On the maximal size of (a,b)-town (mod k) families

Nikola Veselinov, Miroslav Marinov*

Abstract

A family $\mathcal{F} \subseteq \mathscr{P}(n)$ is an (a,b)-town (mod k) if all sets in it have cardinality a (mod k) and all pairwise intersections in it have cardinality b (mod k). For $k=2$ the maximal size of such a family is known for each a,b, while for $k=3$ only $b-a \equiv 2 \pmod 3$ is fully understood. We provide a bound for $k=3$ when $b-a \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$, which turns out to be tight for infinitely many such n. We also give sufficient conditions on the parameters a,b,k,n, which result in a better bound than the one from general settings by Ray-Chaudhuri–Wilson, in particular showing that this bound occurs infinitely often in a sense where all of a,b,n can vary for a fixed k.

1 Introduction

Oddtown in extremal set theory is a problem from the origins of algebraic combinatorics where elegant accessible proofs rely on linear algebra rather than pure combinatorial arguments. Throughout we denote by $[n]$ the set $\{1,2,\ldots,n\}$ for a positive integer n (conventionally, $[0]=\emptyset$), and by $\mathscr{P}(n)$ the power set of $[n]$. The original formulation of Oddtown concerns the maximal size of a family $\mathcal{F} \subseteq \mathscr{P}(n)$ whose sets have odd cardinality and whose intersections of any two distinct sets have even cardinality. This maximum was conjectured to be n by Erdős and proven independently by Berlekamp [3] and Graver [6]. Their methods are applicable in more general settings – the only importance of the integer 2 is that it is a prime. More recently, a counting proof of the Oddtown problem has been established by Petrov [9], although the methods used are based on symmetric difference arguments and so are limited to mod 2.

A different generalization, also attracting considerable attention, is where individual set cardinalities are only required to be non-multiples of k and pairwise intersections to be multiples of k. The extremal size of these families is proven to be n for all prime power moduli, but is yet open for all others. The best known upper bound [2] for mod 6 is $2n - \log_2 n$.

We consider the following generalization.

Definition 1.1. *Let a,b,k be integers with $k \geq 2$ and $0 \leq a,b \leq k-1$. A family of sets $\mathcal{F} \subseteq \mathscr{P}(n)$ is an (a,b)-town* (mod k) *if for any distinct $A,B \in \mathcal{F}$ we have $|A| \equiv a$* (mod k) *and $|A \cap B| \equiv b$* (mod k).

The classical Oddtown is $(1,0)$-town (mod 2). The other three cases of (a,b)-town (mod 2) are also well known – for $(0,1)$ the exact result is $n-1+(n \pmod 2)$, while for $(0,0)$ (*Eventown*) and $(1,1)$ the extremal sizes are $2^{\lfloor \frac{n}{2} \rfloor}$ and $2^{\lfloor \frac{n-1}{2} \rfloor}$, respectively [2, Chapter 1 and 2].

We remark that if $\mathcal{F}$ is an (a,b)-town (mod k), then it is also an (a,b)-town (mod d) for any divisor d of k. All results in this paper can be adapted to mod k if we work with a single prime divisor of k, so throughout we shall work only with prime modulus p.

As it turns out, the maximal size of an (a,b)-town (mod p) is exponential if and only if $a=b$. When $a \neq b$, a classical result by Ray-Chaudhuri–Wilson implies a linear upper bound in our setting.

*nikola.veselinov.veselinov@gmail.com, m.marinov1617@gmail.com

Theorem 1.2. (Modular RW [2, Theorem 5.37]) *Let n be a positive integer, p be a prime and L be a set of $s \leq p-1$ integers. Let $t \geq 0$ be an integer with $t \notin L \pmod p$ and $s+t \leq n$. Let $\mathcal{F} \subseteq \mathscr{P}(n)$ be a family of sets such that $|E| \equiv t \pmod p$ and $|E \cap F| \in L \pmod p$ for any distinct $E, F \in \mathcal{F}$. Then $|\mathcal{F}| \leq \binom{n}{s}$.*

For $s = 1$, $t = a$, $L = \{b\}$ this implies $|\mathcal{F}| \leq n$ for any (a,b)-town $\pmod p$ if p does not divide $a - b$.

Results concerning (a,b)-town are currently limited even for modulo 3. Three of the nine cases have been solved [1], namely, $(1,0)$, $(2,1)$ and $(0,2)$. For $(0,1), (1,2), (2,0)$ we have the linear bound from Theorem 1.2, but we are also able to improve it when $n \equiv 2 \pmod 3$ and partially for $n \equiv 0 \pmod 3$.

Proposition 1.3. *Suppose $\mathcal{F}$ is an (a,b)-town mod 3 family of sets in $[n]$ where $b - a \equiv 1 \pmod 3$ and one of the following holds*:

- $n \equiv 2 \pmod 3$;
- $n \equiv 0 \pmod 3$ *and* $a = 0$, $b = 1$.

Then $|\mathcal{F}| \leq n - 1$.

This bound turns out to be essentially tight – Lahtonen [7] has given a $(2,0)$-town family of size $n-1$ for infinitely many n, namely when $n \equiv 8 \pmod{12}$ and $n-1$ is power of a prime, as well as a $(1,2)$-town of size $n-1$ when $n \equiv 7 \pmod{12}$ is a power of a prime. Furthermore, by taking the complement of each set in his examples, we obtain $(0,1)$-towns of size $n-1$ for infinitely many $n \equiv 1, 2 \pmod 3$.

Regarding $(0,0) \pmod p$, a known bound by Frankl and Odlyzko [4] is $2^{\lfloor \frac{n}{2} \rfloor}$. We are able to give a modified proof to show a similar one holds for $(m,m) \pmod p$.

Proposition 1.4. *Let p be a prime and m be an integer with $1 \leq m \leq p-1$. For any (m,m)-town $\pmod p$ family $\mathcal{F} \subseteq \mathscr{P}(n)$ we have $|\mathcal{F}| \leq 2^{\lfloor \frac{n+1}{2} \rfloor}$.*

Hence so far the bounds on the maximal size for (a,b)-town (mod 3) are as in the table below.

(a,b)	$n \equiv 0 \pmod 3$	$n \equiv 1 \pmod 3$	$n \equiv 2 \pmod 3$
$(0,0)$	Lower: $24^{\lfloor \frac{n}{12} \rfloor}$ Upper: $2^{\lfloor \frac{n}{2} \rfloor}$	Lower: $24^{\lfloor \frac{n}{12} \rfloor}$ Upper: $2^{\lfloor \frac{n}{2} \rfloor}$	Lower: $24^{\lfloor \frac{n}{12} \rfloor}$ Upper: $2^{\lfloor \frac{n}{2} \rfloor}$
(m,m) $m \in \{1,2\}$	Lower: $24^{\lfloor \frac{n-m}{12} \rfloor}$ Upper: $2^{\lfloor \frac{n+1}{2} \rfloor}$	Lower: $24^{\lfloor \frac{n-m}{12} \rfloor}$ Upper: $2^{\lfloor \frac{n+1}{2} \rfloor}$	Lower: $24^{\lfloor \frac{n-m}{12} \rfloor}$ Upper: $2^{\lfloor \frac{n+1}{2} \rfloor}$
$(0,2)$	Tight: $n-2$ [1]	Tight: n [1]	Tight: $n-1$ [1]
$(1,0)$	Tight: n	Tight: n	Tight: n
$(2,1)$	Tight: n [1]	Tight: $n-1$ [1]	Tight: $n-1$ [1]
$(0,1)$	Lower: $\lfloor \frac{n-1}{2} \rfloor$ Upper: $n-1$	Lower: $\lfloor \frac{n-1}{2} \rfloor$ Upper: n	Lower: $\lfloor \frac{n}{2} \rfloor$ Upper: $n-1$
$(1,2)$	Lower: $\lfloor \frac{n}{2} \rfloor$ Upper: n	Lower: $\lfloor \frac{n-1}{2} \rfloor$ Upper: n	Lower: $\lfloor \frac{n-2}{2} \rfloor$ Upper: $n-1$
$(2,0)$	Lower: $\lfloor \frac{n}{2} \rfloor$ Upper: n	Lower: $\lfloor \frac{n}{2} \rfloor$ Upper: n	Lower: $\lfloor \frac{n}{2} \rfloor$ Upper: $n-1$

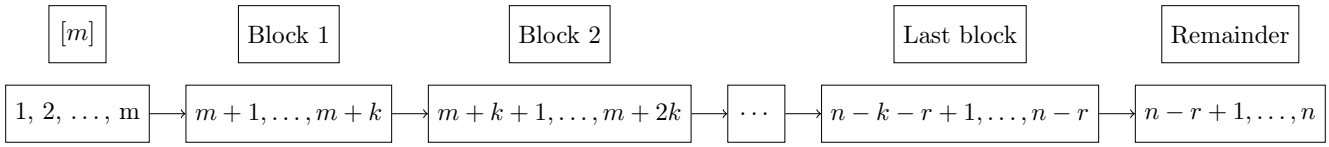
As we already mentioned, we know tight examples for infinitely many, but far not all, $n \equiv 2 \pmod 3$ for $b - a \equiv 1 \pmod 3$. The displayed universal lower bounds for these (a,b) are based on the elementary examples where every element of $[n]$ belongs either to precisely one set or to all sets except one.

When $a = b$ a classical block-type construction of size $2^{\lfloor \frac{n-m}{k} \rfloor}$ is

- If $n \leq k$: $\mathcal{B}(m,k,n) := \{F\}$, where $F = [m]$.
- If $n > k$: for $N = \left\lfloor \frac{n-m}{k} \right\rfloor$ set

$$\mathcal{B}(m,k,n) := \left\{ [m] \cup \left(\bigcup_{i \in F} \{ki+m,\ ki+m-1, \ldots,\ ki+m-(k-1)\} \right) \mid F \subseteq \mathscr{P}(N) \right\}.$$

The sets are formed as unions of $[m]$ and any number of blocks of size k.



This construction is not maximal for moduli greater than 2. Frankl and Odlyzko [4] gave a construction of size $(8k)^{\left\lfloor \frac{n}{4k} \right\rfloor}$ for $(0,0)$-town (mod k) families for arbitrary $k \geq 2$, relying on Hadamard matrices. Furthermore, taking the unions of every set in the latter construction with a disjoint set of size m yields an (m,m)-town of size $(8k)^{\left\lfloor \frac{n-m}{4k} \right\rfloor}$, also asymptotically larger than the block construction. Similar constructions exist for other generalizations of Eventown [5].

Considering $(0,0)$-town (mod 3), the Frankl-Odlyzko construction is a family of size $24^{\left\lfloor \frac{n}{12} \right\rfloor}$. A computer verification confirms that 24 is the maximal size for $n = 12$. See [10] for computed extremal sizes for all cases (a,b) mod k, $k \in \{3,4,5,6\}$, up to $n = 10$ (and up to $n = 13$ for $(0,0)$-town (mod 3)).

Regarding the Ray-Chaudhuri–Wilson bound, we shall prove that there are infinitely many quadruples (a,b,p,n) such that $|\mathcal{F}| \leq n-1$ in a sense where all of a,b,n can vary for a fixed prime p.

Theorem 1.5. *Let p be a prime and a,b be integers with $0 \leq a,b \leq p-1$ and p does not divide b and $a-b$. There are infinitely many n for which any (a,b)-town (mod p) family $\mathcal{F} \subseteq \mathscr{P}(n)$ satisfies $|\mathcal{F}| \leq n-1$.*

We list notation used throughout the paper. For integers x and y write $y \mid x$ to indicate that y divides x, otherwise write $y \nmid x$. For a set $A \in \mathscr{P}(n)$ denote its complement by $A^c = [n] \setminus A$. Denote by $\langle u, v \rangle := \sum_{i=1}^{m} u_i v_i$ the scalar product of $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_m)$. For a set X and subset $A \subseteq X$ the indicator function $\mathbf{1}_A$ of A on X is such that $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ otherwise. For a subset $A \in \mathscr{P}(n)$ denote by $\chi_A := (\mathbf{1}_A(1), \mathbf{1}_A(2), \ldots, \mathbf{1}_A(n))$ its characteristic vector.

It is sometimes useful to augment a characteristic vector by appending a scalar α as its last coordinate. This construction can be used to encode additional information, independent of the vector's corresponding set. The idea is adapted from [1].

Definition 1.6. *Let $\mathbb{F}$ be a field and $\alpha \in \mathbb{F}$ be a scalar. For a set $A \in \mathscr{P}(n)$ we denote by χ_A^α the* α-characteristic vector *of A, where*

$$\chi_A^\alpha := (\mathbf{1}_A(1), \mathbf{1}_A(2), \ldots, \mathbf{1}_A(n), \alpha) \in \mathbb{F}^{n+1}.$$

2 Main proofs

2.1 Uniform upper bound for $a = b$

Recall that in a finite-dimensional vector space V, equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$, a subspace $U \subseteq V$ is *totally isotropic* if $\langle u_1, u_2 \rangle = 0$ for any (not necessarily distinct) $u_1, u_2 \in U$. It is well known that if the form is non-degenerate over V and U is totally isotropic, then $\dim U \leq \left\lfloor \frac{1}{2} \dim V \right\rfloor$. We also need (see e.g. [8]) that any k-dimensional vector space has at most 2^k binary vectors, i.e. such that each of their entries is 0 or 1.

Proof of Proposition 1.4. Let $\alpha \in \mathbb{F}_{p^2}$ be such that $\alpha^2 + m = 0$. Denote by $\chi_i \in \mathbb{F}_{p^2}^{n+1}$ the α-characteristic vector of $F_i \in \mathcal{F}$ and observe that

$$\langle \chi_i, \chi_j \rangle = \alpha^2 + m = 0$$

for all i, j. If U is the span of the χ_i-s, then as U is totally isotropic, we deduce $\dim U \leq \left\lfloor \frac{n+1}{2} \right\rfloor$. Finally, as the first n entries of the α-characteristic vectors are each 0 or 1, we conclude $|\mathcal{F}| \leq 2^{\left\lfloor \frac{n+1}{2} \right\rfloor}$. □

2.2 Upper bounds for $a \neq b$

A direct application of the Modular RW Theorem (Theorem 1.2) shows that the maximal size of an (a, b)-town (mod p) family is bounded above by n when $a \neq b$. Let us present an alternative proof using α-characteristic vectors, whose ideas will be useful for obtaining the stronger results.

Lemma 2.1. *Let $\mathcal{F} \subseteq \mathscr{P}(n)$ be an (a, b)-town* (mod p)*, where p does not divide $a - b$. Then $|\mathcal{F}| \leq n$.*

Proof. Let α be an element of $\mathbb{F}_{p^2}$ such that $\alpha^2 + b = 0$. Let $\mathcal{F} = (F_1, \ldots, F_{|\mathcal{F}|})$. Denote by $\chi_i \in \mathbb{F}_{p^2}^{n+1}$ the α-characteristic vector of $F_i \in \mathcal{F}$ for each $i = 1, \ldots, |\mathcal{F}|$, and observe that

$$\langle \chi_i, \chi_j \rangle = |F_i \cap F_j| + \alpha^2 = \begin{cases} a - b, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let $\lambda_1, \ldots, \lambda_{|\mathcal{F}|}$ be scalars such that $\lambda_1\chi_1 + \ldots + \lambda_{|\mathcal{F}|}\chi_{|\mathcal{F}|} = 0$. Taking the scalar product with χ_i, we obtain $(a - b)\lambda_i = 0$ and hence $\lambda_i = 0$ for all i, i.e. the vectors are linearly independent. Moreover, each of them is orthogonal to $v = (1, 1, \ldots, 1, -a\alpha^{-1})$ for $b \not\equiv 0 \pmod{p}$ and to $e = (0, 0, \ldots, 0, 1)$ for $b \equiv 0 \pmod{p}$, so they lie in a subspace $V \subseteq \mathbb{F}_{p^2}^{n+1}$ of dimension n. The result follows. □

We are now ready to proceed to the improvements of the upper bound to $n - 1$.

Proposition 2.2. *Let a, b, n be non-negative integers and p be a prime. Suppose at least one of the following holds*:

(i) $p \nmid a,\ b,\ a - b,\ n,\ a^2 - nb - a + b$;

(ii) $p \mid a$ *and* $p \nmid b,\ n - 1$.

Then for any (a, b)-town (mod p) *family $\mathcal{F} \subseteq \mathscr{P}(n)$ we have $|\mathcal{F}| \leq n - 1$.*

Note that Theorem 1.5 follows for $p \geq 3$ from this proposition, as for any a and b with $p \nmid b, a - b$ there are at most two forbidden congruence classes for n (mod p), hence at least one attainable.

Proof. In both cases the conditions of Lemma 2.1 hold, so throughout we shall use all notation and computations from its proof. We always have $|\mathcal{F}| \leq n$.

Suppose firstly that $p \nmid a,\ b,\ a - b,\ n,\ a^2 - nb - a + b$. Assume for contradiction that $|\mathcal{F}| = n$. Then $\chi_1, \ldots, \chi_n$ form a basis of V. Consider $u = (1, 1, \ldots, 1, na^{-1}\alpha)$ and observe that $u \in V$ since $p \nmid b$ and u is orthogonal to v. Then there exist $\lambda_1, \ldots, \lambda_n$, at least one being nonzero, such that $u = \lambda_1\chi_1 + \ldots + \lambda_n\chi_n$. Taking the scalar product of both sides with χ_i, we obtain for all $i = 1, \ldots, n$

$$a - na^{-1}b = \lambda_i(a - b) \Leftrightarrow \lambda_i = \frac{a - na^{-1}b}{a - b} = \frac{a^2 - nb}{a(a - b)}.$$

Denote the common value of λ_i by Λ. If $p \mid a^2 - nb$, then $\lambda_i = \Lambda = 0$ for all i, contradiction and we are done. Suppose $p \nmid a^2 - nb$. Now that $u = \Lambda \sum_{i=1}^{n} \chi_i$, we derive another expression for Λ, this time by taking the scalar product of u with itself. This gives

$$n - n^2a^{-2}b = \langle u, u \rangle = \Lambda^2 \left\langle \sum_{i=1}^{n} \chi_i, \sum_{j=1}^{n} \chi_j \right\rangle = \Lambda^2 n(a - b).$$

Therefore, as $p \mid n$,

$$\left(\frac{a^2-nb}{a(a-b)}\right)^2 = \Lambda^2 = \frac{1-na^{-2}b}{a-b} = \frac{a^2-nb}{a^2(a-b)}.$$

Cancelling common non-zero terms on both sides now leads to $a^2 - nb = a - b$ in $\mathbb{F}_{p^2}^{n+1}$. This contradicts the assumption $p \nmid a^2 - nb - a + b$ and completes the proof in this case.

Now suppose $p \mid a$ and $p \nmid b, n-1$. Assume for contradiction that $|\mathcal{F}| = n$. Then $\chi_1, \ldots, \chi_n$ form a basis of V. Consider $e = (0, 0, \ldots, 0, 1)$ and observe that $e \in V$ since $v = (1, 1, \ldots, 1, 0)$ when $p \mid a$. Then there exist $\mu_1, \ldots, \mu_n$, at least one being nonzero, such that $e = \mu_1\chi_1 + \ldots + \mu_n\chi_n$. Taking the scalar product of both sides with χ_i, we obtain $\alpha = \mu_i(a-b) = -\mu_i b$, i.e. $\mu_i = -\alpha b^{-1}$. Denote the common value of μ_i by M. Now that $e = M\sum_{i=1}^{n}\chi_i$, we derive another expression for M, this time by taking the scalar product of e with itself – or equivalently, by comparing the last coordinate in $e = M\sum_{i=1}^{n}\chi_i$. We obtain $1 = Mn\alpha$. Substituting $M = -\alpha b^{-1}$ leads to $1 = -n\alpha^2 b^{-1} = n$ in $\mathbb{F}_{p^2}^{n+1}$. This contradicts the assumption $p \nmid n-1$ and completes the proof. □

We now check how the (a,b)-town property is changed when taking complements.

Definition 2.3. *For a family $\mathcal{F} = \{F_1, \ldots, F_m\} \subseteq \mathscr{P}(n)$ the* substitution family *is $\mathcal{F}_\xi := \{F_1^c, \ldots, F_m^c\}$.*

Since $|\mathcal{F}| = |\mathcal{F}_\xi|$, any bound for $\mathcal{F}_\xi$ is also a bound for $\mathcal{F}$. In what follows it is important that the difference $a-b$ is the same in the original and in the substitution family.

Lemma 2.4. *Let $\mathcal{F} \subseteq \mathscr{P}(n)$ be an (a,b)-town* (mod k). *Then $\mathcal{F}_\xi$ is an $(n-a, n-2a+b)$-town* (mod k).

Proof. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ and $\mathcal{F}_\xi = \{F_1^c, \ldots, F_m^c\}$. We have $|F_i^c| = n - |F_i| \equiv n - a \pmod{k}$ and

$$\left|F_i^c \cap F_j^c\right| \equiv n - |F_i| - |F_j| + |F_i \cap F_j| \equiv n - 2a + b \pmod{k}, \;\; i \neq j.$$ □

Combining Proposition 2.2 and Lemma 2.4 immediately gives even more tuples (a,b,p,n), where the upper bound is less than n.

Proposition 2.5. *Let a, b, n be non-negative integers and p be a prime. Suppose at least one of the following holds*:

(i) $p \nmid n-a,\; n-2a+b,\; a-b,\; n,\; a^2-nb-a+b$;

(ii) $p \mid n-a$ *and* $p \nmid n-2a+b,\; n-1$.

Then for any (a,b)-town (mod k) *family $\mathcal{F} \subseteq \mathscr{P}(n)$ we have $|\mathcal{F}| \leq n-1$.*

It is interesting that the conditions $p \nmid a-b$ and $p \nmid a^2-nb-a+b$ do not change when taking complements, and it would be nice to understand a combinatorial reason behind the second one.

Now we deduce all new improvements for modulo 3.

Proof of Proposition 1.3. Suppose $n \equiv 0 \pmod 3$, then $a = 0, b = 1$ follows from the second part of Proposition 2.2. Now suppose $n \equiv 2 \pmod 3$. Here $a = 0, b = 1$ follows from the second part of Proposition 2.2, while $a = 1, b = 2$ follows from the first part of Proposition 2.2 and $a = 2, b = 0$ follows from the second part of Proposition 2.5. □

Apart from the classical Oddtown $a = 1$, $b = 0$, we can deduce tight bounds when $a = 2$, $b = 1$ for n and p only related by a congruence condition. In fact, only $n \not\equiv 0 \pmod p$ is currently out of reach, so it would be great if a version of Proposition 2.2 with $p \mid n$ can be derived in order to cover this.

Corollary 2.6. *Let $\mathcal{F} \subseteq \mathscr{P}(n)$ be a $(2,1)$-town (mod p). If $n \equiv 3 \pmod{p}$, then $|\mathcal{F}| \leq n$, and if $n \not\equiv 0, 3 \pmod{p}$, then $|\mathcal{F}| \leq n-1$. In all cases the bound is tight.*

Proof. When $a = 2$, $b = 1$, $n \equiv 3 \pmod{p}$ the substitution family $\mathcal{F}_\xi$ is a $(1,0)$-town (mod p), since $n - a \equiv 3 - 2 \equiv 1$ and $n - 2a + b \equiv 3 - 4 + 1 \equiv 0$. By the classical Oddtown problem for a field of characteristic p, we have that n is a tight upper bound for the $(1,0)$-town $\mathcal{F}_\xi$, hence also for $\mathcal{F}$.

When $a = 2$, $b = 1$, $n \not\equiv 0, 3 \pmod{p}$ the bound follows by Proposition 2.2 for $a = 2, b = 1$ and is attained by the family $\{\{1,2\},\{1,3\},\ldots,\{1,n\}\}$. □

We conclude with a short discussion of hypotheses based on modest numerical evidence from [10]. Here the integer k need not be prime.

Conjecture. For $i = 1, 2$ let $\mathcal{F}_i \subseteq \mathscr{P}(n)$ be a (m_i, m_i)-town (mod k) family of maximum size, where $0 \leq m_1, m_2 \leq k-1$. If $m_1 < m_2$, then $|\mathcal{F}_1| \geq |\mathcal{F}_2|$.

We already know that the linear upper bound n holds when the (a,b)-town satisfies $a \not\equiv b \pmod{p}$ for some prime divisor p of k – this holds e.g. when $a \neq b$ and k is squarefree. What remains is where $a \equiv b \pmod{p}$ for every prime divisor $p \mid k$, among which we expect that $a \equiv b \pmod{k}$ is the sole exceptional case of exponential size and the rest are linear.

Conjecture. Let $\mathcal{F} \subseteq \mathscr{P}(n)$ be an (a,b)-town (mod k), where $a \not\equiv b \pmod{k}$. Then $|\mathcal{F}| \leq n$.

References

[1] A. Rast. Families of subsets. International Tournament of Young Mathematicians (ITYM), 2020.

[2] L. Babai and P. Frankl. Linear algebra methods in combinatorics. `people.cs.uchicago.edu/~laci/babai-frankl-book2022.pdf`, 2022. Manuscript, available online.

[3] E. R. Berlekamp. On subsets with intersections of even cardinality. *Canadian Mathematical Bulletin*, 12(4): 471–474, 1969.

[4] P. Frankl and A. M. Odlyzko. On subsets with cardinalities of intersections divisible by a fixed integer. *European Journal of Combinatorics*, 4(3): 215–220, 1983.

[5] P. Frankl and N. Tokushige. Uniform eventown problems. *European Journal of Combinatorics*, 51: 280–286, 2016.

[6] J. E. Graver. Boolean designs and self-dual matroids. *Linear Algebra and its Applications*, 10(2): 111–128, 1975.

[7] J. Lahtonen. Family of sets with $|F_i| \equiv 2 \pmod 3$ and $|F_i \cap F_j| \equiv 0 \pmod 3$. Mathematics Stack Exchange. https://math.stackexchange.com/q/4344920 (version: 2021-12-30).

[8] A. M. Odlyzko. On the ranks of some $(0,1)$-matrices with constant row sums. *Journal of the Australian Mathematical Society*, 31(2): 193–201, 1981.

[9] F. Petrov. List of counting proofs instead of linear algebra method in combinatorics. MathOverflow. https://mathoverflow.net/q/230906 (version: 2021-02-04).

[10] N. Veselinov. Visualization of extremal sizes of (a,b)-town (mod k) families $\mathcal{F} \subseteq \mathscr{P}(n)$. `nikolaveselinov.github.io/abtown-visualization.html`.