UNIQUENESS OF THE ASYMPTOTIC LIMITS FOR RICCI-FLAT MANIFOLDS WITH LINEAR VOLUME GROWTH

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Abstract. Under natural assumptions on curvature and cross section, we establish the uniqueness of asymptotic limits and the exponential convergence rate for complete noncollapsed Ricci-flat manifolds with linear volume growth, which are known to only admit cylindrical asymptotic limits. In dimension four, these assumptions hold automatically, yielding unconditional uniqueness and convergence. In particular, our results show that all asymptotically cylindrical Calabi–Yau manifolds converge exponentially to their asymptotic limits, thereby answering affirmatively a question by Haskins–Hein–Nordström. In dimension four our result strengthens those of Chen–Chen, who proved exponential convergence to its asymptotic limit space for any ALH instanton.

1. introduction

1.1. **Statement of results.** In this paper we study the uniqueness of asymptotic limits and the rate of convergence to them for Ricci flat manifolds with linear volume growth. The motivation comes from a comparison with tangent cones at infinity for Euclidean volume growth which we will elaborate in section 1.2. Let (M, g) be a complete noncompact Riemannian n-manifold with nonnegative Ricci curvature. Gromov precompactness theorem implies that any divergent *translation sequence* (M, g, p_i), meaning $\mathsf{d}_g(p_i, x) \to \infty$ for any fixed $x \in M$, has subsequence pointed Gromov–Hausdorff (pGH in short) converges to a Ricci limit space, which we call an *asymptotic limit*. Calabi and Yau independently showed that the minimal volume growth order is *linear*, see [Yau76]. Then it makes sense to define that M has linear (minimal) volume growth as: For some (hence any) base point $x \in M$

$$\limsup_{r\to\infty} \frac{\mathrm{Vol}_g(B_r(x))}{r} < \infty. \tag{1.1}$$

In addition, we say M is *noncollapsed* if

$$\text{there exists } v > 0 \text{ such that } \mathrm{Vol}_g(B_1(x)) > v \text{ for all } x \in M. \tag{1.2}$$

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Building on earlier work of Sormani [Sor98, Sor00], the second author proved

Theorem 1.1 ([Zhu25, Theorem 1.2])**.** *If (M, g) with $\mathrm{Ric}_g \geq 0$ is noncollapsed (1.2) with linear volume growth (1.1), then every asymptotic limit of M must split as a cylinder $\bar{N} := \mathbb{R} \times N$ with N compact.*

Here, the cross section N may be singular and may depend on the chosen (sub)sequence $\{p_i\}_{i\in\mathbb{N}_0}$. It is a basic question to understand the uniqueness of asymptotic limits. In general there is no uniqueness. There can be non-isometric cross sections [Sor98, Example 27] and even non-homeomorphic cross sections [Zhu25, Theorem 5.1], following the similar construction for tangent cones at infinity [CN13]. Meanwhile, for manifolds asymptotic to a cylinder in a stronger sense, the convergence rate is also of great interests, for instances, in the study of ALH gravitational instantons [CC21], Calabi–Yau manifolds [HHN15] and G_2 manifolds [Nor08].

In this paper we will follow the framework of Cheeger–Tian [CT94] to study the uniqueness and convergence rate to asymptotic limits. Let us first set the stage by introducing some regularity conditions.

Given a Ricci flat manifold N, the infinitesimal deformation of Ricci tensor in the space of smooth symmetric $(0, 2)$ tensors, denoted by $C^\infty(S^2T^*N)$, is given by the following formula

$$\mathrm{DRic}(h) := (\nabla^*\nabla - 2\,\mathrm{Rm})h - \delta^*\delta h - \nabla^2 \operatorname{tr} h + \mathrm{Ric} \circ h. \tag{1.3}$$

We concern with the following infinitesimal Ricci deformation equation.

$$\mathrm{D}\,\mathrm{Ric}(h) = 0 \tag{1.4}$$

Definition 1.2. ([Bes08, section 12]) We say that a (closed) Ricci flat manifold N is *integrable* if any solution $h \in C^\infty(S^2T^*N)$ of (1.4) arises from a deformation of Ricci flat metrics, i.e. if there exists a family of Ricci flat metrics $(g_t)_{t\in(-\varepsilon,\varepsilon)}$ for some $\varepsilon > 0$, such that $h = \dot{g}_0$.

Denote $\Delta^L := \nabla^*\nabla - 2\,\mathrm{Rm}$, which is frequently referred to as the *Lichnerowicz Laplacian.* Now we are in position to state our main theorem.

Theorem A. *Let (M^n, g) be a complete non-compact Ricci flat n-manifold that is noncollapsed (1.2) and has linear volume growth (1.1). Impose additionally the following regularity assumptions on M:*

R.1 Local $L^{n/2}$-curvature bound: There exists a constant Λ such that for any $t > 0$ and $p \in M$

$$\int_{B_{t+1}(p)\setminus B_t(p)} |\mathrm{Rm}|^{n/2}\, d\,\mathrm{vol}_g \;\leq\; \Lambda. \tag{1.5}$$

R.2 There exists one asymptotic limit $(\bar{N} := \mathbb{R} \times N, \bar{g} = dr^2 + g_N)$ such that the cross section N is integrable and has nonnegative Lichnerowicz Laplacian.

Then the asymptotic limit of M is unique. Moreover, there exists $\beta > 0$ such that outside of a compact subset $K \subseteq M$ there exists smooth embedding $\Phi : M \setminus K \to \bar{N}$ and

$$|\Phi^* g - \bar{g}| = O(e^{-\beta r}). \tag{1.6}$$

We will show that (R.1) implies that all asymptotic limits of M are smooth, therefore every cross section N is a closed Ricci flat manifold and (R.2) makes sense. If we know apriori the asymptotic limits are all smooth then we do not need (R.1). Notably, Theorem A answers a question affirmatively by Haskins–Hein–Nordström [HHN15, concluding remarks]: Whether or not all asymptotically cylindrical Calabi–Yau manifolds converge to their asymptotic limits exponentially. In their case the cross sections are integrable and have nonnegative Lichnerowicz Laplacian, see [HHN15, Theorem B].

Although we have a road map to prove our main theorem laid out in [CT94], the technical details are not exactly the same. We point out the most different one here.

Remark 1.3. Compared with Cheeger–Tian [CT94, Theorem 0.13], Theorem A has an extra assumption on the Lichnerowicz Laplacian. This reflects the key difference between cylinders and cones. When we solve the infinitesimal Ricci deformation equation (1.4) on either a cone or a cylinder, we can reduce it to a system of second order ODEs by spectral decomposition, then study the solution by a method developed by Simon [Sim83], now known as the three circles theorem. A crucial difference between cones and cylinders is that for cylinders the characteristic equation of the ODEs may have pure imaginary solutions and multiplicity two zero solutions. Neither can happen for cones and both pathological cases are excluded in the original work of Simon [Sim83, p.525]. Our main technical contribution in this paper is a strategy to deal with multiplicity two zero solutions. For pure imaginary solutions we have to assume they do not exist.

Although the assumptions seem to be strong and restrictive, we actually know that

Remark 1.4. All known examples of closed Ricci flat manifolds are integrable. Moreover, they all admit a spin cover with nonzero parallel spinors, thanks to Dai–Wang–Wei [DWW05], they all have nonnegative Lichnerowicz Laplacian. Both facts are consequences of special holonomy, see [Has12, p. 26] and references therein.

Fortunately, in the very special case $n = 4$, thanks to the codimension 4 theorem of Cheeger–Naber [CN15], all extra assumptions are not needed. Indeed, the codimension 4 theorem implies all asymptotic limits $\mathbb{R} \times N$ have singular set of Hausdorff dimension zero. However the product structure implies that if there is one singular point then there is a singular set of Hausdorff dimension at least 1, so they are all smooth. Then the cross section

N is a Ricci flat 3-manifold. So N is flat hence integrable and trivially has nonnegative Lichnerowicz Laplacian. We get a complete description of the uniqueness of asymptotic limits in dimension 4.

Corollary 1.5. *Let (M^4, g) be a complete non-compact Ricci flat 4-manifold that is noncollapsed (1.2) and has linear volume growth (1.1). Then the asymptotic limit of M is unique and the convergence rate to the limit, defined as in (1.6), is exponential.*

We no longer need the local $L^{n/2}$-curvature bound (R.1) when $n = 4$, but nevertheless it holds true by Jiang–Naber [JN21]. This corollary improves the result of Chen–Chen [CC21, Theorem 4.18], who proved exponential convergence for ALH gravitational instantons with polynomial convergence rate. Our result applies to a larger class of manifolds and does not require any apriori convergence rate.

1.2. **Background and Motivation.** We explain why asymptotic limits for linear (minimal) growth are natural analog of tangent cones at infinity for Euclidean (maximal) volume growth. Recall that (M, g) has nonnegative Ricci curvature. We say M has Euclidean volume growth if the monotone decreasing volume ratio has positive limit at infinity for some (hence any) base point $x \in M$:

$$\lim_{r\to\infty} \frac{\mathrm{Vol}_g(B_r(x))}{r^n} > 0. \tag{1.7}$$

Note that Euclidean volume growth automatically implies noncollapsedness.

Given any scaling sequence $(M, r_i^{-2}g)$ with $r_i \to \infty$, a subsequence pointed GH-converges to a Ricci limit space, called a *tangent cone at infinity* (or an asymptotic cone). When M has Euclidean volume growth, every asymptotic cone is a metric cone $(C(N) = [0,\infty)\times N, dr^2 + r^2 g_N)$ [CC97], with the cross section N depending on the scaling sequence. This theorem is known as volume cone to metric cone theorem. It can be viewed as the (almost) rigidity theorem when the monotone quantity $\mathrm{Vol}_g(B_r(x))/r^n$, equivalently the surface area quotient $\mathcal{H}^{n-1}(\partial B_r(x))/r^{n-1}$, (almostly) becomes a constant.

For asymptotic limits for linear growth, there is a parallel picture. The cylinder structure of asymptotic limits, just like the cone structure of tangent cones at infinity for Euclidean growth, comes from the rigidity when some monotone quantity becomes a constant. This monotone quantity is defined by a *Busemann function.* Let $\gamma : [0,\infty) \to M$ be a ray, the Busemann function b_γ associated to γ is defined to be

$$b_\gamma(x) = \lim_{t\to\infty} t - \mathsf{d}_g(x, \gamma(t)).$$

It is shown by Sormani [Sor98, Theorem 19, Lemma 20] that when M has linear volume growth, then b_γ is proper and $r \to \mathcal{H}^{n-1}(\{b_\gamma = r\})$ is monotone increasing. It is then shown by the second author [Zhu25, Proposition 3.5] that with the extra assumption that M is noncollapsed, $\mathrm{diam}(\{b_\gamma = r\})$

is uniformly bounded. Combining the uniform bound on the diameter of level sets and the monotonicity, we see that the cylinder structure of asymptotic limits indeed follows from the almost rigidity theorem when the monotone quantity $\mathcal{H}^{n-1}(\{b = r\})$ almostly becomes a constant by Sormani [Sor00, Theorem 34], after Cheeger–Colding [CC96].

In view of this analogy, it is quite natural to attempt the uniqueness problem of asymptotic limits by adapting the argument of Colding–Minicozzi [CM14] on the uniqueness of tangent cones at infinity for Euclidean growth. Indeed, one intuition is that if the monotone quantity $\mathcal{H}^{n-1}(\partial B_r(x))/r^{n-1}$ converges fast enough to its limit as $r \to \infty$, then the uniqueness of tangent cone at infinity follows. The statement in the exact same spirit for linear growth is stated by Sormani [Sor00, Remark 43]: If the monotone quantity $\mathcal{H}^{n-1}(\{b_\gamma = r\})$ converges fast enough as $r \to \infty$, then the uniqueness of asymptotic limit follows.

For Ricci flat manifolds, Colding–Minicozzi realized this intuition by a smoothened version of the surface area quotient called $A(r)$ [CM14, Theorem 0.16], which is also a monotone quantity. The smoothened quotient can be viewed as a weighted surface area quotient and the weight is a distance function induced by the Green's function.

However, there are difficulties of implementing the ideas in [CM14] in the linear growth case. The first difficulty of adapting the argument in [CM14] to linear growth setting is that there is no natural globally defined smooth approximation of the Busemann function, hence no global smoothened version of the monotone quantity $\mathcal{H}^{n-1}(\{b_\gamma = r\})$. The most natural smooth approximation of the Busemann function seems to be the local harmonic replacement from the almost splitting theorem [CC97]. Some smoothened version of $\mathcal{H}^{n-1}(\{b_\gamma = r\})$ can be constructed by the local harmonic replacement, but it is not clear how to derive the asymptotics near infinity from local monotone quantities.

The second difficulty is that for Ricci flat metric cones, the cross sections are Einstein manifolds of positive Einstein constant, whereas for Ricci flat cylinders, the cross sections are Ricci flat manifolds. On Ricci flat manifolds, the weighed Einstein–Hilbert functional, which plays a crucial role in [CM14, Section 2], now has trivial first variation. There is also no natural candidate to replace it. Nevertheless we believe the statement align to [CM14] for cylinders still holds.

Conjecture 1.6. Let (M^n, g) be a noncollapsed Ricci flat manifold with linear volume growth. If one asymptotic limit $\mathbb{R} \times N$ is smooth, then the asymptotic limit is unique.

We end this subsection by a remark about the comparison of convergence speed in cylindrical and conical case.

Remark 1.7. A cone metric $dt^2 + t^2 g$ with a change of variable $t = e^r$ is conformal to the cylinder metric $dr^2 + g$. It is immediate to see that polynomial (resp. logarithmic) decay in the conical coordinates corresponds to exponential (resp. polynomial) decay in the cylindrical coordinates.

1.3. **Sketch of proof.** We summarize here the major steps of our proof of Theorem A. Let $(\overline{N}, \bar{g}) := (\mathbb{R} \times N^{n-1}, dr^2 + g_N)$ be a Ricci flat cylinder with compact cross section (N^{n-1}, g_N). Suppose that (M, g) is GH-close to $\overline{N}$ near a sequence of base points. The strategy for proving uniqueness is to construct a diffeomorphism

$$\Phi : (R, R + L') \times N \longrightarrow M$$

from an arbitrarily long tube of $\overline{N}$ into M, and to show that Φ extends to infinity, i.e. $L' = \infty$, with the pull-back metric $\Phi^* g$ converging to $\bar{g}$ as $r \to \infty$.

Heuristically, the convergence arises as follows. We can construct Φ to be an almost isometry on $(R, R+L') \times N$ so that the difference $\Phi^* g - \bar{g}$ is initially small. Meanwhile this difference of two metrics satisfies a harmonic-type equation derived from the Ricci-flatness of $(\overline{N}, \bar{g})$, which up to higher order errors coincides with the infinitesimal Ricci deformation equation (1.4). This equation can be interpreted as an evolution equation along $\mathbb{R}$. The convergence follows from analyzing the asymptotics for the solutions with small initial data using three circles theorem based on [Sim83, Section 5, Theorem 4]. In actuality, to gauge away radially parallel solutions in (1.4), we will not directly consider $\Phi^* g - \bar{g}$, instead we consider $\Phi^* g - \bar{g}_i$ for a sequence $\bar{g}_i$ that are close to $\bar{g}$. But we will leave this technical point to the very end.

The proof is therefore organized into three main steps.

- The first step is to construct the desired diffeomorphism

$$\Phi : (R, R + L') \times N \longrightarrow M$$

  close to be an isometry, such that $\Phi^* g$ is divergence-free with respect to the background metric $\bar{g}$. We require that the closeness here is independent of the size and position of the tube. To construct Φ, we need to analyze the divergence equation (2.1) which can be equivalently expressed as (2.2). Solving (2.2) typically requires constructing the Green's function of the operator $d^* d + 2dd^*$, where d is the exterior derivative and d^* is its adjoint. Further details are provided in Section 2.2.
- The second step is to analyze the infinitesimal Ricci-flat equation (1.4). With (2.1) solved and $L_\bullet g \in \ker \mathrm{D}\,\mathrm{Ric}$, we can assume $\delta h = 0$, then (1.4) reduces to the system

$$\begin{cases} \Delta^L h - \nabla^2(\operatorname{tr} h) = 0, \\ \delta h = 0. \end{cases}$$

The system of equations can be solved by decomposing into radial ($\mathbb{R}$) and tangential (N^{n-1}) direction and there are 5 types of solutions. Compare [CT94, Proposition 4.65]. Let B be a divergence-free 2-tensor on N, then the solutions are:

(1) Exponential growth solutions like $e^{\mu r}B$.
(2) Polynomial growth solutions like rB. No analog of this type on a metric cone.
(3) Radially parallel solutions like B.
(4) Exponential decay solutions like $e^{-\mu r}B$
(5) Oscillatory solutions like $e^{-i\mu r}B$, where $i^2 = -1$. No analog of this type on a metric cone.

- The last step is to exclude all other 4 types of solutions, except for the exponential decay solutions, for the nonlinear equation satisfied by $\Phi^* g - \bar{g}$. In the previous step we already understood the solutions of its linearization. We rely on the assumption (R.2) to exclude the oscillatory solutions, and Ebin–Palais Slice theorem to rule out radially parallel solutions. To handle the remaining case, we establish a refined version of the three circles theorem, used by Cheeger–Tian to handle only the exponential growth solutions, that also accommodates polynomial growth solutions.

1.4. **Notations.** In the sequel, (M^n, g) always denotes a complete, smooth Ricci–flat ($\mathrm{Ric}_g = 0$) Riemannian manifold with linear volume growth. Given a sequence of points $p_i \in M$ diverging to infinity (i.e. exiting every compact set), we assume that

$$(M, g, p_i) \xrightarrow{\text{pGH}} (\overline{N} := \mathbb{R} \times N, \bar{g} := dr^2 + g_N),$$

The product of $\mathbb{R}$ and a compact manifold N is called a cylinder in this paper. In a cylinder, for real numbers $c < d$, we write

$$T_{c,d} := (c, d) \times N \subseteq \overline{N}$$

and $\bar{T}_{c,d}$ for the closed tube between the levels c and d. In particular $\bar{T}_{c,\infty}$ denotes the tube $[c, \infty) \times N$. Given a tensor η We denote its $C^{k,\alpha}$ norm taken with respect to metric g as $|\eta|_{k,\alpha;g}$. If the metric is clear in the context we may omit the background metric in the notation and if $\alpha = 0$ we also omit this subscript. Note that for two metrics g_1, g_2 so that $|g_1 - g_2|_{0,0,g_1} < \frac{1}{2}$, then there us a constant C depends only on the tensor type and k, α, n such that

$$|\eta|_{k,\alpha;g_2} \leq C|\eta|_{k,\alpha}(1 + |g_2 - g_1|_{k,\alpha;g_1}). \tag{1.8}$$

We denote the symmetric tensor product of two tensors η_1 and η_2 as $\eta_1 \boxtimes \eta_2 := \eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1$. In computations, we use $\Psi(x_1, \ldots, x_m | y_1, \ldots y_l)$ to denote a nonnegative function that tends to zero as $x_1 \ldots, x_m \to 0$ and $y_1, \ldots, y_l$ are fixed.

1.5. **Organization.** The paper is organized according to the major steps of the proof. Then in Section 2 we will solve the divergence equation by ODE and Green's function. This provides necessary tools to construct the divergence free gauge and to solve the infinitesimal deformation equation (1.4). Section 3 will devote to the construction of desired diffeomorphism (gauge), combing the structural results of linear volume growth and results in previous sections. Next, we give a complete solution of (1.4) in Section 4. In the last section, Section 5, we will prove the uniqueness of the asymptotic limit by ruling out the existence of solutions of (1.4) without exponential decay.

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2. Solving Divergence free equation

In this section, our goal is to construct (modified) divergence free self-diffeomorphism on an arbitrarily large tube in the cylinder $\overline{N}$, which is Theorem 2.8. Along with the structure results in Section 3 we will complete the construction of the desired diffeomorphism in the next section. Given a self diffeomorphism η on $\overline{N}$, we hope analyze the divergence free condition

$$\delta(\eta^* g) = 0.$$

Taking the first variation, we see that it is necessary to analyze the following equation of a vector field X, given $h \in C^\infty(S^2T^*\overline{N})$:

$$\delta(L_X g_0) = \delta h. \tag{2.1}$$

A straightforward computation shows that $L_X g_0 = \nabla^{\mathrm{sym}} X^\sharp$ and $\delta(L_X g_0) = (d^*d + 2dd^* - 2\,\mathrm{Ric}_{g_0})X^\sharp$; see [PW09, Lemma 2.1] for more details. In the Ricci-flat case, the equation (2.1) reduces to

$$(d^*d + 2dd^*)X^\sharp = \delta h. \tag{2.2}$$

To analyze the invertibility of the operator $d^*d + 2dd^*$ we equip the space of tensors with a weighted Hölder norm.

Definition 2.1. Let E be a vector bundle over $\overline{N} = \mathbb{R} \times N$. Fix a smooth cutoff function $\psi : \mathbb{R} \to [0,1]$ such that $\psi(r) = 1$ for $r \geq 0$ and $\psi(r) = 0$ for $r \leq -1$. For $k \in \mathbb{N}^+$, $\alpha \in (0,1)$, $\rho \in \mathbb{R}$, and $h \in C_0^\infty(E)$, we define the weighted Hölder norm

$$\|h\|_{k,\alpha;\rho} := \sup_{\overline{N}} \big\|\psi(r)\, e^{\rho r} h\big\|_{k,\alpha},$$

where r denotes the coordinate on the $\mathbb{R}$-factor.

We then define $C^{k,\alpha}_{\rho}(E)$ to be the completion of $C^{\infty}_{c}(E)$ with respect to the norm $\|\cdot\|_{k,\alpha;\rho}$. Observe that $C^{k,\alpha}_{0}(E)$ coincides with the classical Hölder space $C^{k,\alpha}(E)$.

2.1. **Formal solutions.** For convenience, instead of dealing with vector fields, we use 1-forms. Any 1-form ω on a cylinder admits a decomposition in radial and tangential components as follows

$$\omega = \eta(r,x) + \kappa(r,x)dr,$$

where $\kappa(r,x)$ is a function and η is a 1-form on $\overline{N}$ that does not contain dr term. In the following we will use primes (e.g. η', κ') to denote the r-derivatives. Direct computation yields

$$dd^*\omega = d_N d_N^*\eta + d_N^*\eta' dr - \kappa'' dr - d_N\kappa'. \tag{2.3}$$

$$d^*d\omega = -\eta'' - d_N^*\eta' dr + d_N^* d_N\eta + d_N\kappa' + d_N^* d_N\kappa dr. \tag{2.4}$$

Using the L^2 orthogonal decomposition into eigenfunctions or eigen 1-forms on the cross section N, we know that any 1-form on $\overline{N}$ can be written as an infinite sum of 1-forms of the following two types and that the operator $d^*d + 2dd^*$ preserves the types:

$$f(r)\phi(x), \tag{2.5}$$

$$k(r)d_N\phi(x) + \ell(r)\phi(x)dr. \tag{2.6}$$

Here, ϕ is a coclosed eigen 1-form of the Hodge Laplacian on N in (2.5) and an eigenfunction of the (Hodge) Laplacian on N in (2.6). Equivalently, in either case ϕ satisfies

$$d_N^*\phi = 0, \quad d_N^* d_N\phi = \mu\phi, \tag{2.7}$$

for some eigenvalue $\mu \geq 0$. With the help of this decomposition we can reduce the equation (2.2) to second order linear ODEs of r, simply by comparing coefficients of similar terms. Then δh will play the role of the inhomogeneous term.

Our strategy to solve (2.2) is to divide δh into two parts and solve them separately; this is possible because the operator $d^*d + 2dd^*$ is linear. The first part corresponds to eigenspaces of $\mu = 0$ in (2.8) and (2.9). It would be a finite sum of type (2.5) and (2.6), because by Hodge theorem on N there are only finitely many harmonic 1-forms. This part can be solved directly by variation of parameters. The second part is a possibly infinite sum of forms corresponding to $\mu > 0$. We will see that it can be inverted by Green functions.

It is enlightening to first consider the homogeneous equation, namely equation (2.2) with $\delta h = 0$. This gives rise to the building blocks of the Green's function. We treat types (2.5) and (2.6) separately. It follows from (2.3) and (2.4) that for type (2.5) we have

$$f''(r) - \mu f(r) = 0, \tag{2.8}$$

and for type (2.6) we have

$$\left(d^*d + 2dd^*\right)\omega = \left(-k'' + 2\mu k - \ell'\right) d_N\phi + \left(-2\ell'' + \mu k' + \mu\ell\right)\phi dr,$$

i.e.,

$$\frac{d}{dr}\begin{bmatrix} k \\ k' \\ \ell \\ \ell' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2\mu & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\mu}{2} & \frac{\mu}{2} & 0 \end{bmatrix}\begin{bmatrix} k \\ k' \\ \ell \\ \ell' \end{bmatrix}. \tag{2.9}$$

Two corresponding fundamental matrix of (2.8) are given by

$$\Psi_0(r) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}, \tag{2.10}$$

$$\Psi_\mu(r) = \begin{bmatrix} e^{\sqrt{\mu}r} & e^{-\sqrt{\mu}r} \\ \sqrt{\mu}e^{\sqrt{\mu}r} & -\sqrt{\mu}e^{-\sqrt{\mu}r} \end{bmatrix}. \tag{2.11}$$

Therefore, for the nonhomogeneous equation

$$f''(r) - \mu f(r) = \alpha_\mu(r),$$

the solution is given by

$$f(r) = \int_0^r (r-s)\alpha_0(s)ds, \quad r > 0 \qquad \mu = 0, \tag{2.12}$$

$$f(r) = -\frac{1}{2\sqrt{\mu}}\int_0^\infty e^{-\sqrt{\mu}|r-s|}\alpha_\mu(s)ds, \quad r > 0 \qquad \mu > 0, \tag{2.13}$$

where $\alpha_\mu(r)$ are coefficients from the decomposition of δh with respect to (2.5).

The characteristic equation of the coefficient matrix in (2.9) is:

$$(\lambda^2 - \mu)^2 = 0.$$

When $\mu = 0$, it has root 0 with algebraic multiplicity 4 and geometric multiplicity 2. When $\mu > 0$, it has roots $-\sqrt{\mu}$ and $\sqrt{\mu}$, each with algebraic multiplicity 2 and geometric multiplicity 1. It indicates that generalized eigenvectors are required.

For $\mu = 0$, general solutions are

$$k(r) = -\frac{A}{2}r^2 + Cr + D, \quad \ell(r) = Ar + B,$$

i.e., the fundamental matrix and its inverse are given by

$$\Phi_0(r) = \begin{bmatrix} 1 & r & 0 & -\frac{r^2}{2} \\ 0 & 1 & 0 & -r \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Phi_0^{-1}(r) = \begin{bmatrix} 1 & -r & 0 & -\frac{r^2}{2} \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & -r \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.14}$$

For $\mu > 0$, we define

$$V = \begin{bmatrix} 1 & 0 & -1 & 0 \\ \sqrt{\mu} & 1 & \sqrt{\mu} & -1 \\ \sqrt{\mu} & -3 & \sqrt{\mu} & 3 \\ \mu & -2\sqrt{\mu} & -\mu & -2\sqrt{\mu} \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{8\sqrt{\mu}} & \frac{1}{8\sqrt{\mu}} & 0 \\ \frac{\sqrt{\mu}}{4} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{4\mu} \\ -\frac{1}{2} & \frac{3}{8\sqrt{\mu}} & \frac{1}{8\sqrt{\mu}} & 0 \\ \frac{\sqrt{\mu}}{4} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{4\mu} \end{bmatrix}.$$

Then, the corresponding fundamental matrix is given by:

$$\Psi_\mu(r) = V \cdot \begin{bmatrix} e^{\sqrt{\mu}r} & re^{\sqrt{\mu}r} & 0 & 0 \\ 0 & e^{\sqrt{\mu}r} & 0 & 0 \\ 0 & 0 & e^{-\sqrt{\mu}r} & re^{-\sqrt{\mu}r} \\ 0 & 0 & 0 & e^{-\sqrt{\mu}r} \end{bmatrix} \tag{2.15}$$

and its inverse is

$$\Psi_\mu^{-1}(r) = \begin{bmatrix} e^{-\sqrt{\mu}r} & -re^{-\sqrt{\mu}r} & 0 & 0 \\ 0 & e^{-\sqrt{\mu}r} & 0 & 0 \\ 0 & 0 & e^{\sqrt{\mu}r} & -re^{\sqrt{\mu}r} \\ 0 & 0 & 0 & e^{\sqrt{\mu}r} \end{bmatrix} \cdot V^{-1}. \tag{2.16}$$

Solutions corresponding to (2.9) are of two types

$$\begin{aligned} \eta_1 &= e^{\sqrt{\mu}r} d_N\phi(x) + \sqrt{\mu} e^{\sqrt{\mu}r}\phi(x)dr, \\ \eta_2 &= -e^{-\sqrt{\mu}r} d_N\phi(x) + \sqrt{\mu} e^{-\sqrt{\mu}r}\phi(x)dr \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \eta_3 &= re^{\sqrt{\mu}r} d_N\phi(x) + (r\sqrt{\mu} - 3)e^{\sqrt{\mu}r}\phi(x)dr, \\ \eta_4 &= -re^{-\sqrt{\mu}r} d_N\phi(x) + (r\sqrt{\mu} + 3)e^{-\sqrt{\mu}r}\phi(x)dr. \end{aligned} \tag{2.18}$$

Consequently, for the nonhomogeneous equation

$$\frac{d}{dr}\begin{bmatrix} k \\ k' \\ \ell \\ \ell' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2\mu & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\mu}{2} & \frac{\mu}{2} & 0 \end{bmatrix}\begin{bmatrix} k \\ k' \\ \ell \\ \ell' \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_\mu(r) \\ 0 \\ \gamma_\mu(r) \end{bmatrix},$$

the solution is given by

$$\begin{bmatrix} k(r) \\ k'(r) \\ \ell(r) \\ \ell'(r) \end{bmatrix} = \Psi_\mu(r)\int^r \Psi_\mu^{-1}(s) \cdot \begin{bmatrix} 0 \\ \beta_\mu(s) \\ 0 \\ \gamma_\mu(s) \end{bmatrix} ds, \tag{2.19}$$

where $\beta_\mu(r)$ and $\gamma_\mu(r)$ are coefficients from the decomposition of δh with respect to (2.6). For convenience, we set

$$\int^r = \int_0^r \quad \text{if } \mu = 0, \quad \text{and} \quad \int^r = \begin{cases} \int_0^r & \text{for} \sqrt{\mu} \\ -\int_r^\infty & \text{for} -\sqrt{\mu}. \end{cases} \quad \text{if } \mu > 0.$$

Direct calculation yields

(2.20)

$$\Psi_\mu^{-1}(s)\cdot\begin{bmatrix}0\\ \beta_\mu(s)\\ 0\\ \gamma_\mu(s)\end{bmatrix}=\begin{bmatrix}e^{-\sqrt{\mu}s} & -se^{-\sqrt{\mu}s} & 0 & 0\\ 0 & e^{-\sqrt{\mu}s} & 0 & 0\\ 0 & 0 & e^{\sqrt{\mu}s} & -se^{\sqrt{\mu}s}\\ 0 & 0 & 0 & e^{\sqrt{\mu}s}\end{bmatrix}\cdot\begin{bmatrix}\frac{3}{8\sqrt{\mu}}\beta_\mu\\ \frac{1}{8}\beta_\mu-\dfrac{1}{4\sqrt{\mu}}\gamma_\mu\\ \frac{3}{8\sqrt{\mu}}\beta_\mu\\ -\frac{1}{8}\beta_\mu-\dfrac{1}{4\sqrt{\mu}}\gamma_\mu\end{bmatrix}$$

$$=\begin{bmatrix}\left(\left(\frac{3}{8\sqrt{\mu}}-\frac{s}{8}\right)\beta_\mu+\frac{s}{4\sqrt{\mu}}\gamma_\mu\right)e^{-\sqrt{\mu}s}\\ \left(\frac{1}{8}\beta_\mu-\frac{1}{4\sqrt{\mu}}\gamma_\mu\right)e^{-\sqrt{\mu}s}\\ \left(\left(\frac{3}{8\sqrt{\mu}}+\frac{s}{8}\right)\beta_\mu+\frac{s}{4\sqrt{\mu}}\gamma_\mu\right)e^{\sqrt{\mu}s}\\ \left(-\frac{1}{8}\beta_\mu-\frac{1}{4\sqrt{\mu}}\gamma_\mu\right)e^{-\sqrt{\mu}s}\end{bmatrix}. \tag{2.21}$$

Lemma 2.2. *Assume that δh is a finite combination of forms as in (2.5) with $\mu\neq 0$. Then $\delta h\in C^{k,\alpha}(T^*\overline{N})$ implies that $X^\sharp$ the solution of (2.2) belongs to $C^{k+2,\alpha}(T^*\overline{N})$. If δh be a finite combination of forms as in (2.6) with $\mu\neq 0$. Then $\delta h\in C^{k,\alpha}(T^*\overline{N})$ implies that $X^\sharp$ the solution of (2.2) satisfies that $X^\sharp$ belongs to $C^{k+2,\alpha}(T^*\overline{N})$.*

Proof. The $C^{k+2,\alpha}$ regularity of solutions follows from standard theory on ODE systems (2.8) and (2.9). For ODE system (2.8), by (2.13), we have

$$\begin{aligned}|f(r)|\leq&\sum_i\|\alpha_{\mu_i}\|_{k,\alpha}\left(e^{-\sqrt{\mu_i}r}\int_0^r e^{\sqrt{\mu_i}s}ds+e^{\sqrt{\mu_i}r}\int_r^\infty e^{-\sqrt{\mu_i}s}ds\right)\\ \leq&C\sum_i\|\alpha_{\mu_i}\|_{k,\alpha}=C\|\delta h\|_{k,\alpha}.\end{aligned}$$

Repeating this argument yields $\|f\|_{k,\alpha}\leq C\|\alpha_\mu\|_{k,\alpha}$.

For ODE system (2.9), by (2.20), we have

$$\begin{aligned}-\int_r^\infty(-\frac{s}{8})\beta_\mu e^{-\sqrt{\mu}s}ds=&\frac{1}{8}\left(-s\int_s^\infty\beta_\mu e^{-\sqrt{\mu}\tau}d\tau\right)\Big|_r^\infty\\ &+\frac{1}{8}\int_r^\infty\int_s^\infty\beta_\mu e^{-\sqrt{\mu}\tau}d\tau ds.\end{aligned}$$

Besides, we have

$$\lim_{s\to\infty}s\int_s^\infty\beta_\mu e^{-\sqrt{\mu}\tau}d\tau=\lim_{s\to\infty}\frac{\int_s^\infty\beta_\mu e^{-\sqrt{\mu}\tau}d\tau}{s^{-1}}=\lim_{s\to\infty}\frac{\beta_\mu e^{-\sqrt{\mu}s}}{s^{-2}}=0.$$

Therefore, we have

$$-\int_r^\infty(-\frac{s}{8})\beta_\mu e^{-\sqrt{\mu}s}ds=\frac{1}{8}r\int_r^\infty\beta_\mu e^{-\sqrt{\mu}\tau}d\tau+\frac{1}{8}\int_r^\infty\int_s^\infty\beta_\mu e^{-\sqrt{\mu}\tau}d\tau ds.$$

Combining with (2.19), we see that

$$e^{\sqrt{\mu}r}\cdot\frac{1}{8}r\int_r^\infty \beta_\mu e^{-\sqrt{\mu}\tau}d\tau \tag{2.22}$$

will be canceled by

$$re^{\sqrt{\mu}r}\left(-\frac{1}{8}\int_r^\infty \beta_\mu e^{-\sqrt{\mu}s}ds\right)$$

while

(2.23)

$$\left|e^{\sqrt{\mu}r}\frac{1}{8}\int_r^\infty\int_s^\infty \beta_\mu e^{-\sqrt{\mu}\tau}d\tau ds\right| \le \|\beta_\mu\|_{k,\alpha}e^{\sqrt{\mu}r}\frac{1}{8}\int_r^\infty\int_s^\infty e^{-\sqrt{\mu}\tau}d\tau ds$$

can be handled similar to the above. Hence, we conclude that even though there are generalized solutions (i.e $re^{\pm\sqrt{\mu}r}$ terms), it will be canceled by the integration by parts. Repeating this argument yields the desired regularity $C^{k,\alpha}$ for system (2.9).

□

For (2.5), condition in Lemma 2.2 is violated precisely when $\mu = 0$ there. Observe that right now ϕ is a harmonic 1-form on N. The corresponding general solutions are spanned by

$$\phi, r\phi. \tag{2.24}$$

For (2.6), condition in Lemma 2.2 is violated precisely when $\mu = 0$ as well. While now ϕ is a harmonic function N, hence a constant. Consequently $k(r)$ will not make an importance. The corresponding general solutions are spanned by

$$rdr, dr. \tag{2.25}$$

Remark 2.3. Recall that $L_X g_0 = \nabla^{\mathrm{sym}}X^\sharp$. So among 1-forms in (2.24, 2.25), dr and ϕ represent the translation invariance of a cylinder and are dual to Killing vector fields. Furthermore, they have no contribution to the equation (2.1). However, direct calculation yields

$$\nabla^{\mathrm{sym}}(rdr) = dr\otimes dr, \quad \nabla^{\mathrm{sym}}(r\phi) = (dr\otimes\phi)^{\mathrm{sym}} = dr\boxtimes\phi.$$

It implies that rdr and $r\phi$ are not dual to Killing vector fields but they provide radially parallel 2-tensors. Hence, we introduce a first order perturbation to break the nontrivial translation invariance of $dr\otimes dr$ and $dr\boxtimes\phi$ generated by rdr and $r\phi$ respectively: fix a parameter $\tau > 0$, we define

$$\delta_\tau := \begin{cases} \delta - \tau\iota_{\partial_r} & \text{on } E_0\ , \\ \delta & \text{otherwise}, \end{cases} \tag{2.26}$$

where E_0 is the span of $\{dr\otimes dr, dr\boxtimes\phi\}$ over $C^\infty(\mathbb{R})$ with harmonic 1-forms ϕ on N.

Here ι is the restriction on the first component of a tensor. It is immediate to verify that if a vector field X has a decomposition as $X^\sharp = \eta(r,x) + \kappa(x,r)dr$, then

$$\iota_{\partial_r} L_X g_0 = \eta' + 2\kappa' dr + d_N \kappa. \tag{2.27}$$

We can check directly that when $\tau > 0$, (the dual of) rdr, $r\phi$ are no longer solutions to the perturbed equation $\delta_\tau(L_X g_0) = 0$. In conclusion, eigenvalues in (2.5) and (2.6) violating condition in Lemma 2.2 has trivial contribution to the perturbed equation $\delta_\tau(L_X g_0) = \delta h$.

2.2. **Construction of the Green functions.** In this subsection we construct the Green functions of the operator $d^*d + 2dd^*$ in two cases, and use Green functions to study the regularity and decay of the solution to (2.1), which is the content of Proposition 2.5. Recall that the second part is a possibly infinite sum of forms corresponding to $\mu > 0$. Based on (2.11), we have

Proposition 2.4. *the Green function restricted on* 1*-forms as in* (2.5) *is*

$$G((t,x),(s,y)) := \sum_{i=1}^{\infty} g_i(t,s)\phi_i(x) \otimes \phi_i(y), \tag{2.28}$$

where

$$g_i(t,s) := \frac{1}{2\sqrt{\mu_i}} \begin{cases} e^{-\sqrt{\mu_i}(t-s)}, & t > s \\ e^{\sqrt{\mu_i}(t-s)}, & t < s \end{cases}.$$

Proof. Let $f(s)\phi(y)$ be an 1-form as in (2.5). We have to verify

$$(d^*d + 2dd^*)_{(t,x)} \left(\int_0^{+\infty} \int_N \langle G((t,x),(s,y)), f(s)\phi(y)\rangle ds \, \mathrm{dvol}(y) \right) = f(t)\phi(x). \tag{2.29}$$

Direct calculation yields that

$$\begin{aligned} &2\sqrt{\mu_i} \int_0^{+\infty} \int_N \langle G((t,x),(s,y)), f(s)\phi(y)\rangle ds \, \mathrm{dvol}(y) \\ &= \left(e^{-\sqrt{\mu_i}t} \left(\int_0^t e^{\sqrt{\mu_i}s} f(s) ds \right) + e^{\sqrt{\mu_i}t} \left(\int_t^{+\infty} e^{-\sqrt{\mu_i}s} f(s) ds \right) \right) \phi(x). \end{aligned}$$

Here we used $\int_N |\phi(y)|^2 \, \mathrm{dvol}(y) = 1$. $\square$

We now discuss the construction of the Green function for (2.9). Recall in (2.17) and (2.18) that η_2 and η_4 tends to zero as $r \to +\infty$; η_1 and η_3 tends to zero as $r \to -\infty$. Hence, we assume that

$$G(t,x;s,y) := \begin{cases} A(s,y) \otimes \eta_2(t,x) + B(s,y) \otimes \eta_4(t,x), & t > s \\ C(s,y) \otimes \eta_1(t,x) + D(s,y) \otimes \eta_3(t,x), & t < s. \end{cases} \tag{2.30}$$

Let $\omega(s,y)$ be an 1-forms as in (2.6). We have to verify

$$(d^*d + 2dd^*)_{(t,x)} \left(\int_0^{+\infty} \int_N \langle G(t,x;s,y), \omega(s,y)\rangle ds \, \mathrm{dvol}(y) \right) = \omega(t,x).$$

Comparing with (2.19), we have

$$
\begin{aligned}
A(s,y) &= \left(-\frac{\sqrt{\mu}s+3}{8\mu\sqrt{\mu}}e^{\sqrt{\mu}s}\right)d_N\phi(y) + \left(-\frac{s}{4\sqrt{\mu}}e^{\sqrt{\mu}s}\right)\phi(y)dr,\\
B(s,y) &= \frac{e^{\sqrt{\mu}s}}{8\mu}d_N\phi(y) + \frac{e^{\sqrt{\mu}s}}{4\sqrt{\mu}}\phi(y)dr,\\
C(s,y) &= \frac{3-\sqrt{\mu}s}{8\mu\sqrt{\mu}}e^{-\sqrt{\mu}s}d_N\phi(y) + \frac{s}{4\sqrt{\mu}}e^{-\sqrt{\mu}s}\phi(y)dr,\\
D(s,y) &= \frac{e^{-\sqrt{\mu}s}}{8\mu}d_N\phi(y) - \frac{e^{-\sqrt{\mu}s}}{4\sqrt{\mu}}\phi(y)dr.
\end{aligned}
$$

Here we used

$$
\int_N |d_N\phi|^2\,\mathrm{dvol} = \mu, \qquad \int_N |\phi dr|^2\,\mathrm{dvol} = 1.
$$

Consequently, when $t > s$,

(2.31)

$$
\begin{aligned}
A(s,y)&\otimes\eta_2(t,x) + B(s,y)\otimes\eta_4(t,x)\\
&= \left(-\frac{t-s}{8\mu} + \frac{3}{8\mu\sqrt{\mu}}\right)e^{-\sqrt{\mu}(t-s)}d_N\phi(x)\otimes d_N\phi(y)\\
&+ \frac{t-s}{8\sqrt{\mu}}e^{-\sqrt{\mu}(t-s)}\phi(x)dr\otimes d_N\phi(y) + \frac{s-t}{4\sqrt{\mu}}e^{-\sqrt{\mu}(t-s)}\phi(y)dr\otimes d_N\phi(x)\\
&+ \left(\frac{t-s}{4} + \frac{3}{4\sqrt{\mu}}\right)e^{-\sqrt{\mu}(t-s)}\phi(x)dr\otimes\phi(y)dr.
\end{aligned}
$$

and when $t < s$,

(2.32)

$$
\begin{aligned}
C(s,y)&\otimes\eta_1(t,x) + D(s,y)\otimes\eta_3(t,x)\\
&= \left(-\frac{s-t}{8\mu} + \frac{3}{8\mu\sqrt{\mu}}\right)e^{-\sqrt{\mu}(s-t)}d_N\phi(x)\otimes d_N\phi(y)\\
&+ \frac{t-s}{8\sqrt{\mu}}e^{-\sqrt{\mu}(s-t)}\phi(x)dr\otimes d_N\phi(y) + \frac{s-t}{4\sqrt{\mu}}e^{-\sqrt{\mu}(s-t)}\phi(y)dr\otimes d_N\phi(x)\\
&+ \left(\frac{s-t}{4} + \frac{3}{4\sqrt{\mu}}\right)e^{-\sqrt{\mu}(s-t)}\phi(x)dr\otimes\phi(y)dr.
\end{aligned}
$$

Proposition 2.5. *Fix $k \in \mathbb{N}$, $\alpha \in (0,1)$ and a weight $\rho \in \mathbb{R}$ with $|\rho| < \sqrt{\mu_1}$. Assume that $\delta h \in C^{k,\alpha}_\rho(T^*\overline{N})$ admits an infinite expansion into eigen 1-forms as in (2.5) or (2.6) with $\mu \neq 0$ and that we have projected away the finite $\mu = 0$ sector. Then there exists a unique solution*

$$
X^\sharp \in C^{k+2,\alpha}_\rho(T^*\overline{N})
$$

to (2.2) *with the estimate*

$$\|X^\sharp\|_{k+2,\alpha;\rho} \;\le\; \frac{C}{(\sqrt{\mu_1}-|\rho|)^p}\,\|\delta h\|_{k,\alpha;\rho}, \tag{2.33}$$

where $C = C(k,\alpha,N,g_N)$ *is independent of* δh *and* $p \in \{1,2\}$*. More precisely,* $p = 1$ *for the type* (2.5) *and* $p \le 2$ *for the type* (2.6)*. In particular, the same regularity as in Lemma 2.2 holds in the case of an infinite expansion.*

Proof. Recall that the Green's function of (2.8) is given by (2.13), while the Green's function of (2.9) is given by (2.19). Moreover, by (2.31, 2.32), we see that (2.19) can be expressed as a finite linear combination of terms of the form

$$P(|t-s|)\,e^{-\sqrt{\mu}\,|t-s|}\cdot \mathbb{T}_\mu,$$

where P is a polynomial of degree at most one and $\mathbb{T}_\mu$ is a differential operator on the cross section N of order at most one.

Next, we consider conjugated Green's functions:

$$e^{\rho t} g_\mu(t,s) e^{-\rho s} = \frac{1}{2\sqrt{\mu}}\, e^{-(\sqrt{\mu}-|\rho|)\,|t-s|},$$

and similarly

$$e^{\rho t} P(|t-s|) e^{-\sqrt{\mu}|t-s|} e^{-\rho s} = P(|t-s|)\, e^{-(\sqrt{\mu}-|\rho|)\,|t-s|}.$$

Note that for $a > 0$ one has

$$\int_{\mathbb{R}} e^{-a|t|}\,dt = \frac{2}{a}, \qquad \int_{\mathbb{R}} (1+|t|) e^{-a|t|}\,dt \le \frac{C}{a^2}.$$

Taking $a = \sqrt{\mu} - |\rho| \ge \sqrt{\mu_1} - |\rho| > 0$ yields the uniform integrability bounds

$$\int_{\mathbb{R}} e^{-(\sqrt{\mu}-|\rho|)\,|t-s|}\,ds \le \frac{2}{\sqrt{\mu_1}-|\rho|},$$
$$\int_{\mathbb{R}} (1+|t-s|) e^{-(\sqrt{\mu}-|\rho|)\,|t-s|}\,ds \le \frac{C}{(\sqrt{\mu_1}-|\rho|)^2}.$$

For type (2.5), the operator-valued kernel is precisely the spectral-calculus kernel

$$\mathbb{G}_1(|t-s|) = \tfrac{1}{2}\,(-\Delta_{1,N})^{-1/2} \circ \mathbb{P}_{-|t-s|},$$

acting on co-closed 1-forms on N. Here $-\Delta_{1,N}$ is the Hodge Laplacian acting on 1-forms and $\mathbb{P}_\tau$ is the Poisson semigroup:

$$\mathbb{P}_\tau := e^{-\tau\,(-\Delta_{1,N})^{1/2}}.$$

On the compact manifold N, standard pseudodifferential estimates [Gru96] imply

$$\big\|\mathbb{G}_1(\tau)\big\|_{C^{k,\alpha}(N)\to C^{k+1,\alpha}(N)} \le C \quad (\tau > 0),$$

and differentiating once in τ gains another derivative in x, giving the mapping $C^{k,\alpha} \to C^{k+2,\alpha}$ with a constant independent of the eigen-index.

For type (2.6), each entry of the Green's function is, up to fixed zeroth/first–order operators in tangential direction, a finite linear combination of

$$(1+\tau)e^{-\sqrt{\mu}\,\tau} \;\cdot\; \Big[\,A_0^{-1/2}P_\tau,\; P_\tau,\; d_N A_0^{-1/2}P_\tau,\; d_N^* A_1^{-1/2}P_\tau\Big], \qquad \tau = |t-s|.$$

Here $A_0 = -\Delta_{0,N}$ on functions and $A_1 = -\Delta_{1,N}$ on 1–forms, while $d_N A_0^{-1/2}$ and $d_N^* A_1^{-1/2}$ are zeroth–order uniformly bounded pseudodifferential operators on $C^{k,\alpha}(N)$. Thus the tangential part still maps $C^{k,\alpha} \to C^{k+1,\alpha}$ uniformly in τ, exactly as in type (2.5). The extra factor $(1+\tau)$ only appears in the r–kernel and is handled during the integration; it does not affect tangential regularity.

Put together and apply Young's inequality in t to the convolution with respect to the integrable kernels, along with the Schauder bounds on N. This yields

$$\|X^\sharp\|_{k+2,\alpha;\rho} \;\le\; \frac{C}{(\sqrt{\mu_1}-|\rho|)^p}\;\|\delta h\|_{k,\alpha;\rho},$$

as claimed in (2.33). Uniqueness on the $\mu > 0$ sector follows from the same estimates applied to homogeneous data.

□

We define

$$C^{k,\alpha}_{0^-}(E) := \bigcap_{\rho<0} C^{k,\alpha}_\rho(E),$$

the space of sections with subexponential growth as $r \to +\infty$. Let $\mathcal{V}_1 \subseteq C^{k,\alpha}_0(T^*\overline{N})$ be the subspace spanned by the $\mu = 0$ modes in (2.5), and let $\mathcal{V}_2 \subseteq C^{k,\alpha}_0(T^*\overline{N})$ be the direct sum of the $\mu > 0$ modes appearing in (2.5) and (2.6). Denote by $\mathcal{W}_1 \subseteq C^{k+2,\alpha}_{0^-}(T\overline{N})$ and $\mathcal{W}_2 \subseteq C^{k+2,\alpha}_0(T\overline{N})$ the corresponding spaces of vector fields.

Theorem 2.6. *Fix $\tau \neq 0$. There exists a bounded linear operator*

$$\big(\delta_\tau L_\bullet g_0\big)^{-1} :\; \mathcal{V}_1 \oplus \mathcal{V}_2 \longrightarrow \mathcal{W}_1 \oplus \mathcal{W}_2$$

such that

$$\big(\delta_\tau L_\bullet g_0\big) \circ \big(\delta_\tau L_\bullet g_0\big)^{-1} = \mathrm{Id} \quad \textit{on } \mathcal{V}_1 \oplus \mathcal{V}_2.$$

In particular, $\big(\delta_\tau L_\bullet g_0\big)^{-1}$ is a bounded right inverse mapping $\mathcal{V}_1$ into $\mathcal{W}_1$ and $\mathcal{V}_2$ into $\mathcal{W}_2$.

2.3. **Construction of gauge on cylinder.** In this subsection we construct a δ_τ-free gauge on the cylinder $\overline{N}$ by taking the flow of the vector field we solved from the divergence equation (2.1). The key is to show that although the vector field may lose the needed exponential decay, the flow still have enough regularity. The precise statement is Theorem 2.8.

Lemma 2.7. *If $X \in C^{k,\alpha}(T\overline{N})$ is defined on a tube $\bar{T}_{R,\infty} \subseteq \overline{N}$ and is tangent to the boundary, then the generated diffeomorphism $\Phi_X(t,\cdot)$ satisfies*

$$\Phi_X(t,\cdot) - \mathrm{Id} : C^{k,\alpha}_{\rho}(S^2T^*\overline{N}) \to C^{k,\alpha}_{\rho}(S^2T^*\overline{N}), \quad \rho < \sqrt{\mu_1}.$$

Proof. The proof follows from the definition

$$\frac{d}{dt}\Phi_X(t,\cdot) = X(\Phi_X(t,\cdot))$$

and integrating w.r.t time parameter t. $\square$

Theorem 2.8. *Let $(\overline{N},\bar{g})$ be a Ricci flat cylinder, and g be a Riemannian metric defined on $\bar{T}_{c,d}$, $d > c \geq 0$. For any $\tau > 0$, there exists a constant $\varepsilon := \varepsilon(\tau)$ such that if*

$$\|g - \bar{g}\|_{k,\alpha} \leq \varepsilon,$$

then there is a diffeomorphism $\Phi : \bar{T}_{c,d} \to \bar{T}_{c,d}$ such that

$$\Phi^* g \in C^{k,\alpha}(S^2T^*\overline{N})$$

and

$$\delta_\tau(\Phi^* g - \bar{g}) = 0.$$

Moreover, if

$$|\delta_\tau(g)|_{k-1,\alpha} < \varepsilon,$$

then

$$|\Phi^* g - g|_{k,\alpha} < \Psi(\varepsilon).$$

Proof. Let $Y = (Y_1, Y_2) \in \mathcal{W}_1 \oplus \mathcal{W}_2$. Thus, $Y_1 = f(r)Y_N$ with Y_N a Killing vector field for the metric g_0. We denote by Φ_Y the diffeomorphism $\Phi_Y := \Phi_{Y_1}(1, \Phi_{Y_2}(1,\cdot))$, where Φ_{Y_i} is the flow generated by Y_i, $i = 1,2$, i.e.

$$\frac{\partial \Phi_{Y_i}(t,\cdot)}{\partial t} = Y_i(\Phi_{Y_i}(t,\cdot)), \quad \Phi_{Y_i}(0,\cdot) = \mathrm{Id}\,.$$

By Lemma 2.7, we already know that $\Phi_{Y_2}(1,\cdot)$ preserves $C^{k,\alpha}(S^2T^*\overline{N})$. Hence, the main point is to check $\Phi_{Y_1}(1,\cdot)$ preserves $C^{k,\alpha}(S^2T^*\overline{N})$ as well.

Ever though function $f(r)$ may have subexponential growth, note that $\Phi_{Y_1}(t,\cdot) = \Phi_{Y_N}(tf(r),\cdot)$ and Φ_{Y_N} is an isometry on N. It is obvious that $\Phi_{Y_1}(1,\cdot)$ satisfies the desired property.

Based on above discussion, we define a well-defined map

$$B : (\mathcal{W}_1 \oplus \mathcal{W}_2) \times C^{k,\alpha}(S^2T^*\overline{N}) \to \mathcal{V}_1 \oplus \mathcal{V}_2$$

given by

$$B(Y_1, Y_2; g) := \delta_\tau(\Phi_Y^* g)$$

It is easy to show that B is differentiable at (Y_1, Y_2, g) and

$$D_{(Y_1,Y_2)}B(0,0;g_0) = \delta_\tau L_{Y_1+Y_2} g_0.$$

By Theorem (2.6) and the implicit function theorem, for $|g - g_0|_{k,\alpha}$ sufficiently small, there is a diffeomorphism Φ satisfying the properties stated in the theorem. $\square$

3. Construction of modified divergence free gauge

In this section we construct the modified divergence–free gauge on a potentially large Tube. The precise statement is corollary 3.4. We leverage the local $L^{n/2}$-curvature bound (R.1). The corresponding section in [CT94] rely on the work of Bando–Kasue–Nakajima [BKN89] to show the existence of tangent cones at infinity with better regularity then what can be derived by the general theory of nonnegative Ricci curvature. In the linear growth case, we develop a very similar structural result using the Busemann function of M and the almost splitting theorem.

We first show that all asymptotic limits of M are smooth, and hence topologically unique.

Theorem 3.1. *Let (M, g) be a noncollapsed Ricci flat manifold with (1.1) and (R.1). Let γ be a ray in M and b be the associated Busemann function. Then for given $0 << L$ and $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$, there exists $R_0(g, \varepsilon, L, k, \alpha)$ such that if $R > R_0$, there is an n-dimensional Ricci flat cylinder $(\overline{N} = \mathbb{R} \times N, \bar{g}_R = dr^2 + g_R)$ and a $C^{k,\alpha}$-approximation map*

$$\phi : (R, R + L) \times N \to b^{-1}((R, R + L)), \tag{3.1}$$

that is a diffeomorphism onto its image. More precisely ϕ satisfies:

$$|\phi^* g - \bar{g}_R|_{k,\alpha} < \varepsilon, \tag{3.2}$$

$$|b(\phi(r, \cdot)) - r|_0 < \varepsilon, \ \forall r \in (0, L), \tag{3.3}$$

$$|\Pi \circ \phi^{-1}(p) - b(p)|_0 < \varepsilon, \ \forall p \in b^{-1}(R, R + L). \tag{3.4}$$

Proof. Notice that it is sufficient to construct

$$\phi : (0, L) \times N \to b^{-1}((R, R + L)),$$

for each $R \geq R_0$ then pre-compose it with the translation $r \to r - R$, translating the interval $(R, R + L)$ to $(0, L)$. We first prove that any asymptotic limit is smooth. Let a sequence $R_i \to \infty$ be such that $(M, g, \gamma(R_i))$ GH-converges to $\mathbb{R} \times N$. By the ε regularity theorem of [And89, Theorem C], see also [CN15, Theorem 6.1], there are at most finitely many singular points in $N \times (-L, L)$ for any $L > 0$. But if there were at least one singular point then its translations along $\mathbb{R}$ factor would all be singular points, a contradiction to the finiteness of singular points. So $\mathbb{R} \times N$ must be smooth.

Then we show that there exists $R_0(g, \varepsilon, L, k, \alpha)$ so that for $R \geq R_0$, there is an n-dimensional Ricci flat cylinder $(\overline{N}_R = \mathbb{R} \times N_R, \bar{g}_R = dr^2 + g_R)$, where the topology of the base manifold may depend on R, and a $C^{k,\alpha}$-approximation map

$$\phi : (0, L) \times N_R \to b^{-1}((R, R + L)) \tag{3.5}$$

with the required properties. In our setting, all Busemann function level sets have uniformly bounded diameter, by the almost splitting theorem and [Zhu25, Proposition 3.8], $b^{-1}((R, R+L))$ is $\Psi(R^{-1}|L, n)$ GH-close to $(0, L)\times N$ with each level set being close to corresponding level set, for some Ricci flat manifold N. If R is large enough, there exists a Gromov–Hausdorff (GH in short) approximation $\phi : (0, L) \times N \to b^{-1}((R, R + L))$ that is also a diffeomorphism onto its image. We can construct this ϕ by gluing local splitting maps so that ϕ is close to the GH approximation that takes $\{r\}\times N$ to the neighborhood of $b^{-1}(R+r)$, as an application of the canonical Reifenberg theorem in [CJN21], see also for example Huang[Hua20]. The convergence of ϕ^*g can be upgraded to $C^{k,\alpha}$-convergence since M is Ricci flat, see Colding [Col97] and Anderson [And90]. In particular, (3.2) holds. To show (3.3) and (3.4), take $T >> R$, and let $b_T^+ = T - R - \mathsf{d}_g(\cdot, \gamma(T))$. We see that b_T^+ converges locally uniformly to $b - R$. Let β^+ be the harmonic replacement of b_T^+ in a ball containing $b^{-1}(R, R + L)$, say B_{L+D}, which is possible thanks to the uniform diameter bound of Busemann function level sets, see [Zhu25]. Then by the construction of GH approximation map we have that

$$|\beta^+(\phi(r, \cdot)) - r| \leq \Psi((T - R)^{-1}|L, n).$$

Note that $|\beta^+ - b_T^+| \leq \Psi((T - R)^{-1}|L, n)$ also holds, so we get C^0-estimate of $|b(\phi(r, \cdot)) - R - r|$. Now if we also take the translation $r \to r - R$ into consideration, this is exactly (3.3). The proof of (3.4) is almost the same. Indeed, by the construction of ϕ, the projection $\Pi \circ \phi$ is $C^{k,\alpha}$-close to β^+ and we have shown that β^+ is C^0-close to $b - R$.

Now suppose that such R_0 does not exist for some fixed ε and L, then there exists a sequence $R_i \to \infty$ so that no Ricci flat cylinder admit a $C^{k,\alpha}$-approximation map to $b^{-1}((R_i, R_i+L))$. However when $i \to \infty$, the sequence $b^{-1}((R_i, R_i + L))$ will converge to $(0, L) \times N$ for some Ricci flat N and a $C^{k,\alpha}$-approximation map can be constructed as described above for large i, a contradiction. In particular $b^{-1}(r)$ is diffeomorphic to N for $r \in (R, R+L)$. For $i \in \mathbb{N}$, applying this argument for $b^{-1}((R + \frac{2i}{3}L, R + L + \frac{2i}{3}L))$, we get that $b^{-1}((R + \frac{2i}{3}L, R + L + \frac{2i}{3}L))$ is diffeomorphic to $(0, L]) \times N_k$ for some N_k. Notice that the adjacent intervals have nontrivial intersection and these intervals covers (R_0, ∞), we infer that any N_R is diffeomorphic to one of N_k, and they are all diffeomorphic to each other. This completes the proof. $\square$

The remainder of this section closely follows [CT94, Section 1]. We will need the slice theorem and related facts on a closed manifold. Let (N, g) be a closed n-dimensional Riemannian manifold, and let δ_g denote the divergence with respect to g. For each $k \in \mathbb{N}_{\geq 1}$ and $\alpha \in (0, 1)$, there exist constants $0 < \tau_1 = \tau_1(g, k, \alpha) < \tau_2 = \tau_2(g, k, \alpha)$ and $c = c(g, k, \alpha) > 0$ with the following properties.

For each Riemannian metric g_1 on N with

$$|g_1 - g|_{k,\alpha;g} < \tau_1(g, k, \alpha),$$

there exists a $C^{k+1,\alpha}$-diffeomorphism $\eta_{g_1}: N \to N$ such that

$$\delta_g(\eta_{g_1}^* g_1) = 0.$$

The map $g_1 \mapsto \eta_{g_1}$ satisfies

$$\mathsf{d}_{k+1,\alpha;g}(\eta_{g_1}, \eta_{g_2}) \le c(k,\alpha,g)|g_1 - g_2|_{k,\alpha;g}.$$

and let Id be the identity map, we have

$$\mathsf{d}_{k+1,\alpha;g}(\eta_{g_1}, \mathrm{Id}) \le c(g,k,\alpha)|\delta_g(g_1)|_{k-1,\alpha;g}.$$

Here $\mathsf{d}_{k,\alpha;g}$ denotes the $C^{k,\alpha}$ norm with respect to g. Finally if there is a diffeomomorphism ζ such that $\delta_g(\zeta^* g_1) = 0$ and

$$|\zeta^* g_1 - g|_{k,\alpha;g} \le \tau_2(g,k,\alpha).$$

then there is an isometry ι of g such that $\iota^* \eta_{g_1}^* g_1 = \zeta^* g_1$, i.e. there is the uniqueness of η_{g_1} up to isometry when $\eta_{g_1}^* g_1$ is sufficiently close to g. For the fixed background metric g, the above τ_1, τ_2 can be chosen such that if $|g_1 - g|_{k,\alpha;g} < \tau_1$, then $|\eta_{g_1}^* g_1 - g|_{k,\alpha;g} \le \tau_2$, and that $\tau_1 < \frac{1}{8}\tau_2$. Meanwhile τ_2 can also be chosen small enough so that if g_1, g_2 satisfies

$$|g_j - g|_{k,\alpha;g} \le \frac{\tau_2}{8},\ |g_2 - g_1|_{k,\alpha;g_j} \le \frac{\tau_2}{8},\ j = 1,2,$$

then

$$\begin{aligned} |g_2 - g_1|_{k,\alpha;g} &\le C(1 + |g - g_j|_{k,\alpha;g_j})|g_2 - g_1|_{k,\alpha;g_j} \\ &\le 2|g_2 - g_1|_{k,\alpha;g_j},\ j = 1,2. \end{aligned}$$

The following lemma is exactly [CT94, Lemma 1.37]; its proof is essentially identical and is therefore omitted. We mention that the desired β_i is defined inductively as $\beta_0 = \mathrm{Id}$, $\beta_{i+1} = \beta_i \circ \eta_{\beta_i^* g_{i+1}}$ and that we have that for $i < N'$

$$|\beta_{i-1}^* \underline{g}_i - g_0| < \tau_1. \tag{3.6}$$

In fact we can choose the constant on the right hand side of (3.6) to be any constant smaller than τ_1.

Lemma 3.2. *Let g_i be a sequence of metrics on N, where $0 \le i < N \le \infty$, such that*

$$|g_{i+1} - g_i|_{k,\alpha;g_i} < \varepsilon < \frac{\tau_2}{8}, i+1 < N,$$

then there exists $N' \le N$ and for $0 \le i < N'$, diffeomorphisms, $\beta_i : N \to N$ such that $\beta_0 = \mathrm{Id}$,

$$\delta_{g_0}(\beta_i^* g_i) = 0$$

and

$$|\beta_i^* g_i - \beta_{i-1}^* g_{i-1}|_{k,\alpha;g_0} \le c(g_0,k,\alpha)\varepsilon.$$

Moreover if $N' < N \le \infty$,

$$|\beta_{N'-1}^* g_{N'-1} - g_0|_{k,\alpha,g_0} \ge \tau_1 - 2\varepsilon.$$

We are now able to state the main result in this section.

Proposition 3.3. *Let (M^n, g) be a complete Ricci flat n-manifold with linear volume growth* (1.1) *and local $L^{n/2}$-curvature bound* (R.1)*. Fix constants $k \in \mathbb{N}_0$, $\alpha \in (0,1)$, $0 < \varepsilon << \chi$ sufficiently small. Then there exists constants $R(k, \alpha, \varepsilon, \chi, g) > 0$ and $L' > 1$, a Ricci flat cylinder $(\overline{N} = \mathbb{R} \times N^{n-1}, \bar{g} = dr^2 + g_0)$, and a $C^{k,\alpha}$-approximation map*

$$\Phi : (R, R + L') \times N \to b^{-1}((R, R + L')), \tag{3.7}$$

which is a diffeomorphism onto its image, such that

$$|\Phi^* g - \bar{g}|_{k,\alpha;\bar{g}} < \chi, \tag{3.8}$$

$$\min_{T_{R,R+L'}} |\Phi^* g - \bar{g}|_{k,\alpha;\bar{g}} < \varepsilon, \tag{3.9}$$

$$|\nabla_{\partial_r}(\Phi^* g)|_{k-1,\alpha;\bar{g}} < \varepsilon, \tag{3.10}$$

$$|\delta_{\bar{g}}(\Phi^* g)|_{k-1,\alpha;\bar{g}} < \varepsilon. \tag{3.11}$$

Moreover, either $L' < \infty$ and for some $c_2(k, \alpha, g) > 0$,

$$|\phi^* g - g_0| > c_2 \chi \quad \text{on } (L' - 1, L') \times N^{n-1} \tag{3.12}$$

or $L' = \infty$.

Proof. Let $\varepsilon_1 \in (0, \varepsilon)$ to be chosen later and $L > 1$. Take $R_0(g, \varepsilon_1, L, k, \alpha)$ as in Theorem 3.1. For $i \in \mathbb{N}_0$ let $I_i = (R + \frac{2i}{3}L, R + L + \frac{2i}{3}L)$, we then have $I_i \cap I_j$ nonempty iff $j = i-1, i, i+1$ and, $I_{i-1} \cap I_i \cap I_{i+1} = \emptyset$. There exists Ricci flat cylinders $(\bar{N}_i, \bar{g}_i) = (\mathbb{R} \times N, dr^2 + g_i)$ and $C^{k,\alpha}$ approximation map $\phi_i : I_i \times (N, g_i) \to b^{-1}(I_i)$. Then on $I_i \cap I_{i+1}$ we have $\phi_{i+1}^{-1} \circ \phi_i : I_i \cap I_{i+1} \times N \to I_i \cap I_{i+1} \times N$ (we may need to shrink both intervals a little bit to ensure the map is well-defined), by triangle inequality we have

$$|(\phi_{i+1}^{-1} \circ \phi_i)^* \bar{g}_{i+1} - \bar{g}_i|_{k,\alpha;\bar{g}_i} \leq \Psi(\varepsilon_1)$$

By restricting the inequality to $dr \otimes dr$ direction, we see that $\Pi \circ (\phi_{i+1}^{-1} \circ \phi_i)$ is $C^{k,\alpha}$ close to an isometry on $I_i \cap I_{i+1}$ with Euclidean metric, which can only be the identity or a reflection. Moreover, we have that

$$\begin{aligned} \left|\Pi \circ \phi_{i+1}^{-1}(\phi_i(r, \cdot)) - r\right|_0 &\leq \left|\Pi \circ \phi_{i+1}^{-1}(\phi_i(r, \cdot)) - b(\phi_i(r, \cdot))\right|_0 + |b(\phi_i(r, \cdot)) - r|_0 \\ &\leq 2\varepsilon_1. \end{aligned}$$

We infer that $\Pi \circ (\phi_{i+1}^{-1} \circ \phi_i)$ can only be close to the identity. Then take a small perturbation of $\phi_{i+1}^{-1} \circ \phi_i$ restricted to any cross section, say $\{\frac{2}{3}L\} \times N$, we get a diffeomorphism $\lambda_{i+1} : N \to N$ such that

$$|\lambda_{i+1}^* g_{i+1} - g_i|_{k,\alpha;g_i} < \Psi(\varepsilon_1), \tag{3.13}$$

$$\mathsf{d}_{k,\alpha;\bar{g}_i}\left((\mathrm{Id}, \lambda_{i+1}^{-1}) \circ (\phi_{i+1}^{-1} \circ \phi_i), \mathrm{Id}\right) \leq \Psi(\varepsilon_1). \tag{3.14}$$

Let $\underline{g}_i := \lambda_1^* \cdots \lambda_i^* g_i$, then $\{\underline{g}_i\}_{i\in\mathbb{N}^+}$ satisfies Lemma 3.2. Take $\chi < \tau_1$, by induction there exists $N' \in \mathbb{N}_0 \cup \{\infty\}$ and for $i < N'$ there exists β_i, such that

$$\delta_{g_0}(\beta_i^* \underline{g}_i) = 0.$$

Set the stopping criterion in the proof of Lemma 3.2 as reaching the largest N' such that for $i < N'$

$$|\beta_{i-1}^* \underline{g}_i - g_0| < \chi. \tag{3.15}$$

Let $\varphi_i = \lambda_i \circ \cdots \lambda_1 \circ \beta_i$ and extend φ_i to $I_i \times N$ as (Id, φ_i), but to simplify the notation in the computation we abuse φ_i to also denote the extended map. We consider the map $\Phi_i := \phi_i \circ \varphi_i : I_i \times N \to b^{-1}(I_i)$. Notice that $\Phi_{i+1} = \phi_{i+1} \circ \lambda_{i+1} \circ \varphi_i \circ \eta_{\beta_i^* \underline{g}_{i+1}}$, together with (3.14) we expect that $\Phi_{i+1}^{-1} \circ \Phi_i$ to be close to the identity when restricted to any cross section $\{r\} \times N$, $r \in I_i \cap I_{i+1}$. Indeed, to simplify the notation, let $\widetilde{\lambda}_{i+1} = \phi_i^{-1} \circ \phi_{i+1} \circ \lambda_{i+1}$, we can compute, for each cross section that

$$\begin{aligned}
\mathsf{d}_{k,\alpha;\varphi_i^* g_i}\left(\Phi_{i+1}^{-1} \circ \Phi_i, \mathrm{Id}\right) &= \mathsf{d}_{k,\alpha;\varphi_i^* g_i}(\eta_{\beta_i^* \underline{g}_{i+1}}^{-1} \circ \varphi_i^{-1} \circ \widetilde{\lambda}_{i+1}^{-1} \circ \varphi_i, \mathrm{Id}) \\
&\leq \mathsf{d}_{k,\alpha;\varphi_i^* g_i}(\eta_{\beta_i^* \underline{g}_{i+1}}^{-1} \circ \varphi_i^{-1} \circ \widetilde{\lambda}_{i+1}^{-1} \circ \varphi_i, \eta_{\beta_i^* \underline{g}_{i+1}}^{-1}) + \mathsf{d}_{k,\alpha;\varphi_i^* g_i}(\eta_{\beta_i^* \underline{g}_{i+1}}^{-1}, \mathrm{Id}).
\end{aligned}$$

Now we estimates the two terms on the right hand side separately. The bound for the second term is straightforward.

$$\begin{aligned}
\mathsf{d}_{k,\alpha;\varphi_i^* g_i}(\eta_{\beta_i^* \underline{g}_{i+1}}^{-1}, \mathrm{Id}) &\leq C(1 + |\varphi_i^* g_i - g_0|_{k,\alpha;g_0}) \mathsf{d}_{k,\alpha;g_0}(\eta_{\beta_i^* \underline{g}_{i+1}}^{-1}, \mathrm{Id}) \\
&\leq C \mathsf{d}_{k+1,\alpha;g_0}(\eta_{\beta_i^* \underline{g}_{i+1}}, \mathrm{Id}) \\
&\leq C |\delta_{g_0}(\beta_i^* \underline{g}_{i+1} - \beta_i^* \underline{g}_i)|_{k-1,\alpha;g_0} \\
&\leq C |\beta_i^* \underline{g}_{i+1} - \beta_i^* \underline{g}_i|_{k,\alpha;g_0} \leq \Psi(\varepsilon_1).
\end{aligned}$$

For the first term, we will use the previous estimates.

$$\begin{aligned}
&\mathsf{d}_{k,\alpha;\varphi_i^* g_i}(\eta_{\beta_i^* \underline{g}_{i+1}}^{-1} \circ \varphi_i^{-1} \circ \widetilde{\lambda}_{i+1}^{-1} \circ \varphi_i, \eta_{\beta_i^* \underline{g}_{i+1}}^{-1}) \\
&\quad \leq C(1 + |\varphi_i^* g_i - g_0|_{k,\alpha;g_0}) \mathsf{d}_{k,\alpha;g_0}(\eta_{\beta_i^* \underline{g}_{i+1}}^{-1} \circ \varphi_i^{-1} \circ \widetilde{\lambda}_{i+1}^{-1} \circ \varphi_i, \eta_{\beta_i^* \underline{g}_{i+1}}^{-1}) \\
&\quad \leq C(1 + |\varphi_i^* g_i - g_0|_{k,\alpha;g_0})^2 (1 + \mathsf{d}_{k+1,\alpha;g_0}(\eta_{\beta_i^* \underline{g}_{i+1}}^{-1}, \mathrm{Id})) \\
&\qquad \times \mathsf{d}_{k,\alpha;\varphi_i^* g_i}(\varphi_i^{-1} \circ \widetilde{\lambda}_{i+1}^{-1} \circ \varphi_i, \varphi_i^{-1} \circ \varphi_i) \\
&\quad \leq C(1 + |\varphi_i^* g_i - g_0|_{k,\alpha;g_0})^2 (1 + \Psi(\varepsilon_1)) \mathsf{d}_{k,\alpha;\overline{g}_i}(\widetilde{\lambda}_{i+1}^{-1} \circ \varphi_i, \varphi_i) \leq \Psi(\varepsilon_1).
\end{aligned}$$

In the last inequality we used (3.14). Putting together, we deduce that

$$\mathsf{d}_{k,\alpha;\varphi^* g_i}\left(\Phi_{i+1}^{-1} \circ \Phi_i, \mathrm{Id}\right) \leq \Psi(\varepsilon_1).$$

Then by Anderson–Cheeger [AC91, p.241], the maps Φ_i can be glued together with small perturbation on each intersection $I_i \cap I_{i+1}$ to form a global map

$$\Phi : \bigcup_{i=0}^{N'-1} I_i \times N \to M.$$

This Φ is the desired map if the ε_1 is chosen to be sufficiently small. Indeed, since on $I_0 \setminus I_1$, $\Phi_0 = \phi_0$, we have (3.9). To verify (3.10), since φ_i does nothing to the $\mathbb{R}$ factor and ϕ_i satisfies (3.10) by construction, we only need to compute the radial derivative on each cross section in $I_i \times N$. It is straightforward that

$$\begin{aligned}
|\nabla_{\partial_r} \Phi_i^* g|_{k-1,\alpha;g_0} &= |\nabla_{\partial_r} (\Phi_i^* g - \varphi_i^* g_i)|_{k-1,\alpha;g_0} \\
&\leq C|\Phi_i^* g - \varphi_i^* g_i|_{k,\alpha;\varphi_i^* g_i}(1 + |\varphi_i^* g_i - g_0|_{k,\alpha;\varphi_i^* g_i}) \\
&\leq C|\phi_i^* g - g_i| \leq \Psi(\varepsilon_1).
\end{aligned}$$

Finally, we verify (3.11). Again, it suffice to verify for each cross section in each $I_i \times N$.

$$\begin{aligned}
|\delta_{g_0}(\varphi_i^* \phi_i^* g)|_{k-1,\alpha;g_0} &= \left|\delta_{g_0}(\varphi_i^* \phi_i^* g - \beta_i^* \underline{g}_i)\right|_{k-1,\alpha;g_0} \\
&\leq C|\varphi_i^* \phi_i^* g - \beta_i^* \underline{g}_i|_{k,\alpha;g_0} \\
&\leq C|\varphi_i^* \phi_i^* g - \varphi_i^* g_i|_{k,\alpha;\varphi_i^* g_i}(1 + |\varphi_i^* g_i - g_0|_{k,\alpha;\varphi_i^* g_i}) \\
&\leq C|\phi_i^* g - g_i|_{k,\alpha;g_i} \leq \Psi(\varepsilon_1).
\end{aligned}$$

Let $L' = \frac{2(N'-1)}{3}L + L$, if N' is finite then we have (3.15), it is immediate to see that (3.21) holds. $\square$

Corporate Proposition 3.3 with Theorem 2.8 we have the desired modified divergence free gauge on an arbitrarily large tube.

Corollary 3.4. *Let (M^n, g) be a complete Ricci flat n-manifold, satisfying linear volume growth, local $L^{n/2}$-curvature bound. Fix constants $k \in \mathbb{N}_0$, $\alpha \in (0,1)$, $\tau > 0$, $0 < \varepsilon << \chi$ sufficiently small. Then there exists constants $R(k, \alpha, \varepsilon, \chi, g) > 0$ and $L' > 1$, a Ricci flat cylinder ($\overline{N} = \mathbb{R} \times N^{n-1}, \bar{g} = dr^2 + g_0$), and a $C^{k,\alpha}$-approximation smooth embedding map*

$$\Phi : (R, R + L') \times N \to b^{-1}((R, R + L')), \tag{3.16}$$

which is a diffeomorphism onto its image, such that

$$|\Phi^* g - \bar{g}|_{k,\alpha} < \chi, \tag{3.17}$$

$$\min_{T_{R,R+L'}} |\Phi^* g - \bar{g}|_{k,\alpha} < \varepsilon, \tag{3.18}$$

$$|\nabla_{\partial_r}(\Phi^* g)|_{k-1,\alpha} < \varepsilon, \tag{3.19}$$

$$\delta_\tau(\Phi^* g) = 0. \tag{3.20}$$

Moreover, either $L' < \infty$ and for some $c > 0$,

$$|\phi^* g - g_0| > c_2\chi \quad \text{on } (L' - 1, L') \times N^{n-1} \tag{3.21}$$

or $L' = \infty$.

Proof. By (2.27) and (3.8), for the diffeomorphism Φ constructed in Proposition 3.3, we obtain

$$\left|\delta_\tau(\Phi^* g)\right|_{k-1,\alpha;\bar{g}} \leq \left|\delta(\Phi^* g)\right|_{k-1,\alpha;\bar{g}} + |\tau|\left|\iota_{\partial_r}(\Phi^* g)\right|_{k-1,\alpha;\bar{g}} \leq (1+4|\tau|)\chi.$$

For fixed $\tau \ll 1$, we may choose χ sufficiently small so that $(1+4|\tau|)\chi < \varepsilon$, where ε is the constant in Theorem 2.8. Consequently, by composing with the flow map from Theorem 2.8, we obtain a refined diffeomorphism that satisfies the properties stated in Corollary 3.4.

□

4. Solution to infinitesimal Ricci deformation equation

The goal of section is to solve the infinitesimal Ricci deformation equation (1.4). Note that since we have solved the divergence equation (2.1), for any solution h of (1.4), δh can be expressed as $L_X g$ for some vector field X, which is always a solution. In conclusion, we only need to consider h with $\delta h = 0$ on the Ricci flat cylinder $\overline{N} = \mathbb{R} \times N$. Eventually, We need to consider the modified divergence free condition $\delta_\tau h = 0$ because that will be the actual condition we work with in the last section and the modified divergence will help to get rid of some radially parallel solutions.

We start with $\delta h = 0$. The deformation equation (1.4) can then be simplified to

$$(\nabla^*\nabla - 2\mathrm{Rm})\mathrm{h} - \nabla^2 \operatorname{tr} \mathrm{h} = 0.$$

We first aim to simplify $\operatorname{tr} h$. Take the trace of this equation and note that when $\mathrm{Ric} = 0$, $\operatorname{tr} \mathrm{Rm}\ \mathrm{h} = 0$. We have

$$\nabla^*\nabla \operatorname{tr} h (= \Delta \operatorname{tr} h) = 0.$$

Equivalently, $\operatorname{tr} h$ is a harmonic function. We will attempt to absorb the trace term into the vector field part of the solution. Formally, we seek a vector field X so that

$$\begin{cases} \operatorname{tr}(L_X g_0) & = \operatorname{tr} h; \\ \delta(L_X g_0) & = 0. \end{cases} \tag{4.1}$$

However, this system does not always have a solution. For example when $\operatorname{tr} h = r$, say $h = r dr \otimes dr$, then there is no X satisfying (4.1). So we will not be able to achieve this goal. Observe that since $\operatorname{tr} h$ is harmonic, it can be written in the form of

$$\operatorname{tr} h = c_0 + \bar{c}_0 r + \sum_{i=1}^{\infty} \left(c_i^+ e^{\sqrt{\mu}_i r} + c_i^- e^{-\sqrt{\mu}_i r} \right) \phi_i$$

where μ_i are eigenvalues of Δ_N and $-\Delta_N \phi_i = \mu_i \phi_i$. If we keep the linear part $h_0 = c_0 + \bar{c}_0 r$, then we can solve (4.1) when replacing h by $\widetilde{h} := h - h_0$.

Lemma 4.1. *There exists a vector field X so that $\operatorname{tr} L_X g = \operatorname{tr} \widetilde{h}$ and $\delta(L_X g) = 0$.*

Proof. We follow closely the argument of Cheeger–Tian to construct solution X of (4.1) with h being $\widetilde{h}$. We first claim the existence of a function v that solves

$$\Delta v = \frac{1}{2}\operatorname{tr}\widetilde{h}, \tag{4.2}$$

and the existence of 2-forms θ and ψ that solve

$$d^*\theta = d\operatorname{tr}\widetilde{h},\ \Delta\psi = \theta, d\psi = 0. \tag{4.3}$$

Then we see that $X = -(dv - d^*\psi)^\flat$ is a desired solution. Indeed, by direct computation we have

$$\operatorname{tr}(L_X g_0) = 2\operatorname{tr}(\nabla X^\sharp) = -2d^* X^\sharp = 2d^*(dv - d^*\psi) = 2d^* dv = 2\Delta v = \operatorname{tr}\widetilde{h},$$

and

$$\begin{aligned}\delta(L_X g_0) &= (d^* d + 2dd^*)(dv - d^*\psi) = 2d(d^* dv) - d^*(dd^*\psi)\\ &= d\operatorname{tr}\widetilde{h} - d^*(\Delta\psi) = d\operatorname{tr}\widetilde{h} - d^*\theta = 0.\end{aligned}$$

We can let v be an infinite sum of $a_i^\pm(r)\phi_i$, $i \geq 1$, then (4.2) reduces to second order ODEs

$$-\left(a_i^\pm\right)''(r) + \mu_i a_i^\pm(r) = \frac{1}{2}c_i^\pm e^{\pm\sqrt{\mu_i}r},$$

By direct calculations, we can choose

$$a_i^\pm(r) = \mp\frac{c_i^\pm}{4\sqrt{\mu_i}}re^{\pm\sqrt{\mu_i}r},$$

such that

$$\begin{aligned}v &= \sum_{i=1}^\infty\left(-\frac{c_i^+}{4\sqrt{\mu_i}}re^{\sqrt{\mu_i}r} + \frac{c_i^-}{4\sqrt{\mu_i}}re^{-\sqrt{\mu_i}r}\right)\phi_i,\\ dv &= \sum_{i=1}^\infty\left(-\frac{c_i^+}{4\sqrt{\mu_i}}\left(1+\sqrt{\mu_i}r\right)e^{\sqrt{\mu_i}r} + \frac{c_i^-}{4\sqrt{\mu_i}}\left(1-\sqrt{\mu_i}r\right)e^{-\sqrt{\mu_i}r}\right)\phi_i dr\\ &+\sum_{i=1}^\infty\left(-\frac{c_i^+}{4\sqrt{\mu_i}}re^{\sqrt{\mu_i}r} + \frac{c_i^-}{4\sqrt{\mu_i}}re^{-\sqrt{\mu_i}r}\right)d_N\phi_i.\end{aligned}$$

Note that for $f \in C^\infty$ and $\alpha, \beta \in \Omega^1$, we have

$$\begin{aligned}d^*(f\alpha\wedge\beta) =& f d^*(\alpha\wedge\beta) - \iota_{\nabla f}(\alpha\wedge\beta)\\ =& f\left((d^*\alpha)\cdot\beta - (d^*\beta)\cdot\alpha + \nabla_{\beta^\flat}\alpha - \nabla_{\alpha^\flat}\beta\right)\\ &- \alpha(\nabla f)\cdot\beta + \beta(\nabla f)\cdot\alpha.\end{aligned}$$

Therefore, we have

$$\begin{aligned}d(e^{\pm\sqrt{\mu}r}\phi) &= \pm\sqrt{\mu}e^{\pm\sqrt{\mu}r}\phi dr + e^{\pm\sqrt{\mu}r}d_N\phi,\\ d^*(e^{\pm\sqrt{\mu}r}dr\wedge d_N\phi) &= e^{\pm\sqrt{\mu}r}d^*(dr\wedge d_N\phi) \mp \sqrt{\mu}e^{\pm\sqrt{\mu}r}\iota_{\frac{\partial}{\partial r}}(dr\wedge d_N\phi)\end{aligned}$$

$$
\begin{aligned}
&= e^{\pm\sqrt{\mu}r}((d^*dr)d_N\phi - (d^*d_N\phi)dr + \nabla_{(d_N\phi)^\flat}dr - \nabla_{\frac{\partial}{\partial r}}d_N\phi) \\
&\mp \sqrt{\mu}e^{\pm\sqrt{\mu}r}d_N\phi \\
&= -\mu e^{\pm\sqrt{\mu}r}\phi dr \mp \sqrt{\mu}e^{\pm\sqrt{\mu}r}d_N\phi \\
&= \mp\sqrt{\mu}d(e^{\pm\sqrt{\mu}r}\phi),
\end{aligned}
$$

where we used facts $(d^*d+dd^*)r = 0$ and $(d^*d+dd^*)\phi = \mu\phi$. Therefore, we can define

$$
\theta := \sum_{i=1}^{\infty}\left(-\frac{c_i^+}{\sqrt{\mu_i}}e^{\sqrt{\mu}_i r}dr\wedge d_N\phi_i + \frac{c_i^-}{\sqrt{\mu_i}}e^{-\sqrt{\mu}_i r}dr\wedge d_N\phi_i\right).
$$

which gives the solution θ to $d^*\theta = d\operatorname{tr}\widetilde{h}$. Meanwhile, we can assume that ψ is in the form of

$$
\psi := \sum_{i=1}^{\infty}\left(b_i^+ dr\wedge d_N\phi_i + b_i^- dr\wedge d_N\phi_i\right)
$$

Due to

$$
\Delta(b(r)dr\wedge d_N\phi) = (-b''(r)+\mu b(r))dr\wedge d_N\phi,
$$

$\Delta\psi = \theta$ reduces to second order ODEs

$$
-\left(b_i^\pm\right)''(r) + \mu_i b_i^\pm(r) = \mp\frac{c_i^\pm}{\sqrt{\mu_i}}e^{\pm\sqrt{\mu}_i r},
$$

By direct calculation, we can choose

$$
b_i^\pm(r) = \frac{c_i^\pm}{2\mu_i}re^{\pm\sqrt{\mu}_i r},
$$

such that

$$
\begin{aligned}
\psi &= \sum_{i=1}^{\infty}\left(\frac{c_i^+}{2\mu_i}re^{\sqrt{\mu}_i r}dr\wedge d_N\phi_i + \frac{c_i^-}{2\mu_i}re^{-\sqrt{\mu}_i r}dr\wedge d_N\phi_i\right), \\
d^*\psi &= \sum_{i=1}^{\infty}\frac{c_i^+}{2\mu_i}\left(-\mu_i re^{\sqrt{\mu}_i r}\phi_i dr - (1+\sqrt{\mu_i}r)e^{\sqrt{\mu}_i r}d_N\phi_i\right) \\
&+ \sum_{i=1}^{\infty}\frac{c_i^-}{2\mu_i}\left(-\mu_i re^{-\sqrt{\mu}_i r}\phi_i dr - (1-\sqrt{\mu_i}r)e^{-\sqrt{\mu}_i r}d_N\phi_i\right).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
dv - d^*\psi =& \sum_{i=1}^{\infty}\left(-\frac{c_i^+}{4\sqrt{\mu_i}}\left(1+\sqrt{\mu_i}r\right)e^{\sqrt{\mu}_i r} + \frac{c_i^-}{4\sqrt{\mu_i}}\left(1-\sqrt{\mu_i}r\right)e^{-\sqrt{\mu}_i r}\right)\phi_i dr \\
&+ \sum_{i=1}^{\infty}\left(-\frac{c_i^+}{4\sqrt{\mu_i}}re^{\sqrt{\mu}_i r} + \frac{c_i^-}{4\sqrt{\mu_i}}re^{-\sqrt{\mu}_i r}\right)d_N\phi_i \\
&- \sum_{i=1}^{\infty}\frac{c_i^+}{2\mu_i}\left(-\mu_i re^{\sqrt{\mu}_i r}\phi_i dr - (1+\sqrt{\mu_i}r)e^{\sqrt{\mu}_i r}d_N\phi_i\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{\infty}\frac{c_i^-}{2\mu_i}\left(-\mu_i r e^{-\sqrt{\mu}_i r}\phi_i dr-(1-\sqrt{\mu_i}r)e^{-\sqrt{\mu}_i r}d_N\phi_i\right)\\
=&\sum_{i=1}^{\infty}\left(-\frac{c_i^+}{4\sqrt{\mu_i}}\,(1-\sqrt{\mu_i}r)\,e^{\sqrt{\mu}_i r}+\frac{c_i^-}{4\sqrt{\mu_i}}\,(1+\sqrt{\mu_i}r)\,e^{-\sqrt{\mu}_i r}\right)\phi_i dr\\
&+\sum_{i=1}^{\infty}\left(\frac{c_i^+}{2\mu_i}(1+\frac{\sqrt{\mu_i}}{2}r)e^{\sqrt{\mu}_i r}+\frac{c_i^-}{2\mu_i}(1-\frac{\sqrt{\mu_i}}{2}r)e^{-\sqrt{\mu}_i r}\right)d_N\phi_i.
\end{aligned}
$$

This will be the dual of the desired vector field $-X$ and recall that $L_X g_0 = \nabla^{\mathrm{sym}} X^\sharp$. So, we have

$$
\begin{aligned}
L_X g_0 =& \sum_{i=1}^{\infty}\left[\mp c_i^{\pm}\frac{\sqrt{\mu_i}}{2}re^{\pm\sqrt{\mu}_i r}\right]\phi_i dr\otimes dr\\
&-\sum_{i=1}^{\infty}\left[\frac{c_i^{\pm}}{2}\left(r\pm\frac{1}{\sqrt{\mu_i}}\right)e^{\pm\sqrt{\mu}_i r}\right]d_N\phi_i\boxtimes dr\\
&-\sum_{i=1}^{\infty}\left(\frac{c_i^{\pm}}{\mu_i}(1\pm\frac{\sqrt{\mu_i}}{2}r)e^{\pm\sqrt{\mu}_i r}\right)\nabla^2\phi_i
\end{aligned}
\tag{4.4}
$$

It is easy to verify that

$$
\begin{aligned}
\operatorname{tr}(L_X g_0) &= \sum_{i=1}^{\infty}\left[\mp c_i^{\pm}\frac{\sqrt{\mu_i}}{2}re^{\pm\sqrt{\mu}_i r}\right]\phi_i+\sum_{i=1}^{\infty}\left(\frac{c_i^{\pm}}{\mu_i}(1\pm\frac{\sqrt{\mu_i}}{2}r)e^{\pm\sqrt{\mu}_i r}\right)\mu_i\phi_i\\
&=\sum_{i=1}^{\infty}c_i^{\pm}e^{\pm\sqrt{\mu}_i r}\phi_i=\operatorname{tr}\widetilde{h},\\
\nabla^2(\operatorname{tr}\widetilde{h}) &= \sum_{i=1}^{\infty}c_i^{\pm}\left(\mu_i\phi_i dr\otimes dr\pm\sqrt{\mu}_i d\phi_i\boxtimes dr+\nabla^2\phi_i\right)e^{\pm\sqrt{\mu}_i r},\\
\Delta^L&\left(\sum_{i=1}^{\infty}\left[\mp c_i^{\pm}\frac{\sqrt{\mu_i}}{2}re^{\pm\sqrt{\mu}_i r}\right]\phi_i dr\otimes dr\right)\\
&=\sum_{i=1}^{\infty}c_i^{\pm}\left(\pm\frac{\sqrt{\mu_i}}{2}(\mu_i r\pm 2\sqrt{\mu_i})\mp\frac{\sqrt{\mu_i}}{2}\mu_i r\right)e^{\pm\sqrt{\mu}_i r}\phi_i dr\otimes dr\\
&=\sum_{i=1}^{\infty}c_i^{\pm}\mu_i e^{\pm\sqrt{\mu}_i r}\phi_i dr\otimes dr,\\
\Delta^L&\left(-\sum_{i=1}^{\infty}\left[\frac{c_i^{\pm}}{2}\left(r\pm\frac{1}{\sqrt{\mu_i}}\right)e^{\pm\sqrt{\mu}_i r}\right]d\phi_i\boxtimes dr\right)\\
&=\sum_{i=1}^{\infty}c_i^{\pm}\left(\frac{\pm 3\sqrt{\mu}_i+\mu_i r}{2}-\frac{\mu_i r\pm\sqrt{\mu}_i}{2}\right)e^{\pm\sqrt{\mu}_i r}d\phi_i\boxtimes dr
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \pm c_i^{\pm} \sqrt{\mu}_i e^{\pm\sqrt{\mu}_i r} d\phi_i \boxtimes dr, \\
&\Delta^L \left(-\sum_{i=1}^{\infty} \left(\frac{c_i^{\pm}}{\mu_i} (1 \pm \frac{\sqrt{\mu_i}}{2} r) e^{\pm\sqrt{\mu}_i r} \right) \nabla^2 \phi_i \right) \\
&= \sum_{i=1}^{\infty} \frac{c_i^{\pm}}{\mu_i} \left(2\mu_i \pm \frac{\mu_i \sqrt{\mu_i}}{2} r - \mu_i (1 \pm \frac{\sqrt{\mu_i}}{2} r) \right) e^{\pm\sqrt{\mu}_i r} \nabla^2 \phi_i \\
&= \sum_{i=1}^{\infty} c_i^{\pm} e^{\pm\sqrt{\mu}_i r} \nabla^2 \phi_i.
\end{aligned}
$$

This completes the proof. □

By the preceding computation, the tensor $L_X g_0$ from (4.4) indeed solves the infinitesimal deformation equation. In general, however, $\nabla^2 \operatorname{tr} h \neq 0$; consequently, some radial factors appearing in (4.4)-for example $r\, e^{\pm\sqrt{\mu_i}\, r}$- fail to satisfy the scalar harmonic equation. This reflects the mismatch of the coefficients in the operator $d^*d + 2\, dd^*$ (i.e., the d^*d and dd^* parts carry different weights). Therefore, imposing the trace-free gauge a priori (equivalently, enforcing $\nabla^2 \operatorname{tr} h = 0$) is essential for decay estimates in Theorem 5.3. Finally,

$$
\operatorname{tr} h = \operatorname{tr}(L_X g_0) + c_0 + \bar{c}_0\, r, \qquad \nabla^2\big(\operatorname{tr} h - \operatorname{tr}(L_X g_0)\big) = 0.
$$

We see that after varying the vector field X in the pure-gauge direction $L_X g_0$, the infinitesimal deformation equation (1.4) reduces to

$$
\begin{cases}
\Delta^L h = 0, \\
\operatorname{tr} h = a + b\, r, \\
\delta h = 0,
\end{cases} \tag{4.5}
$$

for some constants $a, b \in \mathbb{R}$. Here we used that $\nabla^2 \operatorname{tr} h = \nabla^2(a + b\, r) = 0$ on the cylinder, since $a + b\, r$ is affine in r. The goal of this section is to solve (4.5) by separation of variables.

A solution h can be written as an infinite sum of symmetric tensors of the following three types. The first type is

$$
h = f(r) \eta_1 \boxtimes \eta_2. \tag{4.6}
$$

Here η_1, η_2 are 1-forms on the cross section N. They clearly satisfy $\nabla_{\partial_r} \eta_i = 0$ and $\eta_i(\partial_r) = 0$, $i = 1, 2$. The Riemmann curvature tensor on the cylinder M satisfies for X, Y, Z are vector fields tangent to N that

$$
\operatorname{Rm}(X, Y)Z = \operatorname{Rm}_N(X, Y)Z,
$$

and that

$$
\operatorname{Rm}(\cdot, \partial_r) = \operatorname{Rm}(\cdot, \cdot)\partial_r = \langle \operatorname{Rm}, \partial_r \rangle = 0.
$$

It is then a straightforward computation to see that

$$\Delta^L h = (\nabla^*\nabla - 2\mathrm{Rm}) f(r)\eta_1 \boxtimes \eta_2 = (-f''(r) + f(r)\Delta_N^L)\eta_1 \boxtimes \eta_2$$

The second type is

$$h = k(r)\eta \boxtimes dr, \tag{4.7}$$

with

$$\Delta^L h = (\nabla^*\nabla - 2\mathrm{Rm}) k(r)\eta \boxtimes dr = (-k''(r) + k(r)\nabla_N^*\nabla_N)\eta \boxtimes dr.$$

The third type is

$$h = \ell(r)\phi(x) dr \otimes dr, \tag{4.8}$$

with

$$\Delta^L h = (\nabla^*\nabla - 2\mathrm{Rm})\ell(r)\phi dr \otimes dr = (-\ell''(r) + \ell(r)\nabla_N^*\nabla_N)\phi dr \otimes dr.$$

Since N is Ricci flat, $\nabla_N^*\nabla_N$ is in fact the Hodge Laplacian for functions and 1-forms, which is positive semidefinite. We then compute δh and $\operatorname{tr} h$ as follows.

$$\begin{aligned}
\delta(f(r)\eta_1 \boxtimes \eta_2) &= f(r)\delta_N(\eta_1 \boxtimes \eta_2);\\
\delta(k(r)\eta \boxtimes dr) &= -k'(r)\eta + k(r)(\delta_N\eta)dr;\\
\delta(\ell\phi(x) dr \otimes dr) &= -\ell'(r)\phi dr.\\
\operatorname{tr} h &= \ell(r)\phi(x) + f(r)\operatorname{tr}_N(\eta_1 \boxtimes \eta_2).
\end{aligned}$$

The system (4.5) translates into

$$(-f''(r) + f(r)\Delta_N^L)\eta_1 \boxtimes \eta_2 = 0, \tag{4.9}$$

$$(-k''(r) + k(r)\nabla_N^*\nabla_N)\eta \boxtimes dr = 0, \tag{4.10}$$

$$(-\ell''(r) + \ell(r)\nabla_N^*\nabla_N)\phi dr \otimes dr = 0, \tag{4.11}$$

$$f(r)\delta_N(\eta_1 \boxtimes \eta_2) - k'(r)\eta = 0, \tag{4.12}$$

$$-\ell'(r)\phi + k(r)\delta_N\eta = 0, \tag{4.13}$$

$$\ell(r)\phi + f(r)\operatorname{tr}_N(\eta_1 \boxtimes \eta_2) = c + dr. \tag{4.14}$$

Decompose the solutions to this system, as a part of pure-gauge solutions, i.e. the solutions that are of the form $L_Y g$ for some vector field Y, a part of r-function multiples of metric tensor g_N on the cross section, i.e. the pure trace part and a part of r-function multiples of transverse-traceless tensors. Then on N decompose ϕ (resp. η) into sum of eigenfuntions (resp. eigen 1-forms). We consider different eigenvalues in order.

- The eigenvalue $\mu = 0$ for η, i.e. η is a harmonic 1-form. It follows that $\delta_N\eta = 0$ and then from (4.13) we infer that $\ell' = 0$. Plugging it into (4.11), we see that ϕ must be a harmonic function hence a constant. Then $\ell\phi$ is also a constant, and from (4.14) we infer there are two cases, not necessarily strictly exclusive.

- Either $d = 0$, $f(r)$ and $\operatorname{tr}_N(\eta_1 \boxtimes \eta_2)$ are both constant. In this case, we obtain from (4.9) that $\eta_1 \boxtimes \eta_2 \in \ker \Delta_N^L$. Recall that in a closed Ricci flat manifold N, $\ker \Delta_N^L$ has a decomposition into $\mathbb{R}g_N$ and tranverse-traceless tensors, and both are divergence free, so $\delta_N(\eta_1 \boxtimes \eta_2) = 0$. We infer from (4.12) that $k' = 0$. We get solutions of the types $h = \eta \boxtimes dr = L_{r\eta} g$, $h = cdr \otimes dr = L_{cdr} g$, which are radially parallel solutions, and $h = \eta_1 \boxtimes \eta_2$ with $\delta_N(\eta_1 \boxtimes \eta_2) = 0$. The last type admits a unique decomposition into pure trace and traceless part as $h = \frac{\operatorname{tr} h}{n} g_N + (h - \frac{\operatorname{tr} h}{n} g_N)$.
- Or $d \neq 0$, $f = c' + dr$ for some constant c' and $\operatorname{tr}_N(\eta_1 \boxtimes \eta_2)$ is a nonzero constant. Again we obtain from (4.9) that $\eta_1 \boxtimes \eta_2 \in \ker \Delta_N^L$ and in turn that $\delta_N(\eta_1 \boxtimes \eta_2) = 0$. The discussion will be the same as in the previous case. The only difference is we will have the type $h = r\eta_1 \boxtimes \eta_2$ so we will get a term that is a multiple of rg_N.

- The eigenvalue $\mu = 0$ for ϕ, i.e. ϕ is a harmonic function hence constant. Then from (4.11) we have that $\ell'' = 0$, so $\ell' \phi$ is constant. Using (4.13) we infer that k must be constant. If $k \neq 0$, then η is a harmonic 1-form by (4.10). This is the previous situation. If $k = 0$, by (4.12) we deduce that either $f = 0$ or $\delta_N(\eta_1 \boxtimes \eta_2) = 0$. If $f = 0$, then by (4.14) $\ell = c + dr$, and (4.13) implies $d = 0$ as $k = 0$. We obtain a solution $h = cdr \otimes dr = L_{crdr} g$. Now we consider the case $\delta_N(\eta_1 \boxtimes \eta_2) = 0$. By (4.13) we have $\ell' = 0$, so $\ell\phi$ is a constant. Using again (4.14), we return to the same dichotomy $d = 0$ or $d \neq 0$, we can proceed as in previous case.
- The eigenvalues of η, ϕ are both nonzero. From (4.13) we deduce yet two cases as follows.
  - Either $\ell' = k = 0$. In this case, we deduce from (4.12) that either $f = 0$ or $\delta_N(\eta_1 \boxtimes \eta_2) = 0$. If $f = 0$, then since ϕ is not a constant, from (4.14) we see that $\ell = 0$ and $c = d = 0$, as the right hand side does not depend on the cross section. We get 0 solution. If $\delta_N(\eta_1 \boxtimes \eta_2) = 0$, and $f \neq 0$, by (4.14), we must have $\ell\phi + f(r) \operatorname{tr}_N(\eta_1 \boxtimes \eta_2) = 0$ as the right hand side does not depend on the cross section. If $\operatorname{tr}_N(\eta_1 \boxtimes \eta_2) \neq 0$, f is a constant multiple of ℓ, which is a constant. Then again we deduce from (4.9) that $\eta_1 \boxtimes \eta_2 \in \ker \Delta_N^L$, so $\eta_1 \boxtimes \eta_2$ admits a pure trace-traceless decomposition. Or $\operatorname{tr}_N(\eta_1 \boxtimes \eta_2) = 0$, in this case, the only nontrivial equation we have is (4.9) with $\eta_1 \boxtimes \eta_2$ being transverse-traceless. By spectral decomposition into eigenforms of Δ_N^L, we see that $f(r)$ is exactly one of the following depending on the sign of the eigenvalues.

$$
f(r) = \begin{cases} a_i^+ \cos(\sqrt{-\mu_i} r) + a_i^- \sin(\sqrt{-\mu_i} r), & \mu_i < 0 \\ a_0 + \tilde{a}_0 r, & \mu_i = 0 \\ a_i^+ e^{\sqrt{\mu_i} r} + a_i^- e^{-\sqrt{\mu_i} r}, & \mu_i > 0. \end{cases} \tag{4.15}
$$

– Or $\delta_N\eta$ and ϕ are linearly dependent, in which case by the Hodge decomposition on N, we must have that η is a sum of a constant multiple of $d\phi$ and an eigen-1-form $\tilde{\eta}$ with $\delta\tilde{\eta} = 0$. Meanwhile, both η, ϕ share the same $\mu > 0$. However, we then see from (4.12) that $\delta_N(\eta_1 \boxtimes \eta_2) \in \operatorname{Im}\delta$ and $d\phi \in \operatorname{Im} d$ are orthogonal, so $k'(r) = 0$ and in turn either $f(r) = 0$ or $\delta_N(\eta_1 \boxtimes \eta_2) = 0$. By (4.10) we must have $k = 0$ since $\mu > 0$. Note that (4.13) also implies ℓ' is a constant multiple of k. Then $\ell' = 0$ and similarly by (4.11) $\ell = 0$. In the end we either have $f = 0$ and we end up with a 0 solution or we have only nontrivial equation (4.9) as in (4.15) with $\eta_1 \boxtimes \eta_2$ being transverse-traceless.

Summarizing the discussion on (4.5), and adding back trivial solution $L_X g$ such that $\operatorname{tr}(L_X g) = \operatorname{tr} h$ we treated, we have a complete description of the solutions h of (1.4) with $\delta h = 0$. In the next theorem, we separate eigenvalues of Δ_N^L, $\{\mu_i\}_{i\in\mathbb{N}_0}$ into three parts and order them by their absolute values, i.e. $\mu_K < \mu_{K-1} \cdots < \mu_1 < \mu_0 = 0 < \mu_{K+1} < \cdots$. Note that 0 is always an eigenvalue of Δ^L but there does not always exist transverse-traceless eigen-tensors.

Theorem 4.2. *The solution h of (1.4) subject to $\delta h = 0$ can be written uniquely as a sum*

$$
\begin{aligned}
h = {} & L_X g + L_Y g + (a + \tilde{a}r)g_N + a_0 B_0 + \tilde{a}_0 r \tilde{B}_0 \\
& + \sum_{i=1}^{K} \left(a_i^+ \cos(\sqrt{-\mu_i}r) + a_i^- \sin(\sqrt{-\mu_i}r)\right) B_i \\
& + \sum_{i=K+1}^{\infty} \left(a_i^+ e^{\sqrt{\mu_i}r} + a_i^- e^{-\sqrt{\mu_i}r}\right) B_i,
\end{aligned}
\tag{4.16}
$$

where X is the vector field constructed in Lemma 4.1, Y is a vector field dual to $(cr + c')dr + \eta$ with harmonic 1-form η, and B_i is a transverse-traceless eigen-tensor of eigenvalue μ_i, that is, $\Delta_N^L B_i = \mu_i B_i$, $\operatorname{tr}_N B_i = 0$ and $\delta B_i = \delta_N B_i = 0$.

Remark 4.3. Compare with [CT94], in the conical case, the possible repeated root of the characteristic equation is $\mu = 1 - \frac{n}{2}$. When $n \geq 3$ this root still provide a solution with exponential decay in cylindrical coordinates (polynomial decay in conical coordinates).

Now we consider the modified divergence δ_τ introduced in (2.26), the actual condition we have when consider the nonlinear equation in the next section. The δ_τ will also help eliminate the radially parallel pure-gauge solution $L_Y g$, more specifically, $L_{r\eta} g = \eta \boxtimes dr$ for harmonic η, and $L_{rdr} g = dr \otimes dr$. By construction, the perturbation from δ_τ acts only in the $\mu = 0$ eigenspace, i.e., when ϕ is a harmonic function and η is a harmonic 1-form on N. Under the modified divergence free condition the system becomes

$$(\nabla^*\nabla - 2\tau\delta^*\iota_{\partial_r} - 2\mathrm{Rm})h - \nabla^2 \operatorname{tr} h = 0 \tag{4.17}$$

with the constraint $\delta_\tau h = 0$. Writing down the equation for three different types, we get a system of equations as follows with $\delta_N\eta = 0$, $\delta_N^*\eta = 0$, $d\phi = 0$, $\nabla_N^2\phi = 0$:

$$\begin{aligned}(-f'' + f\Delta_N^L)\eta_1 \boxtimes \eta_2 &= \nabla^2 \operatorname{tr}(f\eta_1 \boxtimes \eta_2) + \ell\nabla_N^2\phi + 2\tau k\delta_N^*\eta \\ &= \nabla^2 \operatorname{tr}(f\eta_1 \boxtimes \eta_2);\end{aligned} \tag{4.18}$$

$$\begin{aligned}(-k'' + k\nabla_N^*\nabla_N)\eta \boxtimes dr &= (\ell' + \tau\ell)d\phi \boxtimes dr + \tau k'\eta \boxtimes dr \\ &= \tau k'\eta \boxtimes dr;\end{aligned} \tag{4.19}$$

$$(-\ell'' + \ell\nabla_N^*\nabla_N)\phi dr \otimes dr = (\ell'' + 2\tau\ell')\phi dr \otimes dr \tag{4.20}$$

$$(-\ell' - 2\tau\ell)\phi + k\delta_N\eta = 0; \tag{4.21}$$

$$(-k' - 2\tau k)\eta + f\delta_N(\eta_1 \boxtimes \eta_2) = 0. \tag{4.22}$$

Since ϕ is harmonic, from (4.20) we have $\ell'' + \tau\ell' = 0$. Meanwhile since $\delta_N\eta = 0$ we have from (4.21) that $-\ell' - 2\tau\ell = 0$. Putting together, the only solution is $\ell = 0$.

Since η is harmonic, it is orthogonal to $\delta_N(\eta_1 \boxtimes \eta_2) \in \mathrm{im}\delta$, by (4.22) we have $k' + 2\tau k = 0$ and either f or $\delta_N(\eta_1 \boxtimes \eta_2)$ is zero. Also by (4.19) we have $k'' + \tau k' = 0$. Similarly we have $k = 0$. The only nontrivial equation is (4.18) with $\delta_N(\eta_1 \boxtimes \eta_2) = 0$. Taking its trace we get that

$$-2f'' \operatorname{tr}_N \eta_1 \boxtimes \eta_2 + 2f\Delta_N \operatorname{tr}_N(\eta_1 \boxtimes \eta_2) = 0. \tag{4.23}$$

If $\operatorname{tr}_N \eta_1 \boxtimes \eta_2 = 0$, then we get transverse traceless solutions as in (4.15). If $\operatorname{tr}_N \eta_1 \boxtimes \eta_2 \neq 0$ and $f \neq 0$, we can separate variables to see that

$$\frac{f''}{f} = \frac{\Delta_N \operatorname{tr}_N(\eta_1 \boxtimes \eta_2)}{\operatorname{tr}_N(\eta_1 \boxtimes \eta_2)} := \lambda \geq 0. \tag{4.24}$$

If $\lambda = 0$ then $\operatorname{tr}_N(\eta_1 \boxtimes \eta_2)$ is harmonic hence a (nonzero) constant, and $f(r)$ is an affine function in r, we can perform the pure trace and traceless decomposition. If $\lambda \neq 0$, then $\operatorname{tr}_N(\eta_1 \boxtimes \eta_2)$ is an eigenfunction of Δ_N of positive eigenvalue λ. It follows that $f'' - \lambda f = 0$, so f is an exponential function. Note that the solutions $h = f\eta_1 \boxtimes \eta_2$ have trace of the type $e^{\pm\sqrt{\lambda}r}\phi$ with $-\Delta_N\phi = \lambda\phi$. It follows from the our treatment of trace terms that there exists a vector field X such that $\operatorname{tr}(L_X g) = \operatorname{tr} h$ and X has exponential growth or decay. We also see that $\delta_\tau(L_X g) = \delta(L_X g) = 0$. So $h - L_X g$ is a still a solution to (4.17) with $\delta_\tau(h - L_X g) = 0$ and $\operatorname{tr}(h - L_X g)$ is affine in r because only the solutions with $\lambda = 0$ contribute to the trace.

When neither ϕ nor η is harmonic, we will have the same equations (4.18), (4.10) and (4.11) but different constraints (4.21) and (4.22). The procedure to solve these equations are similar to the case without perturbation. We can show that $\ell = k = 0$ unless $\mu = 4\tau^2 > 0$, which can always be avoided by choosing τ small enough. In conclusion, we have shown that the perturbation removes radially parallel solutions $\eta \boxtimes dr$ for harmonic η, and $dr \otimes dr$.

Theorem 4.4. *For $\tau > 0$ small enough, the solution h of (1.4) subject to $\delta_\tau h = 0$ is of the form (4.16) without $L_Y g$. More precisely, the only radially parallel solutions are $c_0 B_0$ and cg_N for some constants c_0 and c.*

5. Uniqueness of the cross section

In this section, we are going to finish the proof of A. Leveraging the assumption that some cross section is integrable and the Lichnerowicz is laplacian, given the smooth embedding map Φ from Corollary 3.4.

We begin by introducing the notation required for our main results. For any fixed r, we define a inner product for two tensor fields η_1, η_2 on $\overline{N}$ over $\{r\} \times N^{n-1}$ by

$$\langle \eta_1, \eta_2 \rangle_N := \int_N \langle \eta_1, \eta_2 \rangle_{g_N} \,\mathrm{dvol}_N, \tag{5.1}$$

and define a L^2-norm on a tube $T_{a,b}$ by

$$\|\eta\|_{a,b} := \int_a^b |\eta|_N^2 dr. \tag{5.2}$$

Over the each $\bar{T}_{R+kL,R+(k+1)L}$, let π_k be the orthogonal projection on the subspace $\ker \left(\Delta_\tau^L - \nabla^2 \operatorname{tr}\right)|_{\bar{T}_{R+kL,R+(k+1)L}}$ with respect to the inner product (5.2). If there is no confusion we also write π, omitting the subscript.

Consider a divergence free gauge ϕ from Corollary 3.4. It follows that $\phi^* g$ is a Ricci flat metric over a tube $\bar{T}_{R,R+L'} \subseteq \overline{N}$ and satisfies $\delta_\tau(\phi^* g - \bar{g}) = 0$, $\operatorname{tr}(\phi^* g - g_0) = c_0 + \bar{c}_0 r$. According to Theorem 4.4, we know that when subject to $\delta_\tau(\phi^* g - \bar{g}) = 0$, the only radially parallel part in $\pi_k(\phi^* g - \bar{g})$ is of the form $c_0 B_0 + cg_N$. By the following theorem, we can select an alternative Ricci flat metric $\bar{g}_k := dr^2 + g_k$ close to $\bar{g}$ on $\bar{T}_{R+kL,R+(k+1)L} \subseteq \bar{T}_{R,R+L'}$ satisfying $\delta_\tau(g_k - \bar{g}) = 0$ so that there is no radially parallel part in $\pi_k(\phi^* g - \bar{g}_k)$. This is the only place where the integrability enters the proof.

Proposition 5.1. *Let $(\overline{N} = \mathbb{R} \times N, \bar{g} = dr^2 + g_N)$ be a Ricci flat cylinder and N be integrable as a closed Ricci flat manifold. Then for any $\tau > 0$ and any tube $\bar{T}_{R+kL,R+(k+1)L} \subseteq \bar{T}_{R,R+L'}$, there is a Ricci flat metric g_k on satisfying $\delta_\tau(g_k - \bar{g}) = 0$ so that there is no radially parallel part in $\pi_k(\phi^* g - \bar{g}_k)$.*

From now on, to simplify the notation, it does not harm to take $R = 0$ (by translation) in the construction of our diffeomorphism in Corollary 3.4 after choosing L.

Proof. To build the metrics on each tube $\bar{T}_{kL,(k+1)L}$, it suffice to construct metrics g_k on the cross section N. The integrability assumption of N implies that the set of metrics

$$\{g \text{ Riemannian metric on } N | \operatorname{Ric}_g = 0, \delta_{g_N}(g) = 0\}$$

has a smooth manifold structure through Ebin–Palais Slice theorem on a neighborhood $\mathscr{U}$ of g_N, whose tangent space is $\ker \delta_N$ [Has12, Proposition 2.14]. Recall that $\ker \delta_N$ has an L^2-orthogonal decomposition into space of transverse-traceless tensors and the pure trace part

$$\ker \delta_N = \{B \in C^\infty(S^2T^*N) |\, \mathrm{tr}_N B = 0, \delta_N(B) = 0\} \oplus \mathbb{R}g_N.$$

Note that the variations considered in Besse [Bes08, Section 12E] fix the volume so there is no $\mathbb{R}g_N$, but it is obvious that one can lift the volume constraint. Let $\{B_i\}_{i\in\mathbb{N}_0}$ be an orthonormal basis of space of transverse-traceless tensors, we have the smooth map

$$\begin{aligned} F : \mathscr{U} &\to \ker \delta_N \\ g &\mapsto \sum \langle g, B_i\rangle B_i + \frac{1}{n}\langle g, g_N\rangle g_N \end{aligned}$$

with $F(g_N) = g_N$ and $DF_{g_N} = \mathrm{Id}$. Let $\pi_k(\cdot)_0$ be the terms with constant coefficients in $\ker(\Delta^L - \nabla^2 \mathrm{tr})$, which are the only radially parallel terms by the solution formula. We aim to find $g_k \in \mathscr{U}$ such that $\delta_\tau(\phi^* g - \bar{g}_k) = 0$ and $\pi_k(\phi^* g - \bar{g}_k)_0 = 0$. Note that $g_k \in \mathscr{U}$ already implies $\delta_N(g_k) = 0$ so

$$\delta_\tau(\phi^* g - \bar{g}_k) = \delta_\tau(\phi^* g - \bar{g}) + \delta_\tau(\bar{g} - \bar{g}_k) = 0 + \delta_N(g_N - g_k) = 0.$$

Notice also that

$$\pi_k(\phi^* g - \bar{g}_k)_0 = \pi_k(\phi^* g - \bar{g})_0 + \pi_k(\bar{g} - \bar{g}_k)_0 = \pi_k(\phi^* g - \bar{g})_0 + g_N - \pi_k(g_k)_0,$$

it suffices to solve the equation

$$F(g_k) = F(g_N) + \pi_k(\phi^* g - \bar{g})_0.$$

If $\|\phi^* g - \bar{g}\|_{R+kL,R+(k+1)L} \le \chi$ and χ is sufficiently small, by interior estimates we can ensure that $\pi_k(\phi^* g - \bar{g})_0 \in \mathrm{Im} F$. Then the existence of g_k follows from the implicit function theorem. $\square$

Next, we are going to study the asymptotical decay of $\overline{h}_k := \pi_k(\phi^* g - \bar{g}_k) = \pi_k(h_k)$ in the L^2 sense. The following lemma exhibits that the nonlinear equation h_k satisfies is almost the infinitesimal Ricci deformation equation (1.4). Now we denote the background metric $\bar{g} = \bar{g}_0$ for which the assumption (R.2) holds and $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$.

Proposition 5.2. *There exists $C = C(\bar{g}_0, k, \alpha)$ and $\varepsilon_0 > 0$ such that for any metric $\bar{g}_1$ and symmetric 2-tensor h with*

$$|\bar{g}_1 - \bar{g}_0|_{k+2,\alpha} \le \varepsilon_0, \qquad |h|_{k+2,\alpha} \le \varepsilon_0,$$

we have the decomposition

$$\mathrm{Ric}_{\bar{g}_1+h} - \mathrm{Ric}_{\bar{g}_1} = \left(\Delta^L_{\bar{g}_0} h - \nabla^2_{\bar{g}_0} \mathrm{tr}_{\bar{g}_0} h\right) \;+\; \tilde{E}(g_0, g_1, h)$$

with the estimate

$$|\tilde{E}(\bar{g}_0, \bar{g}_1, h)|_{k,\alpha} \;\le\; C\Big(1 + |\bar{g}_1 - \bar{g}_0|_{k+2,\alpha}\Big)|h|^2_{k+2,\alpha}. \tag{5.3}$$

In particular, if $|\bar{g}_1 \bar{-} g_0|_{k+2,\alpha} + |h|_{k+2,\alpha} \le \varepsilon << 1$, then $|\tilde{E}|_{k,\alpha} \le C\,\varepsilon^2$.

Proof. We start from the $\bar{g}_1$-based expansion

$$\mathrm{Ric}_{\bar{g}_1+h} - \mathrm{Ric}_{\bar{g}_1} = \left(\Delta^L_{\bar{g}_1} h - \nabla^2_{\bar{g}_1} \operatorname{tr}_{\bar{g}_1} h\right) \; + \; E(\bar{g}_0, \bar{g}_1, h),$$

where, by the standard argument for Ric as a quasilinear second-order operator,

$$|E(\bar{g}_0, \bar{g}_1, h)|_{k,\alpha} \; \leq \; C\big(1 + |\bar{g}_1 \bar{-} g_0|_{k+2,\alpha}\big)\, |h|^2_{k+2,\alpha}. \tag{5.4}$$

Add and subtract the $\bar{g}_0$-linearization:

$$\mathrm{Ric}_{\bar{g}_1+h} - \mathrm{Ric}_{\bar{g}_1} = \left(\Delta^L_{\bar{g}_0} h - \nabla^2_{\bar{g}_0} \operatorname{tr}_{\bar{g}_0} h\right) \; + \; E(\bar{g}_0, \bar{g}_1, h) \; + \; \left(L_{\bar{g}_1} - L_{\bar{g}_0}\right) h,$$

where $L(h) := \Delta^L h - \nabla^2 \operatorname{tr} h$. Thus $\tilde{E} = E + (L_{\bar{g}_1} - L_{\bar{g}_0})h$ and it remains to bound the operator difference on h.

Claim. There is $C = C(\bar{g}_0, k, \alpha)$ such that

$$|(L_{\bar{g}_1} - L_{\bar{g}_0})h|_{k,\alpha} \; \leq \; C\, |\bar{g}_1 - \bar{g}_0|_{k+2,\alpha}\, |h|_{k+2,\alpha}. \tag{5.5}$$

Granting this, (5.3) follows immediately from (5.4) and (5.5).

Proof of the claim. For convention, we denote Christoffel symbols (as a $(1,2)$ tensor) and Riemannian curvature tensor (as a $(1,3)$ tensor) associated to a metric $\bar{g}$ by $\Gamma_{\bar{g}}$ and $\mathrm{Rm}_{\bar{g}}$, respectively. By the definitions, we know that

$$\begin{aligned} |\Gamma_{\bar{g}_1} - \Gamma_{\bar{g}_0}|_{k,\alpha} &\leq C(k,\alpha) |\bar{g}_1^{-1}|_{k,\alpha} |\bar{g}_1 - \bar{g}_0|_{k+1,\alpha} \\ &\leq C(k,\alpha) \left(1 + |\bar{g}_1 - \bar{g}_0|_{k,\alpha}\right) |\bar{g}_1 - \bar{g}_0|_{k+1,\alpha} \\ &\leq C(\bar{g}_0, k, \alpha) |\bar{g}_1 - \bar{g}_0|_{k+1,\alpha}, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} |\mathrm{Rm}_{\bar{g}_1} - \mathrm{Rm}_{\bar{g}_0}|_{k,\alpha} \leq& C(k,\alpha) \left(|\bar{g}_1 - \bar{g}_0|_{k+2,\alpha} + |\bar{g}_1 - \bar{g}_0|^2_{k+1,\alpha}\right. \\ &\qquad \left. + |\bar{g}_1 - \bar{g}_0|_{k,\alpha} |\mathrm{Rm}_{\bar{g}_0}|_{k,\alpha}\right) \\ \leq& C(\bar{g}_0, k, \alpha) \left(|\bar{g}_1 - \bar{g}_0|_{k+2,\alpha} + |\bar{g}_1 - \bar{g}_0|^2_{k+1,\alpha}\right). \end{aligned} \tag{5.7}$$

Based on (5.6) and (5.7), we obtain

$$\begin{aligned} \left| \left(\nabla^*_{\bar{g}_1} \nabla_{\bar{g}_1} - \nabla^*_{\bar{g}_0} \nabla_{\bar{g}_0}\right) h\right|_{k,\alpha} &\leq C(k,\alpha) \sum_{j=0}^{2} |\bar{g}_1 - \bar{g}_0|_{k+2-j,\alpha} |h|_{k+j,\alpha} \\ &\leq C(k,\alpha) |\bar{g}_1 - \bar{g}_0|_{k+2,\alpha} |h|_{k+2,\alpha}, \end{aligned} \tag{5.8}$$

$$|(\mathrm{Rm}_{\bar{g}_1} - \mathrm{Rm}_{\bar{g}_0}) * h|_{k,\alpha} \leq C(\bar{g}_0, k, \alpha) \left(|\bar{g}_1 - \bar{g}_0|_{k+2,\alpha} + |\bar{g}_1 - \bar{g}_0|^2_{k+1,\alpha}\right) |h|_{k,\alpha}, \tag{5.9}$$

and

$$\begin{aligned} &\left|\left(\nabla^2_{\bar{g}_1} \operatorname{tr}_{\bar{g}_1} - \nabla^2_{\bar{g}_0} \operatorname{tr}_{\bar{g}_0}\right) h\right|_{k,\alpha} \\ &\leq C(k,\alpha) \sum_{j=0}^{2} |\bar{g}_1 - \bar{g}_0|_{k+2-j,\alpha} |h|_{k+j,\alpha} + C(k,\alpha) |\bar{g}_1 - \bar{g}_0|^2_{k+1,\alpha} |h|_{k,\alpha} \\ &\leq C(k,\alpha) \left(|\bar{g}_1 - \bar{g}_0|_{k+2,\alpha} + |\bar{g}_1 - \bar{g}_0|^2_{k+1,\alpha}\right) |h|_{k+2,\alpha}. \end{aligned} \tag{5.10}$$

When $|\bar{g}_1 - \bar{g}_0|_{k+2,\alpha}$ is small enough, combining (5.8), (5.9) and (5.10) yields (5.5). This proves the claim and the proposition. $\square$

Applying Proposition 5.1, we know that there exists $\bar{g}_k$ such that there is no radially parallel part in $\overline{h}_k$. We are going to establish the following decay estimates which is uniform in k. For convenience, we still denote $\overline{h}_k$ by $\overline{h}$. Suppose that $\overline{h}$ is in the form of

$$\overline{h} = \tilde{a} r g_N + \tilde{a}_0 r \tilde{B}_0 + \sum_{i=1}^{\infty} \left(a_i^+ e^{\sqrt{\mu_i} r} + a_i^- e^{-\sqrt{\mu_i} r} \right) B_i, \tag{5.11}$$

i.e., Δ_N^L is non-negative so that $\sum_{i=1}^K \left(a_i^+ \cos(\sqrt{-\mu_i} r) + a_i^- \sin(\sqrt{-\mu_i} r) \right) B_i$ vanishes.

Theorem 5.3. *Suppose that* $\bigcup_{j=1}^3 \bar{T}_{t_j L, (t_j+1)L} \subseteq \bar{T}_{0,L'}$, $t_j \in \mathbb{N}_0$, $t_1 < t_2 < t_3$ *and* $\overline{h}$ *is in the form of* (5.11). *For fixed* $0 < \beta < \sqrt{\mu}_1$ *and* $L >> 1$ *satisfying* $e^{2(\sqrt{\mu}_1 - \beta)L} > 2$, *we have*

$$\|\overline{h}\|_{t_2 L, (t_2+1)L} \leq e^{-\beta' L} \left(\|\overline{h}\|_{t_1 L, (t_1+1)L} + \|\overline{h}\|_{t_3 L, (t_3+1)L} \right), \tag{5.12}$$

where

$$\beta' < \min \left\{ \beta, \frac{1}{2} \log \left(\frac{t_3^2 + L t_3 + \frac{L^2}{3}}{t_2^2 + L t_2 + \frac{L^2}{3}} \right) \right\}. \tag{5.13}$$

Remark 5.4. The advantage of (5.12) is that, on the union

$$\bigcup_{j=0}^{\infty} \bar{T}_{t_j L, (t_j+1)L},$$

for any fixed β' satisfying (5.13), we can choose $\beta'' > \beta'$ slightly larger and obtain the inequality

$$\|\overline{h}\|_{t_j L, (t_j+1)L} \ \leq \ e^{-\beta'' L} \Big(\|\overline{h}\|_{t_{j-1} L, (t_{j-1}+1)L} + \|\overline{h}\|_{t_{j+1} L, (t_{j+1}+1)L} \Big), \quad j \in \mathbb{N}.$$

From this, one shows that there exists $\tilde{L}(\beta', \beta'')$ such that whenever $L > \tilde{L}$, we have

$$\begin{aligned} &\text{either} \quad \|\overline{h}\|_{t_{j-1} L, (t_{j-1}+1)L} \ \geq \ e^{2\beta' L} \, \|\overline{h}\|_{t_j L, (t_j+1)L} \\ &\text{or} \quad \|\overline{h}\|_{t_{j+1} L, (t_{j+1}+1)L} \ \geq \ e^{2\beta' L} \, \|\overline{h}\|_{t_j L, (t_j+1)L}. \end{aligned}$$

Moreover, the following monotonicity properties hold:

$$\begin{aligned} &\|\overline{h}\|_{t_j L, (t_j+1)L} \ \geq \ e^{2\beta' L} \, \|\overline{h}\|_{t_{j-1} L, (t_{j-1}+1)L} \\ \Rightarrow &\|\overline{h}\|_{t_{j+1} L, (t_{j+1}+1)L} \ \geq \ e^{2\beta' L} \, \|\overline{h}\|_{t_j L, (t_j+1)L}, \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} &\|\overline{h}\|_{t_j L, (t_j+1)L} \ \geq \ e^{2\beta' L} \, \|\overline{h}\|_{t_{j+1} L, (t_{j+1}+1)L} \\ \Rightarrow &\|\overline{h}\|_{t_{j-1} L, (t_{j-1}+1)L} \ \geq \ e^{2\beta' L} \, \|\overline{h}\|_{t_j L, (t_j+1)L}. \end{aligned} \tag{5.15}$$

In particular, examining the proof of Theorem 5.3 shows that the restriction

$$\beta' \;\leq\; \tfrac{1}{2}\log\left(\frac{t_3^2+Lt_3+\frac{L^2}{3}}{t_2^2+Lt_2+\frac{L^2}{3}}\right)$$

is only required to control the rB-modes in the monotone increasing case (5.14). Consequently, in (5.15) it suffices to assume merely that

$$\beta' < \sqrt{\mu_1},$$

where μ_1 denotes the first positive eigenvalue on the cross section.

Proof. Without loss of generality, we may assume that symmetric 2-tensors B_i, $i \geq 0$ are orthonormal in the L^2 sense and let $j = 1, 2, 3$.

- B_i, $i \geq 1$. By a direct calculation, we have

$$\begin{aligned}
&\left\|\sum_{i=1}^{\infty}\left(a_i^{+}e^{\sqrt{\mu_i}r}+a_i^{-}e^{-\sqrt{\mu_i}r}\right)B_i\right\|^2_{t_jL,(t_j+1)L}\\
&=\sum_{i=1}^{\infty}\int_{t_jL}^{(t_j+1)L}\int_N\left(a_i^{+}e^{\sqrt{\mu_i}r}+a_i^{-}e^{-\sqrt{\mu_i}r}\right)^2|B_i|^2dr\,\mathrm{dvol}\\
&=\sum_{i=1}^{\infty}|a_i^{+}|^2\frac{e^{2\sqrt{\mu_i}L}-1}{2\sqrt{\mu}_i}e^{2\sqrt{\mu_i}t_jL}+2a_i^{+}a_i^{-}L+|a_i^{-}|^2\frac{1-e^{-2\sqrt{\mu_i}L}}{2\sqrt{\mu}_i}e^{-2\sqrt{\mu_i}t_jL}\\
&:=\sum_{i=1}^{\infty}\left(C_ie^{2\sqrt{\mu_i}t_jL}+D_i+E_ie^{-2\sqrt{\mu_i}t_jL}\right).
\end{aligned}$$

For fixed $\beta < \sqrt{\mu_1}$, we can choose L sufficiently large such that

$$\begin{aligned}
C_ie^{2\sqrt{\mu_i}t_jL} &= C_ie^{2\sqrt{\mu_i}t_{j+1}L}e^{2\sqrt{\mu_i}(t_j-t_{j+1})L}\\
&\leq C_ie^{2\sqrt{\mu_i}t_{j+1}L}e^{2\sqrt{\mu_1}(t_j-t_{j+1})L}\\
&\leq C_ie^{2\sqrt{\mu_i}t_{j+1}L}e^{-2\sqrt{\mu_1}L}\leq\frac{1}{2}e^{-2\beta L}C_ie^{2\sqrt{\mu_i}t_{j+1}L}
\end{aligned}$$

and

$$\begin{aligned}
E_ie^{-2\sqrt{\mu_i}t_jL} &= E_ie^{-2\sqrt{\mu_i}t_{j-1}L}e^{-2\sqrt{\mu_i}(t_j-t_{j-1})L}\\
&\leq E_ie^{-2\sqrt{\mu_i}t_{j-1}L}e^{-2\sqrt{\mu_1}(t_j-t_{j-1})L}\\
&\leq E_ie^{-2\sqrt{\mu_i}t_{j-1}L}e^{-2\sqrt{\mu_1}L}\leq\frac{1}{2}e^{-2\beta L}E_ie^{-2\sqrt{\mu_i}t_{j-1}L}.
\end{aligned}$$

Besides, by Cauchy–Schwarz inequality and Taylor expansion, we have

$$|D_i|\left(1-2e^{-2\beta L}\right)\leq\frac{1}{2}e^{-2\beta L}\left(C_ie^{2\sqrt{\mu_i}t_{j+1}L}+E_ie^{-2\sqrt{\mu_i}t_{j-1}L}\right).$$

Combining them together, we have

$$C_ie^{2\sqrt{\mu_i}t_jL}+D_i+E_ie^{-2\sqrt{\mu_i}t_jL}$$

$$
\begin{aligned}
&\leq \frac{1}{2}e^{-2\beta L}C_i e^{2\sqrt{\mu_i}t_{j+1}L} + \frac{1}{2}e^{-2\beta L}E_i e^{-2\sqrt{\mu_i}t_{j-1}L} \\
&+ \frac{1}{2}e^{-2\beta L}\left(C_i e^{2\sqrt{\mu_i}t_{j+1}L} + E_i e^{-2\sqrt{\mu_i}t_{j-1}L}\right) + 2e^{-2\beta L}D_i \\
&\leq e^{-2\beta L}\left(C_i e^{2\sqrt{\mu_i}t_{j+1}L} + D_i + E_i e^{-2\sqrt{\mu_i}t_{j+1}L}\right) \\
&+ e^{-2\beta L}\left(C_i e^{2\sqrt{\mu_i}t_{j-1}L} + D_i + E_i e^{-2\sqrt{\mu_i}t_{j-1}L}\right).
\end{aligned}
$$

- g_N and B_0. Taking $\tilde{a}_0 r\tilde{B}_0$ as an example. Without loss of generality, we may assume that $\tilde{a}_0 = 1$ and $\int_N |B_0|^2\,\mathrm{dvol}_N = 1$. By a direct calculation, we have

$$
\left\|\tilde{a}_0 r\tilde{B}_0\right\|^2_{t_jL,(t_j+1)L} = \int_{t_j}^{t_j+L}\int_N r^2|B_0|^2 dr\,\mathrm{dvol}_N = Lt_j^2 + L^2t_j + \frac{L^3}{3}. \tag{5.16}
$$

Therefore,

$$
\left\|\tilde{a}_0 r\tilde{B}_0\right\|^2_{t_2L,(t_2+1)L} \leq e^{-2\beta' L}\left(\left\|\tilde{a}_0 r\tilde{B}_0\right\|^2_{t_1L,(t_1+1)L} + \left\|\tilde{a}_0 r\tilde{B}_0\right\|^2_{t_3L,(t_3+1)L}\right)
$$

follows immediately from the definition of β'. Similarly, we can handle $\tilde{a}rg_N$.

Combining them together, we obtain the desired estimate (5.12). □

We now show that the decay estimates obtained in Theorem 5.3 can hold for $\hat{h} := \phi^* g - \hat{g}$ as long as $\hat{g}$ is close enough to $\bar{g}_0$.

Theorem 5.5. *There exists a positive constant $\chi(g) > 0$ such that if $|\hat{g} - \bar{g}_0|_{k,\alpha} < \chi$ and $\hat{h}$ satisfies $|\hat{h} - \bar{g}_0|_{k,\alpha} < \chi$ and*

$$
\delta_\tau(\hat{h}) = 0, \quad \mathrm{tr}(\hat{h}) = c_0 + \bar{c}_0 r, \quad (\pi\hat{h})_0 = 0 \tag{5.17}
$$

then the statement in Theorem 5.3 holds for $\hat{h}$ as well.

Proof. We prove it by contradiction. If the conclusion of Theorem 5.3 is not correct in $\bigcup_{j=1}^3 \bar{T}_{jL,(j+1)L}$, we assume that there exists a sequence of gauges ϕ_i such that $\chi_i \to 0$. We define the renormalized metric by $\hat{h}_i := \frac{h_i}{\|h_i\|_{2L,3L}}$, $h_i := \phi_i^* g - \bar{g}_i$. By the assumption, there exists a constant C independent of i such that

$$
C\|h_i\|_{2L,3L} \geq \|h_i\|_{1L,2L} + \|h_i\|_{3L,4L}.
$$

Combining this with the interior estimate: for any compact subset $K \subseteq \bigcup_{j=1}^3 \bar{A}_{jL,(j+1)L}$, there exists a constant $C(g, K, k, \alpha)$ independent of i such that

$$
|h_i|_{C^{k,\alpha}(K)} \leq C(g, K, k, \alpha)\|h_i\|_{1L,4L},
$$

there is a convergent subsequence $\hat{h}_{t_i} \to \hat{h}_\infty$ in $C^{k,\alpha}$ and $\hat{h}_\infty$ satisfies (4.5) and $(\pi\hat{h}_\infty)_0 = 0$. The desired contradiction follows from Theorem 5.3. Repeating the above argument completes the proof. □

Proof of Theorem A. Following the same strategy in [CT94], we first show that $L' = +\infty$. Otherwise by Corollary 3.4 and Proposition 5.1, we can know that on $[L'-3L, L'] \times N^{n-1}$

$$|\Phi^* g - \bar{g}_0|_{k+1,\alpha} > c\chi, \quad |\bar{g}_1 - \bar{g}_0|_{k+1,\alpha} < \frac{c\chi}{40},$$

which implies that

$$|\Phi^* g - \bar{g}_1|_{k+1,\alpha} > \frac{c\chi}{10}.$$

Recall that by the elliptic estimate and

$$|\nabla_{\bar{g}_0}(\Phi^* g)|_{k,\alpha} < \varepsilon,$$

we always have

$$\begin{aligned} \|h\|_{L'-2L,L'-L} + c_1(n)L^{\frac{1}{2}}\varepsilon &\geq \|h\|_{L'-L,L'}, \\ \|h\|_{L'-2L,L'-L} + c_2(n)L^{\frac{1}{2}}\varepsilon &\leq \|h\|_{L'-L,L'}. \end{aligned} \tag{5.18}$$

In other words, the change of $\|h\|$ over a tube with length L is almost $L^{\frac{1}{2}}\varepsilon$. If there is no rg_N or $r\tilde{B}_0$ in πh, we can apply Theorem 5.3 as [CT94, Proof of Theorem 0.13] did to establish the desired result because in this case $\|h\|$ is exponentially increasing or decreasing which contradicts to (5.18) obviously. Otherwise, if the rate of change of $\|h\|$ is determined by rg_N or $r\tilde{B}_0$, (5.16) in Theorem 5.3 indicates that as L is sufficiently large, the change of $\|h\|$ is $O(L^{\frac{3}{2}})$, contradicting to (5.18). So $L' = \infty$.

Consider the sequence of tubes $\bar{T}_{jL,(j+1)L}$, $j \in \mathbb{N}_0$, together with the associated metrics $\bar{g}_j$ from Proposition 5.1. Set $h_j := \Phi^* g - \bar{g}_j$, it holds that $(\pi h_j)_0 = 0$. By reasoning as above and Remark 5.4, we can assume that (5.15) holds for all j.

Passing to a subsequence $\{t_j\}$, there exists a Ricci-flat cylinder $\bar{g}_\infty$ such that

$$\lim_{j\to\infty} |\bar{g}_{t_j} - \bar{g}_\infty|_{k,\alpha} = 0.$$

We claim that

$$\lim_{j\to\infty} \|h_{t_j}\|_{t_jL,(t_j+1)L} = 0.$$

Indeed, if not, then by Theorem 5.5,

$$\begin{aligned} \|h_{t_{j+i}}\|_{t_{j+i}L,(t_{j+i}+1)L} \geq &\max\left\{\frac{t_{j+i}^2 + Lt_{j+i} + \frac{L^2}{3}}{t_j^2 + Lt_j + \frac{L^2}{3}}, e^{(t_{j+i}-t_j)\sqrt{\mu_1}}\right\} \\ &\times \|h_{t_j}\|_{t_jL,(t_j+1)L}, \end{aligned}$$

and let $i \to \infty$, the above inequality indicates that

$$\|h_{t_{j+i}}\|_{t_{j+i}L,(t_{j+i}+1)L} \to \infty$$

contradicting to the uniform bound:

$$|h_{t_{j+i}}|_{k,\alpha} < \chi, \text{ on } [t_{j+i}L, (t_{j+i}+1)L] \times N^{n-1}, \forall t_{j+i} \in \mathbb{N}_0.$$

Therefore, by standard elliptic estimates, there exists $C > 0$ such that

$$|\Phi^* g - \bar{g}_0|_{k,\alpha} \le Ce^{-2\beta' r}.$$

Here β' may be taken to be any number in $(0, \sqrt{\mu_1})$, by the same reasoning as above. Since $\bar{g}_0$ is a cylindrical metric, we conclude that $\bar{g}_\infty = \bar{g}_0$. This completes the proof.

□

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