

LECTURE NOTES:

BIOLOGICAL PROPAGATION VIA REACTION-DIFFUSION EQUATIONS WITH NONLOCAL DIFFUSION AND FREE BOUNDARY

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ABSTRACT. These notes are based on the lectures given in a mini-course at VIASM (Vietnam Institute for Advanced Study in Mathematics) 2025 Summer School. They give a brief account of the theory (with detailed proofs) for propagation governed by a nonlocal reaction-diffusion model with free boundaries in one space dimension. The main part is concerned with a KPP reaction term, though the basic results on the existence and uniqueness of solutions as well as on the comparison principles are for more general situations. The contents are mostly taken from published recent works of the author with several collaborators, where the kernel function was assumed to be symmetric: $J(x) = J(-x)$. When $J(x)$ is not symmetric, significant differences may arise in the dynamics of the model, as shown in several preprints quoted in the references at the end of these notes, but many of the existing techniques can be easily extended to cover the “weakly non-symmetric case”, and this is done here with all the necessary details.

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1. MAXIMUM PRINCIPLE AND COMPARISON RESULTS

1.1. A maximum principle. Suppose the kernel function $J_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2, \dots, n$) satisfy

$$(\mathbf{J}): J_i \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ is nonnegative, } J_i(0) > 0, \int_{\mathbb{R}} J_i(x) dx = 1, \quad i = 1, 2, \dots, n.$$

The associated nonlocal diffusion operator $\mathcal{L}_i$ is defined by

$$(1.1) \quad \mathcal{L}_i[u](t, x) = \int_{\mathbb{R}} J_i(x - y)u(t, y) dy - u(t, x), \quad i = 1, 2, \dots, n.$$

Let $T > 0$ and $\xi \in C([0, T])$. We define the set of **strict local semi-maximum points** of ξ by

$$\Sigma_{max}^\xi := \{t \in (0, T] : \text{There exists } \epsilon > 0 \text{ such that } \xi(t) > \xi(s) \text{ for } s \in [t - \epsilon, t)\}.$$

Similarly the set of **strict local semi-minimum points** of ξ is given by

$$\Sigma_{min}^\xi := \{t \in (0, T] : \text{There exists } \epsilon > 0 \text{ such that } \xi(t) < \xi(s) \text{ for } s \in [t - \epsilon, t)\}.$$

If ξ is strictly increasing, then clearly $\Sigma_{max}^\xi = (0, T]$, if ξ is nondecreasing, then $\Sigma_{min}^\xi = \emptyset$. In particular, if ξ is a constant function, then $\Sigma_{max}^\xi = \Sigma_{min}^\xi = \emptyset$.

Theorem 1.1 (Maximum Principle). *Let $T, h_0 > 0$, $g, h \in C([0, T])$ satisfy $g(t) < h(t)$ and $-g(0) = h(0) = h_0$. Denote $D_T := \{(t, x) : t \in (0, T], g(t) < x < h(t)\}$ and suppose that for $i, j \in \{1, 2, \dots, n\}$, $\phi_i, \partial_t \phi_i \in C(\overline{D}_T)$, $d_i, c_{ij} \in L^\infty(D_T)$, $d_i \geq 0$, and*

$$(1.2) \quad \begin{cases} \partial_t \phi_i \geq d_i \mathcal{L}_i[\phi_i] + \sum_{j=1}^n c_{ij} \phi_j, & (t, x) \in D_T, \\ \phi_i(t, x) = 0, & t \in (0, T], x \notin [g(t), h(t)], \\ \phi_i(t, g(t)) \geq 0, & t \in \Sigma_{min}^g, \\ \phi_i(t, h(t)) \geq 0, & t \in \Sigma_{max}^h, \\ \phi_i(0, x) \geq 0, & x \in [-h_0, h_0], \end{cases}$$

where $\mathcal{L}_i$ is given by (1.1) with every J_i ($i = 1, \dots, n$) satisfying **(J)**. Then the following conclusions hold:

- (i) If $c_{ij} \geq 0$ on D_T for $i, j \in \{1, \dots, n\}$ and $i \neq j$, then $\phi_i \geq 0$ on $\overline{D}_T$ for $i \in \{1, \dots, n\}$.
- (ii) If in addition $d_{i_0} > 0$ in D_T , $\phi_{i_0}(0, x) \not\equiv 0$ in $[-h_0, h_0]$, then $\phi_{i_0} > 0$ in D_T .

Proof. Since $\phi_i(t, x) = 0$ for $x \notin [g(t), h(t)]$, we have

$$\mathcal{L}_i[\phi_i](t, x) = \int_{g(t)}^{h(t)} J_i(x - y) \phi_i(t, y) dy - \phi_i(t, x), \quad i = 1, \dots, n.$$

Proof of part (i). We prove part (i) in two steps.

Step 1. We first prove that if $(\phi_1, \dots, \phi_n)$ satisfies

$$(1.3) \quad \begin{cases} \partial_t \phi_i > d_i \mathcal{L}_i[\phi_i] + \sum_{j=1}^n c_{ij} \phi_j, & (t, x) \in \overline{D}_T, i \in \{1, \dots, n\} \\ \phi_i(t, g(t)) > 0, & t \in \Sigma_{min}^g, i \in \{1, \dots, n\}, \\ \phi_i(t, h(t)) > 0, & t \in \Sigma_{max}^h, i \in \{1, \dots, n\}, \\ \phi_i(0, x) > 0 & x \in [-h_0, h_0], i \in \{1, \dots, n\}, \end{cases}$$

then $\phi_i > 0$ on $\overline{D}_T$.

Define

$$T_1 = \sup\{0 < t \leq T : \phi_i(s, x) > 0 \text{ for } (s, x) \in D_t, i = 1, \dots, n\}.$$

We have $T_1 > 0$ since $\phi_i(0, x) > 0$ in $[-h_0, h_0]$ and ϕ_i is continuous for $i = 1, \dots, n$. If $T_1 < T$, then there exists $i_0 \in \{1, \dots, n\}$ and $x_1 \in [g(T_1), h(T_1)]$ such that

$$(1.4) \quad \phi_{i_0}(T_1, x_1) = 0, \quad \text{and } \phi_i(t, x) \geq 0 \text{ for } (t, x) \in D_{T_1}, i = 1, \dots, n.$$

We claim that

$$(1.5) \quad \partial_t \phi_{i_0}(T_1, x_1) \leq 0.$$

This fact is evident if $x_1 \in (g(T_1), h(T_1))$. If $x_1 = g(T_1)$, then from (1.3) we can conclude that $T_1 \notin \Sigma_{min}^g$, and hence there exists an increasing sequence $t_k \rightarrow T_1$ such that $g(t_k) \leq g(T_1)$. It follows that for all large k , $x_1 = g(T_1) \in [g(t_k), h(t_k)]$ and hence $\phi_{i_0}(t_k, x_1) \geq 0$. This and the assumption $\partial_t \phi_i \in C(\overline{D}_T)$ clearly imply (1.5). If $x_1 = h(T_1)$, the proof of (1.5) is analogous.

Without loss of generality, we assume from now on $i_0 = 1$. From $c_{1j} \geq 0$ for $j = 2, \dots, n$, (1.4), (1.5) and the first inequality of (1.3), we obtain

$$0 \geq \partial_t \phi_1(T_1, x_1) > d_1(T_1, x_1) \int_{g(T_1)}^{h(T_1)} J_1(x-y) \phi_1(T_1, y) dy + \sum_{j=2}^n c_{1j} \phi_j(T_1, x_1) \geq 0,$$

which is a contradiction. Hence

$$T_1 = T, \phi_i(t, x) > 0 \text{ for } t \in [0, T), x \in [g(t), h(t)], i = 1, \dots, n.$$

It follows that $\phi_i(t, x) \geq 0$ for $(t, x) \in D_T$, $i = 1, \dots, n$. To complete the proof of Step 1, it remains to show

$$\phi_i(T, x) > 0 \text{ for } x \in [g(T), h(T)], i = 1, \dots, n.$$

If there exists $i_0 \in \{1, \dots, n\}$ and $x_0 \in [g(T), h(T)]$ such that $\phi_{i_0}(T, x_0) = 0$, then we can repeat the above argument with $T_1 = T$ to derive a contradiction.

Step 2. We apply the conclusion in Step 1 to show the desired results.

For $i \in \{1, \dots, n\}$, let $\psi_i(t, x) = \phi_i(t, x) + \epsilon e^{At}$ for some positive constants ϵ and A . Then

$$\begin{cases} \psi_i(t, g(t)) = \phi_i(t, g(t)) + \epsilon e^{At} > 0, & t \in \Sigma_{min}^g, \\ \psi_i(t, h(t)) = \phi_i(t, h(t)) + \epsilon e^{At} > 0, & t \in \Sigma_{max}^h, \\ \psi_i(0, x) = \phi_i(0, x) + \epsilon \geq \epsilon > 0, & x \in [-h_0, h_0]. \end{cases}$$

Moreover,

$$\begin{aligned} & \partial_t \psi_i - d_i \mathcal{L}_i[\psi_i] - \sum_{j=1}^n c_{ij} \psi_j \\ &= \partial_t \phi_i - d_i \mathcal{L}_i[\phi_i] - \sum_{j=1}^n c_{ij} \phi_j + \epsilon A e^{At} - d_i \epsilon e^{At} \left[\int_{g(t)}^{h(t)} J_i(x-y) dy - 1 \right] - \epsilon e^{At} \sum_{j=1}^n c_{ij} \\ &\geq \left(A - d_i - \sum_{j=1}^n c_{ij} \right) \epsilon e^{At} > 0 \text{ for } (t, x) \in \overline{D}_T, \end{aligned}$$

provided that $A > \max_{1 \leq i, j \leq n} \{ \|c_{ij}\|_{L^\infty(D_T)} + d_i \}$. It then follows from Step 1 that for any $\epsilon > 0$ and $A > \max_{1 \leq i, j \leq n} \{ \|c_{ij}\|_{L^\infty(D_T)} + d_i \}$,

$$\psi_i(t, x) = \phi_i(t, x) + \epsilon e^{At} > 0 \text{ for } (t, x) \in \overline{D}_T, i = 1, \dots, n.$$

Fix A and let $\epsilon \rightarrow 0$, it gives $\phi_i \geq 0$ on $\overline{D}_T$ for $i = 1, \dots, n$. This completes the proof of (i).

Proof of part (ii). We now prove part (ii), that is, $\phi_{i_0} > 0$ on D_T under the additional conditions

$$(1.6) \quad d_{i_0}(t, x) > 0 \text{ in } D_T, \quad \phi_{i_0}(0, x) \not\equiv 0 \text{ in } [-h_0, h_0].$$

Suppose, on the contrary,

$$\text{there exists a point } (T_1, x_1) \in D_T \text{ such that } \phi_{i_0}(T_1, x_1) = 0.$$

To simplify notations, without loss of generality, let us again assume $i_0 = 1$.

First, we claim that

$$(1.7) \quad \phi_1(T_1, x) = 0 \text{ for } x \in (g(T_1), h(T_1)).$$

If this is not true, then $\phi_1(T_1, \hat{x}_1) > 0$ for some $\hat{x}_1 \in (g(T_1), h(T_1))$. Let I be the maximal open interval containing $\hat{x}_1$ such that $\phi_1(T_1, x) > 0$ for $x \in I$. Then the existence of x_1 implies that at least one of the two boundary points of I must be in $(g(T_1), h(T_1))$. So there exists

$$\tilde{x}_1 \in (g(T_1), h(T_1)) \cap \partial\{x \in (g(T_1), h(T_1)) : \phi_1(T_1, x) > 0\}.$$

Then it follows from $\phi_1(T_1, \tilde{x}_1) = 0$, $c_{ij} \geq 0$ and $\phi_j \geq 0$ for $j \in \{2, \dots, n\}$ that

$$\begin{aligned} 0 &\geq \partial_t \phi_1(T_1, \tilde{x}_1) \geq d_1(T_1, \tilde{x}_1) \int_{g(T_1)}^{h(T_1)} J_1(\tilde{x}_1 - y) \phi_1(T_1, y) dy + \sum_{j=2}^n c_{1j}(T_1, \tilde{x}_1) \phi_j(T_1, \tilde{x}_1) \\ &\geq d_1(T_1, \tilde{x}_1) \int_{g(T_1)}^{h(T_1)} J_1(\tilde{x}_1 - y) \phi_1(T_1, y) dy > 0, \quad [\text{strict inequality due to } J(0) > 0] \end{aligned}$$

which is a contradiction. Hence, $\phi_1(T_1, x) = 0$ for all $x \in [g(T_1), h(T_1)]$.

Define

$$\Phi_i(t, x) := e^{Kt} \phi_i(t, x) \text{ with } K = d_1 + \|c_{11}\|_\infty, \quad i = 1, \dots, n.$$

Then $\Phi_1(t, x)$ satisfies $\Phi_1(t, x) \geq 0$,

$$(1.8) \quad \Phi_1(T_1, x) = 0 \quad \text{for } x \in [g(T_1), h(T_1)]$$

and

$$(1.9) \quad \begin{aligned} \partial_t \Phi_1 &= e^{Kt} (\phi_1)_t + K \Phi_1 \\ &\geq e^{Kt} \left[d_1 \int_{g(t)}^{h(t)} J_1(x-y) \phi_1(t, y) dy + (-d_1 + c_{11}) \phi_1 + \sum_{j=2}^n c_{1j} \phi_j \right] + K \Phi_1 \\ &= d_1 \int_{g(t)}^{h(t)} J_1(x-y) \Phi_1(t, y) dy + (K - d_1 + c_{11}) \Phi_1 + \sum_{j=2}^n c_{1j} \Phi_j \\ &\geq 0 \quad \text{for } (t, x) \in D_T. \end{aligned}$$

We next use (1.8) and (1.9) to drive a contradiction. By (1.6)

$$\Omega_0 := \{x \in (-h_0, h_0) : \Phi_1(0, x) > 0\} \neq \emptyset.$$

Since g and h are continuous and satisfy $g(t) < h(t)$ for all $t \in [0, T]$, for any fixed $y_0 \in \Omega_0$ there is a small constant $t_0 \in (0, T_1)$ such that

$$g(t) < y_0 < h(t) \text{ for } t \in [0, t_0].$$

We claim that

$$(1.10) \quad \Phi_1(t_0, y_0) = 0.$$

If this claim is proved, then by (1.9), $\partial_t \Phi_1(t, y_0) \geq 0$ for $t \in [0, t_0]$, i.e., $\Phi(t, y_0)$ is nondecreasing for $t \in [0, t_0]$, which indicates that $\Phi_1(0, y_0) \leq 0$. However, this contradicts with $y_0 \in \Omega_0$.

Therefore, to complete the proof, it suffices to show (1.10). For clarity, we carry out the proof of (1.10) according to two cases.

Case 1. $\cap_{t \in [t_0, T_1]} (g(t), h(t)) \neq \emptyset$.

In this case, we take

$$y_1 \in \cap_{t \in [t_0, T_1]} (g(t), h(t)),$$

and recall from (1.8) and (1.9) that $\Phi_1(T_1, y_1) = 0$, $\partial_t \Phi_1(t, y_1) \geq 0$ for $t \in [t_0, T_1]$. We then immediately see from $\Phi_1(t_0, y_1) \geq 0$ that $\Phi_1(t_0, y_1) = 0$. Now we may repeat the argument used to prove (1.7) to conclude that $\Phi_1(t_0, x) = 0$ for $x \in [g(t_0), h(t_0)]$. In particular $\Phi_1(t_0, y_0) = 0$, as desired. This completes the proof in Case 1.

Case 2. $\cap_{t \in [t_0, T_1]} (g(t), h(t)) = \emptyset$.

In this case we use a geometric argument in the two-dimensional plane with x and t being the horizontal and vertical axis respectively. Since $g(t) < h(t)$, the continuous path given by

$$\gamma_0 := \{(x, t) : x = \xi(t) = \frac{1}{2}[g(t) + h(t)], \quad t \in [t_0, T_1]\},$$

is contained in the region $G := \{(x, t) : x \in (g(t), h(t)), \quad t \in [t_0, T_1]\}$. Clearly a small tubular neighbourhood of γ_0 still lies in G , and hence we can find a continuous path γ_1 close to γ_0 such that γ_1 lies in G , it consists of finitely many line segments in the xt -plane, and

$$\gamma_1 \cap \{t = T_1\} = (\xi(T_1), T_1), \quad \gamma_1 \cap \{t = t_0\} = (\xi(t_0), t_0).$$

For example, we could take γ_1 the piecewise linear curve connecting the points $p_i \in \gamma_0$, where $p_i = (\xi(s_i), s_i)$ with $s_i = \frac{i}{k}(T_1 - t_0) + t_0$, $i = 0, \dots, k$, for a large enough positive integer k .

Similarly, a small tubular neighbourhood of γ_1 still lies in G , which allows us to find a continuous path γ_2 close to γ_1 with the following two properties:

- (i) $\gamma_2 \subset G$, and $\gamma_2 \cap \{t = T_1\} = (\xi(T_1), T_1)$, $\gamma_2 \cap \{t = t_0\} = (\xi(t_0), t_0)$,
- (ii) γ_2 consists of finitely many line segments which are either vertical or horizontal.

Let the horizontal line segments of γ_2 be denoted by H_i , $i = 1, \dots, m$. Then we can find $t_0 < t_1 < \dots < t_m < t_{m+1} = T_1$ such that $H_i \subset \{t = t_i\}$, $i = 1, \dots, m$. Let V_j denote the vertical line segments of γ_2 that lies between t_{j-1} and t_j , $j = 1, \dots, m+1$, then there exists $x_j \in (g(t_j), h(t_j))$ such that $V_j \subset \{x = x_j\}$, $j = 1, \dots, m+1$. Thus

$$\begin{cases} \text{the two end points of } V_i \text{ are } (x_i, t_{i-1}) \text{ and } (x_i, t_i), & 1 \leq i \leq m+1, \\ \text{the two end points of } H_i \text{ are } (x_i, t_i) \text{ and } (x_{i+1}, t_i), & 1 \leq i \leq m. \end{cases}$$

We show that $\Phi_1(t_m, x) = 0$ for $x \in [g(t_m), h(t_m)]$. Thanks to (1.8) and (1.9), we have $\Phi_1(T_1, x_{m+1}) = 0$ and $\partial_t \Phi_1(t, x_{m+1}) \geq 0$ for $t \in [t_m, T_1]$. This, combined with $\Phi_1(t_m, x_{m+1}) \geq 0$, yields $\Phi_1(t_m, x_{m+1}) = 0$. The arguments leading to (1.7) now infers that $\Phi_1(t_m, x) \equiv 0$ for $x \in [g(t_m), h(t_m)]$. In other words, $\Phi_1(t_{m+1}, x_{m+1}) = 0$ implies $\Phi_1(t_m, x) = 0$ for $x \in [g(t_m), h(t_m)]$.

Repeating the above argument, we can show $\Phi_1(t_i, x_i) = 0$ implies $\Phi_1(t_{i-1}, x) = 0$ for $x \in [g(t_{i-1}), h(t_{i-1})]$, $i = m, \dots, 1$. Thus we have $\Phi_1(t_0, x) = 0$ for $x \in [g(t_0), h(t_0)]$, which clearly implies (1.10). The proof is now complete. $\square$

1.2. An example. A free boundary model for West Nile virus [10]:

$$(1.11) \quad \begin{cases} H_t = d_1 \mathcal{L}_1[H](t, x) + a_1(e_1 - H)V - b_1 H, & x \in (g(t), h(t)), \quad t > 0, \\ V_t = d_2 \mathcal{L}_2[V](t, x) + a_2(e_2 - V)H - b_2 V, & x \in (g(t), h(t)), \quad t > 0, \\ H(t, x) = V(t, x) = 0, & t > 0, \quad x \in \{g(t), h(t)\}, \\ h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_1(y - x) H(t, y) dx, & t > 0, \\ g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_1(y - x) H(t, y) dx, & t > 0, \\ H(0, x) = u_1^0(x), \quad V(0, x) = u_2^0(x), & x \in [-h_0, h_0]. \end{cases}$$

Here $H(t, x)$ and $V(t, x)$ stand for the densities of the infected bird (host) and mosquito (vector) populations at time t and spatial location x , respectively. The interval $[g(t), h(t)]$ is the evolving region of virus infection. The parameters here are all positive constants. The initial functions $u_i^0(x)$ ($i = 1, 2$) satisfy

$$(1.12) \quad \begin{cases} u_i^0 \in C([-h_0, h_0]), \quad u_i^0(-h_0) = u_i^0(h_0) = 0, \\ 0 < u_i^0(x) \leq e_i \text{ for } x \in (-h_0, h_0), \quad i = 1, 2. \end{cases}$$

We can easily apply Theorem 1.1 to obtain the following comparison results.

Corollary 1.2. Assume (J) holds, $T > 0$, $g, h \in C([0, T])$ satisfy $g(t) < h(t)$, and $D_T = \{(t, x) : t \in (0, T], g(t) < x < h(t)\}$. If $H, V, \tilde{H}, \tilde{V} \in C(\overline{D}_T)$ satisfy the following conditions:

- (i) $\Phi_t \in C(\overline{D}_T)$ for $\Phi \in \{H, V, \tilde{H}, \tilde{V}\}$,
- (ii) $0 \leq \Phi \leq e_1$ for $\Phi \in \{H, \tilde{H}\}$, $0 \leq \Psi \leq e_2$ for $\Psi \in \{V, \tilde{V}\}$,
- (iii) for $(t, x) \in D_T$,

$$(1.13) \quad \begin{cases} \tilde{H}_t \geq d_1 \int_{g(t)}^{h(t)} J_1(x - y) \tilde{H}(t, y) dy - d_1 \tilde{H} + a_1(e_1 - \tilde{H}) \tilde{V} - b_1 \tilde{H}, \\ \tilde{V}_t \geq d_2 \int_{g(t)}^{h(t)} J_2(x - y) \tilde{V}(t, y) dy - d_2 \tilde{V} + a_2(e_2 - \tilde{V}) \tilde{H} - b_2 \tilde{V}, \end{cases}$$

- (iv) for $(t, x) \in D_T$, (H, V) satisfies (1.13) but with the inequalities reversed,
- (v) at the boundary,

$$\begin{cases} H(t, g(t)) \leq \tilde{H}(t, g(t)), \quad V(t, g(t)) \leq \tilde{V}(t, g(t)) & \text{for } t \in \Sigma_{min}^g, \\ H(t, h(t)) \leq \tilde{H}(t, h(t)), \quad V(t, h(t)) \leq \tilde{V}(t, h(t)) & \text{for } t \in \Sigma_{max}^h, \end{cases}$$

- (vi) at the initial time,

$$H(0, x) \leq \tilde{H}(0, x), \quad V(0, x) \leq \tilde{V}(0, x) \quad \text{for } x \in [g(0), h(0)],$$

then

$$H(t, x) \leq \tilde{H}(t, x), \quad V(t, x) \leq \tilde{V}(t, x) \quad \text{for } (t, x) \in D_T.$$

Proof. Define

$$\phi_1 := \tilde{H} - H, \quad \phi_2 := \tilde{V} - V,$$

and

$$c_{11} := -(b_1 + a_1 V), \quad c_{12} := a_1(e_1 - \tilde{H}), \quad c_{21} := a_1(e_2 - \tilde{V}), \quad c_{22} := -(b_2 + a_2 H).$$

Then it is easily checked that (ϕ_1, ϕ_2) satisfies (1.2) with $n = 2$. Therefore $\phi_1 \geq 0$ and $\phi_2 \geq 0$ in D_T . $\square$

1.3. A comparison result for a scalar nonlocal free boundary problem. Suppose the kernel function $J(x)$ satisfies the basic condition

$$(\mathbf{J}): J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ is nonnegative, } J(0) > 0, \quad \int_{\mathbb{R}} J(x) dx = 1.$$

The function $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies

$$(\mathbf{f1}): f(t, x, 0) \equiv 0 \text{ and } f(t, x, u) \text{ is continuous in } (t, x, u) \text{ and locally Lipschitz in } u \in \mathbb{R}^+, \text{ i.e., for any } L > 0, \text{ there exists a constant } K = K(L) > 0 \text{ such that}$$

$$|f(t, x, u_1) - f(t, x, u_2)| \leq K|u_1 - u_2| \text{ for } u_1, u_2 \in [0, L], (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

The nonlocal free boundary problem to be considered has the following form:

$$(1.14) \quad \begin{cases} u_t = d \int_{g(t)}^{h(t)} J(x-y)u(t, y)dy - du(t, x) + f(t, x, u), & t > 0, \quad x \in (g(t), h(t)), \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(y-x)u(t, x)dydx, & t > 0, \\ g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(y-x)u(t, x)dydx, & t > 0, \\ u(0, x) = u_0(x), \quad h(0) = -g(0) = h_0, & x \in [-h_0, h_0], \end{cases}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t, x)$, which is always assumed to be identically 0 for $x \in \mathbb{R} \setminus [g(t), h(t)]$; d and μ are positive constants. The initial function $u_0(x)$ satisfies

$$(1.15) \quad u_0(x) \in C([-h_0, h_0]), \quad u_0(-h_0) = u_0(h_0) = 0 \quad \text{and} \quad u_0(x) > 0 \quad \text{in} \quad (-h_0, h_0),$$

with $[-h_0, h_0]$ representing the initial population range of the species.

Theorem 1.3. (*Comparison principle*) Assume that $(\mathbf{J})$ and $(\mathbf{f1})$ hold, u_0 satisfies (1.15) and (u, g, h) satisfies (1.14)¹ for $0 \leq t \leq T \in (0, +\infty)$. Suppose that $\bar{h}, \bar{g} \in C^1([0, T])$ and $\bar{u}_t(t, x), \bar{u}(t, x)$ are continuous for $t \in [0, T]$, $x \in [\bar{g}(t), \bar{h}(t)]$, and

$$(1.16) \quad \begin{cases} \bar{u}_t \geq d \int_{\bar{g}(t)}^{\bar{h}(t)} J(x-y)\bar{u}(t, y)dy - d\bar{u} + f(t, x, \bar{u}), & 0 < t \leq T, \quad x \in (\bar{g}(t), \bar{h}(t)), \\ \bar{u}(t, \bar{g}(t)) \geq 0, \quad \bar{u}(t, \bar{h}(t)) \geq 0, & 0 < t \leq T, \\ \bar{h}'(t) \geq \mu \int_{\bar{g}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J(y-x)\bar{u}(t, x)dydx, & 0 < t \leq T, \\ \bar{g}'(t) \leq -\mu \int_{\bar{g}(t)}^{\bar{h}(t)} \int_{-\infty}^{\bar{g}(t)} J(y-x)\bar{u}(t, x)dydx, & 0 < t \leq T, \\ \bar{u}(0, x) \geq u_0(x), \quad \bar{h}(0) > h_0, \quad \bar{g}(0) < -h_0, & x \in [-h_0, h_0], \\ \bar{u}(0, x) \geq 0, & x \in [\bar{g}(0), \bar{h}(0)]. \end{cases}$$

Then

$$(1.17) \quad u(t, x) < \bar{u}(t, x), \quad g(t) > \bar{g}(t) \quad \text{and} \quad h(t) < \bar{h}(t) \quad \text{for} \quad 0 < t \leq T \quad \text{and} \quad x \in [g(t), h(t)].$$

The triplet $(\bar{u}, \bar{g}, \bar{h})$ above is called an upper solution of (1.14). We can define a lower solution and obtain analogous results by reversing the inequalities in (1.16) and (1.17).

¹Here we implicitly require $g, h \in C^1([0, T])$ and u_t, u are continuous for $t \in [0, T]$, $x \in [g(t), h(t)]$.

Proof. Due to (f1), we can write $f(t, x, \bar{u}(t, x)) = c(t, x)\bar{u}(t, x)$ with $c \in L^\infty$. Hence we can apply Theorem 1.1 with $n = 1$ to conclude that $\bar{u} > 0$ for $0 < t \leq T$, $\bar{g}(t) < x < \bar{h}(t)$, and thus both $\bar{h}$ and $-\bar{g}$ are strictly increasing.

We claim that $h(t) < \bar{h}(t)$ and $g(t) > \bar{g}(t)$ for all $t \in (0, T]$. Clearly, these hold true for small $t > 0$. Suppose by way of contradiction that there exists $t_1 \in (0, T]$ such that

$$h(t) < \bar{h}(t), \quad g(t) > \bar{g}(t) \text{ for } t \in (0, t_1) \text{ and } [h(t_1) - \bar{h}(t_1)][g(t_1) - \bar{g}(t_1)] = 0.$$

Without loss of generality, we may assume that

$$h(t_1) = \bar{h}(t_1) \text{ and } g(t_1) \geq \bar{g}(t_1).$$

We now compare u and $\bar{u}$ over the region

$$\Omega_{t_1} := \{(t, x) \in \mathbb{R}^2 : 0 < t \leq t_1, \quad g(t) < x < h(t)\}.$$

Let $w(t, x) := \bar{u}(t, x) - u(t, x)$. Then for $(t, x) \in \Omega_{t_1}$, we have

$$(1.18) \quad w_t \geq d \int_{g(t)}^{h(t)} J(x-y)w(t, y)dy - dw(t, x) + C(t, x)w(t, x),$$

for some L^∞ function $C(t, x)$. Moreover,

$$w(t, g(t)) > 0, \quad w(t, h(t)) > 0 \text{ for } t \in (0, t_1), \quad w(0, x) \geq 0 \text{ for } x \in [-h_0, h_0].$$

Therefore it follows from Theorem 1.1 that $w(t, x) \geq 0$ in Ω_{t_1} . Moreover, for any $t_0 \in (0, t_1)$, $w(t_0, h(t_0)) > 0$ and so $w(t_0, x) \geq \neq 0$ in $[g(t_0), h(t_0)]$. So we can apply Theorem 1.1 over $t \in [t_0, t_1]$, $x \in [g(t), h(t)]$ to deduce $w(t, x) > 0$ in this range. Since t_0 can be arbitrarily small we obtain

$$w(t, x) = \bar{u}(t, x) - u(t, x) > 0 \text{ for } t \in (0, t_1], \quad x \in [g(t), h(t)].$$

On the other hand, by the definition of t_1 , we have

$$h(t_1) = \bar{h}(t_1), \quad h'(t_1) \geq \bar{h}'(t_1).$$

This leads to the following contradiction:

$$\begin{aligned} 0 &\geq \bar{h}'(t_1) - h'(t_1) \\ &\geq \mu \int_{\bar{g}(t_1)}^{\bar{h}(t_1)} \int_{\bar{h}(t_1)}^{+\infty} J(y-x)\bar{u}(t_1, x)dydx - \mu \int_{g(t_1)}^{h(t_1)} \int_{h(t_1)}^{+\infty} J(y-x)u(t_1, x)dydx \\ &\geq \mu \int_{g(t_1)}^{h(t_1)} \int_{h(t_1)}^{+\infty} J(y-x)[\bar{u}(t_1, x) - u(t_1, x)]dydx > 0. \quad \text{[strict inequality due to } J(0) > 0] \end{aligned}$$

The claim is thus proved, i.e., we always have $h(t) < \bar{h}(t)$ and $g(t) > \bar{g}(t)$ for all $t \in (0, T]$.

We may now use the comparison principle to obtain $\bar{u}(t, x) \geq u(t, x)$ for $t \in [0, T]$, $x \in [g(t), h(t)]$, and $\bar{u}(t, x) > u(t, x)$ for $t \in (t_0, T]$, $x \in [g(t), h(t)]$ for any $t_0 \in (0, T)$. $\square$

Remarks: Theorem 1.1 is a simple variation of Lemma 3.1 in [10]. Theorem 1.3 is a simple variation of Theorem 3.1 in [3].

2. EXISTENCE AND UNIQUENESS

The following theorem is the main result to be proved here.

Theorem 2.1. *Suppose that (J) and (f1)-(f2) hold. Then for any given $h_0 > 0$ and $u_0(x)$ satisfying (1.15), problem (1.14) admits a unique solution $(u(t, x), g(t), h(t))$ defined for all $t > 0$. Moreover, for any $T > 0$, $g \in \mathbb{G}_{h_0, T}$, $h \in \mathbb{H}_{h_0, T}$ and $u \in \mathbb{X}_{u_0, g, h}$.*

Here, and in what follows, for given $h_0, T > 0$ we define

$$\begin{aligned} \mathbb{H}_{h_0, T} &:= \left\{ h \in C([0, T]) : h(0) = h_0, \quad \inf_{0 \leq t_1 < t_2 \leq T} \frac{h(t_2) - h(t_1)}{t_2 - t_1} > 0 \right\}, \\ \mathbb{G}_{h_0, T} &:= \left\{ g \in C([0, T]) : -g \in \mathbb{H}_{h_0, T} \right\}, \\ C_0([-h_0, h_0]) &:= \left\{ u \in C([-h_0, h_0]) : u(-h_0) = u(h_0) = 0 \right\}. \end{aligned}$$

For $g \in \mathbb{G}_{h_0, T}$, $h \in \mathbb{H}_{h_0, T}$ and $u_0 \in C_0([-h_0, h_0])$ nonnegative, we define

$$\begin{aligned}\Omega &= \Omega_{g,h} := \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, g(t) < x < h(t)\}, \\ \mathbb{X} &= \mathbb{X}_{u_0, g, h} := \left\{ \phi \in C(\overline{\Omega}_{g,h}) : \phi \geq 0 \text{ in } \Omega_{g,h}, \phi(0, x) = u_0(x) \text{ for } x \in [-h_0, h_0] \right. \\ &\quad \left. \text{and } \phi(t, g(t)) = \phi(t, h(t)) = 0 \text{ for } 0 \leq t \leq T \right\}.\end{aligned}$$

2.1. An auxiliary initial boundary value problem. For any $T > 0$ and $(g, h) \in \mathbb{G}_{h_0, T} \times \mathbb{H}_{h_0, T}$, we consider the following problem:

$$(2.1) \quad \begin{cases} v_t = d \int_{g(t)}^{h(t)} J(x-y)v(t, y)dy - dv + f(t, x, v), & 0 < t \leq T, x \in (g(t), h(t)), \\ v(t, h(t)) = v(t, g(t)) = 0, & 0 < t \leq T, \\ v(0, x) = u_0(x), & x \in [-h_0, h_0] \end{cases}$$

Lemma 2.2. Suppose that **(J)** and **(f1)**-(**f2**) hold, $h_0 > 0$ and $u_0(x)$ satisfies (1.15). Then (2.1) admits a unique solution, denoted by $V_{g,h}(t, x)$. Moreover $V_{g,h}$ satisfies

$$(2.2) \quad 0 < V_{g,h}(t, x) \leq \max \left\{ \max_{-h_0 \leq x \leq h_0} u_0(x), K_0 \right\} \text{ for } 0 < t \leq T, x \in (g(t), h(t)),$$

where K_0 is defined in the assumption **(f2)**.

Strategy of the proof of Theorem 2.1: By Lemma 2.2, for any $T > 0$ and $(h, g) \in \mathbb{G}_{h_0, T} \times \mathbb{H}_{h_0, T}$, we can find a unique $V_{g,h} \in \mathbb{X}_{u_0, g, h}$ that solves (2.1), and it has the property

$$0 < V_{g,h}(t, x) \leq M_0 := \max \{ \|u_0\|_\infty, K_0 \} \text{ for } (t, x) \in \Omega_{g,h}.$$

A nonlinear mapping: Using $V_{g,h}(t, x)$, we define a mapping $\tilde{\Gamma}$ by

$$\begin{aligned}\tilde{\Gamma}(g, h) &= (\tilde{g}, \tilde{h}), \text{ where, for } 0 < t \leq T, \\ \begin{cases} \tilde{g}(t) = -h_0 - \mu \int_0^t \int_{g(\tau)}^{h(\tau)} \int_{-\infty}^{g(\tau)} J(y-x)V_{g,h}(\tau, x)dydx d\tau, \\ \tilde{h}(t) = h_0 + \mu \int_0^t \int_{g(\tau)}^{h(\tau)} \int_{h(\tau)}^{+\infty} J(y-x)V_{g,h}(\tau, x)dydx d\tau. \end{cases}\end{aligned}$$

Local existence: We will show that if T is small enough, then $\tilde{\Gamma}$ maps a suitable closed subset Σ_T of $\mathbb{G}_{h_0, T} \times \mathbb{H}_{h_0, T}$ into itself, and is a **contraction mapping**. This implies that $\tilde{\Gamma}$ has a unique **fixed point** (g, h) in Σ_T , which gives a solution $(V_{g,h}, g, h)$ of (1.14) defined for $t \in (0, T]$.

Global existence: We will then show that this unique solution defined locally in time can be extended uniquely for all $t > 0$.

Proof of Lemma 2.2: We break the proof into three steps.

Step 1: A parametrized ODE problem.

For given $x \in [g(T), h(T)]$, define

$$(2.3) \quad \begin{aligned} \tilde{u}_0(x) &:= \begin{cases} 0 & \text{if } x \notin [-h_0, h_0], \\ u_0(x) & \text{if } x \in [-h_0, h_0]. \end{cases} \\ t_x &:= \begin{cases} t_{x,g} & \text{if } x \in [g(T), -h_0) \text{ and } g(t_{x,g}) = x, \\ 0 & \text{if } x \in [-h_0, h_0], \\ t_{x,h} & \text{if } x \in (h_0, h(T)] \text{ and } h(t_{x,h}) = x. \end{cases} \end{aligned}$$

Clearly $t_x = T$ for $x = g(T)$ and $x = h(T)$, $t_x < T$ for $x \in (g(T), h(T))$, and

$$x \rightarrow t_x \text{ is continuous over } [g(T), h(T)].$$

For any given $\phi \in \mathbb{X}_{u_0, g, h}$, consider the following ODE initial value problem (with parameter x):

$$(2.4) \quad \begin{cases} v_t = d \int_{g(t)}^{h(t)} J(x-y) \phi(t, y) dy - dv(t, x) + \tilde{f}(t, x, v), & t_x < t \leq T, \\ v(t_x, x) = \tilde{u}_0(x), & x \in (g(T), h(T)), \end{cases}$$

where

$$\tilde{f}(t, x, v) := \begin{cases} 0 & \text{for } v < 0, \\ f(t, x, v) & \text{for } v \geq 0. \end{cases}$$

Clearly $\tilde{f}$ also satisfies **(f1)**-(**f2**). Denote

$$F(t, x, v) := d \int_{g(t)}^{h(t)} J(x-y) \phi(t, y) dy - dv(t, x) + \tilde{f}(t, x, v).$$

Thanks to the assumption **(f1)**, for any $v_1, v_2 \in (-\infty, L]$, we have

$$\left| F(t, x, v_1) - F(t, x, v_2) \right| \leq \left| \tilde{f}(t, x, v_1) - \tilde{f}(t, x, v_2) \right| + d \left| v_1 - v_2 \right| \leq K_1 \left| v_1 - v_2 \right|,$$

where

$$L := 1 + \max \left\{ \|\phi\|_{C(\overline{\Omega}_T)}, K_0 \right\}, \quad K_1 := d + K(L).$$

In other words, the function $F(t, x, v)$ is Lipschitz continuous in v for $v \in (-\infty, L]$ with Lipschitz constant K_1 , uniformly for $t \in [0, T]$ and $x \in (g(T), h(T))$. Additionally, $F(t, x, v)$ is continuous in all its variables in this range. Hence it follows from the Fundamental Theorem of ODEs that, for every fixed $x \in (g(T), h(T))$, problem (2.4) admits a unique solution, denoted by $V_\phi(t, x)$ defined in some interval $[t_x, T_x)$ of t .

We claim that $t \rightarrow V_\phi(t, x)$ can be uniquely extended to $[t_x, T]$. Clearly it suffices to show that if $V_\phi(t, x)$ is uniquely defined for $t \in [t_x, \tilde{T}]$ with $\tilde{T} \in (t_x, T]$, then

$$(2.5) \quad 0 \leq V_\phi(t, x) < L \text{ for } t \in (t_x, \tilde{T}].$$

We first show that $V_\phi(t, x) < L$ for $t \in (t_x, \tilde{T}]$. Arguing indirectly we assume that this inequality does not hold, and hence, in view of $V_\phi(t_x, x) = \tilde{u}_0(x) \leq \|\phi\|_{C(\overline{\Omega}_T)} < L$, there exists some $t^* \in (t_x, \tilde{T}]$ such that $V_\phi(t, x) < L$ for $t \in (t_x, t^*)$ and $V_\phi(t^*, x) = L$. It follows that $(V_\phi)_t(t^*, x) \geq 0$ and $\tilde{f}(t^*, x, V_\phi(t^*, x)) \leq 0$ (due to $L > K_0$). We thus obtain from the differential equation satisfied by $V_\phi(t, x)$ that

$$dL = dV_\phi(t^*, x) \leq d \int_{g(t^*)}^{h(t^*)} J(x-y) \phi(t^*, y) dy \leq d \|\phi\|_{C(\overline{\Omega}_T)} \leq d(L-1).$$

It follows that $L \leq L-1$. This contradiction proves our claim.

We now prove the first inequality in (2.5). Since

$$\tilde{f}(t, x, v) = \tilde{f}(t, x, v) - \tilde{f}(t, x, 0) \geq -K(L)|v| \text{ for } v \in (-\infty, L],$$

we have

$$(V_\phi)_t \geq -K_1 \text{sgn}(V_\phi) V_\phi + d \int_{g(t)}^{h(t)} J(x-y) \phi(t, y) dy \geq -K_1 \text{sgn}(V_\phi) V_\phi \text{ for } t \in [t_x, \tilde{T}].$$

Since $V_\phi(t_x, x) = \tilde{u}_0(x) \geq 0$, the above inequality immediately gives $V_\phi(t, x) \geq 0$ for $t \in [t_x, \tilde{T}]$. We have thus proved (2.5), and therefore the solution $V_\phi(t, x)$ of (2.4) is uniquely defined for $t \in [t_x, T]$.

Step 2: A fixed point problem.

Let us note that $V_\phi(0, x) = u_0(x)$ for $x \in [-h_0, h_0]$, and $V_\phi(t, x) = 0$ for $t \in [0, T)$ and $x \in \partial(g(t), h(t)) = \{g(t), h(t)\}$. Moreover, by the continuous dependence of the unique ODE solution on the initial value and on the parameters in the equation, we also see that $V_\phi(t, x)$ is continuous in $(t, x) \in \overline{\Omega}_T$, and hence $V_\phi \in \mathbb{X}_{u_0, g, h}$. We now define $\Gamma : \mathbb{X}_{u_0, g, h} \rightarrow \mathbb{X}_{u_0, g, h}$ by

$$\Gamma \phi = V_\phi.$$

and notice that ϕ solves (2.1) if it is a fixed point of Γ .

We want to show that Γ is a contraction mapping if T is replaced by a sufficiently small $s \in (0, T]$. For convenience of notation, we define for any $s \in (0, T]$,

$$\Omega_s := \{(t, x) \in \Omega_{g, h} : t \leq s\}, \quad \mathbb{X}_s := \{\psi|_{\overline{\Omega}_s} : \psi \in \mathbb{X}_{u_0, g, h}\}.$$

We then define the mapping $\Gamma_s : \mathbb{X}_s \rightarrow \mathbb{X}_s$ by

$$\Gamma_s \psi = V_\psi.$$

Clearly, if $\Gamma_s \psi = \psi$ then $\psi(t, x)$ solves (2.1) for $t \in (0, s]$, and vice versa.

We show next that for sufficiently small $s > 0$, Γ_s has a unique fixed point in $\mathbb{X}_s$. We will prove this conclusion by the contraction mapping theorem; namely we prove that for such s , Γ_s is a contraction mapping on a closed subset of $\mathbb{X}_s$, and any fixed point of Γ_s in $\mathbb{X}_s$ lies in this closed subset.

Firstly we note that $\mathbb{X}_s$ is a complete metric space with the metric

$$d(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_{C(\overline{\Omega}_s)}.$$

Fix $M > \max \{4\|u_0\|_\infty, K_0\}$ and define

$$\mathbb{X}_s^M := \{\phi \in \mathbb{X}_s : \|\phi\|_{C(\overline{\Omega}_s)} \leq M\}.$$

Clearly $\mathbb{X}_s^M$ is a closed subset of $\mathbb{X}_s$. We show next that there exists $\delta > 0$ small depending on M such that for every $s \in (0, \delta]$, Γ_s maps $\mathbb{X}_s^M$ into itself, and is a contraction mapping.

Let $\phi \in \mathbb{X}_s^M$ and denote $v = \Gamma_s \phi$. Then v solves (2.4) with T replaced by s . It follows that (2.5) holds with $\hat{T}$ replaced by s and V_ϕ replaced by v . We prove that for all small $s > 0$,

$$v(t, x) \leq M \text{ for } t \in [t_x, s], \quad x \in (g(s), h(s)),$$

which is equivalent to $\|v\|_{C(\overline{\Omega}_s)} \leq M$.

Let us observe that due to **(f1)**-**(f2)**, there exists $K_* > 0$ such that

$$f(t, x, u) \leq K_* u \text{ for all } u \in [0, \infty).$$

Now from (2.4) we obtain, for $t \in [t_x, s]$ and $x \in (g(s), h(s))$,

$$v_t \leq d \int_{g(t)}^{h(t)} J(x-y) \phi(t, y) dy + K_* v \leq d \|\phi\|_{C(\overline{\Omega}_s)} + K_* v.$$

It follows that, for such t and x ,

$$e^{-K_* t} v(t, x) - e^{-K_* t_x} v(t_x, x) \leq d \int_{t_x}^t e^{-K_* \tau} d\tau \|\phi\|_{C(\overline{\Omega}_s)},$$

and

$$v(t, x) \leq \|u_0\|_\infty e^{K_* t} + d(t - t_x) e^{K_* t} \|\phi\|_{C(\overline{\Omega}_s)} \leq \|u_0\|_\infty e^{K_* s} + d s e^{K_* s} M.$$

If $\delta_1 > 0$ is small enough such that

$$d \delta_1 e^{K_* \delta_1} \leq \frac{1}{4}, \quad e^{K_* \delta_1} \leq 2,$$

then for $s \in (0, \delta_1]$ we have

$$v(t, x) \leq \frac{1}{4}(8\|u_0\|_\infty + M) \leq M \text{ in } \Omega_s.$$

Thus $v = \Gamma_s \phi \in \mathbb{X}_s^M$, as we wanted. Let us note from the above choice of δ_1 that it only depends on d and K_* .

Next we show that by shrinking δ_1 if necessary, Γ_s is a contraction mapping on $\mathbb{X}_s^M$ when $s \in (0, \delta_1]$. So let $\phi_1, \phi_2 \in \mathbb{X}_s^M$, and denote $V_i = \Gamma_s \phi_i$, $i = 1, 2$. Then $w = V_1 - V_2$ satisfies

$$(2.6) \quad \begin{cases} w_t + c_1(t, x)w = d \int_{g(t)}^{h(t)} J(x-y) (\phi_1 - \phi_2)(t, y) dy, & t_x < t \leq s, \quad x \in (g(t), h(t)), \\ w(t_x, x) = 0, & x \in (g(t), h(t)), \end{cases}$$

where

$$c_1(t, x) := d - \frac{f(t, x, V_1) - f(t, x, V_2)}{V_1 - V_2} \text{ and hence } \|c_1\|_\infty \leq K_1(M) := d + K(M).$$

It follows that, for $t_x < t \leq s$ and $x \in (g(t), h(t))$,

$$w(t, x) = d e^{-\int_{t_x}^t c_1(\tau, x) d\tau} \int_{t_x}^t e^{\int_{t_x}^\xi c_1(\tau, x) d\tau} \int_{g(\xi)}^{h(\xi)} J(x-y) (\phi_1 - \phi_2)(\xi, y) dy d\xi.$$

We thus deduce, for such t and x ,

$$\begin{aligned} |w(t, x)| &\leq d e^{K_1(M)(t-t_x)} \|\phi_1 - \phi_2\|_{C(\bar{\Omega}_s)} \int_{t_x}^t e^{K_1(M)(\xi-t_x)} d\xi \\ &\leq d e^{K_1(M)s} \|\phi_1 - \phi_2\|_{C(\bar{\Omega}_s)} \cdot (t - t_x) e^{K_1(M)(t-t_x)} \\ &\leq s d e^{2K_1(M)s} \|\phi_1 - \phi_2\|_{C(\bar{\Omega}_s)}. \end{aligned}$$

Hence

$$\|\Gamma_s \phi_1 - \Gamma_s \phi_2\|_{C(\bar{\Omega}_s)} = \|w\|_{C(\bar{\Omega}_s)} \leq \frac{1}{2} \|\phi_1 - \phi_2\|_{C(\bar{\Omega}_s)} \quad \text{for } s \in (0, \delta],$$

provided that $\delta \in (0, \delta_1]$ satisfies

$$\delta d e^{2K_1(M)\delta} \leq \frac{1}{2}.$$

For such s we may now apply the Contraction Mapping Theorem to conclude that Γ_s has a unique fixed point V in $\mathbb{X}_s^M$. It follows that $v = V$ solves (2.1) for $0 < t \leq s$.

If we can show that any solution v of (2.1) must satisfy $0 \leq v \leq M$ in Ω_s , then v would coincide with the unique fixed point V of Γ_s in $\mathbb{X}_s^M$, and uniqueness of the local solution to (2.1) is proved.

We next prove such an estimate for v . We note that $v \geq 0$ already follows from (2.5). So we only need to prove $v \leq M$. We actually prove the following stronger inequality

$$(2.7) \quad v(t, x) \leq M_0 := \max \{ \|u_0\|_\infty, K_0 \} < M \text{ for } t \in [t_x, s], x \in (g(s), h(s)).$$

It suffices to show that the above inequality holds with M_0 replaced by $M_0 + \epsilon$ for any given $\epsilon > 0$. We argue by contradiction. Suppose this is not true. Then due to $v(t_x, x) = \tilde{u}_0(x) \leq \|u_0\|_\infty < M_\epsilon := M_0 + \epsilon$, there exists some $t^* \in (t_x, s]$ and $x^* \in (g(s), h(s))$ such that

$$v(t^*, x^*) = M_\epsilon \text{ and } 0 \leq v(t, x) < M_\epsilon \text{ for } t \in [t_x, t^*), x \in (g(s), h(s)).$$

It follows that $v_t(t^*, x^*) \geq 0$ and $f(t^*, x^*, v(t^*, x^*)) \leq 0$. Hence from (2.1) we obtain

$$0 \leq v_t(t^*, x^*) \leq d \int_{g(t^*)}^{h(t^*)} J(x^* - y) v(t^*, y) dy - dv(t^*, x^*).$$

Since $v(t^*, g(t^*)) = v(t^*, h(t^*)) = 0$, for $y \in (g(t^*), h(t^*))$ but close to the boundary of this interval, $v(t^*, y) < M_\epsilon$. It follows that

$$dM_\epsilon = dv(t^*, x^*) \leq d \int_{g(t^*)}^{h(t^*)} J(x^* - y) v(t^*, y) dy < dM_\epsilon \int_{g(t^*)}^{h(t^*)} J(x^* - y) dy \leq dM_\epsilon.$$

This contradiction proves (2.7). Thus v satisfies the wanted inequality and hence coincides with the unique fixed point of Γ_s in $\mathbb{X}_s^M$. We have now proved the fact that for every $s \in (0, \delta]$, Γ_s has a unique fixed point in $\mathbb{X}_s$, which is the unique solution to (2.1) with T replaced by s .

Step 3: Extension and completion of the proof.

From Step 2 we know that (2.1) has a unique solution defined for $t \in [0, s]$ with $s \in (0, \delta]$. Applying Step 2 to (2.1) but with the initial time $t = 0$ replaced by $t = s$ we see that the unique solution can be extended to a slightly bigger interval of t . Moreover, by (2.7) and the definition of δ in Step 2, we see that the new extension can be done by increasing t by at least some $\tilde{\delta} > 0$, with $\tilde{\delta}$ depends only on M_0 and d . Furthermore, from the above proof of (2.7) we easily see that the extended solution v satisfies (2.7) in the newly extended range of t . Thus the extension by $\tilde{\delta}$ for t can be repeated. Clearly by repeating this process finitely many times, the solution of (2.1) will be uniquely extended to $t \in [t_x, T)$. As explained above, now (2.7) holds for $t \in [t_x, T)$, and hence to prove (2.2), it only remains to show $V_{g,h}(t, x) > 0$ for $t \in (0, T)$ and $x \in (g(t), h(t))$. However, due to (f1)-(f2) and (2.7), we may write $f(t, x, V_{g,h}(t, x)) = c(t, x) V_{g,h}(t, x)$ with $c \in L^\infty(\Omega_s)$ for any $s \in (0, T)$. Thus we can use the maximum principle Theorem 2.1 to conclude. $\square$

2.2. Proof of Theorem 2.1. By Lemma 2.2, for any $T > 0$ and $(h, g) \in \mathbb{G}_{h_0, T} \times \mathbb{H}_{h_0, T}$, we can find a unique $V_{g,h} \in \mathbb{X}_{u_0, g, h}$ that solves (2.1), and it has the property

$$0 < V_{g,h}(t, x) \leq M_0 := \max \{ \|u_0\|_\infty, K_0 \} \text{ for } (t, x) \in \Omega_{g,h}.$$

Using such a $V_{g,h}(t, x)$, we define the mapping $\tilde{\Gamma}$ by $\tilde{\Gamma}(g, h) = (\tilde{g}, \tilde{h})$, where, for $0 < t \leq T$,

$$(2.8) \quad \begin{cases} \tilde{h}(t) = h_0 + \mu \int_0^t \int_{g(\tau)}^{h(\tau)} \int_{h(\tau)}^{+\infty} J(y-x) V_{g,h}(\tau, x) dy dx d\tau, \\ \tilde{g}(t) = -h_0 - \mu \int_0^t \int_{g(\tau)}^{h(\tau)} \int_{-\infty}^{g(\tau)} J(y-x) V_{g,h}(\tau, x) dy dx d\tau. \end{cases}$$

To stress the dependence on T , we will write

$$\mathbb{G}_T = \mathbb{G}_{h_0, T}, \quad \mathbb{H}_T = \mathbb{H}_{h_0, T}, \quad \Omega_T = \Omega_{g, h}, \quad \mathbb{X}_T = \mathbb{X}_{u_0, g, h}.$$

To prove this theorem, we will show that if T is small enough, then $\tilde{\Gamma}$ maps a suitable closed subset Σ_T of $\mathbb{G}_T \times \mathbb{H}_T$ into itself, and is a contraction mapping. This clearly implies that $\tilde{\Gamma}$ has a unique fixed point in Σ_T , which gives a solution $(V_{g,h}, g, h)$ of (1.14) defined for $t \in (0, T]$. We will show that any solution (u, g, h) of (1.14) with $(g, h) \in \mathbb{G}_T \times \mathbb{H}_T$ must satisfy $(g, h) \in \Sigma_T$, and hence (g, h) must coincide with the unique fixed point of $\tilde{\Gamma}$ in Σ_T , which then implies that $(u, g, h) = (V_{g,h}, g, h)$ is the unique solution of (1.14).

We will finally show that this unique solution defined locally in time can be extended uniquely for all $t > 0$.

This plan is carried out below in four steps.

Step 1: *Properties of $(\tilde{g}, \tilde{h})$ and a closed subset of $\mathbb{G}_T \times \mathbb{H}_T$.*

Let $(g, h) \in \mathbb{G}_T \times \mathbb{H}_T$. The definitions of $\tilde{h}(t)$ and $\tilde{g}(t)$ indicate that they belong to $C^1([0, T])$ and for $0 < t \leq T$,

$$(2.9) \quad \begin{cases} \tilde{h}'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(y-x) dy V_{g,h}(t, x) dx, \\ \tilde{g}'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(y-x) dy V_{g,h}(t, x) dx. \end{cases}$$

These identities already imply $\tilde{\Gamma}(g, h) = (\tilde{g}, \tilde{h}) \in \mathbb{G}_T \times \mathbb{H}_T$, but in order to show $\tilde{\Gamma}$ is a contraction mapping, we need to prove some further properties of $\tilde{g}$ and $\tilde{h}$, and then choose a suitable closed subset of $\mathbb{G}_T \times \mathbb{H}_T$, which is invariant under $\tilde{\Gamma}$, and on which $\tilde{\Gamma}$ is a contraction mapping.

Since $v = V_{g,h}$ solves (2.1) we obtain by using **(f1)**-**(f2)** and (2.2) that

$$(2.10) \quad \begin{cases} (V_{g,h})_t(t, x) \geq -dV_{g,h}(t, x) - K(M_0)V_{g,h}(t, x), & 0 < t \leq T, \quad x \in (g(t), h(t)), \\ V_{g,h}(t, h(t)) = V_{g,h}(t, g(t)) = 0, & 0 < t \leq T, \\ V_{g,h}(0, x) = u_0(x), & x \in [-h_0, h_0]. \end{cases}$$

It follows that

$$(2.11) \quad V_{g,h}(t, x) \geq e^{-(d+K(M_0))t} u_0(x) \geq e^{-(d+K(M_0))T} u_0(x) \text{ for } x \in [-h_0, h_0], \quad t \in (0, T].$$

By **(J)** there exist constants $\epsilon_0 \in (0, h_0/4)$ and $\delta_0 > 0$ such that

$$(2.12) \quad J(z) \geq \delta_0 \text{ if } |z| \leq \epsilon_0.$$

Using (2.9) we easily see

$$[\tilde{h}(t) - \tilde{g}(t)]' \leq \mu M_0 [h(t) - g(t)] \text{ for } t \in [0, T].$$

We now assume that (g, h) has the extra property that

$$h(T) - g(T) \leq 2h_0 + \frac{\epsilon_0}{4}.$$

Then

$$\tilde{h}(t) - \tilde{g}(t) \leq 2h_0 + T\mu M_0(2h_0 + \frac{\epsilon_0}{4}) \leq 2h_0 + \frac{\epsilon_0}{4} \text{ for } t \in [0, T],$$

provided that $T > 0$ is small enough, depending on $(\mu, M_0, h_0, \epsilon_0)$. We fix such a T and notice from the above extra assumption on (g, h) that

$$h(t) \in [h_0, h_0 + \frac{\epsilon_0}{4}], \quad g(t) \in [-h_0 - \frac{\epsilon_0}{4}, -h_0] \text{ for } t \in [0, T].$$

Combining this with (2.11) and (2.12) we obtain, for such T and $t \in (0, T]$,

$$\begin{aligned} \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(y-x) V_{g,h}(t, x) dy dx &\geq \int_{h(t)-\frac{\epsilon_0}{2}}^{h(t)} \int_{h(t)}^{h(t)+\frac{\epsilon_0}{2}} J(y-x) V_{g,h}(t, x) dy dx \\ &\geq e^{-(d+K(M_0))T} \int_{h_0-\frac{\epsilon_0}{4}}^{h_0} \int_{h_0+\frac{\epsilon_0}{4}}^{h_0+\frac{\epsilon_0}{2}} J(y-x) dy u_0(x) dx \\ &\geq \frac{1}{4} \epsilon_0 \delta_0 e^{-(d+K(M_0))T} \int_{h_0-\frac{\epsilon_0}{4}}^{h_0} u_0(x) dx =: c_0 > 0, \end{aligned}$$

with c_0 depending only on (J, u_0, f) . Thus, for sufficiently small $T = T(\mu, M_0, h_0, \epsilon_0) > 0$,

$$(2.13) \quad \tilde{h}'(t) \geq \mu c_0 \text{ for } t \in [0, T].$$

We can similarly obtain, for such T ,

$$(2.14) \quad \tilde{g}'(t) \leq -\mu \tilde{c}_0 \text{ for } t \in [0, T],$$

for some positive constant $\tilde{c}_0$ depending only on (J, u_0, f) .

We now define, for $s \in (0, T_0] := (0, T(\mu, M_0, h_0, \epsilon_0)]$,

$$\begin{aligned} \Sigma_s := \left\{ (g, h) \in \mathbb{G}_s \times \mathbb{H}_s : \sup_{0 \leq t_1 < t_2 \leq s} \frac{g(t_2) - g(t_1)}{t_2 - t_1} \leq -\mu \tilde{c}_0, \inf_{0 \leq t_1 < t_2 \leq s} \frac{h(t_2) - h(t_1)}{t_2 - t_1} \geq \mu c_0, \right. \\ \left. h(t) - g(t) \leq 2h_0 + \frac{\epsilon_0}{4} \text{ for } t \in [0, s] \right\}. \end{aligned}$$

Our analysis above shows that

$$\tilde{\Gamma}(\Sigma_s) \subset \Sigma_s \text{ for } s \in (0, T_0].$$

Step 2: $\tilde{\Gamma}$ is a contraction mapping on Σ_s for sufficiently small $s > 0$.

Let $s \in (0, T_0]$, $(h_1, g_1), (h_2, g_2) \in \Sigma_s$, and note that Σ_s is a complete metric space under the metric

$$d((h_1, g_1), (h_2, g_2)) = \|h_1 - h_2\|_{C([0, s])} + \|g_1 - g_2\|_{C([0, s])}.$$

For $i = 1, 2$, let us denote

$$V_i(t, x) := V_{h_i, g_i}(t, x) \text{ and } \tilde{\Gamma}(h_i, g_i) := (\tilde{h}_i, \tilde{g}_i).$$

We also define

$$\begin{aligned} H_{min}(t) &:= \min \{h_1(t), h_2(t)\}, \quad H_{max}(t) := \max \{h_1(t), h_2(t)\}, \\ G_{min}(t) &:= \min \{g_1(t), g_2(t)\}, \quad G_{max}(t) := \max \{g_1(t), g_2(t)\}, \\ \Omega_{*s} = \Omega_{G_{min}, H_{max}} &:= \Omega_{g_1, h_1} \cup \Omega_{g_2, h_2}. \end{aligned}$$

For $t \in [0, s]$, we have

$$2h_0 \leq H_{max}(t) - G_{min}(t) \leq 2h_0 + \epsilon_0 \leq 3h_0,$$

and

$$\begin{aligned} &\left| \tilde{h}_1(t) - \tilde{h}_2(t) \right| \\ &\leq \mu \int_0^t \left| \int_{g_1(\tau)}^{h_1(\tau)} \int_{h_1(\tau)}^{+\infty} J(y-x) V_1(\tau, x) dy dx d\tau - \int_{g_2(\tau)}^{h_2(\tau)} \int_{h_2(\tau)}^{+\infty} J(y-x) V_2(\tau, x) dy dx \right| d\tau \\ &\leq \mu \int_0^t \int_{g_1(\tau)}^{h_1(\tau)} \int_{h_1(\tau)}^{+\infty} J(y-x) \left| V_1(\tau, x) - V_2(\tau, x) \right| dy dx d\tau \\ &\quad + \mu \int_0^t \left| \left(\int_{h_1(\tau)}^{h_2(\tau)} \int_{h_1(\tau)}^{+\infty} + \int_{g_2(\tau)}^{g_1(\tau)} \int_{h_1(\tau)}^{+\infty} + \int_{g_2(\tau)}^{h_2(\tau)} \int_{h_1(\tau)}^{h_2(\tau)} \right) J(y-x) V_2(t, x) dy dx \right| d\tau \\ &\leq 3h_0 \mu \|V_1 - V_2\|_{C(\bar{\Omega}_{*s})} s + \mu M_0 (1 + 3h_0 \|J\|_\infty) \|h_1 - h_2\|_{C([0, s])} s + \mu M_0 \|g_1 - g_2\|_{C([0, s])} s \\ &\leq C_0 s \left[\|V_1 - V_2\|_{C(\bar{\Omega}_{*s})} + \|h_1 - h_2\|_{C([0, s])} + \|g_1 - g_2\|_{C([0, s])} \right], \end{aligned}$$

where C_0 depends only on (μ, u_0, J, f) . Let us recall that V_i is always extended by 0 in $([0, \infty) \times \mathbb{R}) \setminus \Omega_{g_i, h_i}$ for $i = 1, 2$.

Similarly, we have, for $t \in [0, s]$,

$$\left| \tilde{g}_1(t) - \tilde{g}_2(t) \right| \leq C_0 s \left[\|V_1 - V_2\|_{C(\bar{\Omega}_s)} + \|h_1 - h_2\|_{C([0,s])} + \|g_1 - g_2\|_{C([0,s])} \right].$$

Therefore,

$$(2.15) \quad \begin{aligned} & \|\tilde{h}_1 - \tilde{h}_2\|_{C([0,s])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C([0,s])} \\ & \leq 2C_0 s \left[\|V_1 - V_2\|_{C(\bar{\Omega}_{*s})} + \|h_1 - h_2\|_{C([0,s])} + \|g_1 - g_2\|_{C([0,s])} \right]. \end{aligned}$$

Next, we estimate $\|V_1 - V_2\|_{C(\bar{\Omega}_{*s})}$. We denote $U = V_1 - V_2$, and for fixed $(t^*, x^*) \in \Omega_{*s}$, we consider three cases separately.

Case 1. $x^* \in [-h_0, h_0]$.

It follows from the equations satisfied by V_1 and V_2 that $U(0, x^*) = 0$ and for $0 < t \leq s$,

$$(2.16) \quad U_t(t, x^*) + c_1(t, x^*)U(t, x^*) = A(t, x^*),$$

where

$$\begin{aligned} c_1(t, x^*) &:= d - \frac{f(t, x^*, V_1(t, x^*)) - f(t, x^*, V_2(t, x^*))}{V_1(t, x^*) - V_2(t, x^*)} \text{ and so } \|c_1\|_\infty \leq d + K(M_0), \\ A(t, x^*) &:= d \int_{g_1(t)}^{h_1(t)} J(x^* - y) V_1(t, y) dy - d \int_{g_2(t)}^{h_2(t)} J(x^* - y) V_2(t, y) dy. \end{aligned}$$

Thus

$$U(t^*, x^*) = e^{-\int_0^{t^*} c_1(\tau, x^*) d\tau} \int_0^{t^*} e^{\int_0^\tau c_1(\tau, x^*) d\tau} A(t, x^*) dt.$$

We have

$$\begin{aligned} |A(t, x^*)| &= d \left| \int_{g_1(t)}^{h_1(t)} J(x^* - y) V_1(t, y) dy - \int_{g_2(t)}^{h_2(t)} J(x^* - y) V_2(t, y) dy \right| \\ &\leq d \int_{g_1(t)}^{h_1(t)} J(x^* - y) |V_1(t, y) - V_2(t, y)| dy + d \left| \left(\int_{g_2(t)}^{g_1(t)} + \int_{h_1(t)}^{h_2(t)} \right) J(x^* - y) V_2(t, y) dy \right| \\ &\leq d \|U\|_{C(\bar{\Omega}_{*s})} + d \|J\|_\infty M_0 [\|h_1 - h_2\|_{C([0,s])} + \|g_1 - g_2\|_{C([0,s])}]. \end{aligned}$$

Thus for some $C_1 > 0$ depending only on (d, u_0, M_0, J) , we have

$$(2.17) \quad \max_{t \in [0, s]} |A(t, x^*)| \leq C_1 \left(\|U\|_{C(\bar{\Omega}_{*s})} + \|h_1 - h_2\|_{C([0,s])} + \|g_1 - g_2\|_{C([0,s])} \right).$$

It follows that

$$(2.18) \quad |U(t^*, x^*)| \leq C_1 s e^{2(d+K(M_0))s} \left(\|U\|_{C(\bar{\Omega}_{*s})} + \|h_1 - h_2\|_{C([0,s])} + \|g_1 - g_2\|_{C([0,s])} \right).$$

Case 2. $x^* \in (h_0, H_{\min}(s))$.

In this case there exist $t_1^*, t_2^* \in (0, t^*)$ such that $x^* = h_1(t_1^*) = h_2(t_2^*)$. Without loss of generality, we may assume that $0 < t_1^* \leq t_2^*$. Now we use (2.16) for $t \in [t_2^*, t^*]$, and obtain

$$U(t^*, x^*) = e^{-\int_{t_2^*}^{t^*} c_1(\tau, x^*) d\tau} \left[U(t_2^*, x^*) + \int_{t_2^*}^{t^*} e^{\int_{t_2^*}^\tau c_1(\tau, x^*) d\tau} A(t, x^*) dt \right].$$

It follows that

$$(2.19) \quad \begin{aligned} |U(t^*, x^*)| &\leq e^{(d+K(M_0))t^*} \left[|U(t_2^*, x^*)| + \int_{t_2^*}^{t^*} e^{(d+K(M_0))t} |A(t, x^*)| dt \right] \\ &\leq e^{(d+K(M_0))s} |U(t_2^*, x^*)| + s e^{2(d+K(M_0))s} \max_{t \in [0, s]} |A(t, x^*)|. \end{aligned}$$

Since $V_1(t_1^*, x^*) = V_2(t_2^*, x^*) = 0$, we have

$$U(t_2^*, x^*) = V_1(t_2^*, x^*) - V_1(t_1^*, x^*) = \int_{t_1^*}^{t_2^*} (V_1)_t(t, x^*) dt,$$

and hence from the equation satisfied by V_1 we obtain

$$\begin{aligned} |U(t_2^*, x^*)| &\leq \int_{t_1^*}^{t_2^*} \left| d \int_{g_1(t)}^{h_1(t)} J(x^* - y) V_1(t, y) dy - dV_1(t, x^*) + f(t, x^*, V_1(t, x^*)) \right| dt \\ &\leq C_2 (t_2^* - t_1^*), \quad \text{for some } C_2 > 0 \text{ depending only on } (d, M_0, f). \end{aligned}$$

If $t_1^* = t_2^*$ then clearly $U(t_2^*, x^*) = 0$. If $t_1^* < t_2^*$, then using $\frac{h_1(t_2^*) - h_1(t_1^*)}{t_2^* - t_1^*} \geq \mu c_0$ we obtain

$$t_2^* - t_1^* \leq |h_1(t_2^*) - h_1(t_1^*)| (\mu c_0)^{-1}.$$

Since

$$0 = h_1(t_1^*) - h_2(t_2^*) = h_1(t_1^*) - h_1(t_2^*) + h_1(t_2^*) - h_2(t_2^*),$$

we have $h_1(t_2^*) - h_1(t_1^*) = h_1(t_2^*) - h_2(t_2^*)$, and thus

$$t_2^* - t_1^* \leq |h_1(t_2^*) - h_1(t_1^*)| (\mu c_0)^{-1} = |h_1(t_2^*) - h_2(t_2^*)| (\mu c_0)^{-1}.$$

Therefore there exists some positive constant $C_3 = C_3(\mu c_0, C_2)$ such that

$$|U(t_2^*, x^*)| \leq C_3 \|h_1 - h_2\|_{C([0, s])}.$$

Substituting this and (2.17) proved in Case 1 above to (2.19), we obtain

$$\begin{aligned} (2.20) \quad |U(t^*, x^*)| &\leq e^{(d+K(M_0))s} C_3 \|h_1 - h_2\|_{C([0, s])} \\ &\quad + C_1 s e^{2(d+K(M_0))s} \left(\|U\|_{C(\bar{\Omega}_{*s})} + \|h_1 - h_2\|_{C([0, s])} + \|g_1 - g_2\|_{C([0, s])} \right). \end{aligned}$$

Case 3. $x^* \in [H_{\min}(s), H_{\max}(s)]$.

Without loss of generality we assume that $h_1(s) < h_2(s)$. Then $H_1(s) = h_1(s)$, $H_2(s) = h_2(s)$ and

$$h_1(t^*) \leq h_1(s) < x^* < H_2(t^*) = h_2(t^*),$$

$$V_1(t, x^*) = 0 \text{ for } t \in [t_2^*, t^*], \quad 0 < h_2(t^*) - h_2(t_2^*) \leq h_2(t^*) - h_1(t^*).$$

We have

$$\begin{aligned} 0 < V_2(t^*, x^*) &= \int_{t_2^*}^{t^*} \left[d \int_{g_2(t)}^{h_2(t)} J(x^* - y) V_2(t, y) dy - dV_2(t, x^*) + f(t, x^*, V_2(t, x^*)) \right] dt \\ &\leq (t^* - t_2^*) [d + K(M_0)] M_0 \\ &\leq [h_2(t^*) - h_2(t_2^*)] (\mu c_0)^{-1} [d + K(M_0)] M_0 \\ &\leq (\mu c_0)^{-1} [d + K(M_0)] M_0 [h_2(t^*) - h_1(t^*)] \\ &\leq C_4 \|h_1 - h_2\|_{C([0, s])}, \end{aligned}$$

with $C_4 := (\mu c_0)^{-1} [d + K(M_0)] M_0$.

We thus obtain

$$(2.21) \quad |U(t^*, x^*)| = V_2(t^*, x^*) \leq C_4 \|h_1 - h_2\|_{C([0, s])}.$$

The inequalities (2.18), (2.20) and (2.21) indicate that, there exists $C_5 > 0$ depending only on $(\mu c_0, d, u_0, J, f)$ such that, whether we are in Cases 1, 2 or 3, we always have

$$(2.22) \quad |U(t^*, x^*)| \leq C_5 \left(\|U\|_{C(\bar{\Omega}_{*s})} s + \|h_1 - h_2\|_{C([0, s])} + \|g_1 - g_2\|_{C([0, s])} \right).$$

Analogously, we can examine the cases $x^* \in (G_2(s), -h_0)$ and $x^* \in (G_1(s), G_2(s))$ to obtain a constant $C_6 > 0$ depending only on $(\mu \tilde{c}_0, d, u_0, J, f)$ such that (2.22) holds with C_5 replaced by C_6 . Setting $C^* := \max\{C_5, C_6\}$, we thus obtain

$$|U(t^*, x^*)| \leq C^* \left(\|U\|_{C(\bar{\Omega}_{*s})} s + \|h_1 - h_2\|_{C([0, s])} + \|g_1 - g_2\|_{C([0, s])} \right) \text{ for all } (t^*, x^*) \in \Omega_{*s}.$$

It follows that

$$\|U\|_{C(\bar{\Omega}_{*s})} \leq C^* \left(\|U\|_{C(\bar{\Omega}_{*s})} s + \|h_1 - h_2\|_{C([0, s])} + \|g_1 - g_2\|_{C([0, s])} \right).$$

Let us recall that the above inequality holds for all $s \in (0, T_0]$ with T_0 given near the end of Step 1. Set $T_1 := \min\left\{T_0, \frac{1}{2C^*}\right\}$. Then we easily deduce

$$\|U\|_{C(\bar{\Omega}_{*s})} \leq 2C^* \left(\|h_1 - h_2\|_{C([0, s])} + \|g_1 - g_2\|_{C([0, s])} \right) \text{ for } s \in (0, T_1].$$

Substituting this inequality into (2.15) we obtain, for $s \in (0, T_1]$,

$$\begin{aligned} & \|\tilde{h}_1 - \tilde{h}_2\|_{C([0,s])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C([0,s])} \\ & \leq 2C_0(2C^* + 1)s [\|h_1 - h_2\|_{C([0,s])} + \|g_1 - g_2\|_{C([0,s])}]. \end{aligned}$$

Thus if we define T_2 by $2C_0(2C^* + 1)T_2 = \frac{1}{2}$, and $T^* := \min\{T_1, T_2\}$, then

$$\|\tilde{h}_1 - \tilde{h}_2\|_{C([0,T^*])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T^*])} \leq \frac{1}{2} [\|h_1 - h_2\|_{C([0,T^*])} + \|g_1 - g_2\|_{C([0,T^*])}],$$

i.e., $\tilde{\Gamma}$ is a contraction mapping on Σ_{T^*} .

Step 3: Local existence and uniqueness.

By Step 2 and the Contraction Mapping Theorem we know that (1.14) has a solution (u, g, h) for $t \in (0, T^*]$. If we can show that $(g, h) \in \Sigma_{T^*}$ holds for any solution (u, g, h) of (1.14) defined over $t \in (0, T^*]$, then it is the unique fixed point of $\tilde{\Gamma}$ in Σ_{T^*} and the uniqueness of (u, g, h) follows.

So let (u, g, h) be an arbitrary solution of (1.14) defined for $t \in (0, T^*]$. Then

$$\begin{cases} h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(y-x)u(t, x)dydx, \\ g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(y-x)u(t, x)dydx. \end{cases}$$

By Lemma 2.2, we have

$$0 < u(t, x) \leq M_0 \text{ for } t \in [0, T^*], x \in (g(t), h(t)).$$

It follows that

$$[h(t) - g(t)]' = \mu \int_{g(t)}^{h(t)} \left[1 - \int_{g(t)}^{h(t)} J(y-x)dy\right] u(t, x)dx \leq \mu M_0 [h(t) - g(t)] \text{ for } t \in (0, T^*].$$

We thus obtain

$$(2.23) \quad h(t) - g(t) \leq 2h_0 e^{\mu M_0 t} \text{ for } t \in (0, T^*].$$

Therefore if we shrink T^* if necessary so that

$$2h_0 e^{\mu M_0 T^*} \leq 2h_0 + \frac{\epsilon_0}{4},$$

then

$$h(t) - g(t) \leq 2h_0 + \frac{\epsilon_0}{4} \text{ for } t \in [0, T^*].$$

Moreover, the proof of (2.13) and (2.14) gives

$$h'(t) \geq \mu c_0, \quad g'(t) \leq -\mu \tilde{c}_0 \text{ for } t \in (0, T^*].$$

Thus indeed $(g, h) \in \Sigma_{T^*}$, as we wanted. This proves the local existence and uniqueness of the solution to (1.14).

Step 4: Global existence and uniqueness.

By Step 3, we see the (1.14) has a unique solution (u, g, h) for some initial time interval $(0, T)$, and for any $s \in (0, T)$, $u(s, x) > 0$ for $x \in (g(s), h(s))$ and $u(s, \cdot)$ is continuous over $[g(s), h(s)]$. This implies that we can treat $u(s, \cdot)$ as an initial function and use Step 3 to extend the solution from $t = s$ to some $T' \geq T$. Suppose $(0, \hat{T})$ is the maximal interval that the solution (u, g, h) of (1.14) can be defined through this extension process. We show that $\hat{T} = \infty$. Otherwise $\hat{T} \in (0, \infty)$ and we are going to derive a contradiction.

Firstly we notice that (2.23) now holds for $t \in (0, \hat{T})$. Since $h(t)$ and $g(t)$ are monotone functions over $[0, \hat{T})$, we may define

$$h(\hat{T}) := \lim_{t \rightarrow \hat{T}} h(t), \quad g(\hat{T}) := \lim_{t \rightarrow \hat{T}} g(t) \quad \text{with } h(\hat{T}) - g(\hat{T}) \leq 2h_0 e^{\mu M_0 \hat{T}}.$$

The third and fourth equations in (1.14), together with $0 \leq u \leq M_0$ indicate that h' and g' belong to $L^\infty([0, \hat{T}))$ and hence with $g(\hat{T})$ and $h(\hat{T})$ defined as above, $g, h \in C([0, \hat{T}])$. It also follows that the

right-hand side of the first equation in (1.14) belongs to $L^\infty(\Omega_{\hat{T}})$, where $\Omega_{\hat{T}} := \{(t, x) : t \in [0, \hat{T}], g(t) < x < h(t)\}$. It follows that $u_t \in L^\infty(\Omega_{\hat{T}})$. Thus for each $x \in (g(\hat{T}), h(\hat{T}))$,

$$u(\hat{T}, x) := \lim_{t \nearrow \hat{T}} u(t, x) \text{ exists,}$$

and $u(\cdot, x)$ is continuous at $t = \hat{T}$. We may now view $u(t, x)$ as the unique solution of the ODE problem in Step 1 of the proof of Lemma 2.2 (with $\phi = u$), which is defined over $[t_x, \hat{T}]$. Since t_x , $J(x - y)$ and $f(t, x, u)$ are all continuous in x , by the continuous dependence of the ODE solution to the initial function and the parameters in the equation, we see that $u(t, x)$ is continuous in $\Omega_{\hat{T}}$. By assumption, $u \in C(\bar{\Omega}_s)$ for any $s \in (0, \hat{T})$. To show this also holds with $s = \hat{T}$, it remains to show that

$$u(t, x) \rightarrow 0 \text{ as } (t, x) \rightarrow (\hat{T}, g(\hat{T})) \text{ and as } (t, x) \rightarrow (\hat{T}, h(\hat{T})) \text{ from } \Omega_{\hat{T}}.$$

We only prove the former as the other case can be shown similarly. We note that as $x \searrow g(\hat{T})$, we have $t_x \nearrow \hat{T}$, and so

$$\begin{aligned} |u(t, x)| &= \left| \int_{t_x}^t \left[d \int_{g(\tau)}^{h(\tau)} J(x - y) u(\tau, y) dy - du(\tau, x) + f(\tau, x, u(\tau, x)) \right] d\tau \right| \\ &\leq (t - t_x) [2d + K(M_0)] M_0 \\ &\rightarrow 0 \text{ as } \Omega_{\hat{T}} \ni (t, x) \rightarrow (\hat{T}, g(\hat{T})). \end{aligned}$$

Thus we have shown that $u \in C(\bar{\Omega}_{\hat{T}})$ and (u, g, h) satisfies (1.14) for $t \in (0, \hat{T}]$. By Lemma 2.2 we have $u(\hat{T}, x) > 0$ for $x \in (g(\hat{T}), h(\hat{T}))$. Thus we can regard $u(\hat{T}, \cdot)$ as an initial function and apply Step 3 to conclude that the solution of (1.14) can be extended to some $(0, \tilde{T})$ with $\tilde{T} > \hat{T}$. This contradicts the definition of $\hat{T}$. Therefore we must have $\hat{T} = \infty$. $\square$

Remark: The material in this section is taken from [3] with some minor variations.

3. SPREADING-VANISHING DICHOTOMY AND CRITERIA

We investigate the long-time dynamics of

$$(3.1) \quad \begin{cases} u_t = d \int_{g(t)}^{h(t)} J(x - y) u(t, y) dy - du + f(u), & t > 0, x \in (g(t), h(t)), \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(y - x) u(t, x) dy dx, & t > 0, \\ g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(y - x) u(t, x) dy dx, & t > 0, \\ u(0, x) = u_0(x), h(0) = -g(0) = h_0, & x \in [-h_0, h_0], \end{cases}$$

where d, μ, h_0 are given positive constants. The initial function $u_0(x)$ satisfies (1.15). The kernel function $J : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the basic condition

$$(\mathbf{J}): J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}), J \geq 0, J(0) > 0, \int_{\mathbb{R}} J(x) dx = 1.$$

The growth term $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies the KPP condition

$$(\mathbf{f}_{\text{KPP}}): \begin{cases} f \in C^1, f(0) = f(1) = 0, f'(0) > 0 > f'(1), \\ f(u)/u \text{ is non-increasing in } (0, \infty). \end{cases}$$

We are going to prove the following two theorems from [3].

Theorem 3.1 (Spreading-vanishing dichotomy). *Suppose $(\mathbf{J})$ and $(\mathbf{f}_{\text{KPP}})$ hold, u_0 satisfies (1.15) and J is symmetric: $J(x) = J(-x)$. Let (u, g, h) be the unique solution of problem (3.1). Then one of the following alternatives must happen for (3.1):*

- (i) Spreading: $\begin{cases} \lim_{t \rightarrow +\infty} (g(t), h(t)) = \mathbb{R}, \\ \lim_{t \rightarrow +\infty} u(t, x) = 1 \text{ locally uniformly in } \mathbb{R}, \end{cases}$
- (ii) Vanishing: $\begin{cases} \lim_{t \rightarrow +\infty} (g(t), h(t)) = (g_\infty, h_\infty) \text{ is a finite interval,} \\ \lim_{t \rightarrow +\infty} u(t, x) = 0 \text{ uniformly for } x \in [g(t), h(t)]. \end{cases}$

Theorem 3.2 (Spreading-vanishing criteria). *Under the conditions of Theorem 3.1, if $d \in (0, f'(0)]$, then spreading always happens. If $d > f'(0)$, then there exists a unique $\ell^* > 0$ such that spreading always happens if $h_0 \geq \ell^*/2$; and for $h_0 \in (0, \ell^*/2)$, there exists a unique $\mu^* > 0$ so that spreading happens exactly when $\mu > \mu^*$.*

As we will see in the proof, ℓ^* depends only on $(f'(0), d, J)$. On the other hand, μ^* depends also on u_0 .

Extension to weakly non-symmetric kernels

It turns out that the symmetry requirement of J in Theorems 3.1 and 3.2 can be significantly relaxed in the above two theorems. For a non-symmetric J satisfying **(J)**, the following two quantities determined by J and $f'(0)$ alone play an important role:

$$c_*^- = \sup_{\nu < 0} \frac{d \int_{\mathbb{R}} J(x) e^{\nu x} dx - d + f'(0)}{\nu}, \quad c_*^+ = \inf_{\nu > 0} \frac{d \int_{\mathbb{R}} J(x) e^{\nu x} dx - d + f'(0)}{\nu},$$

It can be shown that c_*^- is achieved by some $\nu < 0$ when it is finite, and a parallel conclusion holds for c_*^+ . It is easily checked that c_*^- is finite if and only if J satisfies additionally the following **thin-tail** condition at $x = -\infty$,

$$(\mathbf{J}_{\text{thin}}^-) : \text{There exists } \lambda > 0 \text{ such that } \int_0^{+\infty} J(-x) e^{\lambda x} dx < +\infty.$$

Similarly, c_*^+ is finite if and only if J satisfies

$$(\mathbf{J}_{\text{thin}}^+) : \text{There exists } \lambda > 0 \text{ such that } \int_0^{+\infty} J(x) e^{\lambda x} dx < +\infty.$$

If we define

$$(3.2) \quad \begin{cases} c_*^- = -\infty & \text{when } (\mathbf{J}_{\text{thin}}^-) \text{ does not hold,} \\ c_*^+ = +\infty & \text{when } (\mathbf{J}_{\text{thin}}^+) \text{ does not hold,} \end{cases}$$

then the propagation dynamics of the corresponding Cauchy problem of (1.14),

$$(3.3) \quad \begin{cases} U_t = d \int_{\mathbb{R}} J(x-y) U(t, y) dy - dU(t, x) + f(U), & t > 0, x \in \mathbb{R}, \\ U(0, x) = U_0(x) \end{cases}$$

has the properties described in the following result:

Theorem A. ([6]) *Suppose that **(J)** and **(f_{KPP})** hold. Then for any initial function $U_0(x)$ which is continuous and nonnegative with non-empty compact support, the unique solution $U(t, x)$ of (3.3) satisfies*

$$\lim_{t \rightarrow \infty} U(t, x) = \begin{cases} 1 & \text{uniformly for } x \in [a_1 t, b_1 t] \text{ provided that } [a_1, b_1] \subset (c_*^-, c_*^+), \\ 0 & \text{uniformly for } x \leq a_2 t \text{ provided that } c_*^- > -\infty \text{ and } a_2 < c_*^-, \\ 0 & \text{uniformly for } x \geq b_2 t \text{ provided that } c_*^+ < \infty \text{ and } b_2 > c_*^+. \end{cases}$$

Following [1], the conclusions in Theorem A can be interpreted as indicating a leftward spreading speed of c_*^- and rightward spreading speed of c_*^+ for (3.3). The following result of Yagisita [18] (see also Theorem 1.5 in [4]) on traveling waves provides further meanings for c_*^- and c_*^+ .

Theorem B. ([18]) *Suppose that **(J)** and **(f_{KPP})** are satisfied. Then the following conclusions hold.*

(i) *The rightward traveling wave problem*

$$(3.4) \quad \begin{cases} d \int_{\mathbb{R}} J(x-y) \phi(y) dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, & x \in \mathbb{R}, \\ \phi(-\infty) = 1, \quad \phi(+\infty) = 0 \end{cases}$$

has a solution pair $(c, \phi) \in \mathbb{R} \times L^\infty(\mathbb{R})$ with ϕ nonincreasing if and only if $c_^+ < \infty$. Moreover, in such a case, for every $c \geq c_*^+$, (3.4) has a solution $\phi \in C^1(\mathbb{R})$ that is strictly decreasing, and (3.4) has no such solution for $c < c_*^+$.*

(ii) *The leftward traveling wave problem*

$$(3.5) \quad \begin{cases} d \int_{\mathbb{R}} J(x-y)\psi(y) dy - d\psi(x) - c\psi'(x) + f(\psi(x)) = 0, & x \in \mathbb{R}, \\ \psi(-\infty) = 0, \quad \psi(+\infty) = 1, \end{cases}$$

has a solution pair $(c, \psi) \in \mathbb{R} \times L^\infty(\mathbb{R})$ with ψ nondecreasing if and only if $c_*^- > -\infty$. Moreover, in such a case, for each $c \geq -c_*^-$, (3.5) has a solution $\psi \in C^1(\mathbb{R})$ that is strictly increasing, and (3.5) has no such solution for $c < -c_*^-$.

Problem (3.3) and its many variations have been extensively studied in the literature; see, for example, [2, 4, 15, 17] and the references therein as a small sample of these works. It can be shown as in [9] that (3.3) is the limiting problem of (1.14) when $\mu \rightarrow \infty$.

Definition: For a kernel function J satisfying **(J)** we say it is **weakly non-symmetric** if

$$(3.6) \quad -\infty \leq c_*^- < 0 < c_*^+ \leq \infty.$$

Theorem 3.3. *Theorems 3.1 and 3.2 remain valid if $J(x)$ is weakly non-symmetric.*

Remark: If $J(x)$ is not weakly non-symmetric, then fundamental differences arise in the long-time behaviour of (1.14); such a case was considered in [7].

3.1. The associated problem over a fixed spatial interval. For $c \in \mathbb{R}$ and $\Omega = (l_1, l_2)$ a bounded interval, define

$$\mathcal{L}_\Omega^c[\phi](x) := d \int_\Omega J(x-y)\phi(y) dy - d\phi(x) + c\phi'(x) + f'(0)\phi(x), \quad \phi \in C^1(\Omega) \cap C(\bar{\Omega}).$$

It is known [16, 5] that

$$\lambda_p(\mathcal{L}_\Omega^c) := \inf\{\lambda \in \mathbb{R} : \mathcal{L}_\Omega^c[\phi] \leq \lambda\phi, \phi > 0 \text{ in } \Omega \text{ for some } \phi \in C(\bar{\Omega})\}$$

is a principal eigenvalue of $\mathcal{L}_\Omega^c$, which corresponds to a positive eigenfunction. From the definition it is easily seen that

$$\lambda_p(\mathcal{L}_{(l_1, l_2)}^c) = \lambda_p(\mathcal{L}_{(0, l_2-l_1)}^c).$$

Moreover, the following conclusions hold:

Proposition 3.4 ([5, 6]). *Suppose that the kernel J satisfies **(J)** and $c \in \mathbb{R}$. Then $l \rightarrow \lambda_p(\mathcal{L}_{(-l, l)}^c)$ is continuous and strictly increasing in $l \in (0, \infty)$, and*

$$\lim_{l \rightarrow \infty} \lambda_p(\mathcal{L}_{(-l, l)}^c) = \inf_{\nu \in \mathbb{R}} \left[d \int_{\mathbb{R}} J(x) e^{-\nu x} dx + c\nu \right] - d + f'(0).$$

Moreover,

$$\lim_{l \rightarrow \infty} \lambda_p(\mathcal{L}_{(-l, l)}^c) > 0 \text{ if and only if } c \in (c_*^-, c_*^+).$$

Proof. The continuity and monotonicity property of $l \rightarrow \lambda_p(\mathcal{L}_{(-l, l)}^c)$ were proved in [5], the formula for the limit $\lim_{l \rightarrow \infty} \lambda_p(\mathcal{L}_{(-l, l)}^c)$ is given in Theorem 1.2 of [6], and the last conclusion is taken from Proposition 5.1 of [6]. $\square$

Consider the problem

$$(3.7) \quad \begin{cases} V_t = d \int_{-l}^l J(x-y)V(t, y) dy - dV + f(V), & t > 0, x \in (-l, l), \\ V(0, x) = V_0(x), & x \in [-l, l]. \end{cases}$$

By Theorem 1.3 of [6], the following conclusion holds.

Proposition 3.5 ([6]). *Suppose that **(J)** and **(f_{KPP})** hold, and $V_0 \in C([-l, l])$ is nonnegative and not identically 0. Then (3.7) has a unique solution $V(t, x)$ and*

$$\lim_{t \rightarrow \infty} V(t, x) = \begin{cases} 0 & \text{uniformly in } x \in [-l, l] \text{ if } \lambda_p(\mathcal{L}_{(-l, l)}^0) \leq 0, \\ V_l(x) & \text{uniformly in } x \in [-l, l] \text{ if } \lambda_p(\mathcal{L}_{(-l, l)}^0) > 0, \end{cases}$$

where $V_l(x)$ is the unique positive stationary solution of (3.7). Moreover, when $\lambda_p(\mathcal{L}_{\mathbb{R}}^0) > 0$ and hence $\lambda_p(\mathcal{L}_{(-l, l)}^0) > 0$ for all large $l > 0$, we have

$$\lim_{l \rightarrow \infty} V_l(x) = 1 \text{ uniformly for } x \text{ in any bounded interval of } \mathbb{R}.$$

Lemma 3.6. Assume **(J)** and **(f_{KPP})** hold and J is weakly non-symmetric, i.e., (3.6) holds. Then there exists $l_* \geq 0$ such that $\lambda_p(\mathcal{L}_{(-l,l)}^0) > 0$ if and only if $l > l_*$; moreover, $l_* = 0$ when $f'(0) \geq d$, and $l_* > 0$ when $f'(0) < d$.

Proof. We first prove the following conclusion:

$$\lim_{l \rightarrow 0} \lambda_p(\mathcal{L}_{(-l,l)}^0) = f'(0) - d.$$

Since $\lambda_l := \lambda_p(\mathcal{L}_{(-l,l)}^0)$ is a principal eigenvalue, there exists a strictly positive function $\phi_l \in C([-l, l])$ such that

$$d \int_{-l}^l J(x-y) \phi_l(y) dy - d \phi_l(x) + f'(0) \phi_l(x) = \lambda_l \phi_l \quad \text{in } [-l, l].$$

Therefore

$$\begin{aligned} |\lambda_l - f'(0) + d| &= \frac{d \int_{-l}^l \int_{-l}^l J(x-y) \phi_l(y) \phi_l(x) dy dx}{\int_{-l}^l \phi_l^2(x) dx} \leq \frac{d \|J\|_\infty \left(\int_{-l}^l \phi_l(x) dx \right)^2}{\int_{-l}^l \phi_l^2(x) dx} \\ &\leq \frac{d \|J\|_\infty 2l \int_{-l}^l \phi_h^2(x) dx}{\int_{-l}^l \phi_h^2(x) dx} = 2ld \|J\|_\infty \rightarrow 0 \quad \text{as } l \rightarrow 0^+. \end{aligned}$$

By Proposition 3.4, $l \rightarrow \lambda_l$ is continuous and strictly increasing, and due to (3.6), $\lim_{l \rightarrow \infty} \lambda_l > 0$. Therefore,

$$d \in (0, f'(0)] \implies \lambda_l > \lim_{h \rightarrow 0} \lambda_h = f'(0) - d \geq 0 \text{ for every fixed } l > 0,$$

and $d > f'(0)$ implies the existence of a unique $l_* > 0$ such that

$$\lambda_l < 0 \text{ for } l \in (0, l_*), \quad \lambda_{l_*} = 0, \quad \lambda_l > 0 \text{ for } l > l_*.$$

This completes the proof. $\square$

3.2. Proof of Theorem 3.1. Throughout this subsection, we assume that **(J)**, **(f_{KPP})** hold and J is weakly non-symmetric, i.e., (3.6) holds.

Lemma 3.7. If $h_\infty - g_\infty < +\infty$, then $u(t, x) \rightarrow 0$ uniformly in $[g(t), h(t)]$ as $t \rightarrow +\infty$ and $\lambda_p(\mathcal{L}_{(g_\infty, h_\infty)}^0) \leq 0$.

Proof. We first prove that

$$\lambda_p(\mathcal{L}_{(g_\infty, h_\infty)}^0) \leq 0.$$

Suppose that $\lambda_p(\mathcal{L}_{(g_\infty, h_\infty)}^0) > 0$. Then $\lambda_p(\mathcal{L}_{(g_\infty + \epsilon, h_\infty - \epsilon)}^0) > 0$ for small $\epsilon > 0$, say $\epsilon \in (0, \epsilon_1)$. Moreover, for such ϵ , there exists $T_\epsilon > 0$ such that

$$h(t) > h_\infty - \epsilon, \quad g(t) < g_\infty + \epsilon \quad \text{for } t > T_\epsilon.$$

Consider the problem

$$(3.8) \quad \begin{cases} w_t = d \int_{g_\infty + \epsilon}^{h_\infty - \epsilon} J(x-y) w(t, y) dy - dw + f(w), & t > T_\epsilon, \quad x \in [g_\infty + \epsilon, h_\infty - \epsilon], \\ w(T_\epsilon, x) = u(T_\epsilon, x), & x \in [g_\infty + \epsilon, h_\infty - \epsilon]. \end{cases}$$

Since $\lambda_p(\mathcal{L}_{(g_\infty + \epsilon, h_\infty - \epsilon)}^0) > 0$, Proposition 3.5 indicates that the solution $w_\epsilon(t, x)$ of (3.8) converges to the unique steady state $W_\epsilon(x)$ of (3.8) uniformly in $[g_\infty + \epsilon, h_\infty - \epsilon]$ as $t \rightarrow +\infty$.

Moreover, by the maximum principle Theorem 2.1 and a simple comparison argument we have

$$u(t, x) \geq w_\epsilon(t, x) \quad \text{for } t > T_\epsilon \text{ and } x \in [g_\infty + \epsilon, h_\infty - \epsilon].$$

Thus, there exists $T_{1\epsilon} > T_\epsilon$ such that

$$u(t, x) \geq \frac{1}{2} W_\epsilon(x) > 0 \quad \text{for } t > T_{1\epsilon} \text{ and } x \in [g_\infty + \epsilon, h_\infty - \epsilon].$$

Note that since $J(0) > 0$, there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that $J(x) > \delta_0$ if $|x| < \epsilon_0$. Thus for $0 < \epsilon < \min\{\epsilon_1, \epsilon_0/2\}$ and $t > T_{1\epsilon}$, we have

$$\begin{aligned} h'(t) &= \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(y-x)u(t,x)dydx \geq \mu \int_{g_\infty+\epsilon}^{h_\infty-\epsilon} \int_{h_\infty}^{+\infty} J(y-x)u(t,x)dydx \\ &\geq \mu \int_{h_\infty-\epsilon_0/2}^{h_\infty-\epsilon} \int_{h_\infty}^{h_\infty+\epsilon_0/2} \delta_0 \frac{1}{2} W_\epsilon(x)dydx > 0. \end{aligned}$$

This implies $h_\infty = +\infty$, a contradiction to the assumption that $h_\infty - g_\infty < +\infty$. Therefore, we must have

$$\lambda_p(\mathcal{L}_{(g_\infty, h_\infty)}^0) \leq 0.$$

We are now ready to show that $u(t, x) \rightarrow 0$ uniformly in $[g(t), h(t)]$ as $t \rightarrow +\infty$. Let $\bar{u}(t, x)$ denote the unique solution of

$$(3.9) \quad \begin{cases} \bar{u}_t = d \int_{g_\infty}^{h_\infty} J(x-y)\bar{u}(t,y)dy - d\bar{u}(t,x) + f(\bar{u}), & t > 0, x \in [g_\infty, h_\infty], \\ \bar{u}(0, x) = \tilde{u}_0(x), & x \in [g_\infty, h_\infty], \end{cases}$$

where

$$\tilde{u}_0(x) = u_0(x) \text{ if } -h_0 \leq x \leq h_0, \text{ and } \tilde{u}_0(x) = 0 \text{ if } x \notin [-h_0, h_0].$$

By the maximum principle Theorem 2.1, we have $0 \leq u(t, x) \leq \bar{u}(t, x)$ for $t > 0$ and $x \in [g(t), h(t)]$. Since

$$\lambda_p(\mathcal{L}_{(g_\infty, h_\infty)}^0) \leq 0,$$

Proposition 3.5 implies that $\bar{u}(t, x) \rightarrow 0$ uniformly in $x \in [g_\infty, h_\infty]$ as $t \rightarrow +\infty$. Hence $u(t, x) \rightarrow 0$ uniformly in $x \in [g(t), h(t)]$ as $t \rightarrow +\infty$. This completes the proof. $\square$

Lemma 3.8. $h_\infty < +\infty$ if and only if $-g_\infty < +\infty$.

Proof. Arguing indirectly, we assume, without loss of generality, that $h_\infty = +\infty$ and $-g_\infty < +\infty$. By Proposition 3.4, there exists $h_1 > 0$ such that $\lambda_p(\mathcal{L}_{(0, h_1)}^0) > 0$. Moreover, for any $\epsilon > 0$ small, there exists $T_\epsilon > 0$ such that $h(t) > h_1$, $g(t) < g_\infty + \epsilon < 0$ for $t > T_\epsilon$. In particular,

$$\lambda_p(\mathcal{L}_{(g_\infty+\epsilon, h_1)}^0) > \lambda_p(\mathcal{L}_{(0, h_1)}^0) > 0.$$

We now consider the problem

$$\begin{cases} w_t = d \int_{g_\infty+\epsilon}^{h_1} J(x-y)w(t,y)dy - dw + f(w), & t > T_\epsilon, x \in [g_\infty+\epsilon, h_1], \\ w(T_\epsilon, x) = u(T_\epsilon, x), & x \in [g_\infty+\epsilon, h_1]. \end{cases}$$

Similar to the proof of Theorem 3.7, by choosing $\epsilon < \epsilon_0/2$, we have $g'(t) < -c < 0$ for all large t . This is a contradiction to $-g_\infty < +\infty$. $\square$

Lemma 3.9. If $h_\infty - g_\infty = +\infty$, then $\lim_{t \rightarrow +\infty} u(t, x) = 1$ locally uniformly in $\mathbb{R}$.

Proof. Thanks to Lemma 3.8, $h_\infty - g_\infty = +\infty$ implies $h_\infty = -g_\infty = +\infty$. Choose an increasing sequence $\{t_n\}_{n \geq 1}$ satisfying

$$\lim_{n \rightarrow +\infty} t_n = +\infty, \quad \lambda_p(\mathcal{L}_{(g(t_n), h(t_n))}^0) > 0 \text{ for all } n \geq 1.$$

Denote $g_n = g(t_n)$, $h_n = h(t_n)$ and let $\underline{u}_n(t, x)$ be the unique solution of the following problem

$$(3.10) \quad \begin{cases} \underline{u}_t = d \int_{g_n}^{h_n} J(x-y)\underline{u}(t,y)dy - d\underline{u}(t,x) + f(\underline{u}), & t > t_n, x \in [g_n, h_n], \\ \underline{u}(t_n, x) = u(t_n, x), & x \in [g_n, h_n]. \end{cases}$$

By the maximum principle Theorem 2.1 we have

$$(3.11) \quad u(t, x) \geq \underline{u}_n(t, x) \text{ in } [t_n, +\infty) \times [g_n, h_n].$$

Since $\lambda_p(\mathcal{L}_{[g_n, h_n]}^0) > 0$, by Proposition 3.5, problem (3.10) admits a unique positive steady state $\underline{u}_n(x)$ and

$$(3.12) \quad \lim_{t \rightarrow +\infty} \underline{u}_n(t, x) = \underline{u}_n(x) \text{ uniformly in } [g_n, h_n].$$

By Proposition 3.5,

$$\lim_{n \rightarrow \infty} \underline{u}_n(x) = 1 \text{ locally uniformly in } x \in \mathbb{R}.$$

It follows from this fact, (3.11) and (3.12) that

$$(3.13) \quad \liminf_{t \rightarrow +\infty} u(t, x) \geq 1 \quad \text{locally uniformly in } \mathbb{R}.$$

To complete the proof, it remains to prove that

$$(3.14) \quad \limsup_{t \rightarrow +\infty} u(t, x) \leq 1 \quad \text{locally uniformly in } \mathbb{R}.$$

Let $\hat{u}(t)$ be the unique solution of the ODE problem

$$\hat{u}' = f(\hat{u}), \quad \hat{u}(0) = \|u_0\|_\infty.$$

By the maximum principle we have $u(t, x) \leq \hat{u}(t)$ for $t > 0$ and $x \in [g(t), h(t)]$. Since $\hat{u}(t) \rightarrow 1$ as $t \rightarrow \infty$, (3.14) follows immediately. $\square$

Theorem 3.1 clearly follows directly from Lemmas 3.7 and 3.9.

3.3. Proof of Theorem 3.2. Next we look for criteria guaranteeing spreading or vanishing for (1.14). From Lemma 3.6 we see that if

$$(3.15) \quad d \in (0, f'(0)],$$

then $\lambda_p(\mathcal{L}_{(\ell_1, \ell_2)}^0) > 0$ for any finite interval (ℓ_1, ℓ_2) . Combining this with Lemma 3.7 and Theorem 3.1, we immediately obtain the following conclusion:

Lemma 3.10. *When (3.15) holds, spreading always happens for (1.14).*

We next consider the case

$$(3.16) \quad d > f'(0).$$

In this case, by Lemma 3.6, there exists $\ell^* > 0$ such that

$$\lambda_p(\mathcal{L}_I) = 0 \text{ if } |I| = \ell^*, \quad \lambda_p(\mathcal{L}_I) < 0 \text{ if } |I| < \ell^*, \quad \lambda_p(\mathcal{L}_I) > 0 \text{ if } |I| > \ell^*,$$

where I stands for a finite open interval in $\mathbb{R}$, and $|I|$ denotes its length.

Lemma 3.11. *Suppose that (3.16) holds and ℓ^* is defined above. If $h_0 \geq \ell^*/2$ then spreading always happens for (1.14). If $h_0 < \ell^*/2$, then there exists $\underline{\mu} > 0$ such that vanishing happens for (1.14) if $0 < \mu \leq \underline{\mu}$.*

Proof. If $h_0 \geq \ell^*/2$ and vanishing happens, then (g_∞, h_∞) is a finite interval with length strictly bigger than $2h_0 \geq \ell^*$. Therefore $\lambda_p(\mathcal{L}_{(g_\infty, h_\infty)}) > 0$, contradicting the conclusion in Lemma 3.7. Thus when $h_0 \geq \ell^*/2$, spreading always happens for (1.14).

We now consider the case $h_0 < \ell^*/2$. We fix $h_1 \in (h_0, \ell^*/2)$ and consider the following problem

$$(3.17) \quad \begin{cases} w_t(t, x) = d \int_{-h_1}^{h_1} J(x-y)w(t, y)dy - dw + f(w), & t > 0, x \in [-h_1, h_1], \\ w(0, x) = u_0(x), & x \in [-h_0, h_0], \\ w(0, x) = 0, & x \in [-h_1, -h_0) \cup (h_0, h_1] \end{cases}$$

and denote its unique solution by $\hat{w}(t, x)$. The choice of h_1 guarantees that

$$\lambda_1 := \lambda_p(\mathcal{L}_{(-h_1, h_1)}) < 0.$$

Let $\phi_1 > 0$ be the corresponding normalized eigenfunction of λ_1 , namely $\|\phi_1\|_\infty = 1$ and

$$\mathcal{L}_{(-h_1, h_1)}[\phi_1](x) = \lambda_1 \phi_1(x) \text{ for } x \in [-h_1, h_1].$$

By $(\mathbf{f}_{\mathbf{KPP}})$,

$$\begin{aligned} \hat{w}_t(t, x) &= d \int_{-h_1}^{h_1} J(x-y)\hat{w}(t, y)dy - d\hat{w} + f(\hat{w}) \\ &\leq d \int_{-h_1}^{h_1} J(x-y)\hat{w}(t, y)dy - d\hat{w} + f'(0)\hat{w}. \end{aligned}$$

On the other hand, for $C_1 > 0$ and $w_1 = C_1 e^{\lambda_1 t/4} \phi_1$ it is easy to check that

$$\begin{aligned} & d \int_{-h_1}^{h_1} J(x-y) w_1(t, y) dy - dw_1 + f'(0) w_1 - w_{1t}(t, x) \\ &= C_1 e^{\lambda_1 t/4} \left\{ d \int_{-h_1}^{h_1} J(x-y) \phi_1(y) dy - d\phi_1 + f'(0) \phi_1 - \frac{\lambda_1}{4} \phi_1 \right\} \\ &= \frac{3\lambda_1}{4} C_1 e^{\lambda_1 t/4} \phi_1 < 0. \end{aligned}$$

Choose $C_1 > 0$ large such that $C_1 \phi_1 > u_0$ in $[-h_1, h_1]$. Then we can apply the maximum principle Theorem 2.1 to $w_1 - \hat{w}$ to deduce

$$(3.18) \quad \hat{w}(t, x) \leq w_1(t, x) = C_1 e^{\lambda_1 t/4} \phi_1 \leq C_1 e^{\lambda_1 t/4} \quad \text{for } t > 0 \text{ and } x \in [-h_1, h_1].$$

Now define

$$\hat{h}(t) = h_0 + 2\mu h_1 C_1 \int_0^t e^{\lambda_1 s/4} ds \quad \text{and} \quad \hat{g}(t) = -\hat{h}(t) \quad \text{for } t \geq 0,$$

We claim that $(\hat{w}, \hat{h}, \hat{g})$ is an upper solution of (1.14).

Firstly, we compute that for any $t > 0$,

$$\hat{h}(t) = h_0 - 2\mu h_1 C_1 \frac{4}{\lambda_1} \left(1 - e^{\lambda_1 t/4}\right) < h_0 - 2\mu h_1 C_1 \frac{4}{\lambda_1} \leq h_1$$

provided that

$$0 < \mu \leq \underline{\mu} := \frac{-\lambda_1(h_1 - h_0)}{8h_1 C_1}.$$

Similarly, $\hat{g}(t) > -h_1$ for any $t > 0$. Thus by (3.17) we have

$$\hat{w}_t(t, x) \geq d \int_{\hat{g}(t)}^{\hat{h}(t)} J(x-y) \hat{w}(t, y) dy - d\hat{w} + f(\hat{w}) \quad \text{for } t > 0, \quad x \in [\hat{g}(t), \hat{h}(t)].$$

Secondly, due to (3.18), it is easy to check that

$$\int_{\hat{g}(t)}^{\hat{h}(t)} \int_{\hat{h}(t)}^{+\infty} J(y-x) \hat{w}(t, x) dy dx < 2h_1 C_1 e^{\lambda_1 t/4}.$$

Thus

$$\hat{h}'(t) = 2\mu h_1 C_1 e^{\lambda_1 t/4} > \mu \int_{\hat{g}(t)}^{\hat{h}(t)} \int_{\hat{h}(t)}^{+\infty} J(y-x) \hat{w}(t, x) dy dx.$$

Similarly, one has

$$\hat{g}'(t) < -\mu \int_{\hat{g}(t)}^{\hat{h}(t)} \int_{-\infty}^{\hat{g}(t)} J(y-x) \hat{w}(t, x) dy dx.$$

Now it is clear that $(\hat{w}, \hat{h}, \hat{g})$ is an upper solution of (1.14). Hence, by the comparison principle Theorem 2.3, we have

$$u(t, x) \leq \hat{w}(t, x), \quad g(t) \geq \hat{g}(t) \quad \text{and} \quad h(t) \leq \hat{h}(t) \quad \text{for } t > 0, \quad x \in [g(t), h(t)].$$

It follows that

$$h_\infty - g_\infty \leq \lim_{t \rightarrow +\infty} (\hat{h}(t) - \hat{g}(t)) \leq 2h_1 < +\infty.$$

This completes the proof. $\square$

Theorem 3.12. *Suppose that (3.16) holds and $h_0 < \ell^*/2$. Then there exists $\bar{\mu} > 0$ such that spreading happens to (1.14) if $\mu > \bar{\mu}$.*

Proof. Suppose that for any $\mu > 0$, $h_\infty - g_\infty < +\infty$. We will derive a contradiction.

First of all, notice that by Lemma 3.7, we have $\lambda_p(\mathcal{L}_{(g_\infty, h_\infty)}) \leq 0$. This indicates that $h_\infty - g_\infty \leq \ell^*$. To stress the dependence on μ , let (u_μ, g_μ, h_μ) denote the solution of (1.14). By the comparison principle Theorem 2.3, it is easily seen that $u_\mu, -g_\mu, h_\mu$ are increasing in $\mu > 0$. Also denote

$$h_{\mu, \infty} := \lim_{t \rightarrow +\infty} h_\mu(t), \quad g_{\mu, \infty} := \lim_{t \rightarrow +\infty} g_\mu(t).$$

Obviously, both $h_{\mu, \infty}$ and $-g_{\mu, \infty}$ are increasing in μ . Denote

$$H_\infty := \lim_{\mu \rightarrow +\infty} h_{\mu, \infty}, \quad G_\infty := \lim_{\mu \rightarrow +\infty} g_{\mu, \infty}.$$

Recall that since $J(0) > 0$, there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that $J(x) > \delta_0$ if $|x| < \epsilon_0$. Then there exist μ_1, t_1 such that for $\mu \geq \mu_1, t \geq t_1$, we have $h_\mu(t) > H_\infty - \epsilon_0/4$. It follows that

$$\begin{aligned} \mu &= \left(\int_{t_1}^{+\infty} \int_{g_\mu(\tau)}^{h_\mu(\tau)} \int_{h_\mu(\tau)}^{+\infty} J(y-x) u_\mu(\tau, x) dy dx d\tau \right)^{-1} [h_{\mu, \infty} - h_\mu(t_1)] \\ &\leq \left(\int_{t_1}^{t_1+1} \int_{g_{\mu_1}(\tau)}^{h_{\mu_1}(\tau)} \int_{h_{\mu_1}(\tau)+\epsilon_0/4}^{+\infty} J(y-x) u_{\mu_1}(\tau, x) dy dx d\tau \right)^{-1} \ell^* \\ &\leq \left(\delta_0 \int_{t_1}^{t_1+1} \int_{h_{\mu_1}(\tau)-\epsilon_0/2}^{h_{\mu_1}(\tau)} \int_{h_{\mu_1}(\tau)+\epsilon_0/4}^{h_{\mu_1}(\tau)+\epsilon_0/2} u_{\mu_1}(\tau, x) dy dx d\tau \right)^{-1} \ell^* \\ &= \left(\frac{1}{4} \delta_0 \epsilon_0 \int_{t_1}^{t_1+1} \int_{h_{\mu_1}(\tau)-\epsilon_0/2}^{h_{\mu_1}(\tau)} u_{\mu_1}(\tau, x) dx d\tau \right)^{-1} \ell^* < +\infty, \end{aligned}$$

which clearly is a contradiction. $\square$

We can now deduce a sharp criteria in terms of μ for the spreading-vanishing dichotomy.

Lemma 3.13. *Suppose that (3.16) holds and $h_0 < \ell^*/2$. Then there exists $\mu^* \in (0, \infty)$ such that vanishing happens for (1.14) if $0 < \mu \leq \mu^*$ and spreading happens for (1.14) if $\mu > \mu^*$.*

Proof. Define

$$\Sigma = \{\mu : \mu > 0 \text{ such that } h_\infty - g_\infty < +\infty\}.$$

By Lemmas 3.11 and 3.12 we see that $0 < \sup \Sigma < +\infty$. Again we let (u_μ, g_μ, h_μ) denote the solution of (1.14), and set $h_{\mu, \infty} := \lim_{t \rightarrow +\infty} h_\mu(t)$, $g_{\mu, \infty} := \lim_{t \rightarrow +\infty} g_\mu(t)$, and denote $\mu^* = \sup \Sigma$.

As before $u_\mu, -g_\mu, h_\mu$ are increasing in $\mu > 0$. This immediately gives that if $\mu_1 \in \Sigma$, then $\mu \in \Sigma$ for any $\mu < \mu_1$ and if $\mu_1 \notin \Sigma$, then $\mu \notin \Sigma$ for any $\mu > \mu_1$. Hence it follows that

$$(3.19) \quad (0, \mu^*) \subseteq \Sigma, \quad (\mu^*, +\infty) \cap \Sigma = \emptyset.$$

To complete the proof, it remains to show that $\mu^* \in \Sigma$. Suppose that $\mu^* \notin \Sigma$. Then $h_{\mu^*, \infty} = -g_{\mu^*, \infty} = +\infty$. Thus there exists $T > 0$ such that $-g_{\mu^*}(t) > \ell^*, h_{\mu^*}(t) > \ell^*$ for $t \geq T$. Hence there exists $\epsilon > 0$ such that for $\mu \in (\mu^* - \epsilon, \mu^* + \epsilon)$, $-g_\mu(T) > \ell^*/2, h_\mu(T) > \ell^*/2$, which implies $\mu \notin \Sigma$. This clearly contradicts (3.19). Therefore $\mu^* \in \Sigma$. $\square$

4. SEMI-WAVE SOLUTIONS

We want to determine the spreading speed of the nonlocal free boundary problem

$$(4.1) \quad \begin{cases} u_t = d \int_{g(t)}^{h(t)} J(x-y) u(t, y) dy - du + f(u), & t > 0, x \in (g(t), h(t)), \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(y-x) u(t, x) dy dx, & t > 0, \\ g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(y-x) u(t, x) dy dx, & t > 0, \\ u(0, x) = u_0(x), h(0) = -g(0) = h_0, & x \in [-h_0, h_0], \end{cases}$$

where d, μ, h_0 are given positive constants. The initial function $u_0(x)$ satisfies (1.15). The kernel function $J : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the basic condition

$$(\mathbf{J}): J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}), J \geq 0, J(0) > 0, \int_{\mathbb{R}} J(x) dx = 1.$$

The growth term $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies the KPP condition

$$(\mathbf{f}_{\text{KPP}}): \begin{cases} f \in C^1, f(0) = f(1) = 0, f'(0) > 0 > f'(1), \\ f(u)/u \text{ is non-increasing in } (0, \infty). \end{cases}$$

For a non-symmetric J satisfying $(\mathbf{J})$, the following two quantities determined by J and $f'(0)$ alone play an important role:

$$c_*^- = \sup_{\nu < 0} \frac{d \int_{\mathbb{R}} J(x) e^{\nu x} dx - d + f'(0)}{\nu}, \quad c_*^+ = \inf_{\nu > 0} \frac{d \int_{\mathbb{R}} J(x) e^{\nu x} dx - d + f'(0)}{\nu},$$

It can be shown that c_*^- is achieved by some $\nu < 0$ when it is finite, and a parallel conclusion holds for c_*^+ . It is easily checked that c_*^- is finite if and only if J satisfies additionally the following **thin-tail** condition at $x = -\infty$,

$$(\mathbf{J}_{\text{thin}}^-) : \text{There exists } \lambda > 0 \text{ such that } \int_0^{+\infty} J(-x)e^{\lambda x} dx < +\infty.$$

Similarly, c_*^+ is finite if and only if J satisfies

$$(\mathbf{J}_{\text{thin}}^+) : \text{There exists } \lambda > 0 \text{ such that } \int_0^{+\infty} J(x)e^{\lambda x} dx < +\infty.$$

If we define

$$(4.2) \quad \begin{cases} c_*^- = -\infty \text{ when } (\mathbf{J}_{\text{thin}}^-) \text{ does not hold,} \\ c_*^+ = +\infty \text{ when } (\mathbf{J}_{\text{thin}}^+) \text{ does not hold,} \end{cases}$$

then the propagation dynamics of the corresponding Cauchy problem of (4.1),

$$(4.3) \quad \begin{cases} U_t = d \int_{\mathbb{R}} J(x-y)U(t,y) dy - dU(t,x) + f(U), & t > 0, x \in \mathbb{R}, \\ U(0,x) = U_0(x) \end{cases}$$

has the properties described in the following result:

Theorem A. ([6]) *Suppose that $(\mathbf{J})$ and $(\mathbf{f}_{\text{KPP}})$ hold. Then for any initial function $U_0(x)$ which is continuous and nonnegative with non-empty compact support, the unique solution $U(t,x)$ of (4.3) satisfies*

$$\lim_{t \rightarrow \infty} U(t,x) = \begin{cases} 1 \text{ uniformly for } x \in [a_1 t, b_1 t] \text{ provided that } [a_1, b_1] \subset (c_*^-, c_*^+), \\ 0 \text{ uniformly for } x \leq a_2 t \text{ provided that } c_*^- > -\infty \text{ and } a_2 < c_*^-, \\ 0 \text{ uniformly for } x \geq b_2 t \text{ provided that } c_*^+ < \infty \text{ and } b_2 > c_*^+. \end{cases}$$

Following [1], the conclusions in Theorem A can be interpreted as indicating a leftward spreading speed of c_*^- and rightward spreading speed of c_*^+ for (4.3). The following result of Yagisita [18] (see also Theorem 1.5 in [4]) on traveling waves provides further meanings for c_*^- and c_*^+ .

Theorem B. ([18]) *Suppose that $(\mathbf{J})$ and $(\mathbf{f}_{\text{KPP}})$ are satisfied. Then the following conclusions hold.*

(i) *The rightward traveling wave problem*

$$(4.4) \quad \begin{cases} d \int_{\mathbb{R}} J(x-y)\phi(y) dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, & x \in \mathbb{R}, \\ \phi(-\infty) = 1, \quad \phi(+\infty) = 0 \end{cases}$$

has a solution pair $(c, \phi) \in \mathbb{R} \times L^\infty(\mathbb{R})$ with ϕ nonincreasing if and only if $c_^+ < \infty$. Moreover, in such a case, for every $c \geq c_*^+$, (4.4) has a solution $\phi \in C^1(\mathbb{R})$ that is strictly decreasing, and (4.4) has no such solution for $c < c_*^+$.*

(ii) *The leftward traveling wave problem*

$$(4.5) \quad \begin{cases} d \int_{\mathbb{R}} J(x-y)\psi(y) dy - d\psi(x) - c\psi'(x) + f(\psi(x)) = 0, & x \in \mathbb{R}, \\ \psi(-\infty) = 0, \quad \psi(+\infty) = 1, \end{cases}$$

has a solution pair $(c, \psi) \in \mathbb{R} \times L^\infty(\mathbb{R})$ with ψ nondecreasing if and only if $c_^- > -\infty$. Moreover, in such a case, for each $c \geq -c_*^-$, (4.5) has a solution $\psi \in C^1(\mathbb{R})$ that is strictly increasing, and (4.5) has no such solution for $c < -c_*^-$.*

Remark: Problem (4.3) is the limiting problem of (4.1) when $\mu \rightarrow \infty$.

The propagation dynamics of (4.1) depends crucially on the associated semi-wave solutions, which are pairs $(c, \phi) \in (0, +\infty) \times C^1((-\infty, 0])$ and $(\tilde{c}, \psi) \in (0, +\infty) \times C^1([0, \infty))$, determined by the following equations, respectively:

$$(4.6) \quad \begin{cases} d \int_{-\infty}^0 J(x-y)\phi(y) dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, & -\infty < x < 0, \\ \phi(-\infty) = 1, \quad \phi(0) = 0, \end{cases}$$

$$(4.7) \quad c = \mu \int_{-\infty}^0 \int_0^{+\infty} J(y-x)\phi(x) dy dx,$$

$$(4.8) \quad \begin{cases} d \int_0^{+\infty} J(x-y)\psi(y) dy - d\psi(x) - \tilde{c}\psi'(x) + f(\psi(x)) = 0, & 0 < x < +\infty, \\ \psi(0) = 0, \psi(+\infty) = 1. \end{cases}$$

$$(4.9) \quad \tilde{c} = \mu \int_0^{+\infty} \int_{-\infty}^0 J(y-x)\psi(x) dy dx.$$

Note that for $\Phi(t, x) := \phi(x - ct)$ and $\Psi(t, x) = \psi(x + \tilde{c}t)$, (4.6) and (4.8) imply

$$\begin{cases} \Phi_t = d \int_{-\infty}^{ct} J(x-y)\Phi(t, y) dy - d\Phi + f(\Phi), & \Phi(t, ct) = 0 \quad \text{for } x < ct, t > 0, \\ \Psi_t = d \int_{-\tilde{c}t}^{+\infty} J(x-y)\Psi(t, y) dy - d\Psi + f(\Psi), & \Psi(t, -\tilde{c}t) = 0 \quad \text{for } x > -\tilde{c}t, t > 0. \end{cases}$$

We will call ϕ^c a **rightward semi-wave** of (4.3) with speed c if (c, ϕ^c) solves (4.6), and call $\psi^{\tilde{c}}$ a **leftward semi-wave** of (4.3) with speed $\tilde{c}$ if $(\tilde{c}, \psi^{\tilde{c}})$ solves (4.8).

Whether such semi-wave solutions can satisfy additionally (4.7) and (4.9) depends on the following extra properties of $J(x)$, apart from **(J)**,

$$\begin{aligned} (\mathbf{J}_1^+) : & \int_{-\infty}^0 \int_0^{+\infty} J(y-x) dy dx < +\infty, \text{ i.e., } \int_0^{+\infty} xJ(x) dx < +\infty, \\ (\mathbf{J}_1^-) : & \int_0^{+\infty} \int_{-\infty}^0 J(y-x) dy dx < +\infty, \text{ i.e., } \int_0^{+\infty} xJ(-x) dx < +\infty. \end{aligned}$$

We are going to prove the following result.

Theorem 4.1. *Suppose that **(J)** and **(f_{KPP})** are satisfied. Then the following conclusions hold:*

- (a⁺) *Problem (4.6) admits a nonnegative solution $\phi \in C^1((-\infty, 0])$ with $c > 0$ if and only if $c_*^+ \in (0, +\infty]$ and $c < c_*^+$. Moreover, in such a case, (4.6) has a unique solution $\phi = \phi^c$, it is C^1 and $(\phi^c)'(x) < 0$ for $x \in (-\infty, 0]$.*
- (b⁺) *Suppose $c_*^+ \in (0, +\infty]$ and ϕ^c is the unique solution of (4.6) with $c \in (0, c_*^+)$. Then there exists a unique $c_0 \in (0, c_*^+)$ such that $(c, \phi) = (c_0, \phi^{c_0})$ solves (4.7) if and only if **(J₁⁺)** holds.*
- (a⁻) *Problem (4.8) admits a nonnegative solution $\psi \in C^1([0, +\infty))$ with $\tilde{c} > 0$ if and only if $c_*^- \in [-\infty, 0)$ and $\tilde{c} < -c_*^-$. Moreover, in such a case, (4.8) has a unique solution $\psi = \psi^{\tilde{c}}$, it is C^1 and $(\psi^{\tilde{c}})'(x) > 0$ for $x \in [0, +\infty)$.*
- (b⁻) *Suppose $c_*^- \in [-\infty, 0)$, and $\psi^{\tilde{c}}$ is the unique solution of (4.8) with $\tilde{c} \in (0, -c_*^-)$. Then there exists a unique $\tilde{c}_0 \in (0, -c_*^-)$ such that $(c, \psi) = (\tilde{c}_0, \psi^{\tilde{c}_0})$ solves (4.9) if and only if **(J₁⁻)** holds.*

Note that the existence of a solution to (4.6) requires $c_*^+ > 0$. Similarly, the existence of a solution to (4.8) requires $c_*^- < 0$.

The unique speed $c = c_0$ in (b⁺) will determine the asymptotic speed of $h(t)$, and the corresponding ϕ^{c_0} will be called the rightward semi-wave of (4.1). Similarly, the unique speed $\tilde{c} = \tilde{c}_0$ in (b⁻) will determine the asymptotic speed of $g(t)$, and the corresponding $\psi^{\tilde{c}_0}$ will be called the leftward semi-wave of (4.1).

4.1. A maximum principle and its first application.

Lemma 4.2 (Lemma 2.5 in [9]). *Assume that **(J)** holds and $w \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ satisfies*

$$\begin{cases} d \int_{-\infty}^0 J(x-y)w(y) dy - dw(x) + a(x)w'(x) + b(x)w(x) \leq 0, & x < 0, \\ w(x) \geq 0, & x \geq 0, \end{cases}$$

with $a, b \in L_{loc}^\infty(\mathbb{R})$. If $w(x) \geq 0$ and $w(x) \not\equiv 0$ in $(-\infty, 0)$, then $w(x) > 0$ for $x < 0$.

Proof. Suppose that there exists $x_0 < 0$ such that $w(x_0) = 0$. Then $w'(x_0) = 0$ and it follows from the differential-integral inequality satisfied by w that at $x = x_0$,

$$d \int_{-\infty}^0 J(x_0 - y)w(t, y) dy \leq 0,$$

which indicates that $w(y) = 0$ when y is close to x_0 , due to $J(0) > 0$. This implies that $w(x) \equiv 0$ when $x < 0$, since $\{x < 0 \mid w(x) = 0\}$ is now both open and closed in $(-\infty, 0)$. $\square$

This maximum principle will be used frequently. A first application is the following result.

Lemma 4.3. *Suppose that **(J)** and **(f_{KPP})** are satisfied.*

- (i) *Assume $\phi = \phi^c$ is a nonnegative solution of (4.6) with speed $c > 0$. Then $\phi(x) > 0$ for $x < 0$ and $\phi'(x) < 0$ for $x \leq 0$.*

- (ii) Assume $\psi = \psi^{\tilde{c}}$ is a nonnegative solution of (4.8) with speed $\tilde{c} > 0$. Then $\psi(x) > 0$ for $x > 0$ and $\psi'(x) > 0$ for $x \geq 0$.

Proof. We only prove (i), since the proof of (ii) is similar. Since $\phi \geq 0$ and $\phi(-\infty) = 1$, by Lemma 4.2 we have $\phi(x) > 0$ for $x < 0$.

For fixed $\delta > 0$, define $K = \{k \geq 1 : k\phi(x-\delta) \geq \phi(x) \text{ for } x \leq 0\}$. It follows from $\phi(-\infty) = 1$, $\phi(x) > 0$ for $x < 0$ and $\phi(0) = 0$ that $K \neq \emptyset$. Thus $k_* = \inf K \geq 1$ is well-defined.

We claim that $k_* = 1$ (which implies that $\phi(x)$ is decreasing in $(-\infty, 0]$ due to the arbitrariness of $\delta > 0$). Otherwise, suppose that $k_* > 1$. Then $w(x) := k_*\phi(x-\delta) - \phi(x) \geq 0$, and since $w(0) = k_*\phi(-\delta) > 0$ and $\lim_{x \rightarrow -\infty} w(x) = k_* - 1 > 0$, there is $x_0 \in (-\infty, 0)$ such that $w(x_0) = 0$. From the equation satisfied by $\phi(x-\delta)$ and $(\mathbf{f}_{\mathbf{KPP}})$, we have, for $x < 0$,

$$\begin{aligned} 0 &= d \int_{-\infty}^{\delta} J(x-y)k_*\phi(y-\delta) dy - dk_*\phi(x-\delta) + ck_*\phi'(x-\delta) + k_*f(\phi(x-\delta)) \\ &\geq d \int_{-\infty}^{\delta} J(x-y)k_*\phi(y-\delta) dy - dk_*\phi(x-\delta) + ck_*\phi'(x-\delta) + f(k_*\phi(x-\delta)) \\ &\geq d \int_{-\infty}^0 J(x-y)k_*\phi(y-\delta) dy - dk_*\phi(x-\delta) + ck_*\phi'(x-\delta) + f(k_*\phi(x-\delta)), \end{aligned}$$

and it follows that

$$d \int_{-\infty}^0 J(x-y)w(y) dy - dw(x) + cw'(x) + b(x)w(x) \leq 0,$$

where

$$b(x) = \begin{cases} \frac{f(k_*\phi(x-\delta)) - f(\phi(x))}{k_*\phi(x-\delta) - \phi(x)}, & \text{if } k_*\phi(x-\delta) - \phi(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $w(x) \not\equiv 0$ since $k_* > 1$. Then by Lemma 4.2, $w(x) > 0$ for $x < 0$, a contradiction with $w(x_0) = 0$. We have thus proved the claim $k_* = 1$. So $\phi(x)$ is decreasing in $x \in (-\infty, 0]$.

It remains to show $\phi'(x) < 0$ for $x \leq 0$. From (4.6), we get

$$\begin{aligned} (4.10) \quad c\phi'(x) &= d\phi(x) - d \int_{-\infty}^0 J(x-y)\phi(y) dy - f(\phi(x)) \\ &= d\phi(x) - d \int_x^{\infty} J(z)\phi(x-z) dz - f(\phi(x)) \text{ for } x < 0. \end{aligned}$$

Taking the derivative with respect to x on both sides by $\phi \in C^1$, we have

$$c\phi''(x) = d\phi'(x) - d \int_{-\infty}^0 J(x-y)\phi'(y) dy - f'(\phi(x))\phi'(x),$$

where we have used

$$\begin{aligned} \frac{d}{dx} \int_x^{\infty} J(z)\phi(x-z) dz &= -J(x)\phi(0) + \int_x^{\infty} J(z)\phi'(x-z) dz \\ &= \int_{-\infty}^0 J(x-y)\phi'(y) dy. \end{aligned}$$

Thus $w(x) := -\phi'(x) \geq 0$ satisfies

$$d \int_{-\infty}^0 J(x-y)w(y) dy - dw(x) + cw'(x) + f'(\phi(x))w(x) = 0.$$

Since $\phi(-\infty) = 1$ and $\phi(0) = 0$, we have $\phi'(x) \not\equiv 0$, that is, $w(x) \not\equiv 0$. By $(\mathbf{f}_{\mathbf{KPP}})$ and Lemma 4.2, $w(x) = -\phi'(x) > 0$ for $x < 0$. If $w(0) = 0$, that is, $\phi'(0) = 0$, it follows from (4.10) and $\phi(0) = 0$ that

$$0 = -d \int_{-\infty}^0 J(-y)\phi(y) dy < 0,$$

a contradiction. The proof is complete. $\square$

4.2. A perturbed semi-wave problem. For $\delta > 0$, $c > 0$, we consider the auxiliary problem

$$(4.11) \quad \begin{cases} d \int_{-\infty}^{\infty} J(x-y)\phi(y)dy - d\phi + c\phi'(x) + f(\phi(x)) = 0, & -\infty < x < 0, \\ \phi(-\infty) = 1, \quad \phi(x) = \delta, & 0 \leq x < \infty. \end{cases}$$

If $\delta = 0$ then (4.11) is reduced to the semi-wave problem (4.6); therefore (4.11) can be viewed as a perturbed semi-wave problem. As we will see below, the semi-wave solutions and traveling wave solutions of (4.3) can be obtained as the limit of the solution of (4.11) when $\delta \rightarrow 0$, subject to suitable translations in x .

Define

$$(4.12) \quad \tilde{\sigma}(v) := f(v) + cMv - dv \quad \text{for } v \geq 0,$$

where $M > 0$ is a constant. Then the first equation in (4.11) is equivalent to

$$(4.13) \quad -c(e^{-Mx}\phi)' = e^{Mx} \left[d \int_{-\infty}^{\infty} J(x-y)\phi(y)dy + \tilde{\sigma}(\phi(x)) \right].$$

Since f is C^1 , we could choose M large enough such that $\tilde{\sigma}(v)$ is increasing for $v \in [0, 2]$, namely

$$\tilde{\sigma}(v) \geq \tilde{\sigma}(u) \text{ if } u, v \in [0, 2] \text{ and } v \geq u.$$

Lemma 4.4. *Suppose (J) and (f_{KPP}) hold. Let $\delta \in (0, 1)$. Then the problem (4.11) has a solution $\phi(x)$ which is nonincreasing in x , and can be obtained by an iteration process to be specified in the proof.*

Proof. Let

$$\Omega := \{\Gamma \in C(\mathbb{R}) : 0 \leq \Gamma(x) \leq 1 \text{ for all } x \in \mathbb{R}\}.$$

Define an operator $P : \Omega \rightarrow C(\mathbb{R})$ by

$$P[\Gamma](x) = \begin{cases} e^{Mx}\delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} \left[d \int_{-\infty}^{\infty} J(\xi-y)\Gamma(y)dy + \tilde{\sigma}(\Gamma(\xi)) \right] d\xi, & x < 0, \\ \delta, & x \geq 0. \end{cases}$$

Using (4.13) we easily see that (4.11) is equivalent to

$$(4.14) \quad \begin{cases} \phi(x) = P[\phi](x) \text{ for } x \in \mathbb{R}, \\ \phi(-\infty) = 1. \end{cases}$$

We next solve (4.14) in three steps.

Step 1 We show that P has a fixed point in Ω .

Firstly we prove that $P[\delta](x) \geq \delta$ with δ regarded as a constant function. By the definition of P , we have $P[\delta](x) = \delta$ for $x \geq 0$. For $x < 0$,

$$\begin{aligned} P[\delta](x) &= e^{Mx}\delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [d\delta + \tilde{\sigma}(\delta)] d\xi \\ &= e^{Mx}\delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [cM\delta + f(\delta)] d\xi \\ &> e^{Mx}\delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} cM\delta d\xi \\ &= e^{Mx}\delta - e^{Mx}\delta + \delta = \delta \end{aligned}$$

since $f(\delta) > 0$ by (f_{KPP}).

Secondly we show $P[1](x) \leq 1$. Since $\delta > 0$ is small, $P[1](x) = \delta < 1$ for $x \geq 0$. For $x < 0$, we have

$$\begin{aligned} P[1](x) &= e^{Mx}\delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} [d + \tilde{\sigma}(1)] d\xi \\ &= e^{Mx}\delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} cM d\xi = e^{Mx}\delta - e^{Mx} + 1 < 1. \end{aligned}$$

Next we define inductively

$$\Gamma_0(x) := \delta, \quad \Gamma_{n+1}(x) := P[\Gamma_n](x) = P^n[\Gamma_0](x) \quad \text{for } n = 0, 1, 2, \dots, \quad x \in \mathbb{R}.$$

Then

$$\Gamma_0 \leq \Gamma_n \leq \Gamma_{n+1} \leq 1$$

due to the monotonicity of P which is a simple consequence of the fact that $\tilde{\sigma}(v)$ is increasing in $v \in [0, 1]$.

Define

$$\widehat{\Gamma}(x) := \lim_{n \rightarrow \infty} \Gamma_n(x) \in [0, 1].$$

It is clear that $\widehat{\Gamma}(x) = \delta$ for $x \geq 0$. Making use of the Lebesgue dominated convergence theorem and $\Gamma_{n+1}(x) = P[\Gamma_n](x)$, for $x < 0$ we deduce

$$\widehat{\Gamma}(x) = P[\widehat{\Gamma}](x),$$

which also implies that $\widehat{\Gamma}'(x)$ exists and is continuous for $x < 0$. Hence $\widehat{\Gamma}$ is a fixed point of P in Ω .

Step 2. We show that $\widehat{\Gamma}'(x) \leq 0$ for $x < 0$.

It suffices to prove that $\Gamma'_n(x) \leq 0$ for $x < 0$ and each $n = 0, 1, 2, \dots$, since this would imply each Γ_n is nonincreasing and hence $\widehat{\Gamma}(x)$ is nonincreasing for $x < 0$.

It is clear that $\Gamma_0(x) = \delta$ is nonincreasing. Assume $\Gamma'_n(x) \leq 0$ for $x < 0$. We show that $\Gamma'_{n+1}(x) \leq 0$ for $x < 0$.

By the definition, for $x < 0$,

$$\Gamma_{n+1}(x) = e^{Mx}\delta + \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} g_n(\xi) d\xi,$$

where

$$\begin{aligned} g_n(\xi) &= g(\xi; \Gamma_n) := d \int_{-\infty}^{\infty} J(\xi - y) \Gamma_n(y) dy + \tilde{\sigma}(\Gamma_n(\xi)) \\ &= d \int_{-\infty}^{\infty} J(-y) \Gamma_n(y + \xi) dy + \tilde{\sigma}(\Gamma_n(\xi)). \end{aligned}$$

Let us note that $\Gamma'_n(z) \leq 0$ for $z \neq 0$. It follows that $g_n(\xi)$ is differentiable for all $\xi \in \mathbb{R}$, and $g'_n(\xi) \leq 0$ for $\xi \in \mathbb{R}$. Moreover,

$$g_n(0) = g(0; \Gamma_n) \geq g(0; \Gamma_0) = d\delta + \tilde{\sigma}(\delta) = cM\delta + f(\delta) \geq cM\delta,$$

since $\Gamma_n \geq \Gamma_0 = \delta$, $F(\delta) > 0$, and $g(0; \Gamma_n)$ is nondecreasing with respect to Γ_n . Therefore, for $x < 0$,

$$\begin{aligned} (\Gamma_{n+1})'(x) &= \delta M e^{Mx} + M \frac{e^{Mx}}{c} \int_x^0 e^{-M\xi} g_n(\xi) d\xi - \frac{1}{c} g_n(x) \\ &= \delta M e^{Mx} + M \frac{e^{Mx}}{c} \left[\frac{-e^{-M\xi}}{M} g_n(\xi) \Big|_x^0 + \int_x^0 e^{-M\xi} g'_n(\xi) d\xi \right] - \frac{1}{c} g_n(x) \\ &\leq \delta M e^{Mx} + M \frac{e^{Mx}}{c} \left[\frac{-g_n(0)}{M} + \frac{e^{-Mx}}{M} g_n(x) \right] - \frac{1}{c} g_n(x) \\ &= \delta M e^{Mx} - \frac{g_n(0) e^{Mx}}{c} \leq \delta M e^{Mx} - \delta M e^{Mx} = 0. \end{aligned}$$

By the principle of mathematical induction, we have $\Gamma'_n(x) \leq 0$ for $x < 0$ and all $n \geq 1$.

Step 3. We verify $\widehat{\Gamma}(-\infty) = 1$.

By step 2, $\lim_{x \rightarrow -\infty} \widehat{\Gamma}(x) = K$ exists, and $0 \leq K \leq 1$. We claim that

$$(4.15) \quad \lim_{x \rightarrow -\infty} \int_{-\infty}^{\infty} J(x - y) \widehat{\Gamma}(y) dy = K.$$

Indeed, since $\widehat{\Gamma}$ is nonincreasing and $\lim_{x \rightarrow -\infty} \widehat{\Gamma}(x) = K$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} J(x - y) \widehat{\Gamma}(y) dy &= \int_{-\infty}^{\infty} J(-y) \widehat{\Gamma}(y + x) dy \\ &\geq \int_{-\infty}^{-x/2} J(-y) \widehat{\Gamma}(y + x) dy \geq \widehat{\Gamma}(x/2) \int_{-\infty}^{-x/2} J(-y) dy \rightarrow K \end{aligned}$$

as $x \rightarrow -\infty$, and on the other hand

$$\int_{-\infty}^{\infty} J(x - y) \widehat{\Gamma}(y) dy = \int_{-\infty}^{\infty} J(-y) \widehat{\Gamma}(y + x) dy \leq \int_{-\infty}^{\infty} J(-y) K dy = K.$$

Hence (4.15) holds.

If $K \neq 1$, then by $(\mathbf{f}_{\mathbf{KPP}})$, we have $f(K) \neq 0$. Note that $\hat{\Gamma}$ satisfies

$$d \int_{-\infty}^{\infty} J(x-y) \hat{\Gamma}(y) dy - d\hat{\Gamma} + c\hat{\Gamma}'(x) + f(\hat{\Gamma}(x)) = 0, \quad -\infty < x < 0.$$

Letting $x \rightarrow -\infty$ and making use of (4.15), we deduce

$$\lim_{x \rightarrow -\infty} c\hat{\Gamma}'(x) = - \lim_{x \rightarrow -\infty} f(\hat{\Gamma}(x)) = -f(K) \neq 0,$$

which contradicts the fact that $\hat{\Gamma}$ is nonincreasing and bounded. Thus, $\hat{\Gamma}(-\infty) = 1$.

Combining Steps 1-3, we see that (4.11) admits a nonincreasing solution $\hat{\Gamma}$, which is the limit of Γ_n obtained from an iteration process. $\square$

The following result describes the monotonic dependence on c and δ of the solution ϕ to (4.11) obtained in the above lemma. To stress these dependences, we will write $\phi = \phi_{\delta}^c$.

Lemma 4.5. *Suppose $(\mathbf{J})$ and $(\mathbf{f}_{\mathbf{KPP}})$ hold. Let ϕ_{ϵ}^c be the solution of (4.11) obtained through the iteration process in Lemma 4.4, with $c > 0$ and $\delta = \epsilon$. Then*

$$(4.16) \quad \begin{cases} \phi_{\epsilon_1}^c \leq \phi_{\epsilon_2}^c & \text{if } 0 < \epsilon_1 \leq \epsilon_2 \ll 1, \\ \phi_{\epsilon_1}^{c_1} \geq \phi_{\epsilon}^{c_2} & \text{if } 0 < c_1 \leq c_2. \end{cases}$$

Proof. To verify the first inequality in (4.16) for fixed $c > 0$, we adopt the definition of P and ϕ_n in Lemma 4.4, but in order to distinguish them between $\delta = \epsilon_1$ and $\delta = \epsilon_2$, we write $P = P_i$ and $\phi_n = \phi_{i,n}$ for $\delta = \epsilon_i$, $i = 1, 2$. Thus we have

$$\phi_{\epsilon_i}^c(x) = \lim_{n \rightarrow \infty} \phi_{i,n}(x).$$

Since $P[\phi](x)$ is nondecreasing with respect to δ and ϕ , respectively, we have

$$\phi_{1,n+1}(x) = P_1[\phi_{1,n}](x) \leq P_1[\phi_{2,n}](x) \leq P_2[\phi_{2,n}](x) = \phi_{2,n+1}(x)$$

provided that

$$\phi_{1,n}(x) \leq \phi_{2,n}(x).$$

Since $\phi_{1,0}(x) \equiv \epsilon_1 \leq \epsilon_2 \equiv \phi_{2,0}(x)$, the above conclusion combined with the induction method gives $\phi_{1,n}(x) \leq \phi_{2,n}(x)$ for all $n = 0, 1, 2, \dots$, which implies $\phi_{\epsilon_1}^c(x) \leq \phi_{\epsilon_2}^c(x)$, as desired.

We now show the second inequality in (4.16) for fixed $\delta = \epsilon$. To stress the reliance on c_i , we use the notions P^i and ϕ_n^i , respectively, for P and ϕ when $c = c_i$, $i = 1, 2$. From Lemma 4.4, we have for $i = 1, 2$,

$$\phi_{\epsilon}^{c_i}(x) = \lim_{n \rightarrow \infty} \phi_n^i(x) = \lim_{n \rightarrow \infty} P^i[\phi_n^i](x).$$

Due to $c_1 \leq c_2$ and (4.11), we have

$$\begin{aligned} & d \int_{-\infty}^{\infty} J(x-y) \phi_{\epsilon}^{c_1}(y) dy - d\phi_{\epsilon}^{c_1} + c_2(\phi_{\epsilon}^{c_1})'(x) + f(\phi_{\epsilon}^{c_1}(x)) \\ & \leq d \int_{-\infty}^{\infty} J(x-y) \phi_{\epsilon}^{c_1}(y) dy - d\phi_{\epsilon}^{c_1} + c_1(\phi_{\epsilon}^{c_1})'(x) + f(\phi_{\epsilon}^{c_1}(x)) = 0, \end{aligned}$$

which implies that

$$\phi_{\epsilon}^{c_1}(x) \geq P^2[\phi_{\epsilon}^{c_1}](x).$$

Since $P[\phi](x)$ is increasing with respect to ϕ , it follows that

$$\phi_{\epsilon}^{c_1}(x) \geq P^2[\phi_{\epsilon}^{c_1}](x) \geq P^2[\phi_n^{c_1}](x) = \phi_{n+1}^2(x)$$

provided that

$$\phi_{\epsilon}^{c_1}(x) \geq \phi_n^2(x).$$

Recall that $\phi_{\epsilon}^{c_1}(x) \geq \delta \equiv \phi_0^2(x)$. By induction, we obtain that $\phi_{\epsilon}^{c_1}(x) \geq \phi_n^2(x)$ for all $n = 0, 1, 2, \dots$, and so $\phi_{\epsilon}^{c_1}(x) \geq \phi_{\epsilon}^{c_2}(x)$. $\square$

4.3. A dichotomy between semi-waves and traveling waves.

Theorem 4.6. *Suppose (J) and (f_{KPP}) hold. Then for each $c > 0$, (4.3) has either a monotone semi-wave solution with speed c or a monotone traveling wave solution with speed c , but not both. Moreover, one of the following holds:*

- (i) *For every $c > 0$, (4.3) has a monotone semi-wave solution with speed c .*
- (ii) *For every $c > 0$, (4.3) has a monotone traveling wave solution with speed c .*
- (iii) *There exists $C_* \in (0, \infty)$ such that (4.3) has a monotone semi-wave solution with speed c for every $c \in (0, C_*)$, and has a monotone traveling wave solution with speed c for every $c \geq C_*$.*

The following result will be needed to prove Theorem 4.6.

Lemma 4.7. *Suppose (J) and (f_{KPP}) hold. Then for each $c > 0$, (4.3) has either a monotone semi-wave solution with speed c or a monotone traveling wave solution with speed c , but not both.*

Proof. Let ϕ_n^c be the solution of (4.11) defined in Lemma 4.4 with $\delta = \epsilon_n$, $\epsilon_n \searrow 0$ as $n \rightarrow \infty$. Then

$$x_n^c := \max \{x : \phi_n^c(x) = 1/2\}$$

is well defined, and

$$\phi_n^c(x_n^c) = 1/2, \quad \phi_n^c(x) < 1/2 \quad \text{for } x > x_n^c.$$

Moreover, making use of Lemma 4.5, we have

$$(4.17) \quad \begin{cases} 0 > x_n^c \geq x_m^c & \text{if } n \leq m, \\ 0 > x_n^{c_1} \geq x_n^{c_2} & \text{if } c_1 \leq c_2. \end{cases}$$

Define

$$\tilde{\phi}_n^c(x) := \phi_n^c(x + x_n^c), \quad x \in \mathbb{R}.$$

Then $\tilde{\phi}_n^c$ satisfies, for $x < -x_n^c$,

$$(4.18) \quad d \int_{-\infty}^{\infty} J(x-y) \tilde{\phi}_n^c(y) dy - d \tilde{\phi}_n^c(x) + c(\tilde{\phi}_n^c)'(x) + f(\tilde{\phi}_n^c(x)) = 0,$$

and for $x \geq -x_n^c$, $\tilde{\phi}_n^c(x) = \epsilon_n$. Moreover,

$$\tilde{\phi}_n^c(0) = 1/2.$$

Since x_n^c is nonincreasing in n ,

$$x^c := -\lim_{n \rightarrow \infty} x_n^c \in (0, \infty]$$

always exists, and there are two possible cases

- Case 1. $x^c = \infty$
- Case 2. $x^c \in (0, \infty)$.

Clearly, for fixed $c > 0$, $\tilde{\phi}_n^c(x)$ and, by the equation subsequently $(\tilde{\phi}_n^c)'(x)$ (for $x \neq -x_n^c$), are uniformly bounded in n . Then by the Arzela-Ascoli theorem and a standard argument involving a diagonal process of choosing subsequences, we see that $\{\tilde{\phi}_n^c\}_{n \geq 1}$ has a subsequence, still denoted by itself for simplicity of notation, which converges to some $\tilde{\phi}^c \in C(\mathbb{R})$ locally uniformly in $\mathbb{R}$. Moreover, $\tilde{\phi}^c(x)$ is nonincreasing in x with $\tilde{\phi}^c(0) = 1/2$.

If Case 1 happens, we easily see that $\tilde{\phi}^c$ satisfies

$$(4.19) \quad d \int_{-\infty}^{\infty} J(x-y) \tilde{\phi}^c(y) dy - d \tilde{\phi}^c(x) + c(\tilde{\phi}^c)'(x) + f(\tilde{\phi}^c(x)) = 0 \text{ for } x \in \mathbb{R}.$$

In fact, from (4.18), for $x \in \mathbb{R}$ and all large n satisfying $x < -x_n^c$, we have

$$c \tilde{\phi}_n^c(x) - c \tilde{\phi}_n^c(0) = -d \int_0^x \left[\int_{-\infty}^{\infty} J(\xi-y) \tilde{\phi}_n^c(y) dy - d \tilde{\phi}_n^c(\xi) + f(\tilde{\phi}_n^c(\xi)) \right] d\xi.$$

It then follows from the dominated convergence theorem that, for $x \in \mathbb{R}$,

$$c \tilde{\phi}^c(x) - c \tilde{\phi}^c(0) = -d \int_0^x \left[\int_{-\infty}^{\infty} J(\xi-y) \tilde{\phi}^c(y) dy - d \tilde{\phi}^c(\xi) + f(\tilde{\phi}^c(\xi)) \right] d\xi,$$

and (4.19) thus follows by differentiating this equation. Due to the monotonicity and boundedness of $\tilde{\phi}^c(x)$, the arguments in step 3 of the proof of Lemma 4.4 can be repeated to give

$$\lim_{x \rightarrow -\infty} \left[d \int_{-\infty}^{\infty} J(x-y) \tilde{\phi}^c(y) dy - d\tilde{\phi}^c(x) \right] = 0,$$

and so

$$\lim_{x \rightarrow -\infty} [c(\tilde{\phi}^c)'(x) + f(\tilde{\phi}^c(x))] = 0.$$

Denote $K := \lim_{x \rightarrow -\infty} \tilde{\phi}^c(x) \in \mathbb{R}_+$. Then we must have

$$f(K) = \lim_{x \rightarrow -\infty} f(\tilde{\phi}^c(x)) = - \lim_{x \rightarrow -\infty} c(\tilde{\phi}^c)'(x).$$

This is possible only if $f(K) = 0$. By $(\mathbf{f}_{\mathbf{KPP}})$ either $K = 0$ or $K = 1$. Since $\tilde{\phi}^c(x)$ is nonincreasing in x with $\tilde{\phi}^c(0) = 1/2 > 0$, we have $K > 0$ and hence we must have $K = 1$. An analogous analysis can be applied to show $\lim_{x \rightarrow \infty} \tilde{\phi}^c(x) = 0$. Therefore, $\tilde{\phi}^c(x)$ is a monotone traveling wave of (4.3) with speed c .

If Case 2 happens, analogously for fixed $x < x^c$,

$$c\tilde{\phi}^c(x) - c\tilde{\phi}^c(0) = -d \int_0^x \left[\int_{-\infty}^{\infty} J(\xi-y) \tilde{\phi}^c(y) dy - d\tilde{\phi}^c(\xi) + f(\tilde{\phi}^c(\xi)) \right] d\xi,$$

and $\tilde{\phi}^c(x) = 0$ for $x \geq x^c$, which yields

$$\begin{cases} d \int_{-\infty}^{x^c} J(x-y) \tilde{\phi}^c(y) dy - d\tilde{\phi}^c(x) + c(\tilde{\phi}^c)'(x) + f(\tilde{\phi}^c(x)) = 0 & \text{for } x < x^c, \\ \tilde{\phi}^c(x^c) = 0. \end{cases}$$

Let $\phi^c(x) := \tilde{\phi}^c(x + x^c)$ for $x \leq 0$, then $\phi^c(x)$ satisfies

$$\begin{cases} d \int_{-\infty}^0 J(x-y) \phi^c(y) dy - d\phi^c(x) + c(\phi^c)'(x) + f(\phi^c(x)) = 0 & \text{for } x < 0, \\ \phi^c(0) = 0. \end{cases}$$

Moreover, as in Case 1, we can show $\lim_{x \rightarrow -\infty} \phi^c(x) = 1$. Therefore, $\phi^c(x)$ is a monotone semi-wave solution of (4.3) with speed c .

We have thus proved that for any $c > 0$, (4.3) has either a monotone traveling wave solution with speed c or a monotone semi-wave solution with speed c . We show next that for any given $c > 0$, (4.3) cannot have both.

Suppose, on the contrary, there is $c_0 > 0$ such that (4.3) admits a monotone traveling wave solution ψ with speed c_0 and also a monotone semi-wave solution ϕ with speed c_0 . We are going to drive a contradiction.

Let $\tilde{\phi}(x) := k\phi(x)$ for some fixed $k \in (0, 1)$. Then by $(\mathbf{f}_{\mathbf{KPP}})$, $\tilde{\phi}$ satisfies

$$\begin{cases} d \int_{-\infty}^{\infty} J(x-y) \tilde{\phi}(y) dy - d\tilde{\phi}(x) + c\tilde{\phi}(x) + f(k\phi(x)) \geq 0, & x < 0, \\ \tilde{\phi}(-\infty) = k, \quad \tilde{\phi}(x) = 0, & x \geq 0. \end{cases}$$

For $\beta \in \mathbb{R}$, define

$$\psi^\beta(x) := \psi(x + \beta), \quad w^\beta(x) := \psi^\beta(x) - \tilde{\phi}(x), \quad x \in \mathbb{R}.$$

For fixed $x \leq 0$,

$$w^\beta(x) \geq \psi(\beta) - k\phi(x) \geq \psi(\beta) - k \rightarrow 1 - k > 0 \text{ as } \beta \rightarrow -\infty.$$

Therefore there exists $\bar{\beta} \ll -1$ independent of x such that

$$w^\beta(x) > 0 \text{ for } x \leq 0, \quad \beta \leq \bar{\beta}.$$

On the other hand,

$$w^\beta(-1) = \psi(\beta - 1) - k\phi(-1) \rightarrow -k\phi(-1) < 0 \text{ as } \beta \rightarrow \infty.$$

Therefore we can find $\beta^* \in \mathbb{R}$ such that

$$h(\beta) := \inf_{x \leq 0} w^\beta(x) > 0 \text{ for } \beta < \beta^*, \quad h(\beta^*) = 0.$$

Clearly $w^{\beta^*}(-\infty) = 1 - k > 0$ and $w^{\beta^*}(0) = \psi^{\beta^*}(0) > 0$. Therefore due to the continuity of $w^{\beta^*}(x)$ there exists $x^0 \in (-\infty, 0)$ such that $w^{\beta^*}(x^0) = 0$. We can thus conclude that

$$w^{\beta}(x) \geq 0 \text{ for } x \leq 0, \quad \beta \leq \beta^*, \quad \text{and } w^{\beta^*}(x^0) = 0.$$

In particular,

$$(4.20) \quad w^{\beta^*}(x) \geq 0 \text{ for } x \leq 0, \quad w^{\beta^*}(x^0) = 0.$$

By the definition of ψ and ϕ , we see that w^{β^*} satisfies

$$\begin{cases} d \int_{-\infty}^{\infty} J(x-y)w^{\beta^*}(y)dy - dw^{\beta^*}(x) + cw^{\beta^*}'(x) \\ \quad + f(\psi^{\beta^*}(x)) - f(k\phi(x)) \leq 0, & x < 0, \\ w^{\beta^*}(-\infty) = 1 - k > 0, \quad w^{\beta^*}(x) \geq 0, & x \in \mathbb{R}. \end{cases}$$

We have

$$f(\psi^{\beta^*}(x)) - f(k\phi(x)) = C(x)w^{\beta^*}(x)$$

with

$$C(x) := \int_0^1 f'(k\phi(x) + tw^{\beta^*}(x))dt.$$

This allows us to use Lemma 4.2 to conclude that $w^{\beta^*}(x) > 0$ for $x < 0$, which contradicts the second part of (4.20). This completes the proof. $\square$

Proof of Theorem 4.6:

From (4.17) we see that x^c is nondecreasing in c and hence there are three possible cases:

- (1) For any $c > 0$, $x^c < \infty$.
- (2) For any $c > 0$, $x^c = \infty$.
- (3) There is $C_* > 0$ such that $x^c < \infty$ for any $c \in (0, C_*)$, and $x^c = \infty$ for any $c > C_*$.

From the proof of Lemma 4.7, we know that in case (1), (4.3) has a monotone semi-wave with speed c for any $c > 0$; in case (2), it has a monotone traveling wave with speed c for every $c > 0$; in case (3), for each $c \in (0, C_*)$ there is a monotone semi-wave solution with speed c , and for each $c > C_*$, there is a traveling wave with speed c . Therefore to complete the proof it suffices to show that in case (3), (4.3) has a monotone traveling wave solution with speed $c = C_*$.

Let ψ^c be a monotone traveling wave solution of (4.3) with speed $c > C_*$. By a suitable translation we may assume $\psi^c(0) = 1/2$. Since ψ^c is uniformly bounded, by the equation satisfied by ψ^c we see that $(\psi^c)'$ is also uniformly bounded in c for $c > C_*$. Then by the Arzela-Ascoli theorem and a standard argument involving a diagonal process of choosing subsequences, for any sequence $c_n \searrow C_*$, $\{\psi^{c_n}\}_{n=1}^{\infty}$ has a subsequence, still denoted by itself, which converges to some $\psi \in C(\mathbb{R})$ locally uniformly in $\mathbb{R}$ as $n \rightarrow \infty$. Similar to the proof of Lemma 4.7, we can check at once that ψ satisfies

$$\begin{cases} d \int_{-\infty}^{\infty} J(x-y)\psi(y)dy - d\psi(x) + C_*\psi'(x) + f(\psi(x)) = 0, & x \in \mathbb{R}, \\ \psi(0) = 1/2. \end{cases}$$

Making use of the monotonicity of $\psi(x)$ inherited from $\psi_n^c(x)$, we can use the method in Step 3 of the proof of Lemma 4.4 to show that

$$\psi(-\infty) = 1, \quad \psi(\infty) = 0,$$

which implies that ψ is a monotone traveling wave solution of (4.3) with speed $c = C_*$. The proof is now completed. $\square$

Remark: In view of Theorem B, we see that case (1) of Theorem 4.6 happens if and only if $c_*^+ = \infty$; case (2) happens if and only if $c_*^+ \leq 0$; and case (3) happens if and only if $c_*^+ \in (0, \infty)$, and in such a case, $C^* = c_*^+$.

4.4. Uniqueness and strict monotonicity of semi-wave solutions to (4.3).

Theorem 4.8. *Suppose that (J) and (f_{KPP}) hold. Then for any $c > 0$, (4.3) has at most one monotone semi-wave solution $\phi = \phi^c$ with speed c , and when exists, $\phi^c(x)$ is strictly decreasing in x for $x \in (-\infty, 0]$. Moreover, if ϕ^{c_1} and ϕ^{c_2} both exist and $0 < c_1 < c_2$, then $\phi^{c_1}(x) > \phi^{c_2}(x)$ for fixed $x < 0$.*

Proof. Assume that ϕ_1 and ϕ_2 are monotone semi-wave solutions of (4.3) with speed $c > 0$. We want to show that $\phi_1 \equiv \phi_2$.

Claim 1. $\phi'_k(0^-) < 0$ for $k = 1, 2$.

From the equation satisfied by ϕ_k , we deduce, for $k = 1, 2$,

$$\begin{aligned}
 \phi'_k(0^-) &= \lim_{x \rightarrow 0^-} \frac{\phi_k(x)}{x} \\
 (4.21) \quad &= \lim_{x \rightarrow 0^-} \frac{1}{cx} \int_0^x \left[-d \int_{-\infty}^0 J(z-y)\phi_k(y)dy + d\phi_k(z) - f(\phi_k(z)) \right] dz \\
 &= -\frac{1}{c} d \int_{-\infty}^0 J(-y)\phi_k(y)dy < 0.
 \end{aligned}$$

With the help of Claim 1, we are ready to define

$$\rho^* := \inf\{\rho \geq 1 : \rho\phi_1(x) \geq \phi_2(x) \text{ for } x \leq 0\}.$$

Since $\phi_k(-\infty) = 1$ for $k = 1, 2$, $\frac{\phi_2(x)}{\phi_1(x)}$ is uniformly bounded for x in a small left neighbourhood of 0 by Claim 1, we see that $\rho^* \in [1, \infty)$ is well-defined, and $\rho^*\phi_1(x) \geq \phi_2(x)$ for $x \leq 0$.

Claim 2: $\rho^* = 1$.

Otherwise $\rho^* > 1$ and from the definition of ρ^* we can find a sequence $x_n \in (-\infty, 0)$ such that

$$(4.22) \quad \lim_{n \rightarrow \infty} \frac{\phi_2(x_n)}{\phi_1(x_n)} = \rho^* > 1.$$

From $\phi_k(-\infty) = 1$ for $k = 1, 2$ we see that $\{x_n\}$ must be a bounded sequence, and hence by passing to a subsequence, we may assume that $x_n \rightarrow x_* \in (-\infty, 0]$ as $n \rightarrow \infty$. Define

$$V(x) := \rho^*\phi_1(x) - \phi_2(x).$$

Clearly $V(x) \geq 0$ for $x \leq 0$. Our discussion below is organised according to the following two possibilities:

- Case 1. $V(x) > 0$ for all $x < 0$.
- Case 2. There exists $x_0 < 0$ such that $V(x_0) = 0$.

In Case 1, from (4.21) we obtain

$$V'(0^-) = -\frac{1}{c} d \int_{-\infty}^0 J(0-y)V(y)dy < 0.$$

Let us examine the sequence $\{x_n\}$ in (4.22). We have $x_n \rightarrow x_* \in (-\infty, 0]$. If $x_* < 0$ then we deduce $V(x_*) = 0$ which is a contradiction to $V(x) > 0$ for $x < 0$. Therefore we must have $x_* = 0$ and so $x_n \rightarrow 0$ as $n \rightarrow \infty$. It then follows that

$$\lim_{n \rightarrow \infty} \frac{\phi_2(x_n)}{\phi_1(x_n)} = \frac{\phi'_2(0^-)}{\phi'_1(0^-)} < \rho^*,$$

due to $V'(0^-) < 0$ and $(\phi_k)'(0^-) < 0$ for $k = 1, 2$. Thus we always arrive at a contradiction to (4.22) in Case 1.

In Case 2, from the assumption (f_{KPP}), we see that

$$W(x) := \phi_1(x) - (\rho^*)^{-1}\phi_2(x) = (\rho^*)^{-1}V(x)$$

satisfies, for $x \leq 0$,

$$\begin{aligned}
 0 &= d \int_{-\infty}^0 J(x-y)W(y)dy - dW(x) + cW'(x) + f(\phi_1(x)) - (\rho^*)^{-1}f(\phi_2(x)) \\
 &\geq d \int_{-\infty}^0 J(x-y)W(y)dy - dW(x) + cW'(x) + f(\phi_1(x)) - f((\rho^*)^{-1}\phi_2(x)) \\
 &= d \int_{-\infty}^0 J(x-y)W(y)dy - dW(x) + cW'(x) + b(x)W(x),
 \end{aligned}$$

where $b(x)$ is a bounded function. In view of $W(x) \geq 0$ for $x \leq 0$, and $W(-\infty) > 0$, we can apply Lemma 4.2 to conclude that

$$W(x) > 0 \text{ for } x < 0.$$

This is a contradiction to $W(x_0) = (\rho^*)^{-1}V(x_0) = 0$.

We have thus proved $\rho^* = 1$, and so $\phi_1(x) \geq \phi_2(x)$ for $x \leq 0$. By swapping $\phi_1(x)$ with $\phi_2(x)$ we also have $\phi_2(x) \geq \phi_1(x)$ for $x \leq 0$. This completes our proof for uniqueness of the semi-wave solution.

Next we prove the strict monotonicity properties stated in the theorem. Let ϕ^c be a monotone semi-wave solution of (4.3) with speed $c > 0$. The strict monotonicity of $\phi^c(x)$ with respect to $x \leq 0$ clearly follows directly from Lemma 4.3. We show next that for fixed $x < 0$, $\phi^c(x)$ is strictly decreasing with respect to $c > 0$, namely, $\phi^{c_1}(x) > \phi^{c_2}(x)$ for $c_2 > c_1 > 0$. Denote $W(x) := \phi^{c_1}(x) - \phi^{c_2}(x)$. By Lemma 4.5 and the proof of Lemma 4.7 without shifting ϕ_n , we see that $W(x) \geq 0$ for $x \leq 0$. By $(\mathbf{f}_{\mathbf{KPP}})$,

$$f(\phi^{c_1}(x)) - f(\phi^{c_2}(x)) = E(x)W(x)$$

where $E(x)$ is a bounded function. This, combined with $c_1(\phi^{c_1})'(x) - c_2(\phi^{c_2})'(x) > c_1W'(x)$, allows us to apply Lemma 4.2 to conclude that $W(x) > 0$ for $x < 0$. $\square$

4.5. Semi-wave solution with the desired speed.

Theorem 4.9. *Suppose that $(\mathbf{J})$, $(\mathbf{f}_{\mathbf{KPP}})$ hold, $c_*^+ \in (0, \infty]$ and $\phi^c(x)$ is the unique monotone semi-wave solution of (4.3) with speed $c \in (0, c_*^+)$. Then*

$$(4.23) \quad \lim_{c \nearrow c_*^+} \phi^c(x) = 0 \text{ locally uniformly in } (-\infty, 0].$$

Moreover, (4.6) and (4.7) have a solution pair (c, ϕ) with $\phi(x)$ monotone if and only if $(\mathbf{J}_1^+)$ holds. And when $(\mathbf{J}_1^+)$ holds, there exists a unique $c_0 \in (0, c_*^+)$ such that $(c, \phi) = (c_0, \phi^{c_0})$ solves (4.6) and (4.7).

Proof. We first prove (4.23). Since $\phi^c(x)$ is decreasing with respect to c , $\phi(x) := \lim_{c \nearrow c_*^+} \phi^c(x)$ is well-defined, and $\phi(x) \in [0, 1]$ for $x \leq 0$. Moreover, by the uniform boundedness of $(\phi^c)'(x)$ obtained from the equation it satisfies, the convergence of $\phi^c(x)$ to $\phi(x)$ is locally uniform in $(-\infty, 0]$. If $c_*^+ = \infty$, then from

$$\phi^c(x) = \frac{1}{c} \int_0^x \left[-d \int_{-\infty}^0 J(z-y)\phi^c(y)dy + d\phi^c(z) - f(\phi^c(z)) \right] dz$$

we immediately obtain $\phi(x) \equiv 0$. If $c_*^+ < \infty$ then ϕ satisfies

$$\begin{cases} d \int_{-\infty}^0 J(x-y)\phi(y)dy - d\phi(x) + c_*^+ \phi'(x) + f(\phi(x)) = 0, & x < 0, \\ \phi(0) = 0. \end{cases}$$

Note that $\phi(x)$ is nonincreasing since $\phi^c(x)$ is. As in Step 3 of the proof of Lemma 4.4, we can show that $\phi(-\infty) = 1$ or 0. By Theorem 4.6, the Cauchy problem (4.3) admits no monotone semi-wave solution for $c = c_*^+$, and hence necessarily $\phi(-\infty) = 0$. Thus we also have $\phi \equiv 0$, and (4.23) is proved.

Next we show that if $(\mathbf{J}_1^+)$ holds, then (4.6)-(4.7) have a unique solution pair (c_0, ϕ^{c_0}) . It suffices to prove that

$$P(c) := c - M(c), \text{ with } M(c) := \mu \int_{-\infty}^0 \int_0^\infty J(y-x)\phi^c(x)dydx,$$

has a unique root in $(0, c_*^+)$. Let us observe that when $(\mathbf{J}_1^+)$ holds, $M(c)$ is well-defined and strictly decreasing in c by Theorem 4.8. Indeed, an elementary calculation yields

$$\int_{-\infty}^0 \int_0^\infty J(y-x)dydx = \int_0^\infty \int_0^\infty J(x+y)dydx = \int_0^\infty J(y)ydy,$$

which implies that $M(c)$ is well-defined.

Using the uniqueness of ϕ^c , we can apply a similar convergence argument as used above to prove (4.23) to show that $\phi^{c_n} \rightarrow \phi^c$ as $c_n \rightarrow c \in (0, c_*^+)$, which yields the continuity of $\phi^c(x)$ in $c \in (0, c_*^+)$ uniformly for x over any bounded interval of $(-\infty, 0]$. Note that we can easily see that $\phi(x) := \lim_{c_n \rightarrow c} \phi^{c_n}(x)$ satisfies $\phi(-\infty) = 1$ by comparing ϕ^{c_n} to some $\phi^{\hat{c}}$ with $\hat{c} \in (c, c_*^+)$ and using the monotonicity of ϕ^c in c .

Hence $P(c)$ is increasing and continuous in c . For $c \in (0, c_*^+/2)$ close to 0, we have $P(c) \leq c - M(c_*^+/2) < 0$, and for all c close to c_*^+ , $M(c)$ is small and hence $P(c) > 0$. Thus there is a unique $c_0 \in (0, c_*^+)$ such that $P(c_0) = 0$.

Finally we verify that $(\mathbf{J}_1^+)$ holds if (4.6)-(4.7) have a solution pair (c_0, ϕ^{c_0}) . Since

$$c_0 = \mu \int_{-\infty}^0 \int_0^\infty J(y-x) \phi^{c_0}(x) dy dx,$$

we have

$$\int_{-\infty}^0 \int_0^\infty J(y-x) \phi^{c_0}(x) dy dx < \infty.$$

By Theorem 4.8, $\phi^{c_0}(x)$ is decreasing in x . Hence,

$$\int_{-\infty}^0 \int_0^\infty J(y-x) \phi^{c_0}(x) dy dx \geq \phi^{c_0}(-1) \int_{-\infty}^{-1} \int_0^\infty J(y-x) dy dx,$$

and so

$$\begin{aligned} \int_{-\infty}^0 \int_0^\infty J(y-x) dy dx &= \int_{-1}^0 \int_0^\infty J(y-x) dy dx + \int_{-\infty}^{-1} \int_0^\infty J(y-x) dy dx \\ &\leq 1 + \int_{-\infty}^{-1} \int_0^\infty J(y-x) dy dx < \infty. \end{aligned}$$

Therefore, $(\mathbf{J}_1^+)$ holds. $\square$

Theorem 4.1 parts (a^+) and (b^+) clearly follow directly from Theorems 4.6, 4.8 and 4.9. The proof of parts (a^-) and (b^-) is parallel; these conclusions also follow from (a^+) and (b^+) by considering (4.1) with $J(x)$ replaced by $J(-x)$.

Remarks: In the symmetric case $J(x) = J(-x)$, Theorem 4.1 was first proved in [9]. These results have been extended to rather general cooperative systems in [11], and much of our arguments here follow [11] instead of [9].

5. SPREADING SPEED

We are going to determine the spreading speed of the nonlocal free boundary problem (4.1). For a non-symmetric J satisfying $(\mathbf{J})$, the following two quantities determined by J and $f'(0)$ alone play an important role:

$$c_*^- = \sup_{\nu < 0} \frac{d \int_{\mathbb{R}} J(x) e^{\nu x} dx - d + f'(0)}{\nu}, \quad c_*^+ = \inf_{\nu > 0} \frac{d \int_{\mathbb{R}} J(x) e^{\nu x} dx - d + f'(0)}{\nu},$$

It can be shown that c_*^- is achieved by some $\nu < 0$ when it is finite, and a parallel conclusion holds for c_*^+ . It is easily checked that c_*^- is finite if and only if J satisfies additionally the following **thin-tail** condition at $x = -\infty$,

$$(\mathbf{J}_{\text{thin}}^-) : \text{There exists } \lambda > 0 \text{ such that } \int_0^{+\infty} J(-x) e^{\lambda x} dx < +\infty.$$

Similarly, c_*^+ is finite if and only if J satisfies

$$(\mathbf{J}_{\text{thin}}^+) : \text{There exists } \lambda > 0 \text{ such that } \int_0^{+\infty} J(x) e^{\lambda x} dx < +\infty.$$

We define

$$(5.1) \quad \begin{cases} c_*^- = -\infty & \text{when } (\mathbf{J}_{\text{thin}}^-) \text{ does not hold,} \\ c_*^+ = +\infty & \text{when } (\mathbf{J}_{\text{thin}}^+) \text{ does not hold.} \end{cases}$$

We say $J(x)$ is **weakly non-symmetric** if

$$(5.2) \quad -\infty \leq c_*^- < 0 < c_*^+ \leq \infty.$$

Theorem 5.1 (Spreading speed). *Suppose that $(\mathbf{J})$ and $(\mathbf{f}_{\text{KPP}})$ are satisfied, and (5.2) holds. Let (u, g, h) be the unique solution of (4.1), and assume that spreading occurs. Then the following conclusions are valid:*

(i) The spreading speed of the right front $h(t)$ is given by

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = \begin{cases} c_0 & \text{if } (\mathbf{J}_1^+) \text{ holds,} \\ +\infty & \text{if } (\mathbf{J}_1^+) \text{ does not hold,} \end{cases}$$

where (c_0, ϕ^{c_0}) is the solution of (4.6)-(4.7).

(ii) The spreading speed of the left front $g(t)$ is given by

$$\lim_{t \rightarrow \infty} \frac{g(t)}{t} = \begin{cases} -\tilde{c}_0 & \text{if } (\mathbf{J}_1^-) \text{ holds,} \\ -\infty & \text{if } (\mathbf{J}_1^-) \text{ does not hold,} \end{cases}$$

where $(\tilde{c}_0, \psi^{\tilde{c}_0})$ is the solution of (4.8)-(4.9).

(iii) Define $c_0 = \infty$ if $(\mathbf{J}_1^+)$ does not hold and $\tilde{c}_0 = \infty$ if $(\mathbf{J}_1^-)$ does not hold. Then for any constants a and b satisfying $-\tilde{c}_0 < a < b < c_0$, we have

$$\lim_{t \rightarrow \infty} \sup_{[at, bt]} |u(t, x) - 1| = 0.$$

5.1. Comparison principles revisited. The following variations of the comparison principle in Section 1 will be used to prove Theorem 5.1. Their proofs use similar techniques.

Lemma 5.2 (Comparison principle 2). Assume that conditions **(J)** and **(f)** hold, u_0 satisfies (1.15) and (u, g, h) is the unique positive solution of problem (4.1). For $T \in (0, +\infty)$, suppose that $\bar{g} \in C([0, T])$, $\bar{u}(t, x)$ and $\bar{u}_t(t, x)$ are continuous for $t \in [0, T]$, $x \in [\bar{g}(t), h(t)]$ and satisfy $\bar{g}(t) < h(t)$ and

$$\begin{cases} \bar{u}_t \geq d \int_{\bar{g}(t)}^{h(t)} J(x-y) \bar{u}(t, y) dy - d\bar{u} + f(\bar{u}), & 0 < t \leq T, x \in (\bar{g}(t), h(t)), \\ \bar{u}(t, x) \geq 0, & 0 < t \leq T, x \in \{\bar{g}(t), h(t)\}, \\ \bar{g}'(t) \leq -\mu \int_{\bar{g}(t)}^{h(t)} \int_{-\infty}^{\bar{g}(t)} J(y-x) \bar{u}(t, x) dy dx, & 0 < t \leq T, \\ \bar{u}(0, x) \geq u(0, x), \bar{g}(0) \leq g_0, & x \in [g_0, h_0]. \end{cases}$$

Then

$$u(t, x) \leq \bar{u}(t, x) \text{ and } g(t) \geq \bar{g}(t) \text{ for } 0 < t \leq T \text{ and } x \in [g(t), h(t)].$$

Lemma 5.3 (Comparison principle 3). Assume that conditions **(J)** and **(f)** hold, u_0 satisfies (1.15) and (u, g, h) is the unique positive solution of problem (4.1). For $T \in (0, +\infty)$, suppose that $\underline{g}, \underline{h} \in C([0, T])$, $\underline{g}(t) \leq \underline{g}(t) < \underline{h}(t)$, $\underline{u}(t, x)$ and $\underline{u}_t(t, x)$ are continuous for $t \in [0, T]$, $x \in [\underline{g}(t), \underline{h}(t)]$ and satisfy

$$\begin{cases} \underline{u}_t \leq d \int_{\underline{g}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(t, y) dy - d\underline{u} + f(\underline{u}), \underline{u} \geq 0, & 0 < t \leq T, x \in (\underline{g}(t), \underline{h}(t)), \\ \underline{u}(t, \underline{g}(t)) = \underline{u}(t, \underline{h}(t)) = 0, & 0 < t \leq T, \\ \underline{h}'(t) \leq \mu \int_{\underline{g}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(y-x) \underline{u}(t, x) dy dx, & 0 < t \leq T, \\ \underline{u}(0, x) \geq \underline{u}(0, x), \underline{h}(0) \geq \underline{h}(0), & x \in [\underline{g}(0), \underline{h}(0)]. \end{cases}$$

Then

$$u(t, x) \geq \underline{u}(t, x) \text{ and } h(t) \geq \underline{h}(t) \text{ for } 0 < t \leq T \text{ and } x \in [g(t), h(t)].$$

Remark: In the above lemmas the assumption u_t being continuous can be relaxed. If, for each (t, x) , both the one-sided partial derivatives $u_t(t+0, x)$ and $u_t(t-0, x)$ exist, and the differential inequalities hold when u_t is replaced by both one-sided partial derivatives, then the conclusions remain valid (see Remark 2.4 in [11] for symmetric kernels, but the observation there still holds when the symmetry requirement for the kernel functions is dropped).

5.2. Bounds from above.

Lemma 5.4. Suppose that **(J)** and **(f_{KPP})** are satisfied, and (5.2) holds. Let (u, g, h) be the unique solution of (4.1). If $(\mathbf{J}_1^+)$ is satisfied, then $\limsup_{t \rightarrow \infty} \frac{h(t)}{t} \leq c_0$. If $(\mathbf{J}_1^-)$ is satisfied, then $\limsup_{t \rightarrow \infty} \frac{-g(t)}{t} \leq \tilde{c}_0$.

Proof. Since the proofs for the estimates of $h(t)/t$ and $g(t)/t$ are similar, we only present the proof for $g(t)/t$.

For any $\epsilon > 0$, define $\delta := 2\epsilon\tilde{c}_0$ and

$$\begin{aligned}\bar{g}(t) &:= -(\tilde{c}_0 + \delta)t - L, \quad t \geq 0, \\ \bar{u}(t, x) &:= (1 + \epsilon)\psi(x - \bar{g}(t)), \quad x \in [\bar{g}, +\infty), \quad t \geq 0\end{aligned}$$

where $(\tilde{c}_0, \psi)$ satisfies (4.8)-(4.9), and $L > 0$ is a constant to be determined.

A simple comparison to the ODE problem $v' = f(v)$ with $v(0) = \|u_0\|_\infty$ shows that $u(t, x) \leq v(t)$ and hence $\limsup_{t \rightarrow \infty} u(t, x) \leq 1$ uniformly in $x \in [g(t), h(t)]$. Hence there exists $T > 0$ large such that

$$u(T + t, x) \leq 1 + \epsilon/2 \text{ for } t \geq 0 \text{ and } x \in [g(T + t), h(T + t)].$$

Since $\psi(\infty) = 1$, we can choose $L > 0$ large such that $-\bar{g}(0) = L > -2g(T)$ and $\psi(\frac{L}{2}) > \frac{1+\frac{\epsilon}{2}}{1+\epsilon}$. Hence

$$\bar{u}(0, x) = (1 + \epsilon)\psi(x + L) \geq (1 + \epsilon)\psi(\frac{L}{2}) > 1 + \frac{\epsilon}{2} \geq u(T, x) \text{ for } x \in [g(T), h(T)].$$

Moreover, for $t \geq 0$ we have

$$\begin{aligned}\mu \int_{\bar{g}(t)}^{h(t+T)} \int_{-\infty}^{\bar{g}(t)} J(y - x) \bar{u}(t, x) dy dx &\leq \mu \int_{\bar{g}(t)}^{+\infty} \int_{-\infty}^{\bar{g}(t)} J(y - x) \bar{u}(t, x) dy dx \\ &= \mu(1 + \epsilon) \int_0^{+\infty} \int_{-\infty}^0 J(y - x) \psi(x) dy dx \\ &= (1 + \epsilon)\tilde{c}_0 < \tilde{c}_0 + \delta = -\bar{g}'(t).\end{aligned}$$

Using the equation satisfied by ψ , we deduce, for $t \geq 0$ and $x \in [\bar{g}(t), h(t + T)]$,

$$\begin{aligned}\bar{u}_t &= (1 + \epsilon)(\tilde{c}_0 + \delta)\psi'(x - \bar{g}(t)) > (1 + \epsilon)\tilde{c}_0\psi'(x - \bar{g}(t)) \\ &= (1 + \epsilon) \left[d \int_0^{+\infty} J(x - \bar{g}(t) - y) \psi(y) dy - d\psi(x - \bar{g}(t)) + f(\psi(x - \bar{g}(t))) \right] \\ &= d \int_{\bar{g}(t)}^{+\infty} J(x - y) \bar{u}(t, y) dy - d\bar{u}(t, x) + (1 + \epsilon)f(\psi(x - \bar{g}(t))) \\ &\geq d \int_{\bar{g}(t)}^{h(t+T)} J(x - y) \bar{u}(t, y) dy - d\bar{u}(t, x) + f(\bar{u}(t, x)),\end{aligned}$$

where the last inequality follows from $(\mathbf{f}_{\mathbf{KPP}})$.

We may now use the comparison principle (Lemma 5.2) to conclude that $\bar{g}(t) \leq g(t + T)$ and $u(t + T, x) \leq \bar{u}(t, x)$ for $t > 0$, $x \in [g(t + T), h(t + T)]$. Hence

$$\limsup_{t \rightarrow \infty} \frac{-g(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{-\bar{g}(t - T)}{t} = \tilde{c}_0 + \delta = \tilde{c}_0(1 + 2\epsilon).$$

Letting $\epsilon \rightarrow 0$, we obtain the desired conclusion. $\square$

5.3. Bounds from below for compact kernels. We first treat the case of compactly supported kernels. For the general case we will use compactly supported kernels to approximate a general kernel.

Lemma 5.5. *Assume that $(\mathbf{f}_{\mathbf{KPP}})$ holds, J satisfies $(\mathbf{J})$ and has compact support, and so $(\mathbf{J}_1^+)$ and $(\mathbf{J}_1^-)$ are satisfied automatically; then $\liminf_{t \rightarrow \infty} \frac{-g(t)}{t} \geq \tilde{c}_0$, $\liminf_{t \rightarrow \infty} \frac{h(t)}{t} \geq c_0$.*

Proof. We follow the approach used to prove Lemma 3.2 of [9]. Let (c_0, ϕ) and $(\tilde{c}_0, \psi)$ be the unique solution pair for (4.6)-(4.7) and (4.8)-(4.9), respectively. Since $f'(1) < 0$, there is a small $\delta_0 > 0$ such that $f'(u) < 0$ for $u \in [1 - \delta_0, 1]$. For $\epsilon \in (0, \delta_0]$, define

$$\begin{aligned}\underline{h}(t) &:= (1 - 2\epsilon)c_0t + L, \quad \underline{g}(t) := -(1 - 2\epsilon)\tilde{c}_0t - L, \\ \underline{u}(t, x) &:= (1 - \epsilon) [\phi(x - \underline{h}(t)) + \psi(x - \underline{g}(t)) - 1].\end{aligned}$$

We claim that

$$\underline{h}'(t) \leq \mu \int_{\underline{g}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(y - x) \underline{u}(t, x) dy dx \text{ for } t > 0.$$

In fact, by (4.7), we have

$$\begin{aligned}
& \mu \int_{\underline{g}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(y-x) \underline{u}(t, x) dy dx \\
&= \mu(1-\epsilon) \int_{\underline{g}(t)-\underline{h}(t)}^0 \int_0^{+\infty} J(y-x) [\phi(x) + \psi(x + \underline{h}(t) - \underline{g}(t)) - 1] dy dx \\
&= (1-\epsilon)c_0 - \mu(1-\epsilon) \int_{-\infty}^{\underline{g}(t)-\underline{h}(t)} \int_0^{+\infty} J(y-x) \phi(x) dy dx \\
&\quad - \mu(1-\epsilon) \int_{\underline{g}(t)-\underline{h}(t)}^0 \int_0^{+\infty} J(y-x) [1 - \psi(x + \underline{h}(t) - \underline{g}(t))] dy dx.
\end{aligned}$$

By $(\mathbf{J}_1^+)$, for all large $L > 0$, we have

$$\begin{aligned}
0 &\leq \mu(1-\epsilon) \int_{-\infty}^{\underline{g}(t)-\underline{h}(t)} \int_0^{+\infty} J(y-x) \phi(x) dy dx \\
&\leq \mu(1-\epsilon) \int_{-\infty}^{-2L} \int_0^{+\infty} J(y-x) dy dx \\
&< \frac{1}{4} \epsilon c_0.
\end{aligned}$$

Since $\psi(x)$ is increasing, $\psi(\infty) = 1$ and $(\mathbf{J}_1^+)$ holds, we deduce for all large L and all $t \geq 0$,

$$\begin{aligned}
0 &\leq \mu(1-\epsilon) \int_{\underline{g}(t)-\underline{h}(t)}^0 \int_0^{+\infty} J(y-x) [1 - \psi(x + \underline{h}(t) - \underline{g}(t))] dy dx \\
&\leq \mu(1-\epsilon) \int_{\underline{g}(t)-\underline{h}(t)}^{\frac{1}{2}(\underline{g}(t)-\underline{h}(t))} \int_0^{+\infty} J(y-x) dy dx \\
&\quad + \mu(1-\epsilon) \int_{\frac{1}{2}(\underline{g}(t)-\underline{h}(t))}^0 \int_0^{+\infty} J(y-x) [1 - \psi(x + \underline{h}(t) - \underline{g}(t))] dy dx \\
&< \frac{1}{4} \epsilon c_0 + \mu(1-\epsilon)[1 - \psi(L)] \int_{\frac{1}{2}(\underline{g}(t)-\underline{h}(t))}^0 \int_0^{+\infty} J(y-x) dy dx \\
&< \frac{1}{2} \epsilon c_0.
\end{aligned}$$

Therefore, for all large $L > 0$,

$$\mu \int_{\underline{g}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(y-x) \underline{u}(t, x) dy dx \geq (1-\epsilon)c_0 - \epsilon c_0 = h'(t) \text{ for } t > 0.$$

Similarly, we can show, for all large $L > 0$,

$$\underline{g}'(t) \geq -\mu \int_{\underline{g}(t)}^{\underline{h}(t)} \int_{-\infty}^{\underline{g}(t)} J(y-x) \underline{u}(t, x) dy dx \text{ for } t > 0.$$

In the following, we verify

$$\underline{u}_t \leq d \int_{\underline{g}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(t, y) dy - d \underline{u}(t, x) + f(\underline{u}) \text{ for } t > 0, x \in (g(t), h(t)).$$

Let us extend $f(u)$ by defining $f(u) = f'(0)u$ for $u < 0$. Since $f(1) = 0$ and $f'(u) < 0$ for $u \in [1-\epsilon, 1]$, we can choose $\tilde{\epsilon} > 0$ small enough such that

$$(5.3) \quad 2(1-\epsilon)f(1-\frac{\tilde{\epsilon}}{2}) < f(1-\epsilon) \text{ and } f'(u) < 0 \text{ for } u \in [(1-\epsilon)(1-\tilde{\epsilon}), 1]$$

Fix sufficiently large $M > 0$ such that $\phi(-M) > 1 - \frac{\tilde{\epsilon}}{2}$ and $\psi(M) > 1 - \frac{\tilde{\epsilon}}{2}$; then

$$(5.4) \quad \phi(x - \underline{h}(t)), \psi(x - \underline{g}(t)) \in (1 - \frac{\tilde{\epsilon}}{2}, 1) \text{ for } x \in [\underline{g}(t) + M, \underline{h}(t) - M].$$

It follows from the properties of ϕ and ψ that

$$\inf_{x \in [-M, 0]} |\phi'(x)| \geq \epsilon_0 > 0 \text{ and } \inf_{x \in [0, M]} \psi'(x) \geq \epsilon_0 > 0,$$

and so

$$(5.5) \quad \begin{cases} \psi'(x - \underline{g}(t)) \geq \epsilon_0 & \text{for } x \in [\underline{g}(t), \underline{g}(t) + M]; \\ \phi'(x - \underline{h}(t)) \leq -\epsilon_0 & \text{for } x \in [\underline{h}(t) - M, \underline{h}(t)]. \end{cases}$$

By (4.6), we have

$$\begin{aligned} \underline{u}_t &= -(1-\epsilon)(1-2\epsilon)c_0\phi'(x - \underline{h}(t)) + (1-\epsilon)(1-2\epsilon)\tilde{c}_0\psi'(x - \underline{g}(t)) \\ &= (1-\epsilon)2\epsilon c_0\phi'(x - \underline{h}(t)) - (1-\epsilon)2\epsilon\tilde{c}_0\psi'(x - \underline{g}(t)) \\ &\quad + (1-\epsilon) \left[d \int_{-\infty}^0 J(x - \underline{h}(t) - y)\phi(y) dy - d\phi(x - \underline{h}(t)) + f(\phi(x - \underline{h}(t))) \right] \\ &\quad + (1-\epsilon) \left[d \int_0^{+\infty} J(x - \underline{g}(t) - y)\psi(y) dy - d\psi(x - \underline{g}(t)) + f(\psi(x - \underline{g}(t))) \right] \\ &= (1-\epsilon)2\epsilon [c_0\phi'(x - \underline{h}(t)) - \tilde{c}_0\psi'(x - \underline{g}(t))] \\ &\quad + d \int_{\underline{g}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y) dy - d\underline{u}(t, x) \\ &\quad + (1-\epsilon)d \left[\int_{-\infty}^{\underline{g}(t)} J(x - y)[\phi(y - \underline{h}(t)) - 1] dy + \int_{\underline{h}(t)}^{+\infty} J(x - y)[\psi(y - \underline{g}(t)) - 1] dy \right] \\ &\quad + (1-\epsilon) [f(\phi(x - \underline{h}(t))) + f(\psi(x - \underline{g}(t)))] \\ &\leq (1-\epsilon)2\epsilon [c_0\phi'(x - \underline{h}(t)) - \tilde{c}_0\psi'(x - \underline{g}(t))] \\ &\quad + d \int_{\underline{g}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y) dy - d\underline{u}(t, x) + (1-\epsilon) [f(\phi(x - \underline{h}(t))) + f(\psi(x - \underline{g}(t)))] \\ &= d \int_{\underline{g}(t)}^{\underline{h}(t)} J(x - y)\underline{u}(t, y) dy - d\underline{u}(t, x) + f(\underline{u}(t, x)) + \delta(t, x), \end{aligned}$$

where

$$\begin{aligned} \delta(t, x) &:= (1-\epsilon)2\epsilon [c_0\phi'(x - \underline{h}(t)) - \tilde{c}_0\psi'(x - \underline{g}(t))] \\ &\quad + (1-\epsilon) [f(\phi(x - \underline{h}(t))) + f(\psi(x - \underline{g}(t)))] - f(\underline{u}(t, x)). \end{aligned}$$

To verify the desired inequality, it suffices to show that $\delta(t, x) \leq 0$ for $x \in [\underline{g}(t), \underline{h}(t)]$, $t \geq 0$.

Define

$$M_0 := \max_{u \in [0, 1]} |f'(u)|, \quad \hat{\epsilon} := 2\epsilon \min\{c_0, \tilde{c}_0\} \frac{\epsilon_0}{2M_0}.$$

For $x \in [\underline{h}(t) - M, \underline{h}(t)]$, choose large $L > 0$ such that

$$0 > \psi(x - \underline{g}(t)) - 1 \geq \psi(\underline{h}(t) - \underline{g}(t) - M) - 1 \geq \psi(2L - M) - 1 \geq -\hat{\epsilon}.$$

Then

$$\begin{aligned} f(\underline{u}(t, x)) &\geq f((1-\epsilon)\phi(x - \underline{h}(t))) - M_0(1-\epsilon)\hat{\epsilon}, \\ f(\psi(x - \underline{g}(t))) &= f(\psi(x - \underline{g}(t))) - f(1) \leq M_0\hat{\epsilon}. \end{aligned}$$

It now follows from (5.5) and $(\mathbf{f}_{\mathbf{KPP}})$ that

$$\begin{aligned} \delta(t, x) &\leq -(1-\epsilon)2\epsilon c_0\epsilon_0 + (1-\epsilon) [f(\phi(x - \underline{h}(t))) + M_0\hat{\epsilon}] \\ &\quad - f((1-\epsilon)\phi(x - \underline{h}(t))) + M_0(1-\epsilon)\hat{\epsilon} \\ &\leq -(1-\epsilon)2\epsilon c_0\epsilon_0 + 2M_0(1-\epsilon)\hat{\epsilon} \leq 0. \end{aligned}$$

For $x \in [\underline{g}(t), \underline{g}(t) + M]$, choose large $L > 0$ such that

$$0 > \phi(x - \underline{h}(t)) - 1 \geq \phi(\underline{g}(t) - \underline{h}(t) + M) - 1 \geq \phi(-2L + M) - 1 \geq -\hat{\epsilon}.$$

Then

$$\begin{aligned} f(\underline{u}(t, x)) &\geq f((1-\epsilon)\psi(x - \underline{g}(t))) - M_0(1-\epsilon)\hat{\epsilon}, \\ f(\phi(x - \underline{h}(t))) &= f(\phi(x - \underline{h}(t))) - f(1) \leq M_0\hat{\epsilon}. \end{aligned}$$

Therefore by (5.5) and $(\mathbf{f}_{\mathbf{KPP}})$ we have

$$\begin{aligned}\delta(t, x) &\leq -(1-\epsilon)2\epsilon\tilde{c}_0\epsilon_0 + (1-\epsilon)[f(\psi(x-\underline{g}(t))) + M_0\hat{\epsilon}] \\ &\quad - f((1-\epsilon)\psi(x-\underline{g}(t))) + M_0(1-\epsilon)\hat{\epsilon} \\ &\leq -(1-\epsilon)2\epsilon\tilde{c}_0\epsilon_0 + 2M_0(1-\epsilon)\hat{\epsilon} \leq 0.\end{aligned}$$

For $x \in [\underline{g}(t) + M, \underline{h}(t) - M]$ and $t \geq 0$, by (5.4),

$$\underline{u}(t, x) \in [(1-\epsilon)(1-\tilde{\epsilon}), 1-\epsilon].$$

Using (5.3) and (5.4), we have, for such x and $t \geq 0$,

$$\delta(t, x) < (1-\epsilon)[f(1-\frac{\tilde{\epsilon}}{2}) + f(1-\frac{\tilde{\epsilon}}{2})] - f(1-\epsilon) < 0.$$

Since spreading happens and $\lim_{t \rightarrow \infty} u(t, x) = 1$ locally uniformly in $x \in \mathbb{R}$, there exists $T > 0$ large enough such that

$$g(T) < -L = -\underline{g}(0), \quad h(T) > L = \underline{h}(0), \quad u(T, x) > 1-\epsilon > \underline{u}(0, x) \text{ for } x \in [-L, L].$$

By the comparison principle (Theorem 2.3 in Lecture 2), we obtain

$$g(t+T) \leq \underline{g}(t), \quad h(t+T) \geq \underline{h}(t), \quad u(t+T, x) \geq \underline{u}(t, x) \text{ for } t > 0, \quad x \in [\underline{g}(t), \underline{h}(t)].$$

Hence,

$$\begin{aligned}\liminf_{t \rightarrow \infty} \frac{-g(t)}{t} &\geq \lim_{t \rightarrow \infty} \frac{-\underline{g}(t-T)}{t} = (1-2\epsilon)\tilde{c}_0, \\ \liminf_{t \rightarrow \infty} \frac{h(t)}{t} &\geq \lim_{t \rightarrow \infty} \frac{\underline{h}(t-T)}{t} = (1-2\epsilon)c_0.\end{aligned}$$

Letting $\epsilon \rightarrow 0$, we obtain the desired conclusions. The proof is complete. $\square$

5.4. Convergence of semi-wave speeds. Let J satisfy condition $(\mathbf{J})$. Assume a sequence of continuous kernel functions with compact support, denoted by $\{J_n\}$, satisfies, for every $n \geq 1$ and $x \in \mathbb{R}$,

$$(5.6) \quad 0 \leq J_n(x) \leq J_{n+1}(x) \leq J(x), \quad J_n(0) > 0, \quad \lim_{n \rightarrow \infty} J_n = J \quad \text{in } L_{loc}^\infty(\mathbb{R}).$$

Then for each J_n , we may consider the semi-wave problems (4.6)-(4.7) and (4.8)-(4.9) with J replaced by J_n . We note that J_n satisfies $(\mathbf{J})$ except that we only have $0 < \int_{\mathbb{R}} J_n(x) dx \leq 1$.

It is easy to show that as $n \rightarrow \infty$,

$$(5.7) \quad \begin{cases} c_*^-(n) := \sup_{\nu < 0} \frac{d \int_{\mathbb{R}} J_n(x) e^{\nu x} dx - d + f'(0)}{\nu} \rightarrow c_*^-, \\ c_*^+(n) := \inf_{\nu > 0} \frac{d \int_{\mathbb{R}} J_n(x) e^{\nu x} dx - d + f'(0)}{\nu} \rightarrow c_*^+. \end{cases}$$

Therefore, when $c_*^+ > 0$ we have $c_*^+(n) > 0$ for all large n . Moreover, checking the proof of Theorem 4.1 in Section 4, it is easily seen that for every such n , (4.6)-(4.7) with J replaced by J_n has a unique pair of semi-wave solution $(c_n, \phi_n^{c_n})$ with $c_n \in (0, c_*^+(n))$.

Similarly, when $c_*^- < 0$ then for every large n , (4.8)-(4.9) with J replaced by J_n has a unique pair of semi-wave solution $(\tilde{c}_n, \psi_n^{\tilde{c}_n})$ with $\tilde{c}_n \in (0, -c_*^-(n))$.

We have the following result on $\{c_n\}$ and $\{\tilde{c}_n\}$.

Lemma 5.6. *Let J and $\{J_n\}$ be given as above.*

(i) *If $c_*^+ > 0$ and $(c_n, \phi_n^{c_n})$ is the semi-wave solution of (4.6)-(4.7) with J replaced by J_n , which exists for every large n , and let (c_0, ϕ^{c_0}) be the unique semi-wave solution of (4.6)-(4.7) when $(\mathbf{J}_1^+)$ holds, then $c_n \leq c_{n+1}$ and*

$$\lim_{n \rightarrow \infty} c_n = \begin{cases} c_0 & \text{if } (\mathbf{J}_1^+) \text{ holds,} \\ \infty & \text{if } (\mathbf{J}_1^+) \text{ does not hold.} \end{cases}$$

(ii) *Similarly, if $c_*^- < 0$ and $(\tilde{c}_n, \psi_n^{\tilde{c}_n})$ is the semi-wave solution of (4.8)-(4.9) with J replaced by J_n , which exists for every large n , and let $(\tilde{c}_0, \psi^{\tilde{c}_0})$ be the unique semi-wave solution of (4.8)-(4.9) when $(\mathbf{J}_1^-)$ holds, then $\tilde{c}_n \leq \tilde{c}_{n+1}$ and*

$$\lim_{n \rightarrow \infty} \tilde{c}_n = \begin{cases} \tilde{c}_0 & \text{if } (\mathbf{J}_1^-) \text{ holds,} \\ \infty & \text{if } (\mathbf{J}_1^-) \text{ does not hold.} \end{cases}$$

Proof. These follow from [12], where only symmetric kernels are considered, but the proof of the conclusions used here does need the symmetry of the kernel functions.

For part (i), by Lemma 4.2 in [12] we have $c_n \leq c_{n+1}$. The conclusion on $\lim_{n \rightarrow \infty} c_n$ follows from Proposition 4.1 in [12]. The conclusions in part (ii) follow from part (i) by considering the kernel function $J(-x)$. $\square$

5.5. Completion of the proof of Theorem 5.1. Choose a sequence of continuous kernel functions $J_n(x)$ with compact support such that (5.6) holds. By (5.7) we know that $-\infty < c_*^-(n) < 0 < c_*^+(n) < \infty$ for all large n . By passing to a subsequence we may assume that this holds for all $n \geq 1$. We now consider (4.1) with J replaced by J_n . If we define

$$\tilde{J}_n := J_n / \|J_n\|_{L^1(\mathbb{R})}, \quad d_n := d \|J_n\|_{L^1(\mathbb{R})}, \quad f_n(u) := f(u) + (d_n - d)u, \quad \mu_n := \mu \|J_n\|_{L^1(\mathbb{R})},$$

then clearly (4.1) with J replaced by J_n is the same as (4.1) with (J, f, d, μ) replaced by $(\tilde{J}_n, f_n, d_n, \mu_n)$. It is easily checked that $\tilde{J}_n$ satisfies **(J)** and f_n satisfies **(f_{KPP})** except that $f(1) = 0$ is replaced by $f_n(1_n) = 0$ for some unique 1_n close to 1 with $1_n \rightarrow 1$ as $n \rightarrow \infty$. Therefore for each $n \geq 1$ this new problem has a unique solution (u_n, g_n, h_n) . Since by assumption spreading happens to (4.1), it is easy to show that for all large n , spreading also happens to (u_n, g_n, h_n) .

Analogously the corresponding semi-wave problems (4.6)-(4.7) and (4.8)-(4.9) with J replaced by J_n have unique solution pairs $(c_0^n, \phi^{c_0^n})$ and $(\tilde{c}_0^n, \psi^{\tilde{c}_0^n})$, respectively. Moreover, we can apply Lemma 5.5 to conclude that

$$\liminf_{t \rightarrow \infty} \frac{h_n(t)}{t} \geq c_0^n, \quad \liminf_{t \rightarrow \infty} \frac{-g_n(t)}{t} \geq \tilde{c}_0^n.$$

Furthermore, by the comparison principle, we have $h(t) \geq h_{n+1}(t) \geq h_n(t)$, $g(t) \leq g_{n+1}(t) \leq g_n(t)$ and

$$(5.8) \quad u_n(t, x) \leq u_{n+1}(t, x) \leq u(t, x)$$

for all $t > 0$ and $n \geq 1$. Hence, for every large n ,

$$\liminf_{t \rightarrow \infty} \frac{h(t)}{t} \geq c_0^n, \quad \liminf_{t \rightarrow \infty} \frac{-g(t)}{t} \geq \tilde{c}_0^n.$$

We may now apply Lemma 5.6 to conclude that

$$\lim_{n \rightarrow \infty} c_0^n = \begin{cases} \infty & \text{if } (\mathbf{J}_1^+) \text{ does not hold,} \\ c_0 & \text{if } (\mathbf{J}_1^+) \text{ holds.} \end{cases}$$

It follows that

$$\liminf_{t \rightarrow \infty} \frac{h(t)}{t} \geq \begin{cases} \infty & \text{if } (\mathbf{J}_1^+) \text{ does not hold,} \\ c_0 & \text{if } (\mathbf{J}_1^+) \text{ holds.} \end{cases}$$

Combining this with Lemma 5.4, we obtain

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = \begin{cases} \infty & \text{if } (\mathbf{J}_1^+) \text{ does not hold,} \\ c_0 & \text{if } (\mathbf{J}_1^+) \text{ holds.} \end{cases}$$

Similarly we can show

$$\lim_{t \rightarrow \infty} \frac{g(t)}{t} = \begin{cases} -\tilde{c}_0 & \text{if } (\mathbf{J}_1^-) \text{ holds,} \\ -\infty & \text{if } (\mathbf{J}_1^-) \text{ does not hold.} \end{cases}$$

Finally we consider the limit of the density function $u(t, x)$ as $t \rightarrow \infty$. If both $(\mathbf{J}_1^+)$ and $(\mathbf{J}_1^-)$ hold, then by the definition of $\underline{u}$ in the proof of Lemma 5.5, for any small $\epsilon > 0$,

$$\liminf_{t \rightarrow \infty} \min_{(at, bt)} \underline{u}(t, x) \geq 1 - \epsilon, \text{ provided that } -\tilde{c}_0 < a < b < c_0.$$

Consequently, $\liminf_{t \rightarrow \infty} \min_{(at, bt)} u(t, x) \geq 1 - \epsilon$. Since ϵ can be arbitrarily small and $\limsup_{t \rightarrow \infty} u(t, x) \leq 1$ uniformly in $x \in [g(t), h(t)]$, it follows that

$$\lim_{t \rightarrow \infty} \max_{(at, bt)} |u(t, x) - 1| = 0.$$

If neither $(\mathbf{J}_1^+)$ nor $(\mathbf{J}_1^-)$ holds, then we choose a sequence of compactly supported kernels $\{J_n\}$ as at the beginning of this proof, so that the above conclusions for (u, g, h) applies to the corresponding solution (u_n, g_n, h_n) for each large n , and therefore for every small $\delta > 0$

$$\lim_{t \rightarrow \infty} \max_{((-\tilde{c}_0^n + \delta)t, (c_0^n - \delta)t)} |u_n(t, x) - 1_n| = 0.$$

Since $c_0^n \rightarrow \infty$, $\tilde{c}_0^n \rightarrow \infty$ as $n \rightarrow \infty$ and $u(t, x) \geq u_n(t, x)$ for $t > 0$, $x \in [g_n(t), h_n(t)]$, it follows that

$$\liminf_{t \rightarrow \infty} \min_{(at, bt)} u(t, x) \geq \liminf_{t \rightarrow \infty} \min_{(at, bt)} u_n(t, x) = 1_n$$

for any $a < b$.

Since $1_n \rightarrow 1$ as $n \rightarrow \infty$ and $\limsup_{t \rightarrow \infty} u(t, x) \leq 1$ uniformly in x , we thus obtain $\lim_{t \rightarrow \infty} \max_{(at, bt)} |u(t, x) - 1| = 0$ for such a and b .

The remaining cases can be similarly proved, and we omit the details. $\square$

Remarks: In the symmetric case $J(x) = J(-x)$, Theorem 5.1 was first proved in [9]. These results have been extended to rather general cooperative systems in [11].

6. PRECISE RATE OF ACCELERATION

We determine the acceleration rate of the nonlocal free boundary problem (4.1). We will prove the following result, which is taken from [13].

Theorem 6.1. *Suppose that **(J)** and **(f_{KPP})** are satisfied, and **J is symmetric**: $J(x) = J(-x)$. Let (u, g, h) be the unique solution of (4.1), and assume that spreading occurs. Then the following conclusions hold:*

(i) *If*

$$\lim_{|x| \rightarrow \infty} J(x)|x|^\alpha = \lambda \in (0, \infty) \text{ for some } \alpha \in (1, 2],$$

then

$$\begin{cases} \lim_{t \rightarrow \infty} \frac{h(t)}{t \ln t} = \lim_{t \rightarrow \infty} \frac{-g(t)}{t \ln t} = \mu\lambda, & \text{when } \alpha = 2, \\ \lim_{t \rightarrow \infty} \frac{h(t)}{t^{1/(\alpha-1)}} = \lim_{t \rightarrow \infty} \frac{-g(t)}{t^{1/(\alpha-1)}} = \left[\frac{2^{2-\alpha}}{2-\alpha} \mu\lambda \right]^{1/(\alpha-1)}, & \text{when } \alpha \in (1, 2), \end{cases}$$

and for any small $\epsilon > 0$,

$$\lim_{t \rightarrow \infty} u(t, x) = 1 \text{ uniformly for } x \in [(1-\epsilon)g(t), (1-\epsilon)h(t)].$$

(ii) *If*

$$\lim_{|x| \rightarrow \infty} J(x)|x|(\ln |x|)^\beta = \lambda \in (0, \infty) \text{ for some } \beta \in (1, \infty),$$

then

$$\lim_{t \rightarrow \infty} \frac{\ln h(t)}{t^{1/\beta}} = \lim_{t \rightarrow \infty} \frac{\ln[-g(t)]}{t^{1/\beta}} = \left(\frac{2\beta\mu\lambda}{\beta-1} \right)^{1/\beta},$$

namely,

$$-g(t), h(t) = \exp \left\{ \left[\left(\frac{2\beta\mu\lambda}{\beta-1} \right)^{1/\beta} + o(1) \right] t^{1/\beta} \right\} \text{ as } t \rightarrow \infty.$$

Moreover, for any small $\epsilon > 0$,

$$\lim_{t \rightarrow \infty} u(t, x) = 1 \text{ uniformly for } |x| \leq \exp \left[(1-\epsilon) \left(\frac{2\beta\mu\lambda}{\beta-1} \right)^{1/\beta} t^{1/\beta} \right]$$

If $J(x)$ is not symmetric, the rate of acceleration for (4.1) is considered in [8]. We only consider the symmetric case here for simplicity.

Note that when J is symmetric, in Theorem B we have $c_0 = \tilde{c}_0$, and we will simply say **(J₁)** holds when **(J₁⁺)** (and hence **(J₁⁻)**) holds. Moreover, if additionally

$$\lim_{|x| \rightarrow \infty} J(x)|x|^\alpha = \lambda \in (0, \infty) \text{ for some } \alpha > 0,$$

then **(J₁)** holds if and only if $\alpha > 2$, and **(J)** holds only if $\alpha > 1$. If

$$\lim_{|x| \rightarrow \infty} J(x)|x|(\ln |x|)^\beta = \lambda \in (0, \infty) \text{ for some } \beta > 0,$$

then **(J₁)** can never hold, and **(J)** holds only if $\beta > 1$.

We will consider a more general class of symmetric $J(x)$ than those in Theorem 6.1; namely J satisfies **(J)** and either

$$(6.1) \quad \text{for some } \alpha \in (1, 2], \quad J(x) \sim |x|^{-\alpha}, \quad i.e., \quad \begin{cases} \underline{\lambda} := \liminf_{|x| \rightarrow \infty} J(x)|x|^\alpha > 0, \\ \bar{\lambda} := \limsup_{|x| \rightarrow \infty} J(x)|x|^\alpha < \infty, \end{cases}$$

or

$$(6.2) \quad \text{for some } \beta > 1, \quad J(x) \sim \left[|x|(\ln |x|)^\beta \right]^{-1}, \quad i.e., \quad \begin{cases} \underline{\lambda} := \liminf_{|x| \rightarrow \infty} J(x)|x|(\ln |x|)^\beta > 0, \\ \bar{\lambda} := \limsup_{|x| \rightarrow \infty} J(x)|x|(\ln |x|)^\beta < \infty. \end{cases}$$

We will prove some sharp estimates under the above assumptions for J ; Theorem 1.1 is a direct consequence of these more general results.

6.1. Some preparatory results.

Lemma 6.2. For $k > 1$, $\delta \in [0, 1)$, define

$$A = A(k, \delta, J) := \begin{cases} \int_{-k}^{-\delta k} \int_0^\infty J(x-y) dy dx & \text{if (6.1) holds with } \alpha \in (1, 2) \text{ or if (6.2) holds,} \\ \int_{-k}^{-k^\delta} \int_0^\infty J(x-y) dy dx & \text{if (6.1) holds with } \alpha = 2. \end{cases}$$

Then

$$\begin{cases} \liminf_{k \rightarrow \infty} \frac{A}{k^{2-\alpha}} \geq \frac{1 - \delta^{2-\alpha}}{(\alpha-1)(2-\alpha)} \underline{\lambda}, \\ \limsup_{k \rightarrow \infty} \frac{A}{k^{2-\alpha}} \leq \frac{1 - \delta^{2-\alpha}}{(\alpha-1)(2-\alpha)} \bar{\lambda}, \end{cases} \quad \text{if (6.1) holds with } \alpha \in (1, 2),$$

$$\begin{cases} \liminf_{k \rightarrow \infty} \frac{A}{\ln k} \geq (1-\delta) \underline{\lambda}, \\ \limsup_{k \rightarrow \infty} \frac{A}{\ln k} \leq (1-\delta) \bar{\lambda}, \end{cases} \quad \text{if (6.1) holds with } \alpha = 2,$$

$$\begin{cases} \liminf_{k \rightarrow \infty} \frac{A}{k(\ln k)^{1-\beta}} \geq \frac{(1-\delta) \underline{\lambda}}{\beta-1}, \\ \limsup_{k \rightarrow \infty} \frac{A}{k(\ln k)^{1-\beta}} \leq \frac{(1-\delta) \bar{\lambda}}{\beta-1}, \end{cases} \quad \text{if (6.2) holds.}$$

Proof. Case 1: (6.1) holds with $\alpha \in (1, 2)$.

Denote

$$(6.3) \quad D_\delta := \frac{1}{\alpha-1} \int_0^\infty [(y+\delta)^{1-\alpha} - (y+1)^{1-\alpha}] dy.$$

A direct calculation gives

$$D_\delta = \lim_{M \rightarrow \infty} \frac{(M+\delta)^{2-\alpha} - (M+1)^{2-\alpha} + 1 - \delta^{2-\alpha}}{(\alpha-1)(2-\alpha)} = \frac{1 - \delta^{2-\alpha}}{(\alpha-1)(2-\alpha)}.$$

Moreover,

$$\begin{aligned} A &= \int_{-k}^{-\delta k} \int_0^\infty J(x-y) dy dx = \int_{\delta k}^k \int_0^\infty J(x+y) dy dx \\ &= \int_{\delta k}^k \int_0^2 J(x+y) dy dx + \int_{\delta k}^k \int_2^\infty J(x+y) dy dx =: A_1 + A_2, \end{aligned}$$

and by **(J)**,

$$0 \leq A_1 \leq \int_0^2 1 dy \leq 2.$$

Clearly,

$$\begin{aligned} A_2 &= \int_{\delta k}^k \int_2^\infty J(x+y) dy dx = \int_2^\infty \int_{\delta k}^k J(x+y) dx dy \\ &= k^{2-\alpha} \left(\int_{2k^{-1}}^{k^{-1/2}} + \int_{k^{-1/2}}^\infty \right) \int_{\delta+y}^{1+y} \frac{J(kx)}{(kx)^{-\alpha}} x^{-\alpha} dx dy. \end{aligned}$$

We have

$$\begin{aligned} 0 &\leq \int_{2k^{-1}}^{k^{-1/2}} \int_{\delta+y}^{1+y} \frac{J(kx)}{(kx)^{-\alpha}} x^{-\alpha} dx dy \leq \sup_{\xi \geq 1} [J(\xi)\xi^\alpha] \int_{2k^{-1}}^{k^{-1/2}} \int_{\delta+y}^{1+y} x^{-\alpha} dx dy \\ &\leq \frac{\sup_{\xi \geq 1} [J(\xi)\xi^\alpha]}{\alpha-1} \int_{2k^{-1}}^{k^{-1/2}} y^{1-\alpha} dy \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

By this and (6.1), we deduce

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{A_2}{k^{2-\alpha}} &= \limsup_{k \rightarrow \infty} \int_{k^{-1/2}}^{\infty} \int_{\delta+y}^{1+y} \frac{J(kx)}{(kx)^{-\alpha}} x^{-\alpha} dx dy \\ &\leq \bar{\lambda} \int_0^{\infty} \int_{\delta+y}^{1+y} x^{-\alpha} dx dy = \frac{\bar{\lambda}}{\alpha-1} \int_0^{\infty} [(\delta+y)^{1-\alpha} - (1+y)^{1-\alpha}] dy = \bar{\lambda} D_\delta. \end{aligned}$$

Thus,

$$\limsup_{k \rightarrow \infty} \frac{A}{k^{2-\alpha}} = \limsup_{k \rightarrow \infty} \frac{A_2}{k^{2-\alpha}} \leq \bar{\lambda} D_\delta.$$

Similarly,

$$\liminf_{k \rightarrow \infty} \frac{A}{k^{2-\alpha}} = \liminf_{k \rightarrow \infty} \frac{A_2}{k^{2-\alpha}} \geq \underline{\lambda} D_\delta.$$

Case 2: (6.2) holds.

Let A_1 and A_2 be as in Case 1. Clearly, $0 \leq A_1 \leq 2$. A simple calculation gives

$$\begin{aligned} A_2 &= \int_{\delta k}^k \int_{2+x}^{\infty} J(y) dy dx = \int_{\delta k+2}^{k+2} \int_{\delta k}^{y-2} J(y) dx dy + \int_{k+2}^{\infty} \int_{\delta k}^k J(y) dx dy \\ &= \int_{\delta k+2}^{k+2} (y-2-\delta k) J(y) dy + (1-\delta)k \int_{k+2}^{\infty} J(y) dy. \end{aligned}$$

By (6.2), there exists $C > 0$ such that for all large $k > 0$,

$$\int_{\delta k+2}^{k+2} (y-2-\delta k) J(y) dy \leq C \int_{\delta k+2}^{k+2} (\ln y)^{-\beta} dy \leq C(1-\delta)k [\ln(\delta k+2)]^{-\beta},$$

and

$$k \int_{k+2}^{\infty} J(y) dy \leq \bar{\lambda}[1+o_k(1)]k \int_{k+2}^{\infty} y^{-1} (\ln y)^{-\beta} dy = \frac{\bar{\lambda}[1+o_k(1)]k}{\beta-1} [\ln(k+2)]^{1-\beta},$$

where $o_k(1) \rightarrow 0$ as $k \rightarrow \infty$. Hence,

$$\limsup_{k \rightarrow \infty} \frac{A}{k(\ln k)^{1-\beta}} = \limsup_{k \rightarrow \infty} \frac{A_2}{k(\ln k)^{1-\beta}} \leq \frac{(1-\delta)\bar{\lambda}}{\beta-1}.$$

Similarly,

$$\liminf_{k \rightarrow \infty} \frac{A}{k(\ln k)^{1-\beta}} = \liminf_{k \rightarrow \infty} \frac{A_2}{k(\ln k)^{1-\beta}} \geq \frac{(1-\delta)\underline{\lambda}}{\beta-1}.$$

Case 3: (6.1) holds with $\alpha = 2$.

By direct calculation,

$$\begin{aligned} A &= \int_{-k}^{-k^\delta} \int_0^{\infty} J(x-y) dy dx = \int_{k^\delta}^k \int_0^{\infty} J(x+y) dy dx \\ &= \int_{k^\delta}^k \int_0^1 J(x+y) dy dx + \int_{k^\delta}^k \int_1^{\infty} J(x+y) dy dx =: \tilde{A}_1 + \tilde{A}_2, \end{aligned}$$

and by (J),

$$0 \leq \tilde{A}_1 \leq \int_0^1 1 dy = 1.$$

By (6.1), we have

$$\tilde{A}_2 = \int_1^{\infty} \int_{k^\delta+y}^{k+y} J(x) dx dy \leq \bar{\lambda}[1+o_k(1)] \int_1^{\infty} \int_{k^\delta+y}^{k+y} x^{-2} dx dy = \bar{\lambda}[1+o_k(1)] \ln \left(\frac{k+1}{k^\delta+1} \right),$$

where $o_k(1) \rightarrow 0$ as $k \rightarrow \infty$. Therefore,

$$\limsup_{k \rightarrow \infty} \frac{A}{\ln k} = \limsup_{k \rightarrow \infty} \frac{\tilde{A}_2}{\ln k} \leq \lim_{k \rightarrow \infty} \bar{\lambda} \frac{\ln(k+1) - \ln(k^\delta + 1)}{\ln k} = (1 - \delta)\bar{\lambda}.$$

Similarly,

$$\liminf_{k \rightarrow \infty} \frac{A}{\ln k} = \liminf_{k \rightarrow \infty} \frac{\tilde{A}_2}{\ln k} \geq \lim_{k \rightarrow \infty} \underline{\lambda} \frac{\ln(k+1) - \ln(k^\delta + 1)}{\ln k} = (1 - \delta)\underline{\lambda}.$$

The proof is finished. $\square$

Lemma 6.3. *Suppose that J satisfies **(J)** but neither (6.1) nor (6.2) is required. Let $1 < \xi(t) < L(t)$ be functions in $C([0, \infty))$, $\rho \geq 2$ a constant, and define*

$$\phi(t, x) := \min \left\{ 1, \left[1 - \frac{|x|}{L(t)} \right]^\rho \xi(t)^\rho \right\} \text{ for } x \in [-L(t), L(t)], \quad t \in [0, \infty).$$

Then, for any $\epsilon \in (0, 1)$, there exists a constant $\theta^* = \theta^*(\epsilon, J) > 1$, such that

$$(6.4) \quad \int_{-L(t)}^{L(t)} J(x - y) \phi(t, y) dy \geq (1 - \epsilon) \phi(t, x) \text{ for } x \in [-L(t), L(t)], \quad t \geq 0$$

provided that

$$(6.5) \quad L(t) \geq \theta^* \xi(t) \quad \text{for all } t \geq 0.$$

Proof. Since $\|J\|_{L^1} = 1$, there is $L_0 > 0$ depending only on J and ϵ such that

$$(6.6) \quad \int_{-L_0}^{L_0} J(x) dx \geq 1 - \epsilon/2.$$

Define

$$\psi(t, x) := \phi(t, L(t)x) = \min \{ 1, (1 - |x|)^\rho \xi(t)^\rho \}, \quad x \in [-1, 1], \quad t \geq 0.$$

We note that $\rho \geq 2$ implies that $\psi(t, x)$ is a convex function of x when

$$1 - \frac{1}{\xi(t)} \leq |x| \leq 1.$$

Clearly

$$\psi(t, x) = \begin{cases} 1 & \text{for } |x| \leq 1 - \xi(t)^{-1}, \\ [(1 - |x|)\xi(t)]^\rho & \text{for } 1 - \xi(t)^{-1} < |x| \leq 1. \end{cases}$$

It is also easy to check that

$$\frac{|\psi(t, x) - \psi(t, y)|}{|x - y|} \leq M(t) := \rho \xi(t) \text{ for } x, y \in [-1, 1], \quad x \neq y, \quad t \geq 0,$$

which implies

$$(6.7) \quad |\phi(t, x) - \phi(t, y)| = |\psi(t, x/L(t)) - \psi(t, y/L(t))| \leq \frac{M(t)}{L(t)} |x - y|$$

for $x, y \in [-L(t), L(t)]$.

Since $\psi(t, x) > 0$ for $x \in (-1, 1)$, $\psi(t, \pm 1) = 0$, and $\psi(t, x)$ is convex in x for $x \in [-1, -1 + 1/\xi(t)]$ and for $x \in [1 - 1/\xi(t), 1]$, if we extend $\psi(t, x)$ by $\psi(t, x) = 0$ for $|x| > 1$, then

$$\psi(t, x) \text{ is convex for } x \in [1 - 1/\xi(t), \infty) \text{ and for } x \in (-\infty, -1 + 1/\xi(t)].$$

We now verify (6.4) for $x \in [0, L(t)]$; the proof for $x \in [-L(t), 0]$ is parallel and will be omitted. We will divide the proof into two cases:

$$\textbf{(a)} \quad x \in \left[0, \left(1 - \frac{1}{2\xi(t)} \right) L(t) \right] \text{ and } \textbf{(b)} \quad x \in \left[\left(1 - \frac{1}{2\xi(t)} \right) L(t), L(t) \right].$$

Case (a). For

$$x \in \left[0, \left(1 - \frac{1}{2\xi(t)} \right) L(t) \right],$$

a direct calculation gives

$$\int_{-L(t)}^{L(t)} J(x - y) \phi(t, y) dy = \int_{-L(t)-x}^{L(t)-x} J(y) \phi(t, x + y) dy \geq \int_{-L_0}^{L_0} J(y) \phi(x + y) dy,$$

where L_0 is given by (6.6) and we have used

$$L(t) - x \geq \frac{L(t)}{2\xi(t)} \geq L_0, \text{ which is guaranteed if we assume } L(t) \geq 2L_0\xi(t).$$

Then by (6.6), (6.7) and **(J)**,

$$\begin{aligned} & \int_{-L_0}^{L_0} J(y)\phi(t, x+y)dy \\ &= \int_{-L_0}^{L_0} J(y)\phi(t, x)dy + \int_{-L_0}^{L_0} J(y)[\phi(t, x+y) - \phi(t, x)]dy \\ &\geq \int_{-L_0}^{L_0} J(y)\phi(t, x)dy - \frac{M(t)}{L(t)} \int_{-L_0}^{L_0} J(y)|y|dy \\ &\geq (1 - \epsilon/2)\phi(t, x) - \frac{M(t)}{L(t)}L_0. \end{aligned}$$

Clearly

$$M_1(t) := \min_{x \in [0, (1 - \frac{1}{2\xi(t)})L(t)]} \phi(t, x) = \left(\frac{1}{2}\right)^\rho.$$

Then from the above calculations we obtain, for $x \in [0, (1 - \frac{1}{2\xi(t)})L(t)]$,

$$\begin{aligned} \int_{-L(t)}^{L(t)} J(x-y)\phi(t, y)dy &\geq (1 - \epsilon/2)\phi(t, x) - \frac{M(t)}{L(t)}L_0 \\ &= (1 - \epsilon)\phi(t, x) + \frac{\epsilon}{2}\phi(t, x) - \frac{M(t)}{L(t)}L_0 \\ &\geq (1 - \epsilon)\phi(t, x) + \frac{\epsilon}{2}M_1(t) - \frac{M(t)}{L(t)}L_0 \geq (1 - \epsilon)\phi(t, x) \end{aligned}$$

provided that

$$L(t) \geq \frac{2L_0M(t)}{\epsilon M_1(t)} = \frac{2^{\rho+1}L_0\rho}{\epsilon}\xi(t).$$

Case (b). For

$$x \in \left[\left(1 - \frac{1}{2\xi(t)}\right)L(t), L(t) \right],$$

we have, using $-L(t) - x < -L_0$ and $\phi(t, x) = 0$ for $x \geq L(t)$,

$$\begin{aligned} \int_{-L(t)}^{L(t)} J(x-y)\phi(t, y)dy &\geq \int_{-L_0}^{\min\{L_0, L(t)-x\}} J(y)\phi(t, x+y)dy \\ &= \int_{-L_0}^{L_0} J(y)\phi(t, x+y)dy \\ &= \int_0^{L_0} J(y)[\phi(t, x+y) + \phi(t, x-y)]dy. \end{aligned}$$

Since $\phi(t, s)$ is convex in s for $s \geq L(t)[1 - \xi(t)^{-1}]$, and for $x \in \left[\left(1 - \frac{1}{2\xi(t)}\right)L(t), L(t) \right]$, $y \in [0, L_0]$, we have

$$x+y \geq x-y \geq \left(1 - \frac{1}{2\xi(t)}\right)L(t) - L_0 \geq \left(1 - \frac{1}{\xi(t)}\right)L(t) \text{ by our earlier assumption } L(t) \geq 2L_0\xi(t).$$

Then, we can use the convexity of $\phi(t, \cdot)$ and (6.6) to obtain

$$\int_0^{L_0} J(y)[\phi(t, x+y) + \phi(t, x-y)]dy \geq 2\phi(t, x) \int_0^{L_0} J(y)dy \geq (1 - \epsilon/2)\phi(t, x).$$

Thus

$$\int_{-L(t)}^{L(t)} J(x-y)\phi(t, y)dy \geq (1 - \epsilon)\phi(t, x).$$

Summarising, from the above conclusions in cases **(a)** and **(b)**, we see that (6.4) holds if $L(t) \geq \theta^*\xi(t)$ for all $t \geq 0$ with $\theta^* := \frac{2^{\rho+1}L_0\rho}{\epsilon} > 2L_0$. The proof is finished. $\square$

6.2. Lower bounds. From now on, in all our stated results, we will only list the conclusions for $h(t)$; the corresponding conclusions for $-g(t)$ follow directly by considering the problem with initial function $u_0(-x)$, whose unique solution is given by $(\tilde{u}(t, x), \tilde{g}(t), \tilde{h}(t)) = (u(t, -x), -h(t), -g(t))$.

6.2.1. *The case (6.1) holds with $\alpha \in (1, 2)$ and the case (6.2) holds.*

Lemma 6.4. *Assume that J satisfies **(J)** and either (6.1) with $\alpha \in (1, 2)$ or (6.2), f satisfies **(f)**, and spreading happens to (4.1). Then*

$$\begin{cases} \liminf_{t \rightarrow \infty} \frac{h(t)}{t^{1/(\alpha-1)}} \geq \left(\frac{2^{2-\alpha}}{2-\alpha} \mu \lambda \right)^{1/(\alpha-1)} & \text{if (6.1) holds with } \alpha \in (1, 2), \\ \liminf_{t \rightarrow \infty} \frac{\ln h(t)}{t^{1/\beta}} \geq \left(\frac{2\beta\mu\lambda}{\beta-1} \right)^{1/\beta} & \text{if (6.2) holds.} \end{cases}$$

Proof. We construct a suitable lower solution to (4.1), which will lead to the desired estimate by the comparison principle.

Let $\rho > 2$ be a large constant to be determined. For any given small $\epsilon > 0$, define for $t \geq 0$,

$$\begin{cases} \underline{h}(t) := (K_1 t + \theta)^{\frac{1}{\alpha-1}}, \quad \underline{g}(t) := -\underline{h}(t) & \text{if (6.1) holds with } \alpha \in (1, 2), \\ \underline{h}(t) := e^{K_1(t+\theta)^{1/\beta}}, \quad \underline{g}(t) := -\underline{h}(t) & \text{if (6.2) holds,} \end{cases}$$

and

$$\underline{u}(t, x) := K_2 \min \left\{ 1, \left[K_3 \frac{h(t) - |x|}{\underline{h}(t)} \right]^\rho \right\} \text{ for } t \geq 0, \quad |x| \leq \underline{h}(t),$$

where

$$K_1 := \begin{cases} (1-\epsilon)^2(2-\epsilon)^{2-\alpha} D_{\epsilon/(2-\epsilon)}(\alpha-1)\mu\lambda & \text{if (6.1) holds with } \alpha \in (1, 2), \\ \left[(1-\epsilon)^4 \frac{2\beta\mu\lambda}{\beta-1} \right]^{1/\beta} & \text{if (6.2) holds,} \end{cases}$$

$$K_2 := 1 - \epsilon, \quad K_3 := 1/\epsilon, \quad \theta \gg 1 \text{ and } D_{\epsilon/(2-\epsilon)} \text{ is given according to (6.3).}$$

It is easily seen that

$$\underline{u}(t, x) \equiv K_2 = 1 - \epsilon \text{ for } |x| \leq (1 - \epsilon)\underline{h}(t).$$

Moreover, $\underline{u}$ is continuous, and $\underline{u}_t$ exists and is continuous except when $|x| = (1 - \epsilon)\underline{h}(t)$, where $\underline{u}_t$ has a jumping discontinuity. In what follows, we check that $(\underline{u}, \underline{g}, \underline{h})$ defined above forms a lower solution to (4.1). We will do this in three steps.

Step 1. We prove the inequality

$$(6.8) \quad \underline{h}'(t) \leq \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(y-x) \underline{u}(t, x) dy dx,$$

which immediately gives

$$\underline{g}'(t) \geq -\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J(y-x) \underline{u}(t, x) dy dx,$$

due to $\underline{u}(t, x) = \underline{u}(t, -x)$ and $J(x) = J(-x)$.

Using the definition of $\underline{u}$, we have

$$\begin{aligned} \mu \int_{-\underline{h}}^{\underline{h}} \int_{\underline{h}}^{+\infty} J(y-x) \underline{u}(t, x) dy dx &\geq (1-\epsilon) \mu \int_{-(1-\epsilon)\underline{h}}^{(1-\epsilon)\underline{h}} \int_{\underline{h}}^{+\infty} J(y-x) dy dx \\ &= (1-\epsilon) \mu \int_{-(2-\epsilon)\underline{h}}^{-\epsilon\underline{h}} \int_0^{+\infty} J(y-x) dy dx. \end{aligned}$$

Using Lemma 6.2, we obtain for large $\underline{h}$ (guaranteed by $\theta \gg 1$),

$$\int_{-(2-\epsilon)\underline{h}}^{-\epsilon\underline{h}} \int_0^{+\infty} J(y-x) dy dx \geq (1-\epsilon) \lambda D_{\epsilon/(2-\epsilon)} [(2-\epsilon)\underline{h}]^{2-\alpha} \quad \text{if (6.1) holds with } \alpha \in (1, 2),$$

and

$$\begin{aligned} \int_{-(2-\epsilon)\underline{h}}^{-\epsilon\underline{h}} \int_0^{+\infty} J(y-x) dy dx &\geq (1-\epsilon) \frac{(1-\frac{\epsilon}{2-\epsilon})\lambda}{\beta-1} (2-\epsilon)\underline{h} [\ln(2-\epsilon)\underline{h}]^{1-\beta} \\ &= (1-\epsilon)^2 \frac{2\lambda}{\beta-1} \underline{h} [\ln(2-\epsilon)\underline{h}]^{1-\beta} \geq (1-\epsilon)^3 \frac{2\lambda}{\beta-1} \underline{h} (\ln \underline{h})^{1-\beta} \quad \text{if (6.2) holds.} \end{aligned}$$

Therefore, by the definition of K_1 , when (6.1) holds with $\alpha \in (1, 2)$, we have

$$\begin{aligned} & \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(y-x) \underline{u}(t, x) dy dx \\ & \geq (1-\epsilon)^2 \mu \underline{\lambda} D_{\epsilon/(2-\epsilon)} [(2-\epsilon) \underline{h}(t)]^{2-\alpha} \\ & = (1-\epsilon)^2 \mu \underline{\lambda} D_{\epsilon/(2-\epsilon)} (2-\epsilon)^{2-\alpha} (K_1 t + \theta)^{(2-\alpha)/(\alpha-1)} \\ & = \frac{K_1}{\alpha-1} (K_1 t + \theta)^{(2-\alpha)/(\alpha-1)} = \underline{h}'(t); \end{aligned}$$

and when (6.2) holds, we have

$$\begin{aligned} & \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(y-x) \underline{u}(t, x) dy dx \\ & \geq \mu (1-\epsilon)^4 \frac{2\underline{\lambda}}{\beta-1} \underline{h} (\ln \underline{h})^{1-\beta} \\ & = \frac{K_1^\beta}{\beta} \underline{h} (\ln \underline{h})^{1-\beta} = \underline{h}'(t). \end{aligned}$$

This proves (6.8).

Step 2. We prove the following inequality for $t > 0$ and $|x| \in [0, \underline{h}(t)] \setminus \{(1-\epsilon)\underline{h}(t)\}$,

$$(6.9) \quad \underline{u}_t \leq d \int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy - d\underline{u} + f(\underline{u}).$$

From the definition of $\underline{u}$, we see that

$$\underline{u}_t = 0 \quad \text{for } |x| < (1-\epsilon)\underline{h}(t),$$

and for $(1-\epsilon)\underline{h}(t) < |x| < \underline{h}(t)$, if (6.1) holds with $\alpha \in (1, 2)$, then

$$(6.10) \quad \underline{u}_t = K_2 K_3^\rho \rho \left(\frac{\underline{h} - |x|}{\underline{h}} \right)^{\rho-1} \frac{\underline{h}' |x|}{\underline{h}^2} = \frac{K_1 K_2 K_3^\rho \rho}{\alpha-1} \left(\frac{\underline{h} - |x|}{\underline{h}} \right)^{\rho-1} \frac{|x|}{\underline{h}} \underline{h}^{1-\alpha},$$

where we have used $\underline{h}' = \frac{K_1}{\alpha-1} \underline{h}^{2-\alpha}$; and if (6.2) holds, then

$$\underline{u}_t = K_2 K_3^\rho \rho \left(\frac{\underline{h} - |x|}{\underline{h}} \right)^{\rho-1} \frac{\underline{h}' |x|}{\underline{h}^2} = \frac{K_1^\beta K_2 K_3^\rho \rho}{\beta} \left(\frac{\underline{h} - |x|}{\underline{h}} \right)^{\rho-1} \frac{|x|}{\underline{h}} (\ln \underline{h})^{1-\beta},$$

where we have utilized $\underline{h}' = \frac{K_1^\beta}{\beta} \underline{h} (\ln \underline{h})^{1-\beta}$.

Claim. There is $C_1 = C_1(\epsilon) > 0$ such that for $x \in [-\underline{h}(t), \underline{h}(t)]$ and $t \geq 0$,

$$d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(t, y) dy - d\underline{u} + f(\underline{u}) \geq C_1 \left[\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(t, y) dy + \underline{u} \right].$$

The definition of $\underline{u}$ indicates $0 \leq \underline{u}(t, x) \leq K_2 = 1 - \epsilon < 1$. By the properties of f , there exists $\tilde{C}_1 := \tilde{C}_1(\epsilon) \in (0, d)$ such that

$$f(s) \geq \tilde{C}_1 s \quad \text{for } s \in [0, K_2].$$

Using Lemma 6.3 with

$$(L(t), \phi(t, x), \xi(t)) = (\underline{h}(t), \underline{u}(t, x)/K_2, K_3),$$

for any given small $\delta > 0$, we can find large $h_* = h_*(\delta, \epsilon)$ such that for $\underline{h} \geq h_*$ and $|x| \leq \underline{h}$,

$$\int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy \geq (1-\delta) \underline{u}(t, x).$$

Hence, due to $d > \tilde{C}_1$,

$$\begin{aligned}
& d \int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy - d \underline{u}(t, x) + f(\underline{u}(t, x)) \\
& \geq d \int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy + (\tilde{C}_1 - d) \underline{u}(t, x) \\
& \geq \frac{\tilde{C}_1}{3} \int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy + (d - \frac{\tilde{C}_1}{3})(1 - \delta) \underline{u}(t, x) + (\tilde{C}_1 - d) \underline{u}(t, x) \\
& \geq \frac{\tilde{C}_1}{3} \left[\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(t, y) dy + \underline{u}(t, x) \right],
\end{aligned}$$

provided that $\delta = \delta(\epsilon) > 0$ is sufficiently small. Thus the claim holds with $C_1 = \tilde{C}_1/3$.

To verify (6.9), it remains to prove

$$(6.11) \quad \underline{u}_t \leq C_1 \left[\int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy + \underline{u} \right] \quad \text{for } |x| \in [0, \underline{h}(t)] \setminus \{(1-\epsilon)\underline{h}(t)\}.$$

Since $\underline{u}(x, t) \equiv 1 - \epsilon$ for $|x| < (1-\epsilon)\underline{h}(t)$, (6.11) holds trivially for such x . Hence we only need to consider the case of $(1-\epsilon)\underline{h}(t) < |x| < \underline{h}(t)$.

Since $\theta \gg 1$ and $0 < \epsilon \ll 1$, for $x \in [7\underline{h}(t)/8, \underline{h}(t)] \supset [(1-\epsilon)\underline{h}(t), \underline{h}(t)]$, we have

$$\begin{aligned}
& \int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy \geq \int_{-7\underline{h}/8}^{7\underline{h}/8} J(x-y) \underline{u}(t, y) dy \geq K_2 \int_{-7\underline{h}/8}^{7\underline{h}/8} J(x-y) dy \\
& = (1-\epsilon) \int_{-7\underline{h}/8-x}^{7\underline{h}/8-x} J(y) dy \geq (1-\epsilon) \int_{-\underline{h}/4}^{-\underline{h}/8} J(y) dy = (1-\epsilon) \int_{\underline{h}/8}^{\underline{h}/4} J(y) dy.
\end{aligned}$$

Hence, when (6.1) holds with $\alpha \in (1, 2)$, we obtain

$$\int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy \geq \frac{\lambda}{2} \int_{\underline{h}/8}^{\underline{h}/4} y^{-\alpha} dy = \frac{(8^{\alpha-1} - 4^{\alpha-1})\lambda}{2(\alpha-1)} \underline{h}^{1-\alpha} =: C_2 \underline{h}^{1-\alpha},$$

and when (6.2) holds, we have

$$\int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy \geq \frac{\lambda}{2} \int_{\underline{h}/8}^{\underline{h}/4} y^{-1} (\ln y)^{-\beta} dy > \frac{\lambda \underline{h}}{16} y^{-1} (\ln y)^{-\beta} \big|_{y=\underline{h}/4} \geq \frac{\lambda}{4} (\ln \underline{h})^{-\beta} =: \tilde{C}_2 (\ln \underline{h})^{-\beta}.$$

Similar estimates hold for $x \in [-\underline{h}(t), -7\underline{h}(t)/8]$.

Now, if (6.1) holds with $\alpha \in (1, 2)$, then for $|x| \in [(1-C_\epsilon)\underline{h}(t), \underline{h}(t)]$ with

$$C_\epsilon := \left[\frac{C_1 C_2 (\alpha-1)}{K_1 K_2 \rho} \epsilon^\rho \right]^{1/(\rho-1)},$$

we have

$$\begin{aligned}
\underline{u}_t - C_1 \int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy & \leq \frac{K_1 K_2 K_3^\rho \rho}{\alpha-1} \left(\frac{\underline{h}-|x|}{\underline{h}} \right)^{\rho-1} \underline{h}^{1-\alpha} - C_1 C_2 \underline{h}^{1-\alpha} \\
& \leq \left[\frac{K_1 K_2 K_3^\rho \rho}{\alpha-1} C_\epsilon^{\rho-1} - C_1 C_2 \right] \underline{h}^{1-\alpha} = 0,
\end{aligned}$$

and for $(1-\epsilon)\underline{h}(t) < |x| \leq (1-C_\epsilon)\underline{h}(t)$, using the definition of $\underline{u}$, we obtain

$$\begin{aligned}
\underline{u}_t - C_1 \underline{u} & = \left[\frac{K_1 \rho}{\alpha-1} \left(\frac{\underline{h}-|x|}{\underline{h}} \right)^{-1} \frac{|x|}{\underline{h}} \underline{h}^{1-\alpha} - C_1 \right] \underline{u} \\
& \leq \left[\frac{K_1 \rho}{C_\epsilon (\alpha-1)} \underline{h}^{1-\alpha} - C_1 \right] \underline{u} \leq 0
\end{aligned}$$

since $\theta \gg 1$ and $\underline{h}(t) \geq \theta^{1/(\alpha-1)}$, $1-\alpha < 0$. We have thus proved (6.11).

We next deal with the case that (6.2) holds. If $|x|$ satisfies

$$\underline{h}(t) \geq |x| \geq \left[1 - \frac{\tilde{C}_\epsilon}{(\ln \underline{h}(t))^{1/(\rho-1)}} \right] \underline{h}(t),$$

with

$$\tilde{C}_\epsilon := \left[\frac{C_1 \tilde{C}_2 \beta}{K_1^\beta K_2 K_3^\rho} \right]^{1/(\rho-1)} = \left[\frac{C_1 \tilde{C}_2 \beta \epsilon^\rho}{K_1^\beta K_2 \rho} \right]^{1/(\rho-1)},$$

then $|x| \in [7\underline{h}(t)/8, \underline{h}(t)]$ and

$$\begin{aligned} \underline{u}_t - C_1 \int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy &\leq \frac{K_1^\beta K_2 K_3^\rho}{\beta} \left(\frac{\underline{h} - |x|}{\underline{h}} \right)^{\rho-1} (\ln \underline{h})^{1-\beta} - C_1 \tilde{C}_2 (\ln \underline{h})^{-\beta} \\ &\leq \left[\frac{K_1^\beta K_2 K_3^\rho}{\beta} \tilde{C}_\epsilon^{\rho-1} - C_1 \tilde{C}_2 \right] (\ln \underline{h})^{-\beta} = 0. \end{aligned}$$

For $(1-\epsilon)\underline{h} < |x| \leq [1 - \frac{\tilde{C}_\epsilon}{(\ln \underline{h})^{1/(\rho-1)}}] \underline{h}$, from the definition of $\underline{u}$, we deduce

$$\begin{aligned} \underline{u}_t - C_1 \underline{u} &= \left[\frac{K_1^\beta \rho}{\beta} \left(\frac{\underline{h} - |x|}{\underline{h}} \right)^{-1} \frac{|x|}{\underline{h}} (\ln \underline{h})^{1-\beta} - C_1 \right] \underline{u} \\ &\leq \left[\frac{K_1^\beta \rho}{\tilde{C}_\epsilon \beta} (\ln \underline{h})^{1-\beta+(\rho-1)^{-1}} - C_1 \right] \underline{u} \leq 0 \end{aligned}$$

since $\underline{h}(t) \geq e^{K_1 \theta^{1/\beta}} \gg 1$ and we may choose ρ large enough such that $1 - \beta + (\rho-1)^{-1} < 0$. The desired inequality (6.11) is thus proved.

Step 3. Completion of the proof by the comparison principle.

Since spreading happens, there is $t_0 > 0$ large enough such that $[g(t_0), h(t_0)] \supset [-\underline{h}(0), \underline{h}(0)]$, and also

$$\underline{u}(t_0, x) \geq K_2 = 1 - \epsilon \geq \underline{u}(0, x) \quad \text{for } x \in [-\underline{h}(0), \underline{h}(0)].$$

Moreover, from the definition of $\underline{u}$, we see $\underline{u}(x, t) = 0$ for $x = \pm \underline{h}(t)$ and $t \geq 0$. Thus we are in a position to apply the comparison principle to conclude that

$$-\underline{h}(t) \geq g(t_0 + t), \quad \underline{h}(t) \leq h(t_0 + t) \quad \text{for } t \geq 0.$$

The desired conclusion then follows from the arbitrariness of $\epsilon > 0$ and the fact that $D_{\epsilon/(2-\epsilon)} \rightarrow D_0$ as $\epsilon \rightarrow 0$. The proof is finished. $\square$

6.2.2. *The case that (6.1) holds with $\alpha = 2$.*

Lemma 6.5. *If the conditions in Lemma 6.4 are satisfied except that J satisfies (6.1) with $\alpha = 2$, then*

$$(6.12) \quad \liminf_{t \rightarrow \infty} \frac{h(t)}{t \ln t} \geq \mu \underline{\lambda}.$$

Proof. For fixed $\rho \geq 2$, $0 < \epsilon \ll 1$, $0 < \tilde{\epsilon} \ll 1$ and $\theta \gg 1$, define

$$\begin{cases} \underline{h}(t) := K_1(t + \theta) \ln(t + \theta), & t \geq 0, \\ \underline{u}(t, x) := K_2 \min \left\{ 1, \left[\frac{\underline{h}(t) - |x|}{(t + \theta)^{\tilde{\epsilon}}} \right]^\rho \right\}, & t \geq 0, x \in [-\underline{h}(t), \underline{h}(t)], \end{cases}$$

where

$$K_1 := (1 - \epsilon)^3 (1 - \tilde{\epsilon}) \mu \underline{\lambda}, \quad K_2 := 1 - \epsilon.$$

Note that

$$\underline{u}(t, x) = K_2 = 1 - \epsilon \quad \text{for } |x| \leq \underline{h}(t) - (t + \theta)^{\tilde{\epsilon}}.$$

Obviously, for any $t > 0$, $\partial_t \underline{u}(t, x)$ exists for $x \in [-\underline{h}(t), \underline{h}(t)]$ except when $|x| = \underline{h}(t) - (t + \theta)^{\tilde{\epsilon}}$. However, the one-sided partial derivatives $\partial_t \underline{u}(t \pm 0, x)$ always exist.

Step 1. We show that for $\theta \gg 1$,

$$(6.13) \quad \underline{h}'(t) \leq \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(y-x) \underline{u}(t, x) dy dx \quad \text{for } t > 0,$$

which clearly implies, due to $\underline{u}(t, x) = \underline{u}(t, -x)$ and $J(x) = J(-x)$, that

$$-\underline{h}'(t) \geq -\mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J(y-x) \underline{u}(t, x) dy dx \quad \text{for } t > 0.$$

Making use of the definition of $\underline{u}$ and

$$[-2(1-\epsilon)\underline{h}, -[2(1-\epsilon)\underline{h}]^{\tilde{\epsilon}}] \subset [-2\underline{h} + (t + \theta)^{\tilde{\epsilon}}, -(t + \theta)^{\tilde{\epsilon}}]$$

for $\theta \gg 1$, we obtain

$$\begin{aligned} \mu \int_{-\underline{h}}^{\underline{h}} \int_{\underline{h}}^{+\infty} J(y-x) \underline{u}(t, x) dy dx &\geq (1-\epsilon) \mu \int_{-\underline{h}+(t+\theta)^{\tilde{\epsilon}}}^{\underline{h}-(t+\theta)^{\tilde{\epsilon}}} \int_{\underline{h}}^{+\infty} J(y-x) dy dx \\ &= (1-\epsilon) \mu \int_{-2\underline{h}+(t+\theta)^{\tilde{\epsilon}}}^{-(t+\theta)^{\tilde{\epsilon}}} \int_0^{+\infty} J(y-x) dy dx \geq (1-\epsilon) \mu \int_{-2(1-\epsilon)\underline{h}}^{-[2(1-\epsilon)\underline{h}]^{\tilde{\epsilon}}} \int_0^{+\infty} J(y-x) dy dx. \end{aligned}$$

Thanks to Lemma 6.2, for large $\underline{h}$ (which is guaranteed by $\theta \gg 1$),

$$\int_{-2(1-\epsilon)\underline{h}}^{-[2(1-\epsilon)\underline{h}]^{\tilde{\epsilon}}} \int_0^{+\infty} J(y-x) dy dx \geq (1-\epsilon)(1-\tilde{\epsilon})\underline{\lambda} \ln[2(1-\epsilon)\underline{h}].$$

Hence, with $\theta \gg 1$, we have

$$\begin{aligned} \mu \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{+\infty} J(y-x) \underline{u}(t, x) dy dx \\ \geq (1-\epsilon)^2 \mu (1-\tilde{\epsilon}) \underline{\lambda} \ln[2(1-\epsilon)\underline{h}] \\ = (1-\epsilon)^2 \mu (1-\tilde{\epsilon}) \underline{\lambda} \{ \ln(t+\theta) + \ln[\ln(t+\theta)] + \ln[2(1-\epsilon)K_1] \} \\ \geq K_1 \ln(t+\theta) + K_1 = \underline{h}'(t) \text{ for all } t > 0, \end{aligned}$$

which proves (6.13).

Step 2. We show that for $t > 0$ and $x \in [-\underline{h}(t), \underline{h}(t)]$ with $|x| \neq \underline{h}(t) - (t+\theta)^{\tilde{\epsilon}}$,

$$(6.14) \quad \underline{u}_t(t, x) \leq d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(t, y) dy - d \underline{u}(t, x) + f(\underline{u}(t, x))$$

for $\theta \gg 1$.

From the definition of $\underline{u}$, we obtain by direct calculation that, for $t > 0$,

$$(6.15) \quad \underline{u}_t(t, x) = \begin{cases} \rho K_2^{\rho-1} \underline{u}^{1-\rho-1} \left[K_1 \frac{(1-\tilde{\epsilon}) \ln(t+\theta)+1}{(t+\theta)^{\tilde{\epsilon}}} + \frac{\tilde{\epsilon}|x|}{(t+\theta)^{1+\tilde{\epsilon}}} \right] & \text{if } \underline{h}(t) - (t+\theta)^{\tilde{\epsilon}} < |x| \leq \underline{h}(t), \\ 0 & \text{if } 0 \leq |x| < \underline{h}(t) - (t+\theta)^{\tilde{\epsilon}}. \end{cases}$$

Making use of Lemma 6.3 with

$$(L(t), \phi(t, x), \xi(t)) = (\underline{h}(t), \underline{u}(t, x)/K_2, \frac{\underline{h}(t)}{(t+\theta)^{\tilde{\epsilon}}}),$$

for any given small $\delta > 0$, we can find a large $\theta_* = \theta_*(\delta, \epsilon)$ such that for $\theta \geq \theta_*$ and $|x| \leq \underline{h}(t)$,

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(y, t) dy \geq (1-\delta) \underline{u}(x, t).$$

Then, a similar analysis as in the proof of Lemma 6.4 shows that there exists $C_1 > 0$, depending on ϵ and δ , such that for $\theta \gg 1$, $x \in [-\underline{h}(t), \underline{h}(t)]$ and $t \geq 0$,

$$d \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(t, y) dy - d \underline{u} + f(\underline{u}) \geq C_1 \left[\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(t, y) dy + \underline{u} \right].$$

Hence, to verify (6.14), we only need to show that

$$(6.16) \quad \underline{u}_t \leq C_1 \left[\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \underline{u}(t, y) dy + \underline{u} \right] \text{ for } |x| \in [0, \underline{h}(t)] \setminus \{\underline{h}(t) - (t+\theta)^{\tilde{\epsilon}}\}.$$

Clearly, (6.16) holds trivially for $0 \leq |x| < \underline{h}(t) - (t+\theta)^{\tilde{\epsilon}}$ due to $\underline{u}_t = 0$ for such x . We next consider the remaining case $\underline{h}(t) - (t+\theta)^{\tilde{\epsilon}} < |x| < \underline{h}(t)$.

Denote $\eta = \eta(t) := (t+\theta)^{\tilde{\epsilon}}$. Using $\theta \gg 1$ and (6.1), we obtain, for $x \in [\underline{h}(t) - \eta(t), \underline{h}(t)]$,

$$\begin{aligned} \int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy &\geq \int_{-\underline{h}+\eta}^{\underline{h}-\eta} J(x-y) \underline{u}(t, y) dy = K_2 \int_{-\underline{h}+\eta}^{\underline{h}-\eta} J(x-y) dy \\ &= K_2 \int_{-\underline{h}+\eta-x}^{\underline{h}-\eta-x} J(y) dy \geq K_2 \int_{-\underline{h}}^{-\eta} J(y) dy \geq \frac{K_2 \underline{\lambda}}{2} \int_{\eta}^{\underline{h}} y^{-2} dy \\ &= \frac{K_2 \underline{\lambda}}{2} (\eta^{-1} - \underline{h}^{-1}) \geq \frac{(1-\epsilon) \underline{\lambda}}{4} \eta^{-1} =: C_2 (t+\theta)^{-\tilde{\epsilon}}. \end{aligned}$$

The same estimate also holds for $x \in [-\underline{h}(t), -\underline{h}(t) + \eta(t)]$. Therefore, for $|x| \in [\underline{h}(t) - \eta(t), \underline{h}(t)]$, due to $\rho > 2$ and $0 < \tilde{\epsilon} \ll 1$, we have

$$\begin{aligned} & \underline{u}_t(t, x) - C_1 \int_{-\underline{h}}^{\underline{h}} J(x-y) \underline{u}(t, y) dy \\ & \leq \rho K_2^{1/\rho} \underline{u}^{(\rho-1)/\rho} \left[K_1 \frac{(1-\tilde{\epsilon}) \ln(t+\theta) + 1}{(t+\theta)^{\tilde{\epsilon}}} + \frac{\tilde{\epsilon} \underline{h}}{(t+\theta)^{1+\tilde{\epsilon}}} \right] - C_1 C_2 (t+\theta)^{-\tilde{\epsilon}} \\ & \leq 2K_1 \rho K_2^{1/\rho} \underline{u}^{(\rho-1)/\rho} \frac{\ln(t+\theta)}{(t+\theta)^{\tilde{\epsilon}}} - C_1 C_2 (t+\theta)^{-\tilde{\epsilon}} \\ & = \frac{2K_1 \rho K_2 [(\underline{h} - |x|)/(t+\theta)^{\tilde{\epsilon}}]^{\rho-1} \ln(t+\theta) - C_1 C_2}{(t+\theta)^{\tilde{\epsilon}}} \leq 0 \end{aligned}$$

if $|x|$ further satisfies

$$|x| \geq \underline{h}(t) - \left(\frac{C_1 C_2}{2K_1 \rho K_2} \right)^{1/(\rho-1)} \frac{(t+\theta)^{\tilde{\epsilon}}}{[\ln(t+\theta)]^{1/(\rho-1)}} =: \underline{h}(t) - C_3 \frac{(t+\theta)^{\tilde{\epsilon}}}{[\ln(t+\theta)]^{1/(\rho-1)}}.$$

On the other hand, for $\underline{h}(t) - (t+\theta)^{\tilde{\epsilon}} < |x| < \underline{h}(t) - C_3(t+\theta)^{\tilde{\epsilon}}/[\ln(t+\theta)]^{1/(\rho-1)}$, using (6.15) and $0 < \tilde{\epsilon} \ll 1$, $\theta \gg 1$, we deduce

$$\begin{aligned} \underline{u}_t - C_1 \underline{u} & \leq 2K_1 \rho K_2^{1/\rho} \underline{u}^{(\rho-1)/\rho} \frac{\ln(t+\theta)}{(t+\theta)^{\tilde{\epsilon}}} - C_1 \underline{u} \\ & = \underline{u} \left(\frac{2K_1 \rho [(\underline{h} - |x|)/(t+\theta)^{\tilde{\epsilon}}]^{-1/\rho} \ln(t+\theta)}{(t+\theta)^{\tilde{\epsilon}}} - C_1 \right) \\ & \leq \underline{u} \left(\frac{2K_1 \rho [\ln(t+\theta)]^{1+\frac{1}{\rho(\rho-1)}}}{C_3^{1/\rho} (t+\theta)^{\tilde{\epsilon}}} - C_1 \right) < 0. \end{aligned}$$

Hence, (6.16) holds true. This concludes Step 2.

Step 3. We finally prove (6.12).

The definition of $\underline{u}$ clearly gives $\underline{u}(t, \pm \underline{h}(t)) = 0$ for $t \geq 0$. Since spreading happens for (u, g, h) and $K_2 = 1 - \epsilon < 1$, there is a large constant $t_0 > 0$ such that

$$\begin{aligned} & [-\underline{h}(0), \underline{h}(0)] \subset (g(t_0), h(t_0)), \\ & \underline{u}(0, x) \leq K_2 \leq u(t_0, x) \quad \text{for } x \in [-\underline{h}(0), \underline{h}(0)]. \end{aligned}$$

It follows that

$$\begin{aligned} & [-\underline{h}(t), \underline{h}(t)] \subset [g(t+t_0), h(t+t_0)] & \text{for } t \geq 0, \\ & \underline{u}(t, x) \leq u(t+t_0, x) & \text{for } t \geq 0, x \in [-\underline{h}(t), \underline{h}(t)], \end{aligned}$$

which implies

$$\liminf_{t \rightarrow \infty} \frac{h(t)}{t \ln t} \geq (1-\epsilon)^3 (1-\tilde{\epsilon}) \mu \lambda.$$

Since $\epsilon > 0$ and $\tilde{\epsilon} > 0$ can be arbitrarily small, we thus obtain (6.12) by letting $\epsilon \rightarrow 0$ and $\tilde{\epsilon} \rightarrow 0$. This completes the proof of the lemma. $\square$

6.3. Upper bounds. Recall that we will only state and prove the conclusions for $h(t)$, as the corresponding conclusion for $-g(t)$ follows directly by considering the problem with initial function $u_0(-x)$.

Lemma 6.6. *Assume that J satisfies **(J)** and one of the conditions (6.1) and (6.2), f satisfies **(f)**, and spreading happens to (4.1). Then*

$$(6.17) \quad \begin{cases} \limsup_{t \rightarrow \infty} \frac{h(t)}{t^{1/(\alpha-1)}} \leq \left(\frac{2^{2-\alpha}}{2-\alpha} \mu \bar{\lambda} \right)^{1/(\alpha-1)} & \text{if (6.1) holds with } \alpha \in (1, 2), \\ \limsup_{t \rightarrow \infty} \frac{h(t)}{t \ln t} \leq \mu \bar{\lambda} & \text{if (6.1) holds with } \alpha = 2, \\ \limsup_{t \rightarrow \infty} \frac{\ln h(t)}{t^{1/\beta}} \leq \left(\frac{2\beta \mu \bar{\lambda}}{\beta-1} \right)^{1/\beta} & \text{if (6.2) holds.} \end{cases}$$

Proof. For any given small $\epsilon > 0$, define, for $t \geq 0$,

$$\bar{h}(t) := \begin{cases} (Kt + \theta)^{1/(\alpha-1)} & \text{if (6.1) holds with } \alpha \in (1, 2], \\ K(t + \theta) \ln(t + \theta) & \text{if (6.1) holds with } \alpha = 2, \\ e^{K(t+\theta)^{1/\beta}} & \text{if (6.2) holds,} \end{cases}$$

$$\bar{u}(t, x) := 1 + \epsilon, \quad x \in [-\bar{h}(t), \bar{h}(t)],$$

where $\theta \gg 1$ and

$$(6.18) \quad K := \begin{cases} (1 + \epsilon)^3 \frac{2^{2-\alpha}}{2 - \alpha} \mu \bar{\lambda} & \text{if (6.1) holds with } \alpha \in (1, 2), \\ (1 + \epsilon)^3 \mu \bar{\lambda} & \text{if (6.1) holds with } \alpha = 2, \\ \left[\frac{2(1 + \epsilon)^3 \beta \mu \bar{\lambda}}{\beta - 1} \right]^{1/\beta} & \text{if (6.2) holds,} \end{cases}$$

We verify that for $t > 0$,

$$(6.19) \quad \bar{h}'(t) \geq \mu \int_{-\bar{h}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{+\infty} J(y - x) \bar{u}(t, x) dy dx,$$

which clearly implies

$$-\bar{h}'(t) \leq -\mu \int_{-\bar{h}(t)}^{\bar{h}(t)} \int_{-\infty}^{-\bar{h}(t)} J(y - x) \bar{u}(t, x) dy dx$$

since $\bar{u}(t, x) = \bar{u}(t, -x)$ and $J(x) = J(-x)$.

Using $\bar{u} = 1 + \epsilon$, we have

$$\begin{aligned} \mu \int_{-\bar{h}}^{\bar{h}} \int_{\bar{h}}^{+\infty} J(y - x) \bar{u}(t, x) dy dx &= (1 + \epsilon) \mu \int_{-\bar{h}}^{\bar{h}} \int_{\bar{h}}^{+\infty} J(y - x) dy dx \\ &= (1 + \epsilon) \mu \int_{-2\bar{h}}^0 \int_0^{+\infty} J(y - x) dy dx. \end{aligned}$$

By Lemma 6.2 with $\delta = 0$, we see that for large $\bar{h}$, which is guaranteed by $\theta \gg 1$,

$$\begin{cases} \int_{-2\bar{h}}^0 \int_0^{+\infty} J(y - x) dy dx \leq (1 + \epsilon) \frac{\bar{\lambda}}{(\alpha - 1)(2 - \alpha)} (2\bar{h})^{2-\alpha}, & \text{if (6.1) holds with } \alpha \in (1, 2), \\ \int_{-2\bar{h}}^0 \int_0^{+\infty} J(y - x) dy dx \leq (1 + \epsilon) \bar{\lambda} \ln(2\bar{h}), & \text{if (6.1) holds with } \alpha = 2, \\ \int_{-2\bar{h}}^0 \int_0^{+\infty} J(y - x) dy dx \leq (1 + \epsilon) (2\bar{h}) [\ln(2\bar{h})]^{1-\beta} \frac{\bar{\lambda}}{\beta - 1} & \text{if (6.2) holds.} \end{cases}$$

Therefore, when (6.1) holds with $\alpha \in (1, 2)$, by the definition of K , we have

$$\begin{aligned} \mu \int_{-\bar{h}}^{\bar{h}} \int_{\bar{h}}^{+\infty} J(y - x) \bar{u}(t, x) dy dx &\leq (1 + \epsilon)^2 \mu \frac{\bar{\lambda}}{(\alpha - 1)(2 - \alpha)} (2\bar{h})^{2-\alpha} \\ &= (1 + \epsilon)^2 \mu \frac{\bar{\lambda}}{(\alpha - 1)(2 - \alpha)} 2^{2-\alpha} (Kt + \theta)^{(2-\alpha)/(\alpha-1)} \\ &\leq \frac{K}{\alpha - 1} (Kt + \theta)^{(2-\alpha)/(\alpha-1)} = \bar{h}'(t). \end{aligned}$$

When (6.1) holds with $\alpha = 2$, we similarly obtain, due to $\theta \gg 1$,

$$\begin{aligned} \mu \int_{-\bar{h}}^{\bar{h}} \int_{\bar{h}}^{+\infty} J(y - x) \bar{u}(t, x) dy dx &\leq (1 + \epsilon)^2 \mu \bar{\lambda} \ln(2\bar{h}) \\ &= (1 + \epsilon)^2 \mu \bar{\lambda} \{ \ln(t + \theta) + \ln[\ln(t + \theta)] + \ln 2K \} \\ &\leq K \ln(t + \theta) + K = \bar{h}'(t). \end{aligned}$$

Finally, when (6.2) holds, we have

$$\begin{aligned} \mu \int_{-\bar{h}}^{\bar{h}} \int_{\bar{h}}^{+\infty} J(y-x) \bar{u}(t, x) dy dx &\leq (1+\epsilon)^2 \mu (2\bar{h}) [\ln(2\bar{h})]^{1-\beta} \frac{\bar{\lambda}}{\beta-1} \\ &\leq (1+\epsilon)^3 \mu (2\bar{h}) (\ln \bar{h})^{1-\beta} \frac{\bar{\lambda}}{\beta-1} \\ &= \frac{K^\beta}{\beta} \underline{h} (\ln \underline{h})^{1-\beta} = \underline{h}'(t). \end{aligned}$$

Thus (6.19) always holds.

Recalling that $\bar{u} \geq 1$ is a constant, we get, for $t > 0$, $x \in [-\bar{h}(t), \bar{h}(t)]$,

$$\bar{u}_t(t, x) \equiv 0 \geq d \int_{-\bar{h}}^{\bar{h}} J(x-y) \bar{u}(t, y) dy - d \bar{u}(t, x) + f(\bar{u}(t, x)).$$

Note that condition **(f)** implies, by simple comparison with ODE solutions,

$$\limsup_{t \rightarrow \infty} \max_{x \in [g(t), h(t)]} u(t, x) \leq 1;$$

hence there is $t_0 > 0$ such that

$$u(t_0, x) \leq 1 + \epsilon = \bar{u}(t_0, x) \quad \text{for } x \in [g(t_0), h(t_0)] \subset [-\bar{h}(0), \bar{h}(0)]$$

with the last part holding for large θ .

We are now in a position to use the comparison principle (Theorem 1.3) to conclude that

$$\begin{aligned} [g(t+t_0), h(t+t_0)] &\subset [-\bar{h}(t), \bar{h}(t)] \quad \text{for } t \geq 0, \\ u(t+t_0, x) &\leq \bar{u}(t, x) \quad \text{for } t \geq 0, \quad x \in [g(t+t_0), h(t+t_0)]. \end{aligned}$$

By the arbitrariness of $\epsilon > 0$, we get (6.17). The proof is finished. $\square$

Proof of Theorem 6.1: The conclusions for $g(t)$ and $h(t)$ follow directly from the above lower and upper bounds. The conclusion on $\lim_{t \rightarrow \infty} u(t, x)$ follows from the definitions of the lower and upper solutions.

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