RATIONAL QUIVER REPRESENTATIONS: TAME AND WILD

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Abstract. We study representation finite K-rational quivers over fields of characteristic 0 and their indecomposable representations, exploiting that all Brauer obstructions for descent of representations are trivial in this case. Contrasting the tame case, we give an example of a simple quiver of wild representation type, where we realize every possible Brauer obstruction of a given Galois extension L/K in the category of quiver representations over L.

Contents

Introduction

In [11] the author introduced the notion of rational quivers and their representations. Initially, the motivation was to gain a different perspective on the rational structures on Harish-Chandra modules previously studied by Harder [4], Harris [5, 6] and the author [7], which in turn were motivated by applications in number theory [8, 9, 10]. In [11] the author showed that Gelfand's equivalence between certain blocks of Harish-Chandra modules for $\mathrm{SL}_2(\mathbf{R})$ over the complex numbers naturally extends to an equivalence over any field of characteristic 0, if we furnish the Gelfand quiver with a suitable natural $\mathbf{Q}$-rational structure.

In this article, we investigate various properties of K-rational quivers and their representations. After a quick recap of the main definitions and results of [11] in Section 1, we consider K-rational quivers of finite representation type in Section 2. We generalize one direction of Gabriel's celebrated ADE classification result [3, 1, 2]: If the underlying quiver Γ of a K-rational quiver Γ_K is of type A, D or E, then Γ_K is of finite representation type (cf. Theorem 2.3).

In Remark 2.5 we show that the converse is not true: The quiver Γ underlying a general K-rational quiver Γ_K of finite representation type is not

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necessarily of type ADE. In particular, if we assume that the rational structure on Γ_K is given by an action of the Galois group of a finite extension L/K, then the base change Γ_L of Γ_K to L need not be of finite representation type if Γ_K was of finite representation type.

As an illustration of our generalization of Gabriel's Theorem we discuss in Section 3 a non-abelian Galois action in the context of triality and describe all indecomposable representations of the resulting K-rational quiver explicitly.

A crucial fact in this context is that there is no Brauer obstruction in Gabriel's situation of an ADE type quiver (cf. Corollary 2.2). Section 4 is dedicated to the 2-loop quiver with trivial rational structure for a given Galois extension L/K in characteristic 0. We show that every (non-trivial) class in the Brauer group arises as an obstruction for the descent of an absolutely irreducible representation of the 2-loop quiver in the extension L/K.

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1. Rational quiver representations

We recall the main definitions from [11].

Rational structures on quivers.

Definition 1.1 (K-rational quiver, [11]). Let $\Gamma = (V, E, s, t)$ be a classical quiver, consisting of a finite set of vertices V, a finite set of edges E, and two maps $s, t\colon E \to V$ for the source and target of an edge. For a Galois extension L/K, a *K-rational structure* on Γ is a group homomorphism $\rho\colon \mathrm{Gal}(L/K) \to \mathrm{Aut}(\Gamma)$ such that the induced actions on V and E are continuous (for the discrete topology on V and E). This means that for any $\sigma \in \mathrm{Gal}(L/K)$ and $e \in E$, the relations

$$s(\sigma e) = \sigma s(e) \quad \text{and} \quad t(\sigma e) = \sigma t(e) \tag{1}$$

hold. The pair $\Gamma_K = (\Gamma, \rho)$ is called a *K-rational quiver.*

Definition 1.2 (Representation of a rational quiver, [11]). A *K-rational representation M* of a K-rational quiver Γ_K (with respect to L/K) consists of:

(1) A family $M = (M(v))_{v\in V}$ of L-vector spaces.
(2) For each $\sigma \in \mathrm{Gal}(L/K)$, a family of σ-linear isomorphisms $\varphi_{v,\sigma}\colon M(v) \to M(\sigma v)$ satisfying the cocycle condition $\varphi_{\tau v,\sigma} \circ \varphi_{v,\tau} = \varphi_{v,\sigma\tau}$.
(3) A family of L-linear maps $\phi_e\colon M(s(e)) \to M(t(e))$ for each edge $e \in E$, which are commute with the Galois action.

Associated étale K-species.

Definition 1.3 (K-species). A *K-species* $S = (L_i, {}_iM_j)_{i,j\in I}$ is a finite collection $(L_i)_{i\in I}$ of division rings containing K in their center, together with an L_i, L_j-bimodule ${}_iM_j$ for each pair $i, j \in I$, which is finite-dimensional over K.

This classical definition was refined by the author as follows:

Definition 1.4 (Étale K-species, [11]). An *étale K-species* is a K-species $S = (L_i, {}_iM_j)_{i,j\in I}$ where each L_i is a finite separable extension of K and each ${}_iM_j$ is a commutative L_i, L_j-bialgebra which is finite and étale as a K-algebra.

To any K-rational quiver Γ_K we can associate functorially an étale K-species $S(\Gamma_K) = (L_i, {}_iM_j)_{i,j\in I}$ as follows (cf. Definition 2.13 in [11]).

- The index set I of the species is the set of orbits of the Galois group $G = \mathrm{Gal}(L/K)$ on the set of vertices V, i.e., $I = G\backslash V$.
- For each orbit $i \in I$, choose a representative vertex $v_i \in i$. The associated division algebra L_i is the fixed field $L^{G_{v_i}}$, where G_{v_i} is the stabilizer of v_i in G. This is a finite separable extension of K.
- For any two orbits $i, j \in I$, let $E_{ij} = \{e \in E \mid s(e) \in i, t(e) \in j\}$. The L_i-L_j-bimodule is defined as the direct sum over the orbits in $G\backslash E_{ij}$:
$$ {}_iM_j := \bigoplus_{\varepsilon\in G\backslash E_{ij}} L^{G_{e_\varepsilon}}, $$
where e_ε is a representative of the edge orbit ε.

We showed in [11] that this construction gives rise to a categorical anti-equivalence between K-rational quivers and étale K-species (cf. Theorem 2.17 in loc. cit.) preserving representations, i. e. the category of representations of Γ_K is canonically equivalent to the category of representations of the associated K-species $S(\Gamma_K)$ (cf. Theorem 2.35 in loc. cit.)

2. K-rational quivers of finite representation type

Let Γ be a connected quiver of type A, D or E. Let L/K denote a Galois extension of fields of characteristic 0 and let L^{alg} denote an algebraic closure of L. By Gabriel's Theorem (cf. [3, 1, 2]), the indecomposable representations of Γ over L^{alg} are in bijection with set Δ^+ of positive roots of the root system Δ associated to the Dynkin diagram underlying Γ. Moreover, the endomorphism ring of each indecomposable representation V is known to isomorphic to L^{alg}.

The explicit description of the indecomposable representation via dimension vectors shows that each indecomposable representation of Γ descends to a representation over L and by Hilbert 90, this descent is unique up to isomorphim. Now every indecomposable representation V of Γ over L decomposes over L^{alg} into a direct sum of indecomposable representations, which each descend to L, hence V must agree with one of them. This shows

Theorem 2.1. *Let Γ be a connected quiver of type A, D, or E. Then the indecomposable representaitons of Γ over any field L of charactereristic 0 are in canonical bijection with the positive roots of the underyling root system Δ. The endomorphism rings of each indecomposable representation over L is L and each irreducible representation over L is absolutely irreducible.*

Proof. The main statement was proven in the preceeding discussion. The remaining claims follow with Proposition 1.21 in [11]. □

Corollary 2.2. *Let Γ_K denote a connected quiver over K which is of type A, D or E, which splits inside a Galois extension L/K. Then there are no*

Brauer obstructions for descent of irreducibles for intermediate fields inside the extension L/K.

Combining Theorem 2.1 with Corollary 2.2 we obtain

Theorem 2.3. *Let Γ_K denote a quiver over K whose underlying quiver is of Γ is of type A, D or E. Then Γ_K is representation finite and the set of isomorphism classes of indecomposable representations over K is in canonical bijection with the set of* $\mathrm{Gal}(L/K)$*-orbits acting on the set $\Delta_+ \subseteq \Delta$ of positive roots.*

The Galois action in the statement is the canonical extension to the root system Δ of the given action of $\mathrm{Gal}(L/K)$ on the underlying Dynkin diagram.

Proof. Assume first that Γ is of type ADE. The classification of indecomposables for Γ_K is a straightforward application of Galois descent: Let $\mathfrak{O} \subseteq \Delta_+$ denote a Galois orbit. Fix a representative $\alpha \in \mathfrak{O}$, write $G_\alpha \subseteq \mathrm{Gal}(L/K)$ for its stabilizer. Then the indecomposable representation associated to α is defined over L^{G_α} and its restriction of scalars inside the extension L^{G_α} to K is easily seen to be an indecomposable representation of Γ_K over K, independent of the choice of representative α. Moreover, every indecomposable representation over K is of this form, again by the same argument as above. $\square$

Remark 2.4. We remark that the implication Γ of type ADE implies Γ_K is of finite representation type also is a consequence of Dlab–Ringel's extension of Gabriel's Theorem to K-species (cf. [2]) combined with Theorem 2.35 in [11]: The category of representations of Γ_K is equivalent to that of the associated K-species $S(\Gamma_K)$. By Dlab–Ringel, this K-species is of finite representation type if and only if its associated valued graph is a Dynkin diagram (i. e. of type A, B, C, D, E, F, or G). Considering the possible autorphisms of diagrams of types A, D and E and the resulting foldings, the folding process can never create a graph of Euclidean (affine) or wild type, for the quiver structure on Γ imposes additional constraints (e. g. the central arrow of a quiver of type A_{2n} forces the Galois action to be trivial etc.). Consequently, the species $S(\Gamma_K)$ is seen to have a Dynkin diagram as valued graph.

Remark 2.5 (A finite type K-rational quiver of infinite type over L). There is no converse to Theorem 2.3 in the spirit of Gabriel's ADE characterization. Consider the Kronecker quiver Γ, which has two vertices v_1, v_2 and two parallel edges $a, b : v_1 \to v_2$. This quiver is of tame infinite representation type over any field. Let L/K be a quadratic extension with Galois group $G = \{1, \tau\}$. We define a K-rational quiver Γ_K via a K-rational structure on Γ by letting τ act trivially on the vertices and by permuting the edges: $\tau(v_i) = v_i$, $\tau(a) = b$, and $\tau(b) = a$. The associated étale K-species $S(\Gamma_K)$ has two nodes, corresponding to the two fixed vertex orbits $\{v_1\}$ and $\{v_2\}$, with associated field K for both. The two edges form a single orbit whose stabilizer is trivial, yielding a (K, K)-bimodule L. The resulting valued graph is that of type B_2:

$$K \xrightarrow[(2,2)]{L} K$$

By the Dlab–Ringel Theorem [2], this species is of finite representation type. Thus, the rational quiver Γ_K is of finite type over K, despite its underlying quiver Γ being of infinite type over L.

3. A rational quiver of type D_4 and triality

We illustrate explore Theorem 2.3 with an example involving a non-abelian Galois group acting on a quiver of finite representation type in the context of triality: We consider a quiver of type D_4 and a rational structure based on a Galois extension with group S_3 and classify the indecomposable representations.

The K-rational quiver. Let K be a field with characteristic 0. Let L/K be a Galois extension with Galois group $G = \mathrm{Gal}(L/K) \cong S_3$, the symmetric group on three letters.

We consider the quiver Γ with underlying graph the Dynkin diagram D_4. We label the central 'root' vertex v_0 and the outer 'leaf' vertices v_1, v_2, v_3. We orient the quiver by having all arrows point outwards from the center, with three edges $e_i : v_0 \to v_i$ for $i = 1, 2, 3$.

We define a K-rational structure on Γ by letting S_3 act naturally on the indices $\{1, 2, 3\}$. For any permutation $\sigma \in S_3$:

- Action on vertices: $\sigma v_0 = v_0$ and $\sigma v_i = v_{\sigma^{-1}(i)}$ for $i \in \{1, 2, 3\}$.
- Action on edges: $\sigma e_i = e_{\sigma^{-1}(i)}$ for $i \in \{1, 2, 3\}$.

This action ρ_{S_3} on Γ gives rise to a K-rational quiver $\Gamma_K = (\Gamma, \rho_{S_3})$ in the sense of Definition 1.1.

The associated étale K-species. The associated K-species $S(\Gamma_K)$ is given by the following data.

1. **The index set and fields:** The two vertex orbits are $i_0 = \{v_0\}$ and $i_1 = \{v_1, v_2, v_3\}$.
    - For i_0, the stabilizer is S_3, so the field is $L_{i_0} = L^{S_3} = K$.
    - For i_1, the stabilizer of the representative v_1 is $H \cong S_2$. The field is $L_{i_1} = L^H = M$.
2. **The bimodule:** The single edge orbit $\{e_1, e_2, e_3\}$ connects i_0 to i_1. The stabilizer of the representative e_1 is H. The resulting (K, M)-bimodule is ${}_{i_0}M_{i_1} = L^H = M$.

The resulting étale K-species $S(\Gamma_K)$ is of type G_2:

$$K \xrightarrow{\quad {}_K M_M = M \quad} M$$

Since the species is of Dynkin type G_2, it has 6 indecomposable representations, corresponding to the 6 positive roots of G_2. This matches the number of Galois orbits of the 12 positive roots of D_4, see below.

The Galois action on the D_4 root system. In order to apply Theorem 2.3 in the given situation, we first describe the Galois action on the root system D_4.

We denote the simple roots corresponding to the vertices $\{v_0, v_1, v_2, v_3\}$ by $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. The action of an element $\sigma \in S_3$ on the quiver induces an action on these simple roots:

$$\sigma(\alpha_0) = \alpha_0 \quad \text{and} \quad \sigma(\alpha_i) = \alpha_{\sigma^{-1}(i)} \text{ for } i \in \{1, 2, 3\}.$$

This action extends to the entire set of 12 positive roots. For clarity, we will identify each positive root with the dimension vector of its corresponding indecomposable representation, written in the basis (v_0, v_1, v_2, v_3). The action of σ sends a representation with dimension vector (d_0, d_1, d_2, d_3) to one with dimension vector $(d_0, d_{\sigma^{-1}(1)}, d_{\sigma^{-1}(2)}, d_{\sigma^{-1}(3)})$.

The 12 positive roots (dimension vectors) of D_4 are:

- $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$
- $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 0, 0, 1)$
- $(1, 1, 1, 0)$, $(1, 1, 0, 1)$, $(1, 0, 1, 1)$
- $(1, 1, 1, 1)$
- $(2, 1, 1, 1)$

Applying the permutation action of S_3 to these 12 dimension vectors partitions them into exactly 6 orbits:

- **Orbit 1** (size 1)**:** $(1, 0, 0, 0)$ is fixed by the action.
- **Orbit 2** (size 3)**:** $\{(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ are permuted.
- **Orbit 3** (size 3)**:** $\{(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1)\}$ are permuted.
- **Orbit 4** (size 3)**:** $\{(1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1)\}$ are permuted.
- **Orbit 5** (size 1)**:** $(1, 1, 1, 1)$ is fixed.
- **Orbit 6** (size 1)**:** $(2, 1, 1, 1)$ is fixed.

This partitioning leads by Theorem 2.3 directly to the main classification result for the K-rational triality quiver.

Theorem 3.1. *The K-rational quiver Γ_K of type D_4 with S_3-action has exactly* 6 *indecomposable representations over the field K. Each indecomposable corresponds to one of the* 6 *Galois orbits of the* 12 *positive roots of D_4.*

These 6 indecomposable K-rational representations are described as follows:

1. **The simple representation at v_0:** Corresponds to Orbit 1. It is the absolutely irreducible representation M_K over K with dimension vector $(1, 0, 0, 0)$, realized as $M(v_0) = L$ and $M(v_i) = 0$ for $i = 1, 2, 3$ with the tautological Galois actions.
2. **The indecomposable from the simple leaves:** Corresponds to Orbit 2. Its base change to L is the direct sum of the three simple representations at the leaf vertices. Over K it is the direct sum with the appropriate Galois action permuting the three summands.
3. **The indecomposable from the projective leaves:** Corresponds to Orbit 3. Its base change to L is the direct sum of the three representations with dimension vectors $(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1)$.
4. **The indecomposable of type** $(1, 1, 1, 0)$**:** Corresponds to Orbit 4. Its base change to L is the direct sum of the three representations in that orbit.

(5) **The projective indecomposable at v_0:** Corresponds to Orbit 5. It is the abolutely irreducible representation over K with dimension vector $(1, 1, 1, 1)$.
(6) **The injective indecomposable at v_0:** Corresponds to Orbit 6. It is the absolutely irreducible representation with dimension vector $(2, 1, 1, 1)$.

This completes the classification.

The classification for the associated species. A representation of $S(\Gamma_K)$ is a pair (V_K, V_M) of a K-vector space and an M-vector space, with a K-linear map $\hat{f} : V_K \to V_M$. In this notation, the indecomposable representations are:

(1) **Simple at K-node:** $(K, 0)$ with $\hat{f} = 0$. Corresponds to the simple representation at v_0 associated to orbit 1.
(2) **Simple at M-node:** $(0, M)$ with $\hat{f} = 0$. Corresponds to orbit 2 for the simple representations at $\{v_1, v_2, v_3\}$.
(3) **Projective cover of simple at K:** (K, M) with $\hat{f} : K \hookrightarrow M$ the canonical inclusion. Corresponds to the orbit of representations with dimension vector $(1, 1, 0, 0)$ and its permutations (orbit 3).
(4) **Injective hull of simple at M:** (M, M) with $\hat{f} = \mathrm{id}_M$. Corresponds to the orbit of representations with dimension vector $(1, 1, 1, 0)$ and its permutations (orbit 4).
(5) **Intermediate indecomposable:** Let $\ker(\mathrm{Tr}_{M/K})$ be the 2-dim K-subspace of M of trace-zero elements. The representation is $(\ker(\mathrm{Tr}_{M/K}), M)$ with $\hat{f}$ being the inclusion. Corresponds to the Galois-fixed representation with dimension vector $(2, 1, 1, 1)$ (orbit 5).
(6) **Injective hull of simple at K:** (M, K) with $\hat{f} = \mathrm{Tr}_{M/K}$. Corresponds to the Galois-fixed projective indecomposable at v_0 with dimension vector $(1, 1, 1, 1)$ (orbit 6).

As expected, the 6 indecomposable rational representations of Γ_K are in perfect correspondence with the 6 indecomposable representations of its associated étale K-species of type G_2.

4. The wild case: Realizing arbitrary Brauer obstructions

Let K denote a field of characteristic 0. Let D be a central division algebra over K of degree n, and let L/K be a Galois splitting field for D of degree n with Galois group $\mathrm{Gal}(L/K)$. We fix a $\mathrm{Gal}(L/K)$-equivariant L-algebra isomorphism $\Phi \colon L \otimes_K D \xrightarrow{\cong} M_n(L)$. By Albert's Theorem, we may choose two elements $d_1, d_2 \in D$ that generate D as a K-algebra (see, for instance, [12, Prop. 11.2]).

We give a construction that realizes D as the endomorphism ring of a K-rational representation of two 2-loop quiver with trivial rational structure.

- **Quiver Γ:** A single vertex v and two loops, e_1 and e_2, at v:

$$e_1 \circlearrowleft v \circlearrowright e_2$$

- **K-rational structure:** We endow Γ with the trivial rational structure for L/K.

Remark 4.1. The choice of a quiver with one vertex and two loops is crucial. Its path algebra over K is the free algebra $K\langle e_1, e_2\rangle$, and it is of wild representation type. The wildness implies that for any finite-dimensional K-algebra A—including our division algebra D—there exists some representation of the quiver whose image algebra is A.

However, this existence result from representation theory is too abstract, since we need more information about the representation.

We define an absolutely irreducible representation M of Γ over the field L as follows. Put $M(v) = L^n$ and define the maps for the loops via the splitting isomorphism Φ:

$$\phi_{e_1} := \Phi(1 \otimes d_1) \in M_n(L) \quad \text{and} \quad \phi_{e_2} := \Phi(1 \otimes d_2) \in M_n(L),$$

considered as endomorphisms of $M(v) = L^n$.

Proposition 4.2. *The representation M is absolutely irreducible over L.*

Proof. Since d_1 and d_2 generate D as a K-algebra, the matrices ϕ_{e_1} and ϕ_{e_2} generate the L-algebra $\Phi(L \otimes_K D) = M_n(L)$. An invariant subspace of L^n is therefore an $M_n(L)$-submodule. As the standard module L^n is irreducible over $M_n(L)$, M is an irreducible representation. This argument is valid over any extension of L, so M is absolutely irreducible. $\square$

Let M_K be the K-rational representation corresponding to M obtained by restriction of scalars, i. e. we consider $M(v)$ as K-vector space $M_K(v)$ and ϕ_{e_i} as K-linear endomorphisms of $M_K(v)$.

Theorem 4.3. *The endomorphism ring of M_K in the category of K-rational representations is the opposite algebra of D:*

$$\mathrm{End}_{\Gamma_K}(M_K) \cong D^{\mathrm{op}}.$$

Proof. The K-rational representation M_K corresponds to the pair

$$(M, (\varphi_\sigma)_{\sigma\in\mathrm{Gal}(L/K)}),$$

where the semi-linear maps φ_σ are the standard entry-wise action of $\mathrm{Gal}(L/K)$ on L^n. An endomorphism of M_K is an endomorphism of the underlying module over the K-path algebra $P_K(\Gamma)$. The endomorphism ring is the centralizer of the image of this path algebra in $\mathrm{End}_K(L^n)$.

Let $A = K\langle \phi_{e_1}, \phi_{e_2}\rangle \subset M_n(L)$ be the K-algebra generated by the representation maps. By construction, $A \cong D$. The endomorphism ring is the centralizer of this simple subalgebra:

$$\mathrm{End}_{\Gamma_K}(M_K) = C_{\mathrm{End}_K(L^n)}(A).$$

By the Double Centralizer Theorem, this centralizer is isomorphic to $A^{\mathrm{op}} \cong D^{\mathrm{op}}$, hence $\mathrm{End}_{\Gamma_K}(M_K) \cong D^{\mathrm{op}}$. $\square$

This construction realizes the Brauer obstruction given by D: M does not descend to K unless $D = K$, the corresponding obstruction is $[D] \in H^2(\mathrm{Gal}(L/K); L^\times)$.

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