by the set of letters  $W$  that is the symmetric difference of the sets  $U$  and  $V$ :

$$W = U \triangle V := (U \cup V) \setminus (U \cap V) \quad (1)$$

Stated in words (pun not intended!), the word obtained by combining two words is the union of their sets of letters, but with all letters common to the two sets dropped. This prescription can be used to combine any number of

| Length | Parity proof                 | Ray-basis symbol                                          |
|--------|------------------------------|-----------------------------------------------------------|
| 1      | $j, q, r', s'$               | $30_2-15_4$ (irreducible)                                 |
| 3      | $cdy, def', efl', fgo'$      | $90_2-45_4 = 30_2-15_4 \oplus 30_2-15_4 \oplus 30_2-15_4$ |
| 5      | $abkrf', ablt'o'$            | $150_2-75_4 = 30_2-15_4 \oplus \dots$ (5 times)           |
| 7      | $abegkri'$                   | $150_230_4-105_4$ (irreducible)                           |
| 7      | $bdklszy$                    | $180_215_4-105_4$ (irreducible)                           |
| 9      | $fghilmsup'$                 | $180_245_4-135_4$ (irreducible)                           |
| 9      | $abhilmnsq'$                 | $210_230_4-135_4 = 42_26_4-27_4 \oplus \dots$ (5 times)   |
| 11     | $abfghikmnsj', defghikmsun'$ | $195_245_415_6-165_4$ (irreducible)                       |
| 13     | $bcddefghiknszq'$            | $225_230_415_615_8-195_4$ (irreducible)                   |
| 15     | $bcdelnaszd'e'h'k'n'p'q'$    | $165_2120_415_6-225_4 = 170_210_4-95_4 + \dots$ (3 times) |

**Table 8** Parity proofs of the 120-cell with all word lengths from 1 to 15 obtained by combining words in Table 7. Proofs with the same ray-basis symbol are listed together, with the symbol being shown in the last column. If the proof is reducible, the smaller proofs within it are indicated (see the text for an explanation of this point). The symbol  $\oplus$  indicates that the smaller proofs have no bases in common, whereas  $+$  (which occurs only in the last entry) indicates that they do. See Table 9 for a blown-up version of the proof  $cdy$ . The dots ... in the last column indicate that the ray-basis symbol to their left has to be added to itself the number of times shown in brackets.

| A         | C         | J          | L          | A         | D <sub>1</sub> | I <sub>1</sub> | L          | C         | D <sub>1</sub> | I <sub>1</sub> | J          |
|-----------|-----------|------------|------------|-----------|----------------|----------------|------------|-----------|----------------|----------------|------------|
| <b>1</b>  | <b>50</b> | <b>245</b> | <b>286</b> | <b>1</b>  | <b>65</b>      | <b>222</b>     | <b>289</b> | 46        | 64             | 224            | 253        |
| <i>2</i>  | <i>51</i> | <i>246</i> | <i>287</i> | <i>2</i>  | <i>66</i>      | <i>223</i>     | <i>290</i> | <b>47</b> | <b>65</b>      | <b>225</b>     | <b>254</b> |
| 3         | 52        | 247        | 288        | 3         | 67             | 224            | 291        | <i>48</i> | <i>66</i>      | <i>211</i>     | <i>255</i> |
| <b>4</b>  | <b>53</b> | <b>248</b> | <b>289</b> | <b>4</b>  | <b>68</b>      | <b>225</b>     | <b>292</b> | 49        | 67             | 212            | 241        |
| <i>5</i>  | <i>54</i> | <i>249</i> | <i>290</i> | <i>5</i>  | <i>69</i>      | <i>211</i>     | <i>293</i> | <b>50</b> | <b>68</b>      | <b>213</b>     | <b>242</b> |
| 6         | 55        | 250        | 291        | 6         | 70             | 212            | 294        | <i>51</i> | <i>69</i>      | <i>214</i>     | <i>243</i> |
| <b>7</b>  | <b>56</b> | <b>251</b> | <b>292</b> | <b>7</b>  | <b>71</b>      | <b>213</b>     | <b>295</b> | 52        | 70             | 215            | 244        |
| <i>8</i>  | <i>57</i> | <i>252</i> | <i>293</i> | <i>8</i>  | <i>72</i>      | <i>214</i>     | <i>296</i> | <b>53</b> | <b>71</b>      | <b>216</b>     | <b>245</b> |
| 9         | 58        | 253        | 294        | 9         | 73             | 215            | 297        | <i>54</i> | <i>72</i>      | <i>217</i>     | <i>246</i> |
| <b>10</b> | <b>59</b> | <b>254</b> | <b>295</b> | <b>10</b> | <b>74</b>      | <b>216</b>     | <b>298</b> | 55        | 73             | 218            | 247        |
| <i>11</i> | <i>60</i> | <i>255</i> | <i>296</i> | <i>11</i> | <i>75</i>      | <i>217</i>     | <i>299</i> | <b>56</b> | <b>74</b>      | <b>219</b>     | <b>248</b> |
| 12        | 46        | 241        | 297        | 12        | 61             | 218            | 300        | <i>57</i> | <i>75</i>      | <i>220</i>     | <i>249</i> |
| <b>13</b> | <b>47</b> | <b>242</b> | <b>298</b> | <b>13</b> | <b>62</b>      | <b>219</b>     | <b>286</b> | 58        | 61             | 221            | 250        |
| <i>14</i> | <i>48</i> | <i>243</i> | <i>299</i> | <i>14</i> | <i>63</i>      | <i>220</i>     | <i>287</i> | <b>59</b> | <b>62</b>      | <b>222</b>     | <b>251</b> |
| 15        | 49        | 244        | 300        | 15        | 64             | 221            | 288        | <i>60</i> | <i>63</i>      | <i>223</i>     | <i>252</i> |

**Table 9** The three-letter proof  $cdy$ . The generators  $c, d$  and  $y$  are shown in bold font at the top and their orbits of fifteen bases below them. The entire set of bases in this table gives a  $90_2 - 45_4$  parity proof, but the proof is reducible and breaks up into the three smaller  $30_2 - 15_4$  proofs which are indicated in bold, italic and regular font. Each of the smaller proofs has a five-fold symmetry (i.e., adding three to the rays of any basis yields the next basis in it). The smaller proofs have no bases in common, so  $cdy$  is written as a direct sum ( $\oplus$ ) of them in Table 8.

words in Table 7. If the word that results is of odd length it represents a parity proof, otherwise it must be dropped. It is not hard to see that the words in the nullspace of  $M$  form a group under the law of composition given in Eq.(1), with the null word being the identity element and every word being its own inverse.

To ease the task of getting directly at the odd length words, all but one of the words in Table 7 have been chosen to be of even length. This makes it possible to construct odd length words by combining any number of even length words to get an even length word and finally combining it with  $j$  to get an odd length word.

An example may help make the construction clear. If one combines the 6-letter word  $abdekr$  with the 4-letter word  $dgji'$ , one gets the 8-letter word  $abegjkri'$  and combining it with  $j$  gives the 7-letter word  $abegkri'$ , which represents a parity proof. The ray-basis symbol of this proof can be worked out as follows:

$$abegkri' \rightarrow (B_1B_2D_1D_2E_2H_1I_1I_2K_1K_2)_2(AL)_4 \rightarrow 150_230_4 - 105_4 \quad (2)$$

In the first step we replaced each letter (or generator) by its profile as given in Table 6 and collected all pentadecagons occurring the same number of times over the profiles within brackets, using subscripts to indicate the number of times they occurred over the profiles. In the second step we replaced the string of letters within each pair of brackets by fifteen times the number of letters and added the number of bases (which is fifteen times the number of letters in the word) after the dash.

Table 8 shows examples of parity proofs of all lengths from one to fifteen obtained by combining the words in Table 7. The question arises as to which parity proofs of the 120-cell are minimal (i.e., cannot be reduced to shorter proofs by leaving out some of their letters). The answer can be given by using a result from coding theory, since the words describing parity proofs can be interpreted as the codewords of a linear binary code.

Recall that the matrix  $M$  has dimensions  $m \times n$ , where  $m$  is the number of pentadecagons and  $n$  is the number of generators. The nullspace of  $M$  has  $k(< n)$  linearly independent vectors from which all the others can be built up. These  $k$  vectors span the rowspace of an  $(n, k)$  linear binary code whose odd length<sup>9</sup> codewords correspond to fifteen-fold symmetric parity proofs. Thus the question of determining which parity proofs are minimal is the same as that of determining which odd length codewords of an  $(n, k)$  linear binary code are minimal (i.e., cannot be converted into smaller odd length codewords by replacing some of their 1's by 0's). The answer to this question is partially given by the following result of Ashikhmin and Barg[24]:

**Proposition 1.** The weight  $l$  of a minimal codeword in an  $(n, k)$  linear binary code must satisfy  $l \leq n - k + 1$ .

---

<sup>9</sup> By an odd-length codeword we mean one having an odd number of 1's in it. The length of a codeword is often termed its weight in coding theory.

For the 120-cell,  $n = 45$  and  $k = 30$ , so a minimal codeword must have length  $l \leq 16$ , implying that no parity proofs of length 17 or higher can be minimal. One can use the MacWilliams identities [25,26] to calculate the number of parity proofs of any length and finds that the number of proofs of length 1,3,5,7 or 9 is 4,48,564,5116 or 42576, respectively. The number grows rapidly after that and peaks at well over a hundred million for proofs of length 23 before falling off at the high end (see Table 1 of Supplementary Materials for a complete list). The longest proofs have a length of 39 and there are 1212 of them. As mentioned above, no parity proof of length 17 or larger can be minimal. As for proofs of length 15 or less, a detailed examination is needed in each case to determine if it is minimal or not.

Fifteen-fold symmetric parity proofs, whether minimal or not, sometimes break up into a number of smaller parity proofs. We will term them reducible if they do, and irreducible if they do not. The smaller proofs within a reducible proof do not involve entire pentadecagons of rays and therefore cannot be described as words made up of the lower case letters we have introduced. We will describe them simply by their ray-basis symbols. Table 8 shows how all the reducible proofs break up into smaller proofs, with the direct sum symbol,  $\oplus$ , being used in those cases where the smaller proofs have no bases in common and the ordinary sum,  $+$ , being used in the cases they do. Table 9 shows a blown up version of the reducible proof *cdy*, which makes clear how its bases split up into those of three smaller proofs, each of which has a five-fold symmetry. The other reducible proofs in Table 8 also split up into a number of smaller proofs of five-fold symmetry, making them easy to pick out.

The smaller proofs in a reducible proof could have still smaller quantum contextual sets within them. These sets are simply sets of bases whose rays cannot be assigned the value 0 or 1 in a noncontextual fashion in such a way that every basis has exactly one ray assigned the value 1 in it. Only a detailed examination in each case can tell if this is so or not.

## 5 Gosset's polytope $4_{21}$

Gosset's polytope  $4_{21}$  is a uniform polytope in eight dimensions with 240 vertices lying on the surface of a 7-sphere. It has 120 rays that form 2025 bases, with each ray occurring in 135 bases, so its ray-basis symbol is  $120_{135} - 2025_8$ .

The triacontagonal projection arranges the 120 rays, in sets of fifteen, at the vertices of eight concentric regular pentadecagons which we label by the letters A through H in order of decreasing circumradius; see Table 10 for the numbering of the rays and their coordinates in the plane of the projection.

| Pentadecagon | Circumradius | Angle | Rays    |
|--------------|--------------|-------|---------|
| A            | 1.0000       | 0°    | 1-15    |
| B            | 0.8135       | 6°    | 16-30   |
| C            | 0.6723       | 6°    | 31-45   |
| D            | 0.6180       | 0°    | 46-60   |
| E            | 0.5027       | 6°    | 61-75   |
| F            | 0.4159       | 6°    | 76-90   |
| G            | 0.3383       | 0°    | 91-105  |
| H            | 0.2091       | 0°    | 106-120 |

**Table 10** Pentadecagons of Gosset’s polytope, labeled A through H in order of decreasing circumradius. The rays on the pentadecagons are numbered in counter-clockwise order, beginning from the ray lying on or just above the positive x-axis. The starting angles of rays on all pentadecagons are indicated in the third column. The rays on all pentadecagons are spaced 24° apart.

The 2025 bases of Gosset’s polytope break up into 135 orbits of fifteen bases each, with the orbits having 33 different basis profiles. Table 2 of the Supplementary Materials lists a generator for each orbit, along with its basis profile. Since there are 135 generators but only 33 profiles, many profiles have more than one generator associated with them.

An examination of the 33 basis profiles shows that the sixteen generators associated with eight of them directly give parity proofs (two of which are shown as the first two entries in Table 12). However, a large number of other parity proofs can be constructed by putting the generators together in the right combinations. We indicate how this can be done using a technique identical to that we used for the 120-cell.

Using the information in Table 2 of the Supplementary Materials, one can set up a  $8 \times 135$  dimensional matrix  $M$  whose  $i, j$ -th element is the number of times pentadecagon  $i$  occurs in the basis profile of generator  $j$ . We used Maple to determine that the nullspace of  $M$  has dimension 131 and found a set of linearly independent vectors in it. These vectors, which can be written as words of one to five letters (representing generators), are given in Table 3 of the Supplementary Materials. By combining these words together in all possible ways using the procedure laid out earlier for the 120-cell, and retaining only the odd length words, one gets a total of  $2^{130}$  parity proofs for Gosset’s polytope.

We now exhibit some of these proofs. To spare the reader from the necessity of looking up the Supplementary Materials file, we list a few of the generators of Gosset’s polytope in Table 11 and some of the proofs that can be constructed out of them in Table 12. A calculation similar to that in Eq.(2) can be carried out for each of the parity proofs in Table 12 to obtain its ray-basis symbol, if one takes the basis profiles of the relevant generators from Table 11.

| Generator                                    | Basis Profile |
|----------------------------------------------|---------------|
| $a_1 = \{1, 4, 21, 76, 95, 109, 111, 115\}$  | AABFGHHH      |
| $b_1 = \{1, 4, 35, 37, 112, 114, 116, 118\}$ | AACCHHHH      |
| $c_1 = \{1, 5, 32, 50, 105, 113, 115, 119\}$ | AACDGHHH      |
| $d_1 = \{1, 4, 35, 74, 76, 97, 111, 112\}$   | AACEFGHH      |
| $e_1 = \{1, 4, 51, 59, 91, 94, 114, 116\}$   | AADDGGHH      |
| $e_2 = \{1, 5, 46, 50, 100, 101, 115, 116\}$ | AADDGGHH      |
| $h_1 = \{1, 18, 21, 42, 87, 106, 108, 119\}$ | ABBCFHHH      |
| $i_4 = \{1, 20, 27, 46, 86, 101, 106, 115\}$ | ABBDGHHH      |
| $m_1 = \{1, 18, 43, 51, 73, 78, 101, 111\}$  | ABCDEFGH      |
| $m_8 = \{1, 20, 41, 51, 70, 76, 103, 108\}$  | ABCDEFGH      |
| $n_5 = \{1, 27, 32, 71, 73, 81, 88, 96\}$    | ABCEFFFG      |
| $c'_1 = \{16, 19, 31, 50, 51, 66, 76, 93\}$  | BBCDDEFG      |
| $e'_2 = \{16, 40, 41, 43, 58, 61, 74, 81\}$  | BCCCDEEF      |

**Table 11** Shown above are 12 of the 135 generators of Gosset's polytope, with their letter labels and basis profiles. Each letter of a basis profile indicates the pentadecagon to which the corresponding ray of the generator belongs.

| Parity proof           | Pentadecagon counts           | Ray-Basis symbol                |
|------------------------|-------------------------------|---------------------------------|
| $b_1$                  | $(AC)_2(H)_4$                 | $30_2 15_4 - 15_8$              |
| $e_1$                  | $(ADGH)_2$                    | $60_2 - 15_8$                   |
| $a_1 c_1 e'_2$         | $(BDEFG)_2(AC)_4(H)_6$        | $75_2 30_4 15_6 - 45_8$         |
| $a_1 h_1 n_5$          | $(CEG)_2(ABF)_4(H)_6$         | $45_2 45_4 15_6 - 45_8$         |
| $c_1 h_1 i_4$          | $(CDFG)_2(AB)_4(H)_8$         | $60_2 30_4 15_8 - 45_8$         |
| $a_1 c_1 d_1 h_1 m_1$  | $(DE)_2(BCFG)_4(A)_8(H)_{12}$ | $30_2 60_4 15_8 15_{12} - 75_8$ |
| $a_1 c_1 h_1 m_8 c'_1$ | $(E)_2(CDFG)_4(AB)_6(H)_{10}$ | $15_2 60_4 30_6 15_{10} - 75_8$ |

**Table 12** Some parity proofs of Gosset's polytope. The transition from the parity proofs in the first column to the pentadecagon counts in the second and the ray-basis symbols in the third can be made in the way illustrated in Eq.(2) if one replaces each letter of the parity proof by its basis profile as given in Table 11.

We next determine which proofs of Gosset's polytope are minimal. Since  $n = 135$  and  $k = 131$  in this case, it follows from Proposition 1 that only proofs of length  $l \leq n - k + 1 = 5$  can be minimal. One finds from the MacWilliams identities that the number of proofs of length 1, 3 and 5 are 16, 25812 and a little over 21 million, respectively, with the number growing rapidly after that and going to zero beyond a length of 133 (see Table 4 of the Supplementary Materials for full details). While Proposition 1 asserts categorically that proofs of length 7 or longer are not minimal, it is ambivalent about proofs of length 5 or less. Only a direct examination of any proof of length 5 or less can reveal if it is minimal. We have determined that all the proofs in Table 12 are minimal and irreducible (i.e., they do not have smaller parity proofs in them).

We end by giving an example of a word of two letters that is not a parity proof but still very interesting. The word is  $e_1 e_2$ , and its bases are listed in Table 13. Each of the letters gives a parity proof of fifteen bases that is both minimal and irreducible. However, the word as a whole is reducible because it has three smaller parity proofs embedded in it, each of which inherits its bases from both the letters. Two of these proofs are shown in bold and italic

| A         | A         | D         | D         | G          | G          | H          | H          | A         | A         | D         | D         | G          | G          | H          | H          |
|-----------|-----------|-----------|-----------|------------|------------|------------|------------|-----------|-----------|-----------|-----------|------------|------------|------------|------------|
| <b>1</b>  | <b>4</b>  | <b>51</b> | <b>59</b> | <b>91</b>  | <b>94</b>  | <b>114</b> | <b>116</b> | 1         | 5         | 46        | 50        | 100        | 101        | 115        | 116        |
| <i>2</i>  | <i>5</i>  | <i>52</i> | <i>60</i> | <i>92</i>  | <i>95</i>  | <i>115</i> | <i>117</i> | 2         | 6         | 47        | 51        | 101        | 102        | 116        | 117        |
| <b>3</b>  | <b>6</b>  | <b>53</b> | <b>46</b> | <b>93</b>  | <b>96</b>  | <b>116</b> | <b>118</b> | 3         | 7         | 48        | 52        | 102        | 103        | 117        | 118        |
| <i>4</i>  | <i>7</i>  | <i>54</i> | <i>47</i> | <i>94</i>  | <i>97</i>  | <i>117</i> | <i>119</i> | <b>4</b>  | <b>8</b>  | <b>49</b> | <b>53</b> | <b>103</b> | <b>104</b> | <b>118</b> | <b>119</b> |
| 5         | 8         | 55        | 48        | 95         | 98         | 118        | 120        | <i>5</i>  | <i>9</i>  | <i>50</i> | <i>54</i> | <i>104</i> | <i>105</i> | <i>119</i> | <i>120</i> |
| <b>6</b>  | <b>9</b>  | <b>56</b> | <b>49</b> | <b>96</b>  | <b>99</b>  | <b>119</b> | <b>106</b> | 6         | 10        | 51        | 55        | 105        | 91         | 120        | 106        |
| <i>7</i>  | <i>10</i> | <i>57</i> | <i>50</i> | <i>97</i>  | <i>100</i> | <i>120</i> | <i>107</i> | 7         | 11        | 52        | 56        | 91         | 92         | 106        | 107        |
| <b>8</b>  | <b>11</b> | <b>58</b> | <b>51</b> | <b>98</b>  | <b>101</b> | <b>106</b> | <b>108</b> | 8         | 12        | 53        | 57        | 92         | 93         | 107        | 108        |
| <i>9</i>  | <i>12</i> | <i>59</i> | <i>52</i> | <i>99</i>  | <i>102</i> | <i>107</i> | <i>109</i> | <b>9</b>  | <b>13</b> | <b>54</b> | <b>58</b> | <b>93</b>  | <b>94</b>  | <b>108</b> | <b>109</b> |
| 10        | 13        | 60        | 53        | 100        | 103        | 108        | 110        | <i>10</i> | <i>14</i> | <i>55</i> | <i>59</i> | <i>94</i>  | <i>95</i>  | <i>109</i> | <i>110</i> |
| <b>11</b> | <b>14</b> | <b>46</b> | <b>54</b> | <b>101</b> | <b>104</b> | <b>109</b> | <b>111</b> | 11        | 15        | 56        | 60        | 95         | 96         | 110        | 111        |
| <i>12</i> | <i>15</i> | <i>47</i> | <i>55</i> | <i>102</i> | <i>105</i> | <i>110</i> | <i>112</i> | 12        | 1         | 57        | 46        | 96         | 97         | 111        | 112        |
| <b>13</b> | <b>1</b>  | <b>48</b> | <b>56</b> | <b>103</b> | <b>91</b>  | <b>111</b> | <b>113</b> | 13        | 2         | 58        | 47        | 97         | 98         | 112        | 113        |
| <i>14</i> | <i>2</i>  | <i>49</i> | <i>57</i> | <i>104</i> | <i>92</i>  | <i>112</i> | <i>114</i> | <b>14</b> | <b>3</b>  | <b>59</b> | <b>48</b> | <b>98</b>  | <b>99</b>  | <b>113</b> | <b>114</b> |
| 15        | 3         | 50        | 58        | 105        | 93         | 113        | 115        | <i>15</i> | <i>4</i>  | <i>60</i> | <i>49</i> | <i>99</i>  | <i>100</i> | <i>114</i> | <i>115</i> |

**Table 13** Parity proofs contained in the word  $e_1e_2$ . The generators  $e_1$  and  $e_2$  are shown at the top of the blocks and the bases they give rise to are shown below them. The bases in the left and right blocks each give a  $30_2 - 15_4$  parity proof. However, three smaller parity proofs can be obtained by combining bases from both the blocks. One of these proofs is shown in bold font and another in italic font, and they both have the ray-basis symbol  $36_2 - 9_8$ . The third proof, which also has this symbol, has some bases in common with the other two proofs but also has three bases that are its own. If the bases are numbered 1 to 30 going up to down and left to right, the bases associated with the third proof are 1,4,6,9,11,14,17,22 and 27.

font, and they have no bases in common. The third proof has some bases in common with the other two proofs in addition to some bases of its own. All three proofs involve nine bases and are the smallest parity proofs in Gosset's polytope (in that there are no proofs with a smaller number of bases). For comparison, the smallest parity proofs in the 600-cell and 120-cell consist of thirteen and fifteen bases, respectively.

The list of proofs in Table 12 can be extended to proofs of larger length by making use of the data given in Tables 2 and 3 of the Supplementary Materials. Repeated application of Eqs.(1) and (2) to this data allows progressively longer proofs to be constructed.

## 6 Discussion

This paper shows how the triacontagonal projections of the 600-cell, 120-cell and  $4_{21}$  introduced by Coxeter can be used to obtain parity proofs of the BKS theorem. The special feature of these proofs is that they involve both rays and bases that are multiples of fifteen, as a consequence of their being embedded in a Kochen-Specker diagram derived from a triacontagonal projection. Each proof can be represented by a word made up of distinct letters, each of which represents a set of fifteen bases whose members are related to each other by

rotations about the center of the projection. A word makes it possible to write down the ray-basis symbol of the associated proof without first recovering its bases. Many words can be broken down into smaller parity proofs involving fewer bases.

The approach of this paper can be contrasted to other approaches [18]-[22] that have been used to obtain parity proofs in these polytopes. Parity proofs in all three polytopes were first obtained in [18]-[20] by using an algorithm that systematically picked out the proofs, beginning from the smallest and proceeding to larger ones. These methods were improved and generalized in [21] and [22], which made it possible to obtain a much fuller picture of the quantum contextual sets in these polytopes (of which the parity proofs are only a small subset).

A rather different approach was taken in [23], in which an algebraic method was used to obtain parity proofs in all three polytopes. We used just this method in Sec.4 to obtain the fifteen-fold parity proofs of the 120-cell, but with the difference that whereas [23] considered the incidence matrix between the individual rays and bases, we considered the same matrix between pentadecagons (which are sets of fifteen rays) and generators (which give rise to orbits of fifteen bases). Thus our approach can be considered a coarse-grained version of their fine-grained one. The fine-grained approach yields **all** the parity proofs in these polytopes, whereas our coarse-grained approach, which applies only to polytopes with a triacontagonal projection, picks out only proofs with a fifteen-fold symmetry about the center of the projection. The proofs obtained with our approach are just a small subset of the proofs obtained using the fine-grained approach. However, as a compensation, we can give a far more detailed account of the proofs in this limited class, as we have tried to illustrate in the results given in the last three sections.

Perhaps the most important message of this paper is that Coxeter's triacontagonal projections lead directly to parity proofs in all three polytopes if only one realizes that the thirty-fold symmetry of the projection translates into a fifteen-fold symmetry of both the rays **and the bases** about the center of the projection (which is also the center of the Kochen-Specker diagram). This leads directly to parity proofs of fifteen bases in all three polytopes, without our having to hunt through the bases to find them.

We end by mentioning a couple of open questions about the polytopes studied in this paper. The first is that the largest noncontextual sets in these polytopes are not known. Another way of stating the question is this: What is the largest number of bases with the property that every ray in them can be assigned the value 0 or 1 in such a way that every basis has exactly one ray assigned the value 1 (and all the others assigned the value 0) in it? This question is interesting because the gap between the number of bases in this set and the total number of bases of the polytope provides a measure of how

contextual (or nonclassical) the bases of the polytope are. It was shown in [16] that there is a two-way connection between a quantum contextual set and a perfect quantum strategy in a nonlocal two player game based on that set. Determining the largest noncontextual set would allow us to determine how large a quantum advantage the polytope provides in such a game. Unfortunately, we know of no good (i.e., efficient) algorithm for solving this problem.

A second question pertains to rigidity. The rays (or the equivalent projectors) of a polytope are said to be rigid if any set of projectors (not necessarily of rank one and possibly in a dimension greater than that of the polytope) that satisfy the same orthogonality relations as them can be mapped into them in a one-to-one fashion by a suitable unitary transformation. Proving the rigidity of a set of rays is not an easy task and not many examples of such sets are known. Rigidity is of interest because only a KS set that is rigid can be Bell self-tested[31]-[33]. Therefore it is of interest to add to the known store of rigid KS sets. An important advance in this direction was recently made in [27], where two rigid KS sets were identified in  $\mathbb{C}^3$ , with one of them (of 31 vectors) being conjectured to be the smallest such set in this dimension.

We would like to raise the question of whether the rays of the three polytopes studied here are rigid. Let us first consider the 600-cell. If each ray is considered to be an element of  $\mathbb{RP}^3$ , it is determined by three real numbers and the 60 rays are determined by 180 real numbers. However the 75 bases of the 600-cell imply 450 orthogonality relations between its rays, and since the number of constraints imposed by orthogonality (450) greatly exceeds the number of ray components (180), it would seem that the rays are uniquely determined. However, there could be rays in  $\mathbb{RP}^n$  or  $\mathbb{CP}^n$  for  $n > 3$  that obey the same orthogonality relations as the rays of the 600-cell but are not unitarily equivalent to them, which would demonstrate that the 600-cell is not rigid. We show in the Appendix that of a subset of the rays of  $E_8$  (or Gosset's polytope) satisfy all the orthogonality relations of the rays of the 600-cell but are not unitarily equivalent to them, thereby establishing that the 600-cell is not rigid.

However we cannot make a statement about the rigidity of either the 120-cell or Gosset's polytope. The number of orthogonalities in these polytopes (particularly the latter) is so large and imposes so many constraints that we feel that rigidity may well be forced. However, this is just a conjecture and we have not come close to proving it. We leave it as an open problem to be considered. If these polytopes are rigid, that would certainly enhance their interest as far as quantum contextuality is concerned.

One final point should be made. The bases of all three polytopes can be obtained as the products of powers of three unitary operators acting on the computational basis[18]-[20]. If gates corresponding to the basic unitary operators can be designed, it would be possible to realize these polytopes (i.e., their bases) experimentally in the laboratory. This might be worth keeping in

mind for the time when the technology for doing it becomes available.

### Appendix: Proof that the 600-cell is not rigid

We follow Moody and Patera [34] in obtaining the 240 root vectors of  $E_8$  (or the vertices of Gosset's polytope) using icosians, which are quantities related to the symmetries of the regular icosahedron. We first construct the 120 root vectors of  $H_4$  (which comprise the vertices of a regular 600-cell) in two different ways. Let  $\beta = (1 + \sqrt{5})/2$  be the golden ratio and  $\alpha = (1 - \sqrt{5})$  its conjugate. Next, let  $\mathbb{Z}[\alpha] = \{m + n\alpha : m, n \in \mathbb{Z}\}$  denote the "golden ring" consisting of all linear combinations of 1 and  $\alpha$  with integer coefficients. In particular, we notice there is a bijection  $\phi : \mathbb{Z}[\alpha] \rightarrow \mathbb{Z}^2$  which maps  $m + n\alpha$  to  $(m, n)$  and that this bijection extends to a map  $\phi : \mathbb{Z}[\alpha]^4 \rightarrow \mathbb{Z}^8$ .

Let  $G \subset GL(4, \mathbb{R})$  be the linear group generated by all even permutations and arbitrary sign changes of the four coordinates of a vector  $(w, x, y, z) \in \mathbb{R}^4$ . This group has  $|G| = 12 \cdot 16 = 192$  elements. The roots of  $H_4$  can be constructed as the union of the orbits of  $(2, 0, 0, 0)$ ,  $(1, 1, 1, 1)$  and  $(0, \alpha, 1, \beta) = (0, \alpha, 1, -\alpha + 1)$  under the action of  $G$ . The 120 vectors thus obtained represent the vertices of a 600-cell on a sphere of radius 2. The vectors come in pairs that are the negatives of each other, and if we keep just one member of each pair we get the 60 rays of the 600-cell. Let us call this set of 60 vectors (or rays)  $H_4^a$ . If we multiply each of the vectors of  $H_4^a$  by  $\alpha$  we get a set  $H_4^b$  which serves as the set of vertices of a second 600-cell, scaled by a factor of  $\alpha$  relative to the first. Analogous to  $H_4^a$ , we may represent the respective orbits by  $(2\alpha, 0, 0, 0)$ ,  $(\alpha, \alpha, \alpha, \alpha)$  and  $(0, 1 + \alpha, \alpha, 1)$ .

We define the 240 root vectors of  $E_8$  as  $\phi(H_4^a \cup H_4^b)$ . In more detail, we recall that every vector of either  $H_4^a$  or  $H_4^b$  has components that are elements of the golden ring  $\mathbb{Z}[\alpha]$  and so can be written as  $v = (m_1 + n_1\alpha, m_2 + n_2\alpha, m_3 + n_3\alpha, m_4 + n_4\alpha)$ . Using the map  $\phi$  defined above, we map this to the 8-dimensional vector with components  $\phi(v) = (m_1, m_2, m_3, m_4, n_1, n_2, n_3, n_4)$ . The 60 vectors obtained in this way from  $H_4^a$  along with the 60 vectors obtained from  $H_4^b$ , together with all their negatives, make up the 240 root vectors of  $E_8$  (and also the vertices of Gosset's polytope), cf. [34].

The 60 vectors of  $H_4^a$  are unitarily equivalent to the 60 rays of the 600-cell introduced in Sec.3 and thus satisfy all the orthogonalities represented by the 75 bases of the 600-cell. If we take four vectors of  $H_4^a$  that are mutually orthogonal and turn them into vectors of  $E_8$  by the construction above, we find that the vectors we get in  $E_8$  are also mutually orthogonal (in an 8-dimensional space.) However, the converse of this statement is not true, i.e., if one takes four vectors of  $E_8$  that are mutually orthogonal and looks at the vectors of  $H_4^a$  (or  $H_4^b$ ) that correspond to them, we will find that these latter vectors are generally not mutually orthogonal.

Here is an example that illustrates this. Consider the following four vectors of  $H_4^a$ :

$$v_1 = (2, 0, 0, 0), \quad v_2 = (0, \alpha, 1, \beta), \quad v_3 = (0, 1, \beta, \alpha), \quad v_4 = (0, \beta, \alpha, 1)$$

The four vectors of  $E_8$  that correspond to these are

$$\begin{aligned} \phi(v_1) &= (2, 0, 0, 0, 0, 0, 0, 0), & \phi(v_2) &= (0, 0, 1, 1, 0, 1, 0, -1), \\ \phi(v_3) &= (0, 1, 1, 0, 0, 0, -1, 1), & \phi(v_4) &= (0, 1, 0, 1, 0, -1, 1, 0). \end{aligned}$$

It can be checked that the vectors  $\{v_1, v_2, v_3, v_4\}$  are mutually orthogonal, as are their images  $\{\phi(v_1), \phi(v_2), \phi(v_3), \phi(v_4)\} \subset E_8$ . Now consider the two additional vectors  $v_5 = (0, \alpha, 1, -\beta)$  and  $v_6 = (\alpha, \beta, 1, 0)$  of  $H_4^a$  whose images in  $E_8$  are  $\phi(v_5) = (0, 0, 1, -1, 0, 1, 0, 1)$  and  $\phi(v_6) = (0, 1, 1, 0, 1, -1, 0, 0)$ . One may check that while the vectors  $\{\phi(v_1), \phi(v_2), \phi(v_5), \phi(v_6)\} \subset E_8$  are mutually orthogonal, their preimages  $\{v_1, v_2, v_5, v_6\} \subset H_4^a$  are not because the vectors  $v_1$  and  $v_6$  are not orthogonal (and neither are  $v_2$  and  $v_5$ ).

To summarize, we have constructed a bijective mapping between the rays of  $E_8$  and  $H_4^a \cup H_4^b$  with the property that all the orthogonalities between rays in  $H_4^a$  are maintained between the corresponding rays of  $E_8$ , but with some orthogonalities between rays in  $E_8$  not being preserved as orthogonalities between the corresponding rays of  $H_4^a$ . This demonstrates that the mapping we have constructed between  $H_4^a$  and  $E_8$  is not unitary (or orthogonal) and, therefore, that the 600-cell is not rigid.

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## Supplementary Material for “Triacontagonal proofs of the Bell-Kochen-Specker theorem”

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| Length | Proofs   | Length | Proofs    |
|--------|----------|--------|-----------|
| 1      | 4        | 21     | 115880440 |
| 3      | 48       | 23     | 127058600 |
| 5      | 564      | 25     | 96649232  |
| 7      | 5116     | 27     | 51402240  |
| 9      | 42576    | 29     | 19478640  |
| 11     | 363392   | 31     | 5267856   |
| 13     | 2371056  | 33     | 975268    |
| 15     | 10593552 | 35     | 127856    |
| 17     | 33022968 | 37     | 14708     |
| 19     | 73615584 | 39     | 1212      |

**Table 1** Fifteen-fold symmetric parity proofs of the 120-cell. Shown above are the number of proofs for each word of odd length from 1 to 39. The total number of odd length proofs is  $2^{29}$ .

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| Basis profile | Generator(s)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| AABFGHHH      | $a_1 = \{1, 4, 21, 76, 95, 109, 111, 115\}, a_2 = \{1, 4, 21, 81, 100, 114, 115, 116\}, a_3 = \{1, 4, 21, 86, 105, 106, 115, 119\}$                                                                                                                                                                                                                                                                                                                                                                      |
| AACCHHHH      | $b_1 = \{1, 4, 35, 37, 112, 114, 116, 118\}$                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
| AACDGHHH      | $c_1 = \{1, 5, 32, 50, 105, 113, 115, 119\}, c_2 = \{1, 5, 41, 46, 96, 112, 116, 118\}$                                                                                                                                                                                                                                                                                                                                                                                                                  |
| AACEFGHH      | $d_1 = \{1, 4, 35, 74, 76, 97, 111, 112\}, d_2 = \{1, 4, 35, 75, 88, 105, 106, 112\}, d_3 = \{1, 4, 37, 72, 89, 95, 109, 118\},$<br>$d_4 = \{1, 4, 37, 73, 86, 103, 118, 119\}, d_5 = \{1, 5, 32, 73, 85, 96, 118, 119\}, d_6 = \{1, 5, 32, 74, 87, 94, 108, 119\},$<br>$d_7 = \{1, 5, 41, 74, 76, 92, 108, 112\}, d_8 = \{1, 5, 41, 75, 78, 105, 112, 113\}$                                                                                                                                            |
| AADDGGHH      | $e_1 = \{1, 4, 51, 59, 91, 94, 114, 116\}, e_2 = \{1, 5, 46, 50, 100, 101, 115, 116\}$                                                                                                                                                                                                                                                                                                                                                                                                                   |
| AADEFGGH      | $f_1 = \{1, 4, 51, 72, 89, 91, 105, 106\}, f_2 = \{1, 4, 51, 73, 76, 91, 103, 111\}, f_3 = \{1, 4, 59, 74, 86, 94, 97, 119\},$<br>$f_4 = \{1, 4, 59, 75, 88, 94, 95, 109\}, f_5 = \{1, 5, 46, 64, 87, 91, 94, 116\}, f_6 = \{1, 5, 48, 64, 89, 91, 105, 113\},$<br>$f_7 = \{1, 5, 48, 69, 89, 95, 96, 118\}, f_8 = \{1, 5, 48, 74, 89, 100, 101, 108\}, f_9 = \{1, 5, 50, 69, 76, 92, 95, 115\}$                                                                                                         |
| AAEEFFGG      | $g_1 = \{1, 4, 72, 74, 81, 89, 97, 100\}, g_2 = \{1, 4, 73, 75, 81, 88, 100, 103\}, g_3 = \{1, 5, 64, 73, 76, 85, 91, 92\},$<br>$g_4 = \{1, 5, 69, 75, 78, 87, 94, 95\}, g_5 = \{1, 5, 73, 75, 78, 85, 100, 101\}$                                                                                                                                                                                                                                                                                       |
| ABBCFH HH     | $h_1 = \{1, 18, 21, 42, 87, 106, 108, 119\}, h_2 = \{1, 19, 27, 37, 78, 112, 113, 114\}, h_3 = \{1, 20, 27, 32, 81, 113, 114, 115\}$                                                                                                                                                                                                                                                                                                                                                                     |
| ABBDFFGH      | $i_1 = \{1, 18, 20, 51, 89, 101, 106, 108\}, i_2 = \{1, 19, 21, 56, 85, 96, 106, 119\}, i_3 = \{1, 19, 27, 46, 88, 96, 106, 112\},$<br>$i_4 = \{1, 20, 27, 46, 86, 101, 106, 115\}$                                                                                                                                                                                                                                                                                                                      |
| ABBEFFGH      | $j_1 = \{1, 18, 20, 64, 81, 89, 97, 113\}, j_2 = \{1, 18, 20, 65, 83, 89, 95, 118\}, j_3 = \{1, 18, 21, 64, 81, 87, 102, 116\},$<br>$j_4 = \{1, 18, 21, 65, 78, 87, 95, 111\}, j_5 = \{1, 19, 21, 64, 76, 85, 102, 109\}, j_6 = \{1, 19, 21, 65, 78, 85, 100, 114\},$<br>$j_7 = \{1, 19, 27, 74, 76, 83, 102, 112\}, j_8 = \{1, 20, 27, 70, 76, 83, 95, 115\}$                                                                                                                                           |
| ABCCEFHH      | $k_1 = \{1, 18, 42, 43, 73, 83, 118, 119\}, k_2 = \{1, 19, 34, 37, 64, 89, 109, 113\}, k_3 = \{1, 20, 32, 35, 65, 85, 114, 118\},$<br>$k_4 = \{1, 21, 41, 42, 71, 76, 108, 109\}$                                                                                                                                                                                                                                                                                                                        |
| ABCDDGHH      | $l_1 = \{1, 18, 42, 50, 59, 97, 113, 119\}, l_2 = \{1, 19, 37, 50, 56, 103, 113, 119\}, l_3 = \{1, 20, 32, 51, 57, 94, 108, 114\},$<br>$l_4 = \{1, 21, 42, 48, 57, 100, 108, 114\}, l_5 = \{1, 27, 42, 48, 59, 91, 113, 114\}$                                                                                                                                                                                                                                                                           |
| ABCDEF GH     | $m_1 = \{1, 18, 43, 51, 73, 78, 101, 111\}, m_2 = \{1, 19, 34, 51, 69, 89, 96, 106\}, m_3 = \{1, 19, 43, 46, 64, 88, 94, 109\},$<br>$m_4 = \{1, 19, 43, 51, 69, 78, 94, 114\}, m_5 = \{1, 19, 43, 56, 74, 83, 94, 119\}, m_6 = \{1, 20, 35, 56, 75, 85, 101, 106\},$<br>$m_7 = \{1, 20, 41, 46, 65, 86, 103, 118\}, m_8 = \{1, 20, 41, 51, 70, 76, 103, 108\}, m_9 = \{1, 20, 41, 56, 75, 81, 103, 113\},$<br>$m_{10} = \{1, 21, 41, 56, 71, 81, 96, 116\}, m_{11} = \{1, 27, 42, 46, 72, 87, 91, 106\}$ |
| ABCEEFFG      | $n_1 = \{1, 18, 43, 64, 73, 81, 88, 92\}, n_2 = \{1, 19, 34, 65, 74, 83, 89, 100\}, n_3 = \{1, 20, 35, 64, 70, 76, 85, 97\},$<br>$n_4 = \{1, 21, 41, 65, 71, 78, 86, 105\}, n_5 = \{1, 27, 32, 71, 73, 81, 88, 96\}, n_6 = \{1, 27, 32, 72, 74, 81, 87, 102\},$<br>$n_7 = \{1, 27, 37, 70, 72, 78, 87, 95\}, n_8 = \{1, 27, 37, 71, 73, 78, 86, 101\}, n_9 = \{1, 27, 42, 71, 73, 76, 83, 91\}$                                                                                                          |
| ABDDDGGH      | $o_1 = \{1, 18, 50, 51, 59, 101, 102, 116\}, o_2 = \{1, 21, 48, 56, 57, 95, 96, 111\}$                                                                                                                                                                                                                                                                                                                                                                                                                   |
| ABDDEFFG      | $p_1 = \{1, 18, 50, 59, 65, 88, 92, 95\}, p_2 = \{1, 19, 46, 50, 65, 88, 100, 103\}, p_3 = \{1, 19, 50, 51, 69, 76, 102, 103\},$<br>$p_4 = \{1, 20, 46, 57, 64, 86, 94, 97\}, p_5 = \{1, 20, 56, 57, 75, 83, 94, 95\}, p_6 = \{1, 21, 48, 57, 64, 86, 102, 105\},$<br>$p_7 = \{1, 27, 48, 59, 70, 88, 95, 96\}, p_8 = \{1, 27, 48, 59, 74, 86, 101, 102\}$                                                                                                                                               |
| ACCCDEHH      | $q_1 = \{1, 34, 41, 42, 59, 75, 109, 113\}, q_2 = \{1, 35, 42, 43, 48, 69, 114, 118\}, q_3 = \{1, 41, 42, 43, 46, 72, 109, 118\}$                                                                                                                                                                                                                                                                                                                                                                        |
| ACCCEEFH      | $r_1 = \{1, 32, 34, 35, 65, 74, 87, 111\}, r_2 = \{1, 34, 35, 37, 64, 70, 87, 116\}, r_3 = \{1, 34, 35, 42, 69, 75, 87, 106\}$                                                                                                                                                                                                                                                                                                                                                                           |
| ACCDDEGH      | $s_1 = \{1, 32, 34, 51, 57, 73, 96, 111\}, s_2 = \{1, 34, 41, 51, 59, 70, 96, 116\}, s_3 = \{1, 35, 37, 50, 56, 71, 101, 116\},$<br>$s_4 = \{1, 35, 43, 48, 56, 74, 101, 111\}$                                                                                                                                                                                                                                                                                                                          |
| ACCDEEFG      | $t_1 = \{1, 32, 35, 50, 65, 71, 88, 105\}, t_2 = \{1, 34, 37, 57, 64, 73, 86, 92\}, t_3 = \{1, 34, 41, 59, 65, 74, 86, 92\},$<br>$t_4 = \{1, 34, 42, 57, 73, 75, 83, 100\}, t_5 = \{1, 35, 42, 50, 69, 71, 76, 97\}, t_6 = \{1, 35, 43, 48, 64, 70, 88, 105\},$<br>$t_7 = \{1, 41, 43, 51, 70, 72, 78, 105\}, t_8 = \{1, 41, 43, 56, 72, 74, 81, 92\}$                                                                                                                                                   |
| ACDDDEGG      | $u_1 = \{1, 32, 50, 51, 57, 72, 102, 105\}, u_2 = \{1, 37, 50, 56, 57, 72, 92, 95\}, u_3 = \{1, 42, 46, 50, 57, 72, 97, 100\}$                                                                                                                                                                                                                                                                                                                                                                           |
| BBBBFFHH      | $v_1 = \{16, 19, 23, 27, 82, 88, 106, 117\}$                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
| BBBCEFFH      | $w_1 = \{16, 19, 23, 43, 63, 80, 88, 109\}, w_2 = \{16, 19, 27, 37, 62, 82, 90, 114\}$                                                                                                                                                                                                                                                                                                                                                                                                                   |
| BBBDFFGH      | $x_1 = \{16, 18, 20, 51, 52, 77, 99, 106\}, x_2 = \{16, 18, 20, 52, 53, 79, 95, 118\}, x_3 = \{16, 20, 24, 51, 57, 80, 99, 114\}$                                                                                                                                                                                                                                                                                                                                                                        |
| BBBDEFFG      | $y_1 = \{16, 18, 20, 52, 63, 81, 90, 97\}, y_2 = \{16, 19, 23, 50, 66, 79, 88, 100\}, y_3 = \{16, 19, 27, 53, 74, 76, 82, 93\}$                                                                                                                                                                                                                                                                                                                                                                          |
| BBCCDDHH      | $z_1 = \{16, 18, 36, 43, 49, 53, 118, 119\}, z_2 = \{16, 19, 31, 34, 51, 52, 106, 117\}, z_3 = \{16, 23, 41, 43, 49, 58, 109, 118\}$                                                                                                                                                                                                                                                                                                                                                                     |
| BBCCDEFH      | $a'_1 = \{16, 18, 36, 43, 51, 62, 77, 111\}, a'_2 = \{16, 19, 31, 43, 51, 62, 80, 114\}, a'_3 = \{16, 19, 34, 37, 52, 63, 90, 109\},$<br>$a'_4 = \{16, 19, 37, 40, 50, 66, 90, 119\}, a'_5 = \{16, 19, 40, 43, 53, 74, 80, 119\}$                                                                                                                                                                                                                                                                        |
| BBCCEEFF      | $b'_1 = \{16, 18, 36, 43, 61, 63, 81, 88\}, b'_2 = \{16, 20, 40, 41, 66, 75, 81, 90\}$                                                                                                                                                                                                                                                                                                                                                                                                                   |
| BBCDDEFG      | $c'_1 = \{16, 19, 31, 50, 51, 66, 76, 93\}, c'_2 = \{16, 19, 34, 52, 53, 74, 79, 100\}, c'_3 = \{16, 20, 40, 53, 57, 75, 80, 95\},$<br>$c'_4 = \{16, 20, 41, 47, 51, 66, 76, 99\}$                                                                                                                                                                                                                                                                                                                       |
| BBDDDDGG      | $d'_1 = \{16, 23, 49, 50, 57, 58, 97, 100\}$                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
| BCCDEEFF      | $e'_1 = \{16, 34, 36, 37, 57, 61, 63, 86\}, e'_2 = \{16, 40, 41, 43, 58, 61, 74, 81\}$                                                                                                                                                                                                                                                                                                                                                                                                                   |
| BCCDDDEG      | $f'_1 = \{16, 34, 36, 49, 53, 57, 75, 100\}, f'_2 = \{16, 37, 40, 50, 57, 58, 61, 95\}, f'_3 = \{16, 41, 43, 47, 51, 58, 62, 105\}$                                                                                                                                                                                                                                                                                                                                                                      |
| CCCCDDEE      | $g'_1 = \{31, 32, 33, 34, 51, 52, 67, 73\}$                                                                                                                                                                                                                                                                                                                                                                                                                                                              |

**Table 2** Generators of Gosset's polytope. The generators are shown in the second column and their basis profiles in the first column. There are 135 generators but only 33 profiles, so many of the profiles have more than one generator associated with them. All generators with the same profile are labeled by the same letter and distinguished from each other by their subscripts. Each generator gives rise to fifteen bases and these bases, taken together, make up all the 2025 bases of Gosset's polytope.

|                     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
|---------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1-letter words (16) | $b_1, e_1, e_2, g_1, g_2, g_3, g_4, g_5, v_1, z_1, z_2, z_3, b'_1, b'_2, d'_1, g'_1$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |
| 2-letter words (22) | $a_1 a_2, a_1 a_3, a_1 x_1, a_1 x_2, a_1 x_3, c_1 c_2,$<br>$d_1 c'_1, d_1 c'_2, d_1 c'_3, d_1 c'_4, d_1 d_2, d_1 d_3,$<br>$d_1 d_4, d_1 d_5, d_1 d_6, d_1 d_7, d_1 d_8, h_1 h_2,$<br>$h_1 h_3, h_1 r_1, h_1 r_2, h_1 r_3$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| 3-letter words (56) | $a_1 c_1 w_1, a_1 c_1 w_2, a_1 c_1 e'_1, a_1 c_1 e'_2, a_1 h_1 l_1, a_1 h_1 l_2,$<br>$a_1 h_1 l_3, a_1 h_1 l_4, a_1 h_1 l_5, a_1 h_1 n_1, a_1 h_1 n_2, a_1 h_1 n_3,$<br>$a_1 h_1 n_4, a_1 h_1 n_5, a_1 h_1 n_6, a_1 h_1 n_7, a_1 h_1 n_8, a_1 h_1 n_9,$<br>$c_1 d_1 f_1, c_1 d_1 f_2, c_1 d_1 f_3, c_1 d_1 f_4, c_1 d_1 f_5, c_1 d_1 f_6,$<br>$c_1 d_1 f_7, c_1 d_1 f_8, c_1 d_1 f_9, c_1 d_1 a'_1, c_1 d_1 a'_2, c_1 d_1 a'_3,$<br>$c_1 d_1 a'_4, c_1 d_1 a'_5, c_1 h_1 i_1, c_1 h_1 i_2, c_1 h_1 i_3, c_1 h_1 i_4,$<br>$c_1 h_1 t_1, c_1 h_1 t_2, c_1 h_1 t_3, c_1 h_1 t_4, c_1 h_1 t_5, c_1 h_1 t_6,$<br>$c_1 h_1 t_7, c_1 h_1 t_8, d_1 h_1 j_1, d_1 h_1 j_2, d_1 h_1 j_3, d_1 h_1 j_4,$<br>$d_1 h_1 j_5, d_1 h_1 j_6, d_1 h_1 j_7, d_1 h_1 j_8, d_1 h_1 s_1, d_1 h_1 s_2,$<br>$d_1 h_1 s_3, d_1 h_1 s_4$ |
| 4-letter words (26) | $a_1 c_1 d_1 f'_1, a_1 c_1 d_1 f'_2, a_1 c_1 d_1 f'_3, a_1 c_1 d_1 y_1,$<br>$a_1 c_1 d_1 y_2, a_1 c_1 d_1 y_3, a_1 c_1 h_1 o_1, a_1 c_1 h_1 o_2,$<br>$a_1 d_1 h_1 k_1, a_1 d_1 h_1 k_2, a_1 d_1 h_1 k_3, a_1 d_1 h_1 k_4,$<br>$a_1 d_1 h_1 p_1, a_1 d_1 h_1 p_2, a_1 d_1 h_1 p_3, a_1 d_1 h_1 p_4,$<br>$a_1 d_1 h_1 p_5, a_1 d_1 h_1 p_6, a_1 d_1 h_1 p_7, a_1 d_1 h_1 p_8,$<br>$c_1 d_1 h_1 q_1, c_1 d_1 h_1 q_2, c_1 d_1 h_1 q_3, c_1 d_1 h_1 u_1,$<br>$c_1 d_1 h_1 u_2, c_1 d_1 h_1 u_3$                                                                                                                                                                                                                                                                                                                  |
| 5-letter words (11) | $a_1 c_1 d_1 h_1 m_1, a_1 c_1 d_1 h_1 m_2, a_1 c_1 d_1 h_1 m_3,$<br>$a_1 c_1 d_1 h_1 m_4, a_1 c_1 d_1 h_1 m_5, a_1 c_1 d_1 h_1 m_6,$<br>$a_1 c_1 d_1 h_1 m_7, a_1 c_1 d_1 h_1 m_8, a_1 c_1 d_1 h_1 m_9,$<br>$a_1 c_1 d_1 h_1 m_{10}, a_1 c_1 d_1 h_1 m_{11}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |

**Table 3** The 131 words representing a linearly independent set of vectors in the nullspace of the matrix  $M^T$  of Gosset's polytope. The 16 single letter words all give  $30_2 - 15_4$  parity proofs and the 3- and 5-letter words represent parity proofs of 45 and 75 bases, respectively. All the parity proofs in this table are both minimal and irreducible.

| Length | Proofs                                  | Length | Proofs                                  |
|--------|-----------------------------------------|--------|-----------------------------------------|
| 1      | 16                                      | 69     | 180520872122823098245862570199127030680 |
| 3      | 25812                                   | 71     | 155821839317285933878889210179604711024 |
| 5      | 21653868                                | 73     | 119534561668054963002852653943661854832 |
| 7      | 8652003024                              | 75     | 81455804005150246833959629030543329328  |
| 9      | 1953439358160                           | 77     | 49274358540367715978978010339417431984  |
| 11     | 279696462166032                         | 79     | 26436389051355999506519101553074086512  |
| 13     | 27345712822595216                       | 81     | 12565444178730938040258584478235056624  |
| 15     | 1922273370347815632                     | 83     | 5283911436898023021008844055037541520   |
| 17     | 100919351992599919440                   | 85     | 1962595676562122833284741838569192272   |
| 19     | 4073954893763530856880                  | 87     | 642656964391499727733011021261899760    |
| 21     | 129396567337947795186928                | 89     | 185116714972832401789642506963347440    |
| 23     | 3294242253859564715609424               | 91     | 46787741146979619043486256387889200     |
| 25     | 68256699499928681794401616              | 93     | 10346237289631303832281044955692720     |
| 27     | 1165808870376411660238444048            | 95     | 1995097493028567348842572411414000      |
| 29     | 16591240524716074401300742544           | 97     | 334230250120765120581947447514800       |
| 31     | 198559684989345477675990749008          | 99     | 48436171064707869561000746697120        |
| 33     | 2014177410611613275385516844560         | 101    | 6042532231834847059425191542336         |
| 35     | 17437021583294837611603671701504        | 103    | 645318976215369969425403092912          |
| 37     | 129599484740704840232429138427040       | 105    | 58622383187330985956893984176           |
| 39     | 831290622095236372451283545241360       | 107    | 4496691357165952988389892464            |
| 41     | 4622786874090582681904187711459600      | 109    | 288778344038199652858660080             |
| 43     | 22376745765946773959473914330329680     | 111    | 15373130518005207330609072              |
| 45     | 94615209875003228427091933876846800     | 113    | 670509485297303171419696                |
| 47     | 350540162395363486324657008310384400    | 115    | 23628938383825022644112                 |
| 49     | 1141043998001234207494166219135912080   | 117    | 661582418589474325200                   |
| 51     | 3270992794270204725504176218418242608   | 119    | 14417050288814344240                    |
| 53     | 8274804703066715294324920218378861296   | 121    | 238298351913548848                      |
| 55     | 18505472335949199660124342087904997008  | 123    | 2890197197138544                        |
| 57     | 36639907632581122125005175119122150032  | 125    | 24613298133232                          |
| 59     | 64307213688276510679781711925729104720  | 127    | 138432535088                            |
| 61     | 100150578694856860867732458243810022096 | 129    | 469443904                               |
| 63     | 138508301615365274584106443908685446288 | 131    | 829828                                  |
| 65     | 170205393715804635473398600245423862256 | 133    | 540                                     |
| 67     | 185909554872758116099101213433344650760 |        |                                         |

**Table 4** Fifteen-fold symmetric parity proofs of Gosset's polytope. Shown above are the number of proofs for each word of odd length from 1 to 133. The total number of proofs is  $2^{130}$ .