

Nonresonant renormalization scheme for twist-2 operators in $\mathcal{N} = 1$ SUSY $SU(N)$ Yang-Mills theory

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ABSTRACT: The short-distance asymptotics of the generating functional for n -point correlators of twist-2 operators in $\mathcal{N} = 1$ supersymmetric (SUSY) $SU(N)$ Yang-Mills (SYM) theory were recently calculated in [1, 2]. This calculation depends on a change of basis for renormalized twist-2 operators, in which $-\gamma(g)/\beta(g)$ reduces to $\gamma_0/(\beta_0 g)$ at all orders in perturbation theory, where γ_0 is diagonal, $\gamma(g) = \gamma_0 g^2 + \dots$ is the anomalous-dimension matrix, and $\beta(g) = -\beta_0 g^3 + \dots$ is the beta function. The method is founded on a new geometric interpretation of operator mixing [3], assuming that the eigenvalues of the matrix γ_0/β_0 meet the nonresonant condition $\lambda_i - \lambda_j \neq 2k$, with the eigenvalues λ_i ordered nonincreasingly and $k \in \mathbb{N}^+$. This nonresonant condition was numerically verified for i, j up to 10^4 in [1, 2]. In this work, we employ techniques initially developed in [4] to present a number-theoretic proof of the nonresonant condition for twist-2 operators, fundamentally based on the classic result that Harmonic numbers are not integers.

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1 Introduction

Recently, the ultraviolet (UV) asymptotics of the generating functional of correlators of twist-2 operators in $SU(N)$ SYM theory was explicitly computed for the first time [1, 2]. This result establishes strong UV constraints on the anticipated nonperturbative solution of large- N SYM theory and could serve as an essential guide in the search for this solution [1, 2].

Moreover, the aforementioned computation has also led to a refinement of the 't Hooft topological expansion in large- N $SU(N)$ pure YM theory [5] that is intimately connected with the corresponding nonperturbative effective theory of glueballs [5, 6].

A critical tool for performing this calculation is a change of basis of renormalized twist-2 operators, in which the renormalized mixing matrix $Z(\lambda)$ defined in Eq. (1.5) becomes diagonal and one-loop exact to all orders of perturbation theory. This is achieved through a new geometric interpretation of operator mixing [3], which we outline below.

As noted in the introduction of [7], a change of renormalization scheme can, in general, involve both a reparametrization of the coupling—which alters the beta function $\beta(g) = -\beta_0 g^3 + \dots$, with $g \equiv g(\mu)$ the renormalized coupling—and a change in the basis of the operators that mix under renormalization, which modifies the anomalous dimension matrix $\gamma(g) = \gamma_0 g^2 + \dots$.

In this paper, we focus exclusively on a change of the operator basis [3], while holding the renormalization scheme for $\beta(g)$ fixed, for instance, in the $\overline{\text{MS}}$ scheme.

Naturally, this change of basis also influences the ratio $-\frac{\gamma(g)}{\beta(g)}$, with $\beta(g)$ fixed, in a manner that we will detail shortly.

In the case of operator mixing, the renormalized Euclidean correlators

$$\langle \mathcal{O}_{k_1}(x_1) \dots \mathcal{O}_{k_n}(x_n) \rangle = G_{k_1 \dots k_n}^{(n)}(x_1, \dots, x_n; \mu, g(\mu)) \quad (1.1)$$

satisfy the Callan-Symanzik equation

$$\begin{aligned} & \left(\sum_{\alpha=1}^n x_\alpha \cdot \frac{\partial}{\partial x_\alpha} + \beta(g) \frac{\partial}{\partial g} + \sum_{\alpha=1}^n D_{\mathcal{O}_\alpha} \right) G_{k_1 \dots k_n}^{(n)} + \\ & + \sum_a \left(\gamma_{k_1 a}(g) G_{a k_2 \dots k_n}^{(n)} + \gamma_{k_2 a}(g) G_{k_1 a k_3 \dots k_n}^{(n)} \dots + \gamma_{k_n a}(g) G_{k_1 \dots a}^{(n)} \right) = 0, \end{aligned} \quad (1.2)$$

with the solution

$$\begin{aligned} & G_{k_1 \dots k_n}^{(n)}(\lambda x_1, \dots, \lambda x_n; \mu, g(\mu)) \\ & = \sum_{j_1 \dots j_n} Z_{k_1 j_1}(\lambda) \dots Z_{k_n j_n}(\lambda) \lambda^{-\sum_{i=1}^n D_{\mathcal{O}_{j_i}}} G_{j_1 \dots j_n}^{(n)}(x_1, \dots, x_n; \mu, g(\frac{\mu}{\lambda})), \end{aligned} \quad (1.3)$$

where $D_{\mathcal{O}_i}$ is the canonical dimension of $\mathcal{O}_i(x)$, and

$$\left(\frac{\partial}{\partial g} + \frac{\gamma(g)}{\beta(g)} \right) Z(\lambda) = 0 \quad (1.4)$$

in matrix notation, and

$$Z(\lambda) = P \exp \left(\int_{g(\mu)}^{g(\frac{\mu}{\lambda})} \frac{\gamma(g')}{\beta(g')} dg' \right). \quad (1.5)$$

The question arises whether a basis of renormalized operators exists where $Z(\lambda)$ becomes diagonal, so that Eq. (1.3) is greatly simplified, reducing to a single term.

In essence, to address this question, we interpret [3] a finite change of renormalization scheme—that is, a change in the basis of renormalized operators expressed in matrix notation

$$\mathcal{O}'(x) = S(g) \mathcal{O}(x) \quad (1.6)$$

as a formal real-analytic invertible gauge transformation $S(g)$ ¹ [3]. Under the action of $S(g)$, the matrix

$$A(g) = -\frac{\gamma(g)}{\beta(g)} = \frac{1}{g} \left(\frac{\gamma_0}{\beta_0} + \dots \right) \quad (1.7)$$

associated with the differential equation for $Z(\lambda)$

$$\left(\frac{\partial}{\partial g} - A(g) \right) Z(\lambda) = 0 \quad (1.8)$$

can be seen as a connection $A(g)$

$$A(g) = \frac{1}{g} \left(A_0 + \sum_{n=1}^{\infty} A_{2n} g^{2n} \right), \quad (1.9)$$

with a regular singularity at $g = 0$ that transforms as

$$A'(g) = S(g)A(g)S^{-1}(g) + \frac{\partial S(g)}{\partial g} S^{-1}(g), \quad (1.10)$$

with

$$\mathcal{D} = \frac{\partial}{\partial g} - A(g) \quad (1.11)$$

as the corresponding covariant derivative. As a result, $Z(\lambda)$ can be interpreted as a Wilson line that transforms as

$$Z'(\lambda) = S(g(\mu))Z(\lambda)S^{-1}(g(\frac{\mu}{\lambda})). \quad (1.12)$$

Theorem 1. [3] *If the matrix $\frac{\gamma_0}{\beta_0}$ is diagonalizable and nonresonant, i.e., its eigenvalues, ordered nonincreasingly, $\lambda_1, \lambda_2, \dots$ satisfy*

$$\lambda_i - \lambda_j \neq 2k, \quad i > j, \quad k \in \mathbb{N}^+, \quad (1.13)$$

then a formal holomorphic gauge transformation $S(g)$ exists that puts $A(g)$ into the canonical nonresonant form

$$A'(g) = \frac{\gamma_0}{\beta_0} \frac{1}{g} \quad (1.14)$$

which is one-loop exact to all orders of perturbation theory. Consequently, $Z(\lambda)$ is diagonalizable as well, with eigenvalues

$$Z_{\mathcal{O}_i}(\lambda) = \left(\frac{g(\mu)}{g(\frac{\mu}{\lambda})} \right)^{\frac{\gamma_{0\mathcal{O}_i}}{\beta_0}}, \quad (1.15)$$

where $\gamma_{0\mathcal{O}_i}$ are the eigenvalues of γ_0 .

In this paper, we demonstrate for twist-2 operators in $SU(N)$ SYM theory that the matrix $\frac{\gamma_0}{\beta_0}$, which is already known to be diagonal [8, 9], fulfills the aforementioned nonresonant condition, thereby proving the existence of the corresponding diagonal nonresonant renormalization scheme.

¹Obviously, in this context the gauge transformation $S(g)$ only depends on the coupling g and it has nothing to do with the spacetime gauge group of the theory.

2 Plan of the paper

Section 3 defines the twist-2 operators in $SU(N)$ SYM theory and presents their one-loop anomalous dimensions [10].

Section 4 reviews key number-theoretic concepts, such as the p -adic order and the classical proof demonstrating that the Harmonic numbers H_n are not integers.

Section 5 provides the proof of the nonresonant condition for all twist-2 operators in $SU(N)$ SYM theory.

3 Anomalous dimensions of twist-2 operators in SYM theory

3.1 Twist-2 operators in SYM theory

In the standard basis [10–12], the gauge-invariant collinear twist-2 operators in the light-cone gauge that respectively appear as components of the balanced and unbalanced superfields [13]², are given by

$$\begin{aligned}
O_s^A &= \frac{1}{2} \partial_+ \bar{A}^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-2} C_{s-2}^{\frac{5}{2}} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \partial_+ A^a \\
\tilde{O}_s^A &= \frac{1}{2} \partial_+ \bar{A}^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-2} C_{s-2}^{\frac{5}{2}} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \partial_+ A^a \\
O_s^\lambda &= \frac{1}{2} \bar{\lambda}^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-1} C_{s-1}^{\frac{3}{2}} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \lambda^a \\
\tilde{O}_s^\lambda &= \frac{1}{2} \bar{\lambda}^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-1} C_{s-1}^{\frac{3}{2}} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \lambda^a \\
M_s &= \frac{1}{2} \partial_+ A^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-1} P_{s-1}^{(2,1)} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \lambda^a \\
\bar{M}_s &= \frac{1}{2} \bar{\lambda}^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-1} P_{s-1}^{(1,2)} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \partial_+ \bar{A}^a,
\end{aligned} \tag{3.1}$$

²We refer to composite superfields made of two elementary superfields of opposite chirality as balanced, and to those made of two elementary superfields with the same chirality as unbalanced.

where O_s^A and O_s^λ are even spin operators while $\tilde{O}_s^A$ and $\tilde{O}_s^\lambda$ are odd spin operators. For the unbalanced operators

$$\begin{aligned}
S_s^A &= \frac{1}{2\sqrt{2}} \partial_+ \bar{A}^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-2} C_{s-2}^{\frac{5}{2}} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \partial_+ \bar{A}^a \\
\bar{S}_s^A &= \frac{1}{2\sqrt{2}} \partial_+ A^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-2} C_{s-2}^{\frac{5}{2}} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \partial_+ A^a \\
S_s^\lambda &= \frac{1}{2\sqrt{2}} \bar{\lambda}^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-1} C_{s-1}^{\frac{3}{2}} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \bar{\lambda}^a \\
\bar{S}_s^\lambda &= \frac{1}{2\sqrt{2}} \lambda^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-1} C_{s-1}^{\frac{3}{2}} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \lambda^a \\
T_s &= \frac{1}{2} \lambda^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-1} P_{s-1}^{(1,2)} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \partial_+ \bar{A}^a \\
\bar{T}_s &= \frac{1}{2} \partial_+ A^a (i \vec{\partial}_+ + i \overleftarrow{\partial}_+)^{s-1} P_{s-1}^{(2,1)} \left(\frac{\vec{\partial}_+ - \overleftarrow{\partial}_+}{\vec{\partial}_+ + \overleftarrow{\partial}_+} \right) \bar{\lambda}^a,
\end{aligned} \tag{3.2}$$

with S_s^A and S_s^λ being even spin operators and where $C_l^\alpha(x)$ are the Gegenbauer polynomials [9]. These operators represent the restriction to components with the maximal spin projection s along the p_+ direction of linear combinations of twist-2 operators of the form

$$\begin{aligned}
O_s^{A\mathcal{T}=2} &= \text{Tr } F_{(\rho_1}^\mu \overleftarrow{D}_{\rho_2} \dots \overrightarrow{D}_{\rho_{s-1}} F_{\rho_s)\mu} - \text{traces} \\
\tilde{O}_s^{A\mathcal{T}=2} &= \text{Tr } \tilde{F}_{(\rho_1}^\mu \overleftarrow{D}_{\rho_2} \dots \overrightarrow{D}_{\rho_{s-1}} F_{\rho_s)\mu} - \text{traces} \\
O_s^{\lambda\mathcal{T}=2} &= \text{Tr } \bar{\chi} \gamma_{(\rho_1} \overleftarrow{D}_{\rho_2} \dots \overrightarrow{D}_{\rho_{s-1}} \chi - \text{traces} \\
\tilde{O}_s^{\lambda\mathcal{T}=2} &= \text{Tr } \bar{\chi} \gamma_{(\rho_1} \gamma_5 \overleftarrow{D}_{\rho_2} \dots \overrightarrow{D}_{\rho_{s-1}} \chi - \text{traces} \\
M_s^{\mathcal{T}=2} &= \text{Tr } F_{(\rho_1}^\nu \overleftarrow{D}_{\rho_2} \dots \overrightarrow{D}_{\rho_{s-1}} \sigma_\nu \lambda - \text{traces} \\
\bar{M}_s^{\mathcal{T}=2} &= \text{Tr } \bar{\lambda} \sigma_\nu \overleftarrow{D}_{(\rho_{s-1}} \dots \overrightarrow{D}_{\rho_2} F_{\rho_1)}^\nu - \text{traces} \\
\mathbb{S}_s^{A\mathcal{T}=2} &= \text{Tr } (F_{\mu(\nu} + i \tilde{F}_{\mu(\nu} \overleftarrow{D}_{\rho_1} \dots \overrightarrow{D}_{\rho_{s-2}} (F_{\lambda)\sigma} + i \tilde{F}_{\lambda)\sigma}) - \text{traces} \\
S_s^{\lambda\mathcal{T}=2} &= \text{Tr } \bar{\chi} \sigma_{\mu(\rho_1} \overleftarrow{D}_{\rho_2} \dots \overrightarrow{D}_{\rho_{s-1}} \chi - \text{traces} \\
T_{s+\frac{1}{2}}^{\mathcal{T}=2} &= \text{Tr } F_{(\rho_1}^\nu \overleftarrow{D}_{\rho_2} \dots \overrightarrow{D}_{\rho_{s-1}} \sigma_{\rho_s)\nu} \chi - \text{traces},
\end{aligned} \tag{3.3}$$

including all possible combinations of right and left derivatives [14, 15], where the parentheses indicate symmetrization of the enclosed indices and the trace subtraction ensures that any two-index contraction vanishes.

To the leading order in perturbation theory, appropriate linear combinations of these twist-2 operators are conserved [14, 15], and they automatically transform as primary operators under the conformal group [14–16].

3.2 Balanced and unbalanced superfields

For the balanced superfields we get [13]

$$\mathbb{W}_s(x, \theta, \bar{\theta}) \sim \mathbb{S}_{s+1}^{(2)} + \theta \bar{M}_{s+1} + \bar{\theta} M_{s+1} + \theta \bar{\theta} \mathbb{S}_{s+2}^{(1)}, \tag{3.4}$$

where $\mathbb{S}^{(i)} = \{S^{(i)}, \tilde{S}^{(i)}\}$ include both even- and odd-spin operators. For even spin with $s \geq 2$

$$\begin{aligned} S_s^{(1)} &= \frac{6}{s-1} O_s^A - O_s^\lambda \\ S_s^{(2)} &= \frac{6}{s+2} O_s^A + O_s^\lambda \end{aligned} \quad (3.5)$$

and for odd spin with $s \geq 3$

$$\begin{aligned} \tilde{S}_s^{(1)} &= -\frac{6}{s-1} \tilde{O}_s^A - \tilde{O}_s^\lambda \\ \tilde{S}_s^{(2)} &= -\frac{6}{s+2} \tilde{O}_s^A + \tilde{O}_s^\lambda, \end{aligned} \quad (3.6)$$

where $\tilde{O}_s^A$ is not defined for $s = 1$, whereas $\tilde{O}_1^\lambda$ is defined, and $\tilde{S}_1^{(2)} = \tilde{O}_1^\lambda$.

These operators also diagonalize the anomalous dimension matrix to order g^2 , where SYM theory is conformally invariant in the conformal scheme [9].

Similarly, we get for the unbalanced superfields [13]

$$\mathbb{W}_s^+(x, \theta, \bar{\theta}) \sim T_{s-1} + \theta S_s^A + \bar{\theta} \tilde{S}_s^\lambda + \theta \bar{\theta} T_s + \theta \bar{\theta} \partial_+ T_{s-1} \quad (3.7)$$

and

$$\mathbb{W}_s^-(x, \theta, \bar{\theta}) \sim \bar{T}_{s-1} + \theta \bar{S}_s^A + \bar{\theta} S_s^\lambda + \theta \bar{\theta} \bar{T}_s + \theta \bar{\theta} \partial_+ \bar{T}_{s-1}. \quad (3.8)$$

These operators also diagonalize the anomalous dimension matrix to order g^2 [13].

3.3 Anomalous dimensions

The maximal-spin components of the operators $\mathcal{O}_s$ mentioned above only mix with derivatives along the p_+ direction of operators of the same type but with lower spin and identical canonical dimensions [8, 9]. We define the bare operators for $s \geq k$ as

$$\mathcal{O}_s^{B(k)} = (i\partial_+)^k \mathcal{O}_s^B \quad (3.9)$$

which, at the leading order of perturbation theory and for $k > 0$, are conformal descendants of the corresponding primary conformal operator $\mathcal{O}_s^{B(0)} = \mathcal{O}_s^B$. As a result of operator mixing, we obtain for the renormalized operators [8, 9]

$$\mathcal{O}_s^{(k)} = \sum_{s \geq i \geq 2} Z_{si} \mathcal{O}_i^{B(k+s-i)}, \quad (3.10)$$

where the bare mixing matrix Z is lower triangular³ [8, 9].

Therefore, the anomalous-dimension matrix $\gamma(g)$ is generally lower triangular, though γ_0 is diagonal. The eigenvalues of γ_0 for $\mathcal{O}_s = S_s^{(1)}, S_s^{(2)}, \tilde{S}_s^{(1)}, \tilde{S}_s^{(2)}, M_s, \bar{M}_s$ are given by [10]

$$\gamma_0 \mathcal{O}_s = \frac{1}{4\pi^2} \left(\tilde{\gamma}_{\mathcal{O}_s}^0 - \frac{3}{2} \right) \mathcal{O}_s \quad (3.11)$$

³ Z , which in dimensional regularization depends on g and ϵ , should not be confused with $Z(\lambda)$.

with

$$\begin{aligned}\tilde{\gamma}_{0S_s^{(1)}} &= \psi(s+2) + \psi(s-1) - 2\psi(1) - \frac{2(-1)^s}{(s+1)s(s-1)} \\ \tilde{\gamma}_{0S_s^{(2)}} &= \psi(s+3) + \psi(s) - 2\psi(1) + \frac{2(-1)^s}{(s+2)(s+1)s}\end{aligned}\quad (3.12)$$

and [10, 12]

$$\begin{aligned}\tilde{\gamma}_{0\bar{S}_s^{(i)}} &= \tilde{\gamma}_{0S_s^{(i)}} \\ \tilde{\gamma}_{0M_s} &= \tilde{\gamma}_{0S_s^{(2)}}\end{aligned}\quad (3.13)$$

Besides [10],

$$\gamma_{0\bar{\mathcal{O}}_1} = \frac{1}{4\pi^2} \frac{2}{3} = \frac{1}{6\pi^2}. \quad (3.14)$$

For the operators $\mathcal{O} = S^A, \bar{S}^A, S^\lambda, \bar{S}^\lambda, T, \bar{T}$, γ_0 is diagonal, with eigenvalues [10]

$$\gamma_{0\mathcal{O}_s} = \frac{1}{4\pi^2} \left(\tilde{\gamma}_{0\mathcal{O}_s} - \frac{3}{2} \right), \quad (3.15)$$

where [10]

$$\begin{aligned}\tilde{\gamma}_{0S_s^A} &= 2\psi(s+1) - 2\psi(1) \\ \tilde{\gamma}_{0S_s^\lambda} &= 2\psi(s+1) - 2\psi(1)\end{aligned}\quad (3.16)$$

and [10, 12]

$$\tilde{\gamma}_{0T_s} = \begin{cases} \psi(s+1) - \psi(1), & s = 2, 4, \dots \\ \psi(s+2) - \psi(1), & s = 1, 3, \dots \end{cases} \quad (3.17)$$

We have numerically verified that the initial 10^4 eigenvalues of $\frac{\gamma_{0\mathcal{O}_s}}{\beta_0}$ are nonresonant, with $\beta_0 = \frac{3}{(4\pi)^2}$.

The eigenvalues of γ_0 are naturally ordered in increasing sequence with increasing s , which is contrary to the ordering in Theorem 1. However, it can be easily seen from the proof in [3] that for this case, the nonresonant condition takes the form

$$\lambda_j - \lambda_i \neq 2k, \quad j > i, \quad k \in \mathbb{N}^+, \quad (3.18)$$

with $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$

4 Number-theoretic concepts

4.1 p -adic order

The p -adic order of an integer n is defined as the exponent of the highest power of a prime number p that divides n [17]. Specifically, the p -adic order of an integer is the function

$$\nu_p(n) = \begin{cases} \max\{k \in \mathbb{N} : p^k \text{ divides } n\} & \text{if } n \neq 0 \\ \infty & \text{if } n = 0. \end{cases} \quad (4.1)$$

For example, $\nu_3(24) = \nu_3(3 \times 2^3) = 1$ and $\nu_2(24) = 3$.

This concept can be extended to rational numbers through the property [17]

$$\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b). \quad (4.2)$$

Consequently, rational numbers may have a negative p -adic order, whereas integers can only have non-negative values for any prime p . Additional properties include [17]

$$\begin{aligned} \nu_p(a \cdot b) &= \nu_p(a) + \nu_p(b) \\ \nu_p(a + b) &\geq \min\{\nu_p(a), \nu_p(b)\}. \end{aligned} \quad (4.3)$$

When $\nu_p(a) \neq \nu_p(b)$

$$\nu_p(a + b) = \min\{\nu_p(a), \nu_p(b)\} \quad (4.4)$$

[17], a fact which is crucial for the subsequent proof.

4.2 Harmonic numbers and Bertrand's postulate

Bertrand's postulate⁴ [18, 19] asserts that for any real number $x \geq 2$, there is at least one prime number p satisfying

$$\frac{x}{2} + 1 \leq p \leq x. \quad (4.5)$$

This implies that for any prime $p \in [\frac{x}{2} + 1, x]$, its double, $2p$, cannot lie within the same interval, since $2p \geq x + 2$.

We will apply Bertrand's postulate to prove the classical result that Harmonic numbers, H_n ,

$$H_n = \sum_{k=1}^n \frac{1}{k}, \quad (4.6)$$

are never integers for any $n \geq 2$.

4.3 Standard argument

Let p be a prime number within the interval

$$\frac{n}{2} + 1 \leq p \leq n. \quad (4.7)$$

For such a prime p , the term $\frac{1}{p}$ appears in the summation of Eq. (4.6). However, no term with $k > p$ can have p as a prime factor, as its prime factorization would have to include at least $2p$, which falls outside the specified interval. Thus, with the exception of $\frac{1}{p}$, the denominator k of every term $\frac{1}{k}$ in the sum is divisible only by primes other than p . Consequently, if we write the sum as

$$\sum_{k=1}^n \frac{1}{k} = \frac{1}{p} + \frac{a}{b}, \quad (4.8)$$

⁴It is actually a theorem.

then the denominator b is not divisible by p , i.e., $\gcd(b, p) = 1$. This leads to the conclusion that Harmonic numbers are not integers. To be more precise,

$$\nu_p\left(\frac{1}{p}\right) = -1 \quad \text{while} \quad \nu_p\left(\frac{a}{b}\right) = \nu_p(a) > 0. \quad (4.9)$$

Then, observing that $\nu_p\left(\frac{1}{p}\right) \neq \nu_p\left(\frac{a}{b}\right)$, it follows from Eq. (4.4)

$$\nu_p\left(\frac{1}{p} + \frac{a}{b}\right) = \min\left(\nu_p\left(\frac{1}{p}\right), \nu_p\left(\frac{a}{b}\right)\right) = -1 \quad (4.10)$$

and, finally,

$$\nu_p(H_n) = \nu_p\left(\frac{1}{p} + \frac{a}{b}\right) = \min\left(\nu_p\left(\frac{1}{p}\right), \nu_p\left(\frac{a}{b}\right)\right) = -1. \quad (4.11)$$

Thus, because the p -adic order of H_n is negative, H_n cannot be an integer.

This line of reasoning will be referred to as the *standard argument*, since it will be used multiple times in the subsequent sections.

4.4 Generalized argument

We now consider sums that are more complex than the Harmonic numbers

$$\Omega_n = \sum_{k=1}^n \frac{c_k}{k}. \quad (4.12)$$

We start with the case where the coefficients c_k can assume positive or negative values, such as ± 1 or ± 2 .

The coefficients ± 1 clearly do not affect the standard argument, as 1 is coprime with any prime p . If p is found by Bertrand's postulate, there is no $k \neq p$ in the sum that has p in its prime factorization. Therefore, Ω_n is not an integer according to the standard argument.

Slightly more caution is required when ± 2 appears in the numerators. For $n \geq 3$, Bertrand's postulate guarantees the existence of a prime $p \geq 3$ satisfying Eq. (4.7) such that

$$\Omega_n = \frac{c_p}{p} + \frac{a}{b} \quad (4.13)$$

where $\gcd(b, p) = \gcd(c_p, p) = 1$. In this scenario, c_p and p are indeed coprime and, as before, no other denominator k in the sum shares p as a prime factor, even if other coefficients c_k take values $\pm 1, \pm 2$.

For instance, let us suppose there are terms with denominators

$$k_1 = p' \quad (4.14)$$

and

$$k_2 = 2p' \quad (4.15)$$

such that $c_{k_2} = \pm 2$; then the terms $\frac{c_{k_1}}{k_1} = \frac{c_{p'}}{p'}$ and $\frac{\pm 2}{k_2} = \frac{\pm 1}{p'}$ would combine to form $\frac{c_{p'}}{p'}$, where $c_{p'}$ could potentially be zero. However, according to Eqs. (4.14) and (4.15), p' is not

one of the primes identified by Bertrand's postulate. Therefore, its potential absence from the sum does not impact the standard argument, which remains valid.

More broadly, if $c_k \in \mathbb{Z} \setminus \{0\}$ and if a prime p can be found that satisfies Eq. (4.7) and $\gcd(c_p, p) = 1$, then the standard argument remains applicable. This is because, according to Bertrand's postulate, all terms in the sum other than $\frac{c_p}{p}$ will combine into a fraction whose denominator is not divisible by p . It is clear by the nature of the sum in Eq. (4.12) that there is an implicit condition on n that must be satisfied, namely, given a certain $c_k \in \mathbb{Z} \setminus \{0\}$, the value of n must be such that a suitable p exists for the requirement $\gcd(c_p, p) = 1$ to be fulfilled. This condition on n is something that must be checked on a case by case situation for every proof given the sequence c_k .

Finally, we examine the most complex case, where some coefficients c_k are permitted to be zero. For the argument to hold, there must be at least one prime p satisfying Eq. (4.7) for which $c_p \neq 0$ and $\gcd(c_p, p) = 1$. When this condition is met, even with an arbitrary number of zero coefficients c_k , the standard argument can still be applied to demonstrate that K_n is not an integer.

Evidently, verifying this condition requires a direct, case-by-case inspection of the sum.

5 Proof of the nonresonant condition

This section provides the proof that the eigenvalues of the anomalous dimensions for the aforementioned twist-2 operators are nonresonant, as defined in Eq. (3.18).

We begin by recalling that the digamma function can be expressed as

$$\psi(n+1) = H_n - \gamma, \quad (5.1)$$

where γ is the Euler-Mascheroni constant.

5.1 Nonresonant condition for unbalanced twist-2 operators

Using Eq. (5.1), we express the anomalous dimension of S_s^A and S_s^λ in a more suitable form

$$\gamma_{0n}^{S^A} = \gamma_{0n}^{S^\lambda} = \frac{2}{(4\pi)^2} \left(2H_n - \frac{3}{2} \right), \quad (5.2)$$

with $n = 2, 4, 6, \dots$

Lemma 1. *The sequence $\gamma_{0n}^{S^A}$ is monotonically increasing*

$$\gamma_{0n+1}^{S^A} \geq \gamma_{0n}^{S^A} \quad (5.3)$$

Proof.

$$\gamma_{0n+1}^{S^A} - \gamma_{0n}^{S^A} = \frac{4}{(4\pi)^2} \frac{1}{n+1} > 0. \quad (5.4)$$

□

Therefore, the sequence $\gamma_{0n}^{S^A}$ (and $\gamma_{0n}^{S^\lambda}$) is increasing and matches the ordering in Eq. (3.18).

Theorem 2. *The eigenvalues of $\frac{\gamma_0^{S^{A,\lambda}}}{\beta_0}$ are nonresonant*

$$\frac{\gamma_{0n}^{S^{A,\lambda}} - \gamma_{0m}^{S^{A,\lambda}}}{\beta_0} \neq 2k, \quad k \in \mathbb{N}^+, \quad \forall n > m \geq 2, \quad (5.5)$$

where $\beta_0 = \frac{3}{(4\pi)^2}$.

Proof.

Let us set $n = m + x$, with $x \geq 1$ being a natural number. Equation (5.5) can then be written as

$$\Delta_m^{S^{A,\lambda}}(x) = \frac{\gamma_{0m+x}^{S^{A,\lambda}} - \gamma_{0m}^{S^{A,\lambda}}}{\beta_0} = \frac{4}{3} \sum_{k=m+1}^{m+x} \frac{1}{k} = \frac{4}{3} \Sigma_m(x), \quad (5.6)$$

with

$$\Sigma_m(x) = \sum_{k=m+1}^{m+x} \frac{1}{k}. \quad (5.7)$$

We parametrize x as

$$x = m + t \quad t \geq 0, \quad (5.8)$$

which leaves out the first part of the proof for all possible values of $x < m$; we will consider these later. Hence,

$$\Delta_m^{S^{A,\lambda}}(m+t) = \frac{4}{3} \sum_{k=m+1}^{2m+t} \frac{1}{k}. \quad (5.9)$$

By Bertrand's postulate, there again exists a prime p in the interval

$$m+1 + \frac{t}{2} \leq p \leq 2m+t. \quad (5.10)$$

Therefore, applying the standard argument yields

$$\nu_p(\Sigma_m(m+t)) = -1. \quad (5.11)$$

Thus, using Eqs. (4.2) and (4.3)

$$\begin{aligned} \nu_p\left(\frac{4}{3}\Sigma_m(m+t)\right) &= \nu_p\left(\frac{4}{3}\right) + \nu_p(\Sigma_m(m+t)) \\ &= \nu_p(4) - \nu_p(3) - 1. \end{aligned} \quad (5.12)$$

Then, for $m > 2$ or for $m = 2$ with $t > 0$, Eq. (5.10) implies $p \geq 5$, so that $\nu_p(4) = 0$ and

$$\nu_p\left(\frac{4}{3}\Sigma_m(m+t)\right) < 0. \quad (5.13)$$

For the special case $m = 2$ and $t = 0$, we find $\Sigma_2(2) = \frac{7}{12}$ and $\Delta_2^{S^{A,\lambda}}(2) = \frac{7}{9}$. We thus conclude that for $x \geq m$, the p -adic order of $\nu_p(\Delta_m^{S^{A,\lambda}}(x)) < 0$, which means it cannot be an integer.

The preceding part of the proof did not consider the case where $x < m$. We now address this case, beginning by demonstrating as in Proposition 1 that

$$\Sigma_m(x) < \log(2) < 1 \quad \forall x < m. \quad (5.14)$$

Hence, Eq. (5.14) implies

$$\Delta_m^{S^{A,\lambda}}(x) = \frac{4}{3}\Sigma_m(x) < \frac{4}{3}\log(2) < 0.93 \quad \forall x < m. \quad (5.15)$$

Therefore, $\Delta_m^{S^{A,\lambda}}(x)$ is strictly less than 1.

In conclusion, $\Delta_m^{S^{A,\lambda}}$ cannot be an integer for $x \geq m$, nor can it be an integer for $x < m$. This completes the proof of the theorem. $\square$

Using Eq. (5.1), we now write the anomalous dimension of T_s in a more suitable form

$$\gamma_{0n}^T = \begin{cases} \frac{2}{(4\pi)^2} \left(H_n - \frac{3}{2}\right), & n = 2, 4, \dots \\ \frac{2}{(4\pi)^2} \left(H_{n+1} - \frac{3}{2}\right), & n = 1, 3, \dots \end{cases} \quad (5.16)$$

Since the even and odd anomalous dimensions share the same functional dependence, we can parametrize them as $n = 2l$ and $n = 2l - 1$ for even and odd spins respectively to obtain the same functional form. We can therefore work directly with

$$\gamma_{0n}^T = \frac{2}{(4\pi)^2} \left(H_{2n} - \frac{3}{2}\right), \quad n = 1, 2, 3, \dots, \quad (5.17)$$

for all values of n .

Lemma 2. *The sequence γ_{0n}^T is monotonically increasing*

$$\gamma_{0n+1}^T \geq \gamma_{0n}^T \quad (5.18)$$

Proof.

$$\gamma_{0n+1}^T - \gamma_{0n}^T = \frac{2}{(4\pi)^2} \left(\frac{1}{2n+1} + \frac{1}{2n+2} \right) > 0. \quad (5.19)$$

$\square$

Therefore, the sequence γ_{0n}^T is increasing and matches the ordering in Eq. (3.18).

Theorem 3. *The eigenvalues of $\frac{\gamma_0^T}{\beta_0}$ are nonresonant*

$$\frac{\gamma_{0n}^T - \gamma_{0m}^T}{\beta_0} \neq 2k, \quad k \in \mathbb{N}^+, \quad \forall n > m \geq 2, \quad (5.20)$$

where $\beta_0 = \frac{3}{(4\pi)^2}$.

Proof.

Let us set $n = m + x$, where $x \geq 1$ is a natural number. Equation (5.20) can then be written as

$$\Delta_m^T(x) = \frac{\gamma_{0m+x}^T - \gamma_{0m}^T}{\beta_0} = \frac{2}{3} \sum_{k=2m+1}^{2m+2x} \frac{1}{k} = \frac{2}{3} \Sigma'_m(x), \quad (5.21)$$

with

$$\Sigma'_m(x) = \sum_{k=2m+1}^{2m+2x} \frac{1}{k}. \quad (5.22)$$

We parametrize x as

$$x = m + t \quad t \geq 0, \quad (5.23)$$

which excludes values of $x < m$ from this part of the proof. Thus,

$$\Delta_m^T(m+t) = \frac{2}{3} \sum_{k=2m+1}^{4m+2t} \frac{1}{k}. \quad (5.24)$$

By Bertrand's postulate, a prime exists in the interval

$$2m+1+t \leq p \leq 4m+2t. \quad (5.25)$$

Applying the standard argument gives

$$\nu_p(\Sigma'_m(m+t)) = -1. \quad (5.26)$$

Using Eqs. (4.2) and (4.3), we find

$$\begin{aligned} \nu_p\left(\frac{2}{3}\Sigma'_m(m+t)\right) &= \nu_p\left(\frac{2}{3}\right) + \nu_p(\Sigma'_m(m+t)) \\ &= \nu_p(2) - \nu_p(3) - 1. \end{aligned} \quad (5.27)$$

For $m > 1$, or for $m = 1$ with $t > 0$, Eq. (5.25) implies $p \geq 5$, meaning $\nu_p(2) = \nu_p(3) = 0$ and

$$\nu_p\left(\frac{2}{3}\Sigma'_m(m+t)\right) < 0. \quad (5.28)$$

For the case $m = 1$ and $t = 0$, $\Sigma'_1(1) = \frac{7}{12}$ and $\Delta_1^T(1) = \frac{7}{18}$. We conclude that for $x \geq m$, the p -adic order of $\nu_p(\Delta_m^T(x))$ is negative, so it cannot be an integer.

We now consider the case where $x < m$. By an argument identical to the one in Proposition 1 we see that

$$\Sigma'_m(x) < \log(2) < 1 \quad \forall x < m. \quad (5.29)$$

Equation (5.29) therefore implies

$$\Delta_m^T(x) = \frac{2}{3}\Sigma'_m(x) < \frac{2}{3}\log(2) < 0.5 \quad \forall x < m. \quad (5.30)$$

Thus, $\Delta_m^T(x)$ is strictly less than 1.

We conclude that Δ_m^T cannot be an integer for $x \geq m$, nor for $x < m$. The theorem is therefore proved. $\square$

5.2 Nonresonant condition for balanced twist-2 operators of even spin

We now examine the anomalous dimension of balanced operators $S_s^{(1)}$ with even spin

$$\gamma_{0n}^{S^{(1)}} = \frac{2}{(4\pi)^2} \left(2H_{n-2} + \frac{3}{n} - \frac{3}{2} \right) \quad (5.31)$$

for $n = 2, 4, 6, \dots$

Lemma 3. *The sequence $\gamma_{0n}^{S^{(1)}}$ is monotonically increasing*

$$\gamma_{0n+1}^{S^{(1)}} \geq \gamma_{0n}^{S^{(1)}} \quad (5.32)$$

Proof.

The difference between consecutive eigenvalues can be written as

$$\gamma_{0n+1}^{S^{(1)}} - \gamma_{0n}^{S^{(1)}} = \frac{2}{(4\pi)^2} \frac{2n(n+1) - (n^2 + n - 3)}{n(n^2 - 1)} > 0. \quad (5.33)$$

Therefore, $\gamma_{0n}^{S^{(1)}}$ is monotonically increasing and aligns with the ordering in Eq. (3.18). $\square$

Theorem 4. *The eigenvalues of $\frac{\gamma_0^{S^{(1)}}}{\beta_0}$ are nonresonant*

$$\frac{\gamma_{0n}^{S^{(1)}} - \gamma_{0m}^{S^{(1)}}}{\beta_0} \neq 2k, \quad k \in \mathbb{N}^+, \quad \forall n > m \geq 2, \quad (5.34)$$

where $\beta_0 = \frac{3}{(4\pi)^2}$.

Proof.

The proof follows a similar structure to that for unbalanced twist-2 operators, though with additional care as outlined in section 4.4.

We again set $n = m + x$, where $x > 0$ is a natural number. The difference of eigenvalues can then be expressed as

$$\begin{aligned} \Delta_m^{S^{(1)}}(x) &= \frac{\gamma_{0m+x}^{S^{(1)}} - \gamma_{0m}^{S^{(1)}}}{\beta_0} \\ &= \frac{2}{3} \left(2 \sum_{k=m-1}^{m-2+x} \frac{1}{k} + \frac{3}{m+x} - \frac{3}{m} \right) \\ &= \frac{2}{3} K_m(x) \end{aligned} \quad (5.35)$$

with

$$K_m(x) = 2 \sum_{k=m-1}^{m-2+x} \frac{1}{k} + \frac{3}{m+x} - \frac{3}{m}. \quad (5.36)$$

This sum clearly matches the form of Eq. (4.12), with coefficients $c_k = \pm 2, \pm 3$ and no gaps. As before, we set

$$x = m + t \quad t \geq 0, \quad (5.37)$$

which excludes a finite number of values $x < m$. For $x \geq m$, we have

$$\Delta_m^{S(1)}(m+t) = \frac{2}{3} \left(2 \sum_{k=m-1}^{2m-2+t} \frac{1}{k} + \frac{3}{2m+t} - \frac{3}{m} \right). \quad (5.38)$$

The generalized argument applies directly for a prime in the interval

$$m + \frac{t}{2} \leq p \leq 2m - 2 + t, \quad (5.39)$$

when $m + \frac{t}{2} > 3$. For $m \leq 3 - \frac{t}{2}$, we must check a small number of cases directly. Notably, for $m = 2$ and $t = 1$, we find $\Delta_2^{S(1)}(3) = 1$. In all other instances, $\Delta_m^{S(1)}(m+t)$ is not an integer. We conclude that $\Delta_m^{S(1)}(m+t)$ is never an integer greater than 1.

We now consider the values of $x < m$. In this case, similar to section 5.1, we show in Proposition 2 that the bound below holds

$$K_m(x) < 2 \log(2), \quad \forall x < m. \quad (5.40)$$

From Eq. (5.40), it follows that

$$\Delta_m^{S(1)}(x) = \frac{2}{3} K_m(x) < \frac{4}{3} \log(2) < 0.93 \quad \forall x < m. \quad (5.41)$$

We conclude that $\Delta_m^{S(1)}$ cannot be an integer greater than 1 for $x \geq m$ and cannot be an integer for $x < m$, which proves the theorem. $\square$

Next, we study the anomalous dimension of balanced operators $S_s^{(2)}$ of even spin

$$\gamma_{0n}^{S(2)} = \frac{2}{(4\pi)^2} \left(2H_n + \frac{2(-1)^n}{(n+2)(n+1)n} - \frac{3}{2} \right) \quad (5.42)$$

with $n = 2, 4, 6, \dots$

Lemma 4. *The sequence $\gamma_{0n}^{S(2)}$ is monotonically increasing*

$$\gamma_{0n+1}^{S(2)} \geq \gamma_{0n}^{S(2)} \quad (5.43)$$

Proof.

The difference of consecutive eigenvalues can be written as

$$\gamma_{0n+1}^{S(2)} - \gamma_{0n}^{S(2)} = \frac{2}{(4\pi)^2} \left(\frac{2}{n+1} + \frac{2(-1)^{n+1}}{(n+3)(n+2)(n+1)} - \frac{2(-1)^n}{(n+2)(n+1)n} \right) > 0. \quad (5.44)$$

Therefore, $\gamma_{0n}^{S(2)}$ increases monotonically and matches the ordering in Eq. (3.18). $\square$

Theorem 5. *The eigenvalues of $\frac{\gamma_0^{S(2)}}{\beta_0}$ are nonresonant*

$$\frac{\gamma_{0n}^{S(2)} - \gamma_{0m}^{S(2)}}{\beta_0} \neq 2k, \quad k \in \mathbb{N}^+, \quad \forall n > m \geq 2, \quad (5.45)$$

where $\beta_0 = \frac{3}{(4\pi)^2}$.

Proof.

The proof is analogous to the previous cases. We set $n = m + x$ for a natural number $x > 0$. The difference of eigenvalues is

$$\begin{aligned}\Delta_m^{S(2)}(x) &= \frac{\gamma_{0m+x}^{S(2)} - \gamma_{0m}^{S(2)}}{\beta_0} \\ &= \frac{2}{3} \left(2 \sum_{k=m+1}^{m+x} \frac{1}{k} + \frac{2(-1)^{m+x}}{(m+x+2)(m+x+1)(m+x)} - \frac{2(-1)^m}{(m+2)(m+1)m} \right) \\ &= \frac{2}{3} J_m(x)\end{aligned}\tag{5.46}$$

with

$$J_m(x) = 2 \sum_{k=m+1}^{m+x} \frac{1}{k} + \frac{2(-1)^{m+x}}{(m+x+2)(m+x+1)(m+x)} - \frac{2(-1)^m}{(m+2)(m+1)m}.\tag{5.47}$$

This sum is of the form in Eq. (4.12). Setting $x = m + t$ for $t \geq 0$, we find

$$\Delta_m^{S(2)}(m+t) = \frac{2}{3} \left(2 \sum_{k=m+1}^{2m+t} \frac{1}{k} + \frac{2(-1)^{2m+t}}{(2m+t+2)(2m+t+1)(2m+t)} - \frac{2(-1)^m}{(m+2)(m+1)m} \right).\tag{5.48}$$

The generalized argument applies to a prime in the interval $m+1 + \frac{t}{2} \leq p \leq 2m+t$. Since the denominators of the fractional terms are polynomials in m and t , they will be coprime with p for sufficiently large m . It can be verified that $\Delta_m^{S(2)}(m+t)$ is never an integer.

For the remaining values $x < m$, we demonstrate in Proposition 3 the bound

$$J_m(x) < 2 \log(2), \quad \forall x < m.\tag{5.49}$$

Equation (5.49) then implies

$$\Delta_m^{S(2)}(x) = \frac{2}{3} J_m(x) < \frac{4}{3} \log(2) < 0.93 \quad \forall x < m.\tag{5.50}$$

Therefore, $\Delta_m^{S(2)}$ cannot be an integer for $x \geq m$ or for $x < m$, which proves the theorem. $\square$

From Eq. (3.13), we conclude that all other anomalous dimensions of balanced operators of even spin also satisfy the nonresonant condition.

5.3 Nonresonant condition for balanced twist-2 operators of odd spin

We now address the anomalous dimension of odd-spin balanced operators. First, for $S_s^{(1)}$,

$$\gamma_{0n}^{S(1)} = \frac{2}{(4\pi)^2} \left(2H_{n-2} + \frac{2}{n-1} - \frac{1}{n} + \frac{2}{n+1} - \frac{3}{2} \right)\tag{5.51}$$

with $n = 3, 5, 7, \dots$

Lemma 5. *The sequence $\gamma_{0n}^{S(1)}$ is monotonically increasing*

$$\gamma_{0n+1}^{S(1)} \geq \gamma_{0n}^{S(1)}\tag{5.52}$$

Proof.

The explicit difference is

$$\gamma_{0n+1}^{S(1)} - \gamma_{0n}^{S(1)} = \frac{2}{(4\pi)^2} \left(\frac{2}{n-1} + \frac{2}{n} - \frac{2}{n+1} - \frac{2}{n+2} \right) > 0. \quad (5.53)$$

Thus, $\gamma_{0n}^{S(1)}$ is a monotonically increasing sequence, matching the ordering in Eq. (3.18). $\square$

Theorem 6. *The eigenvalues of $\frac{\gamma_0^{S(1)}}{\beta_0}$ are nonresonant*

$$\frac{\gamma_{0n}^{S(1)} - \gamma_{0m}^{S(1)}}{\beta_0} \neq 2k, \quad k \in \mathbb{N}^+, \quad \forall n > m \geq 3, \quad (5.54)$$

where $\beta_0 = \frac{3}{(4\pi)^2}$.

Proof.

Following the established procedure, for $n = m + x$ with $x > 0$ a natural number, the difference of eigenvalues is

$$\Delta_m^{S(1)}(x) = \frac{2}{3} \left(2 \sum_{k=m-1}^{m+x-2} \frac{1}{k} + \frac{2(-1)^{m+x-1}}{(m+x)(m+x-1)} - \frac{2(-1)^{m-1}}{m(m-1)} \right) = \frac{2}{3} L_m(x). \quad (5.55)$$

Setting $x = m + t$ for $t \geq 0$ leaves a finite number of cases $x < m$. The generalized argument applies to the case $x \geq m$, confirming non-integrality for large enough m . A direct check handles the few remaining small m cases.

For $x < m$, we show in Proposition 4 that the following bound holds

$$L_m(x) < 2 \log(2) + \frac{3}{2}, \quad \forall x < m. \quad (5.56)$$

This implies that

$$\Delta_m^{S(1)}(x) = \frac{2}{3} L_m(x) < \frac{4}{3} \log(2) + 1 < 1.93, \quad \forall x < m. \quad (5.57)$$

We conclude that $\Delta_m^{S(1)}$ cannot be an integer greater than 1 and in particular it cannot be an even integer, thus proving the nonresonance condition. $\square$

Finally, for the operators $S_s^{(2)}$ of odd spin,

$$\gamma_{0n}^{S(2)} = \frac{2}{(4\pi)^2} \left(2H_{n+1} - \frac{2(-1)^n}{(n+2)(n+1)} - \frac{3}{2} \right) \quad (5.58)$$

with $n = 1, 3, 5, \dots$

Lemma 6. *The sequence $\gamma_{0n}^{S(2)}$ is monotonically increasing*

$$\gamma_{0n+1}^{S(2)} \geq \gamma_{0n}^{S(2)} \quad (5.59)$$

Proof.

The difference is

$$\gamma_{0n+1}^{S(2)} - \gamma_{0n}^{S(2)} = \frac{2}{(4\pi)^2} \left(\frac{2}{n+1} + \frac{2}{n+2} - \frac{2(-1)^{n+1}}{(n+3)(n+2)} + \frac{2(-1)^n}{(n+1)n} \right) > 0. \quad (5.60)$$

Thus, $\gamma_{0n}^{S(2)}$ is monotonically increasing and aligns with the ordering in Eq. (3.18). $\square$

Theorem 7. *The eigenvalues of $\frac{\gamma_0^{S(2)}}{\beta_0}$ are nonresonant*

$$\frac{\gamma_{0n}^{S(2)} - \gamma_{0m}^{S(2)}}{\beta_0} \neq 2k, \quad k \in \mathbb{N}^+, \quad \forall n > m \geq 1, \quad (5.61)$$

where $\beta_0 = \frac{3}{(4\pi)^2}$.

Proof.

The proof strategy remains the same. The difference of eigenvalues

$$\begin{aligned} \Delta_m^{S(2)}(x) &= \frac{2}{3} \left(2 \sum_{k=m+2}^{m+x+1} \frac{1}{k} - \frac{2(-1)^{m+x}}{(m+x+2)(m+x+1)} + \frac{2(-1)^m}{(m+2)(m+1)} \right) \\ &= \frac{2}{3} U_m(x). \end{aligned} \quad (5.62)$$

is not an integer for $x \geq m$ by the generalized argument. For the remaining cases $x < m$, we show in Proposition 5 the bound holds

$$U_m(x) < 2 \log(2), \quad \forall x < m. \quad (5.63)$$

This implies

$$\Delta_m^{S(2)}(x) = \frac{2}{3} U_m(x) < \frac{4}{3} \log(2) < 0.93, \quad \forall x < m. \quad (5.64)$$

We conclude that $\Delta_m^{S(2)}$ cannot be an integer, which completes the proof. $\square$

As in the even-spin case, Eq. (3.13) implies that all other anomalous dimensions of balanced operators of odd spin satisfy the nonresonant condition.

6 Conclusions

We have shown that the eigenvalues of the (diagonal) matrices $\frac{\gamma_0}{\beta_0}$ for the twist-2 operators in SUSY $\mathcal{N} = 1$ $SU(N)$ Yang-Mills theory satisfy the nonresonant condition in **Theorem 1**. Consequently, a nonresonant diagonal scheme exists for all twist-2 operators in this

theory, wherein the renormalized mixing matrices $Z(\lambda)$ from Eq. (1.5) are one-loop exact with eigenvalues [3]

$$Z_{\mathcal{O}_i}(\lambda) = \left(\frac{g(\mu)}{g(\frac{\mu}{\lambda})} \right)^{\frac{\gamma_{\mathcal{O}_i}}{\beta_0}}. \quad (6.1)$$

It can be concluded that the UV asymptotics of the generating functional for correlators, as computed in [1, 2], is applicable to all twist-2 operators within SUSY $\mathcal{N} = 1$ SU(N) Yang-Mills theory.

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A Bounds on sums

A.1 Integral sandwich lemma

Lemma 7. *Let $f : [a - 1, b + 1] \rightarrow \mathbb{R}$ be decreasing and $a, b \in \mathbb{Z}$ with $a \leq b$. Then*

$$\int_a^{b+1} f(x) dx \leq \sum_{k=a}^b f(k) \leq \int_{a-1}^b f(x) dx. \quad (A.1)$$

Proof. For each $k \in \{a, \dots, b\}$ and $x \in [k, k + 1]$, decreasingness gives $f(x) \leq f(k)$, from which we get

$$\int_k^{k+1} f(x) dx \leq f(k) \int_k^{k+1} dx \leq f(k). \quad (A.2)$$

Summing over $k = a, \dots, b$ yields the left inequality in (A.1). Similarly, for $x \in [k - 1, k]$ we have $f(x) \geq f(k)$, so

$$\int_{k-1}^k f(x) dx \geq f(k), \quad (A.3)$$

and summing over $k = a, \dots, b$ gives the right inequality. $\square$

Corollary 1. *Taking $f(x) = 1/x$ (decreasing on $(0, \infty)$) in (A.1) gives, for integers $1 \leq a \leq b$,*

$$\log \frac{b+1}{a} \leq \sum_{k=a}^b \frac{1}{k} \leq \log \frac{b}{a-1}. \quad (A.4)$$

Proof. Integrate $\frac{1}{x}$ to obtain $\int_u^v \frac{dx}{x} = \log v - \log u$. $\square$

A.2 Upper bound for $\Sigma_m(x)$ with $x < m$

Proposition 1. *For all $m \geq 2$ and $x < m$*

$$\Sigma_m(x) = \sum_{k=m+1}^{m+x} \frac{1}{k} < \log 2, \quad (A.5)$$

Proof. From Eq. (A.4),

$$\Sigma_m(x) \leq \log \left(1 + \frac{x}{m} \right) \leq \log 2 \quad (A.6)$$

since $x < m$. $\square$

A.2.1 An alternative route

By noticing that the anomalous dimensions are all monotonically increasing functions we immediately establish that for $x < m$

$$\Sigma_m(x) < \Sigma_m(m-1). \quad (\text{A.7})$$

Now, as it was noticed in [4], we show that also $\Sigma_m(m-1)$ is monotonic in m

$$\begin{aligned} \Sigma_{m+1}(m) - \Sigma_m(m-1) &= \sum_{k=m+2}^{2m+1} \frac{1}{k} - \sum_{k=m+1}^{2m-1} \frac{1}{k} \\ &= \frac{1}{2m+1} + \frac{1}{2m} - \frac{1}{m+1} \\ &= \frac{3m+1}{4m^3+6m^2+2m} > 0 \end{aligned} \quad (\text{A.8})$$

Therefore we can use the bound [4]

$$\Sigma_m(m-1) \leq \lim_{m \rightarrow \infty} \Sigma_m(m-1) = \log 2 \quad (\text{A.9})$$

A.3 Upper bound for $K_m(x)$ with $x < m$

Proposition 2. *For all $m \geq 2$*

$$K_m(x) \leq 2 \log 2. \quad (\text{A.10})$$

Proof. We start from the definition of $K_m(x)$

$$K_m(x) = 2 \sum_{k=m-1}^{m-2+x} \frac{1}{k} + \frac{3}{m+x} - \frac{3}{m} \quad (\text{A.11})$$

then by applying Eq. (A.4) on the first term

$$K_m(x) \leq 2 \log \left(1 + \frac{x}{m-2} \right) + \frac{3}{m+x} - \frac{3}{m}, \quad (\text{A.12})$$

since $1 \leq x < m$ we can easily bound this sum as

$$\frac{3}{m+x} - \frac{3}{m} \leq 0 \quad (\text{A.13})$$

and so

$$K_m(x) \leq 2 \log 2 \quad (\text{A.14})$$

thus completing the proof. $\square$

A.4 Upper bound for $J_m(x)$ for $x < m$

Proposition 3. *For all $m \geq 2$ and $x < m$*

$$J_m(x) < 2 \log 2. \quad (\text{A.15})$$

Proof. From the definition of $J_m(x)$

$$J_m(x) = 2 \sum_{k=m+1}^{m+x} \frac{1}{k} + \frac{2(-1)^{m+x}}{(m+x+2)(m+x+1)(m+x)} - \frac{2(-1)^m}{(m+2)(m+1)m} \quad (\text{A.16})$$

we use as above Eq. (A.4) on the first term

$$J_m(x) \leq 2 \log \left(1 + \frac{x}{m} \right) + \frac{2(-1)^{m+x}}{(m+x+2)(m+x+1)(m+x)} - \frac{2(-1)^m}{(m+2)(m+1)m} \quad (\text{A.17})$$

we use the fact that for $x < m$

$$\log \left(1 + \frac{x}{m} \right) \leq \log \left(1 + \frac{m-1}{m} \right) = \log \left(2 - \frac{1}{m} \right) = \log 2 + \log \left(1 - \frac{1}{2m} \right). \quad (\text{A.18})$$

Using $\log(1-u) \leq -u$ for $u \in (0,1)$,

$$2 \log \left(1 - \frac{x}{2m} \right) \leq 2 \log 2 - \frac{1}{m}. \quad (\text{A.19})$$

For the rational terms we have that in the worst case scenario both $-(-1)^m$ and $(-1)^{m+x}$ yield a positive term and so we consider

$$\begin{aligned} J_m(x) &\leq 2 \log \left(1 - \frac{x}{m} \right) + \frac{2}{(m+x+2)(m+x+1)(m+x)} + \frac{2}{(m+2)(m+1)m} \\ &\leq 2 \log 2 - \frac{1}{m} + \frac{3}{m(m^2+5m+6)} \\ &= 2 \log 2 - \frac{m^2+5m+3}{m(m^2+5m+6)} \\ &\leq 2 \log 2 \end{aligned} \quad (\text{A.20})$$

where in the second line we have put $x = 1$ as a majorant and we have also used Eq. (A.31). $\square$

A.5 Upper bound for $L_m(x)$ with $x < m$

Proposition 4. For all $m \geq 3$ and $x < m$

$$L_m(x) \leq 2 \log 2 + \frac{3}{2}. \quad (\text{A.21})$$

Proof. From the definition of $L_m(x)$

$$L_m(x) = 2 \sum_{k=m-1}^{m+x-2} \frac{1}{k} + \frac{2(-1)^{m+x-1}}{(m+x)(m+x-1)} - \frac{2(-1)^{m-1}}{m(m-1)} \quad (\text{A.22})$$

then by applying Eq. (A.4) on the first term

$$L_m(x) \leq 2 \log \left(1 + \frac{x}{m-2} \right) + \frac{2(-1)^{m+x-1}}{(m+x)(m+x-1)} - \frac{2(-1)^{m-1}}{m(m-1)} \quad (\text{A.23})$$

we use the fact that for $x < m$

$$\log\left(1 + \frac{x}{m-2}\right) \leq \log\left(1 + \frac{m-1}{m-2}\right) = \log\left(2 + \frac{1}{m-2}\right) = \log 2 + \log\left(1 + \frac{1}{2(m-2)}\right). \quad (\text{A.24})$$

Using $\log(1+u) \leq u$ for $u \in (0, 1)$,

$$2 \log\left(1 + \frac{x}{m-2}\right) \leq 2 \log 2 + \frac{1}{m-2}. \quad (\text{A.25})$$

For the rational terms we have that in the worst case scenario both $-(-1)^{m-1}$ and $(-1)^{m+x-1}$ yield a positive term and so we consider

$$\begin{aligned} L_m(x) &\leq 2 \log 2 + \frac{1}{m-2} + \frac{2}{(m+x)(m+x-1)} + \frac{2}{m(m-1)} \\ &\leq 2 \log 2 + \frac{1}{m-2} + \frac{2}{(m+1)m} + \frac{2}{m(m-1)} \\ &= 2 \log 2 + \frac{m(m+4) - 9}{(m-2)(m^2-1)} \\ &\leq 2 \log 2 + \frac{3}{2} \end{aligned} \quad (\text{A.26})$$

□

A.6 Upper bound for $U_m(x)$ for $x < m$

Proposition 5. *For all $m \geq 2$ and $x < m$*

$$U_m(x) < 2 \log 2. \quad (\text{A.27})$$

Proof. From the definition of $U_m(x)$

$$U_m(x) = 2 \sum_{k=m+2}^{m+x+1} \frac{1}{k} - \frac{2(-1)^{m+x}}{(m+x+2)(m+x+1)} + \frac{2(-1)^m}{(m+2)(m+1)} \quad (\text{A.28})$$

we use as above Eq. (A.4) on the first term

$$U_m(x) \leq 2 \log\left(1 + \frac{x}{m+1}\right) - \frac{2(-1)^{m+x}}{(m+x+2)(m+x+1)} + \frac{2(-1)^m}{(m+2)(m+1)} \quad (\text{A.29})$$

we use the fact that for $x < m$

$$\log\left(1 + \frac{x}{m+1}\right) \leq \log\left(1 + \frac{m-1}{m+1}\right) \leq \log\left(1 + \frac{m-1}{m}\right) = \log\left(2 - \frac{1}{m}\right) = \log 2 + \log\left(1 - \frac{1}{2m}\right). \quad (\text{A.30})$$

Using $\log(1-u) \leq -u$ for $u \in (0, 1)$,

$$2 \log\left(1 - \frac{x}{m+1}\right) \leq 2 \log 2 - \frac{1}{m}. \quad (\text{A.31})$$

For the rational terms we have that in the worst case scenario both $(-1)^m$ and $(-1)^{m+x}$ yield a positive term and so we consider

$$\begin{aligned}
U_m(x) &\leq 2 \log \left(1 + \frac{x}{m+1} \right) + \frac{2}{(m+x+2)(m+x+1)} + \frac{2}{(m+2)(m+1)} \\
&\leq 2 \log 2 - \frac{1}{m} + \frac{2}{m^2 + 4m + 3} \\
&= 2 \log 2 - \frac{m^2 + 2m + 3}{m^3 + 4m^2 + 3m} \\
&\leq 2 \log 2
\end{aligned} \tag{A.32}$$

where in the second line we have put $x = 1$ as a majorant and we have also used Eq. (A.31). $\square$

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