Improved upper bounds on color reversal by local inversions*

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Abstract. We study the problem of color reversal in bicolored graphs under local inversions. A *bicoloration* of a graph $G = (V, E)$ is a mapping $\beta : V \to \{-1, 1\}$. A *local inversion* at a vertex $v \in V$ consists of reversing the colors of all neighbors of v and replacing the subgraph induced by these neighbors with its complement, while leaving v and the rest of G unchanged. Sabidussi (Discrete Mathematics, 1987) showed that any bicolored graph on n vertices without isolated vertices can be color-reversed (that is, all vertex colors flipped while preserving the underlying graph) in at most $6n + 3$ local inversions, and that any bicolored graph can be transformed into another bicolored graph on the same underlying graph in at most $9n$ local inversions. We improve both bounds: we prove that the first task can be accomplished in at most $4n-3$ local inversions, and the second in at most $\lfloor \frac{11n-3}{2} \rfloor$ local inversions. Furthermore, we show that for stars and complete graphs, color reversal can be performed with at most $3n$ local inversions.

Keywords: local inversion · local complementation · bicolored graph · color reversal

1 Introduction

Local transformations of graphs form a common language across structural graph theory, algebraic graph theory, and quantum information. Among these, *local complementation*—toggling adjacency within the open neighborhood of a chosen vertex—plays a central role. It appears in Bouchet's theory of isotropic systems and the structure of circle graphs [2,3], underpins the vertex–minor relation and rank-width [10], and captures equivalences between quantum graph states under local Clifford operations [9,8]. Closely related, Seidel switching is well known for its applications to combinatorial and algebraic aspects of graphs [13]. Local complementation falls under the broad umbrella of graph

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modification problems, which have been extensively studied in algorithmic and parameterized complexity [1,5,7].

We work with simple, undirected, labelled graphs $G = (V, E)$ whose vertices carry a binary color (encoded by $\beta : V \to \{-1, +1\}$). A single move, called *local inversion*, at a vertex $v \in V$ applies local complementation at v to the underlying graph and simultaneously flips the colors of all neighbors of v. Iterating moves along a string $w \in V^*$ transforms the bicolored graph $B = (G, \beta)$ into B_w, where V^* denotes the set of all finite sequences of vertices from V. For $S \subseteq V$, we write B^S for the bicolored graph that keeps the underlying graph G and flips the colors of vertices in S only; in particular, B^V performs a *global color reversal*. Our central quantity is the *color reversal number* $cr(G)$: the minimum ℓ such that, for every initial coloring β, there exists a string w with $|w| \leq \ell$ and $B_w = B^V$. Note that if there exists a string w such that $B_w = B^V$, then for every bicolored graph B' on G, we also have $B'_w = (B')^V$. Therefore, it is independent of the initial coloring.

In 1987, Sabidussi [11] introduced the color-reversal problem under local inversions. He proved that a bicolored graph without isolated vertices can be color-reversed using at most $6n + 3$ local inversions. He also studied a more general problem: given two bicolored graphs B and B' with the same underlying graph G, can one transform B into B' using local inversions? He showed that, for any such pair, this can be done in at most $9n$ local inversions. Brijder and Hoogeboom [4] applied these results in their study of the group structure of pivot and loop complementation on graphs and set systems.

Our contributions. We improve both bounds of Sabidussi [11].

- For every connected n-vertex graph G with $n \geq 2$,

$$cr(G) \leq \begin{cases} 4n - 4, & \text{if } n \text{ is even,} \\ 4n - 3, & \text{if } n \text{ is odd,} \end{cases}$$

  obtained via parity-aware decompositions and repeated applications of constant-length strings (Theorem 1).
- For any two bicolored graphs $B = (G, \beta)$ and $B' = (G, \beta')$ on the same connected n-vertex graph G with $n \geq 2$, there exists a string w with $|w| \leq \left\lfloor \frac{11n-3}{2} \right\rfloor$ such that $B_w = B'$ (Theorem 2).
- For star graphs and complete graphs G with at least two vertices, we have $cr(G) \leq 3n$ (Theorem 3, Theorem 4).

Theorem 1 and Theorem 2 crucially use the perfect forest theorem of Scott [12], which states that every graph of even order has a spanning forest in which each tree is an induced subgraph of the graph and every vertex has odd degree within its tree. See Caro et al. [6] for two shorter proofs of the theorem. All constructions that we provide are explicit and run in polynomial time.

The rest of the paper is organized as follows. Section 2 introduces the preliminaries, providing definitions and notations for graphs, bicolored graphs, local complementation, and local inversion. It also states several known results and recalls the Perfect Forest Theorem. Section 3 presents the proof of Theorem 1, which establishes an improved bound on $cr(G)$, and of Theorem 2, which bounds the number of moves required to transform one bicolored graph into another. Section 4 focuses on special graph families, namely star graphs and complete graphs, and shows that stronger, family-specific bounds can be obtained through explicit constructions, leading to Theorem 3 and Theorem 4. Finally, Section 5 discusses a few open problems.

2 Preliminaries

We consider only simple, undirected, and labelled graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Two graphs G and G' on the same vertex set are *equivalent* if and only if $E(G) = E(G')$; equivalently, for all distinct $u, v \in V(G)$, $\{u, v\} \in E(G)$ if and only if $\{u, v\} \in E(G')$. For a subset $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S. The *open neighborhood* of a vertex v in G is denoted by $N_G(v)$. A path, a complete graph, and a star on t vertices are denoted by P_t, K_t, and S_t, respectively.

A *bicoloration* of a graph $G = (V, E)$ is a mapping $\beta : V \to \{-1, 1\}$. A *bicolored graph* is a pair $B = (G, \beta)$, where G is a graph and β is a bicoloration of G. Let $G = (V, E)$ be a graph and let $a \in V$. The *local complement* of G at a is the graph $G_a = (V, E_a)$ obtained by toggling adjacency among the neighbors of a; formally, for distinct $x, y \in V$,

$$\{x,y\} \in E_a \iff \begin{cases} \{x,y\} \in E \text{ and } (x \notin N_G(a) \text{ or } y \notin N_G(a)), \\ \text{or} \\ \{x,y\} \notin E \text{ and } x, y \in N_G(a). \end{cases}$$

Fig. 1 illustrates the construction of G_a from G.

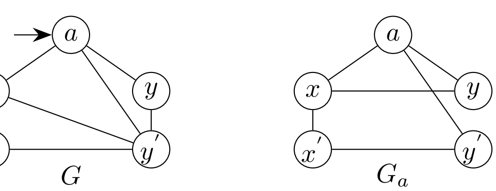

Fig. 1. Local complement of a graph G with respect to vertex a.

For a graph G, a *string* on the alphabet $V(G)$ is a finite sequence of vertices of $V(G)$. The set $V(G)^*$ denotes the set of all finite sequences over $V(G)$. We use ε to denote the *empty string*. Let $w \in V(G)^*$ be any string. We define the local complement of G with respect to w, denoted G_w, inductively as follows:

- Base case: $G_\varepsilon = G$.
- Inductive case: if $w = w'a$ with $a \in V(G)$, then $G_w = (G_{w'})_a$.

Fig. 2 shows an example of local complementation with respect to the string $w = abca$.

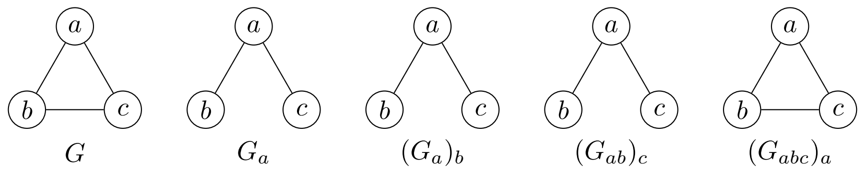

Fig. 2. Local complement of a graph with respect to the string *abca*.

Given a bicoloration β, the bicoloration with respect to a vertex a, denoted β_a, is defined by inverting the color of each neighbor of a:

$$\beta_a(x) = \begin{cases} -\beta(x), & \text{if } x \in N_G(a), \\ \beta(x), & \text{otherwise.} \end{cases}$$

Let $B = (G, \beta)$ be a bicolored graph and let $a \in V(G)$. Writing G_a and β_a as above, the *local inversion* of B at a, denoted B_a, is (G_a, β_a). Let $B = (G, \beta)$ be a bicolored graph and let $w \in V(G)^*$ be a string. The local inversion of B with respect to w is defined inductively as follows:

- Base case: $B_\varepsilon = B$.
- Inductive case: if $w = w'a$ with $a \in V(G)$, then $B_w = (B_{w'})_a$.

For example, see Fig. 3.

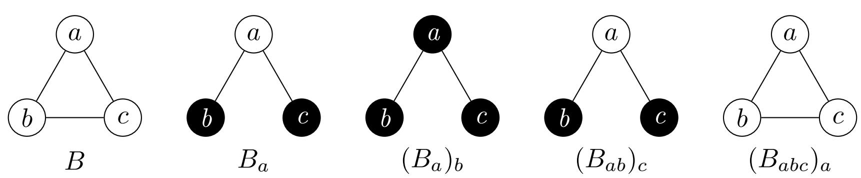

Fig. 3. Local inversion of a graph with respect to the string *abca*.

We now state some known results that will be used in the following sections. Proposition 1 means that it does not make sense to repeatedly apply local inversion on a vertex.

Proposition 1 ([11]). *For any bicolored graph $B = (G, \beta)$, $B_{aa} = B$, where a is any vertex in G.*

Let $B = (G, \beta)$ be a bicolored graph with $\beta : V(G) \to \{-1, 1\}$, and let $A \subseteq V(G)$. Define the operator h_A by

$$\big(h_A(\beta)\big)(v) = \begin{cases} -\beta(v), & \text{if } v \in A, \\ \beta(v), & \text{if } v \notin A. \end{cases}$$

We write $B^A = (G, h_A(\beta))$ for the bicolored graph obtained from B by flipping the colors of all vertices in A and leaving all others unchanged. We write B^a for $B^{\{a\}}$.

Proposition 2 states that in a bicolored graph, the endpoints of an edge can be color-reversed using 6 local inversions, while the underlying graph remains unchanged.

Proposition 2 ([11]). *Let $B = (G, \beta)$ be a bicolored graph. If $ab \in E(G)$ and $w = ababab$ (of length 6), then $B_w = B^{\{a,b\}}$.*

Proposition 3 states that one vertex of a triangle in a bicolored graph can be color-reversed in 7 moves.

Proposition 3 ([11]). *Let $B = (G, \beta)$ be a bicolored graph. If $a, b, c \in V(G)$ form a triangle and $w = abacbac$ (of length 7), then $B_w = B^a$.*

Proposition 4 handles the base cases when G has 2 or 3 vertices.

Proposition 4 ([11]). *A bicolored graph on K_2 with vertices a, b can be color-reversed using 2 local inversions, for example, with the string ab. A bicolored graph on a triangle abc can be color-reversed using 9 local inversions, for example, with the string $abababcac$. Similarly, a bicolored graph on a path P_3 with vertices a, b, c, where b is the degree-2 vertex, can be color-reversed using 9 local inversions, for example, with the string $ababacacb$.*

An *odd tree* is a tree in which every vertex has odd degree. A *perfect forest* of a graph G is a spanning forest of G in which each tree is an induced subgraph of G and is an odd tree.

Proposition 5 (Perfect Forest Theorem [6,12]). *Let G be a connected graph on n vertices, where n is even. Then G has a perfect forest, and such a forest can be computed in polynomial time.*

3 General bounds

In this section, we prove Theorem 1 and Theorem 2. We start with some simple lemmas derivable from the results by Sabidussi [11]. Lemma 1 states that in a bicolored graph, the end points of an induced P_3 can be color-reversed in 8 local inversions, without changing the underlying graph.

Lemma 1. *Let $B = (G, \beta)$ be a bicolored graph, and let $a, b, c \in V(G)$ with $a, b \in N_G(c)$ and $ab \notin E(G)$. Let $w = cabababc$ (of length 8). Then $B_w = B^{\{a,b\}}$.*

Proof. Let $w' = ababab$ (of length 6). Then $w = cw'c$. In G_c, since $a, b \in N_G(c)$ and $ab \notin E(G)$, the edge ab appears, so a, b, c form a triangle in G_c. In B_c, relative to B, the colors of the neighbors of c are flipped, while the colors of all other vertices remain unchanged. By Proposition 2, $(G_c)_{w'} = G_c$, and in $(B_c)_{w'}$ the colors of a and b are flipped (relative to B_c). Thus $B_{cw'}$ has underlying graph G_c, with the colors of all vertices in the neighborhood of c flipped except those of a and b, while the colors of the remaining vertices stay unchanged (relative to B). Finally, $(B_{cw'})_c$ has underlying graph $G_{cc} = G$ (by Proposition 1), and the colors of all neighbors of c except a and b revert to their original values. Therefore, only the colors of a and b are flipped relative to B. ☐

Lemma 2 states that an endvertex of an induced P_3 in a bicolored graph can be color-reversed using 7 local inversions.

Lemma 2. *Let $B = (G, \beta)$ be a bicolored graph, and let $a, b, c \in V(G)$ with $a, b \in N_G(c)$ and $ab \notin E(G)$. Let $w = cabacba$ (of length 7). Then $B_w = B^a$.*

Proof. Let $w' = abacbac$ (of length 7). Then

$$cw'c = (c)(abacbac)(c) = (cabacba)(cc) \sim cabacba = w,$$

where the simplification $cc \sim \epsilon$ follows from Proposition 1.

In G_c, since $a, b \in N_G(c)$ and $ab \notin E(G)$, the edge ab appears, so a, b, c form a triangle in G_c. In B_c, relative to B, the colors of the neighbors of c are flipped, while the colors of all other vertices remain unchanged. By Proposition 3, $(G_c)_{w'} = G_c$, and in $(B_c)_{w'}$ the color of a is flipped (relative to B_c).

Thus, in $B_{cw'}$, the underlying graph is G_c, with the colors of all vertices in $N_G(c)$ flipped, except for a, while the colors of all other vertices remain unchanged (relative to B).

Finally, $(B_{cw'})_c$ has underlying graph $G_{cc} = G$ (by Proposition 1), and the colors of all neighbors of c are flipped again. Therefore, only the color of a is flipped relative to B, yielding $B_w = B^a$. ☐

Proposition 3 and Lemma 2 together imply Lemma 3, which states that only 7 moves are sufficient to color-reverse any vertex in a nontrivial connected graph.

Lemma 3. *Let G be a connected graph on $n \geq 3$ vertices. Let a be any vertex in G. Let B be any bicolored graph of G. Then there is a string w of length 7 such that $B_w = B^a$.*

Lemma 4 states that the edges of an odd tree can be partitioned into copies of P_3 together with one K_2, with certain useful properties.

Lemma 4. *Let T be a rooted odd tree on n vertices. Then there exists a partition $\mathcal{P}$ of the edges of T into $(n-2)/2$ copies of P_3 together with one K_2, such that the following conditions are satisfied:*

i. In each P_3 of $\mathcal{P}$, the two endvertices are children (with respect to the given root) of its center.
ii. One of the endvertices of the K_2 in $\mathcal{P}$ is the root.
iii. Each vertex of T is an endvertex of exactly one path in $\mathcal{P}$.
iv. The partition can be computed in polynomial time.

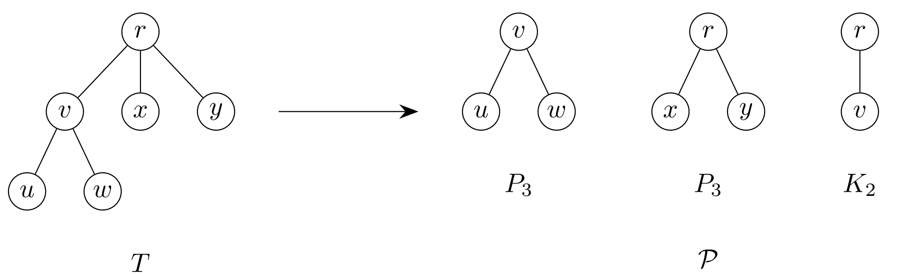

Fig. 4. Partition of T into P_3s and K_2

Proof. Clearly, n is even. We proceed by induction on n. The base case $n = 2$ is trivial.

Assume $n \geq 4$ and that the claim holds for all smaller even values of n. Let r be the root of T, and let v be a vertex farthest from r that is adjacent to a leaf. Then v has no non-leaf descendant. Since v is not a leaf and every degree in T is odd, we have $\deg(v) \geq 3$. Hence, v has at least two leaf neighbors; let u and w be two of them.

Delete the path u–v–w (i.e., remove the edges uv and vw) and then delete the isolated vertices u and w to obtain a tree T' on $n - 2$ vertices. All degrees in T' remain odd (the degree of v decreases by 2, and the degrees of all other vertices are unchanged). Moreover, T' has an even number of vertices. By the induction hypothesis, $E(T')$ can be partitioned into $(n-4)/2$ copies of P_3 together with one K_2. Adding the P_3 $u-v-w$ to this partition yields the desired partition $\mathcal{P}$.

Since v is the center of the P_3 $u-v-w$, statement (i) follows inductively. For every P_3 in $\mathcal{P}$ that contains r, by (i), r must be its center. Since r has odd degree, it must also be an endvertex of the K_2 in $\mathcal{P}$, proving (ii). At each step, the process deletes the endvertices of the newly created P_3. Therefore, each vertex is the endvertex of at most one path in $\mathcal{P}$. As no vertex remains at the end, each vertex is the endvertex of exactly one path in $\mathcal{P}$, establishing statement (iii). Finally, the construction of $\mathcal{P}$ can clearly be carried out in polynomial time, proving statement (iv). See Fig. 4 for a demonstration. ◻

The next two lemmas state that the vertices of an induced odd tree T in a graph can be color-reversed using $4|V(T)| - 4$ local inversions, by means of a string w that ends (Lemma 5) or starts (Lemma 6) with a designated vertex r of T.

Lemma 5. *Let G be a graph on n vertices, and let T be an induced odd tree of G on at least 4 vertices. Let r be any vertex of T, and let $B = (G, \beta)$ be any bicolored graph of G. Then there exists a string $w \in V(T)^*$ such that the following conditions hold:*

i. $|w| = 4|V(T)| - 4$,
ii. $B_w = B^{V(T)}$,
iii. w *ends with* r.

Proof. Root T at r. By Lemma 4, there exists a partition $\mathcal{P}$ of $E(T)$ into $(n-2)/2$ copies of P_3 together with one K_2, such that in each P_3 the endvertices are children of its center, and every vertex of T is an endvertex of exactly one path in $\mathcal{P}$. Moreover, r is one of the endvertices of the K_2 in $\mathcal{P}$. Let rv denote this K_2.

Since T has at least 4 vertices, either r or v has degree greater than 1 in T.

First, assume that r has degree greater than 1. Then r also belongs to some P_3 in $\mathcal{P}$. Since r can be an endvertex of at most one path in $\mathcal{P}$, it must serve as the center of every P_3 containing it. Let xry be one such P_3.

Apply Lemma 1 on B successively with each P_3 in $\mathcal{P}$, except xry. Let the resulting bicolored graph be B', and let w' be the corresponding string used. In B', every vertex of T is color-flipped, except x, y, r, and v.

Define

$$w_1 = vrvrvr, \quad w_2 = rxyxyxyr, \quad w'' = (vrvrv)(xyxyxyr).$$

By Proposition 1,
$w_1 w_2 = (vrvrvr)(rxyxyxyr) \sim (vrvrv)(rr)(xyxyxyr) \sim (vrvrv)(xyxyxyr)$
$= w''$.

By Proposition 2, in B'_{w_1} the colors of r and v (with respect to B') are flipped without altering the underlying graph G. Then, by Lemma 1, in $(B'_{w_1})_{w_2}$ the colors of x and y are flipped, again without changing G. Hence

$$B'_{w_1 w_2} = B'_{w''}$$

is a bicolored graph in which the colors of all vertices of T are flipped with respect to B. Note that $w = w'w''$ ends with w'', which in turn ends with r.

Now, assume that r has degree 1. Then v has degree greater than 1 in T. Let xvy be a P_3 containing v in $\mathcal{P}$. In this case, the same argument works with

$$w_1 = vxyxyxyv, \quad w_2 = vrvrvr, \quad w'' = (vxyxyxy)(rvrvr).$$

Finally, we count the length of w. We applied w'' with $|w''| = 12$, and we used Lemma 1 on $(|V(T)|-4)/2$ paths, contributing $8 \cdot (|V(T)|-4)/2$. Therefore,

$$|w| = 8 \cdot \frac{|V(T)| - 4}{2} + 12 = 4|V(T)| - 4.$$

This completes the proof. □

Lemma 6, where the only difference from Lemma 5 is that w starts with r, can be proved in a similar fashion.

Lemma 6. *Let G be a graph on n vertices, and let T be an induced odd tree of G on at least 4 vertices. Let r be any vertex of T, and let $B = (G, \beta)$ be any bicolored graph of G. Then there exists a string $w \in V(T)^*$ such that the following conditions hold:*

i. $|w| = 4|V(T)| - 4$,
ii. $B_w = B^{V(T)}$,
iii. *w starts with r.*

The next two lemmas state that if G' is a nontrivial connected induced subgraph of a graph G with $n' = |V(G')|$ even, then the vertices of G' can be color-reversed (in G) using $4n' - 4$ local inversions, by means of a string that ends (Lemma 7) or starts (Lemma 8) with a designated vertex $v \in V(G')$.

Lemma 7. *Let G be a graph and let G' be any connected induced subgraph of G with $n' \geq 4$ vertices, where n' is even. Let v be any vertex of G', and let $B = (G, \beta)$ be any bicolored graph of G. Then there exists a string $w \in V(G')^*$ such that the following conditions hold:*

i. $|w| \leq 4n' - 4$,
ii. $B_w = B^{V(G')}$,
iii. *w ends with v.*

Proof. By Proposition 5, G' admits a perfect forest F. Let $T_1, T_2, \ldots, T_t$ (with $t \geq 1$) be the trees in F. Without loss of generality, assume that $v \in T_t$.

For each tree T_i, we construct a string w_i as follows. If T_i is a K_2, say xy, then set $w_i = xyxyxy$. In particular, if T_t is a K_2, say uv, then $w_t = uvuvuv$. If T_i has more than two vertices, then w_i is the string obtained from Lemma 5. In particular, if T_t is a tree with more than two vertices, then w_t is the string from Lemma 5, which ends in v.

Let $w = w_1 w_2 \cdots w_t$. Clearly, w ends with v. By Proposition 2 and Lemma 5, the underlying graph of B_w is still G, and the colors of all vertices in G' are flipped relative to B, while the colors of all other vertices remain unchanged.

It remains to prove the bound on $|w|$. Suppose t' trees in F are K_2s, for some $0 \leq t' \leq t$. Then, by Proposition 2 and Lemma 5,

$$\begin{aligned}
|w| &\leq 6t' + 4(n' - 2t') - 4(t - t') && t'K_2 \text{ covers } 2t' \text{ vertices} \\
&= 4n' - 4t + 2t' \\
&\leq 4n' - 4t + 2t && \text{since } t' \leq t \\
&= 4n' - 2t.
\end{aligned}$$

If $t \geq 2$, we obtain $|w| \leq 4n' - 4$ as required. If $t = 1$, then G' itself is a tree. Since G' has at least four vertices, T_1 cannot be a K_2, and the bound follows from Lemma 5. $\square$

Lemma 8, where the only difference from Lemma 7 is that w starts with v, can be proved in a similar fashion.

Lemma 8. *Let G be a graph and let G' be any connected induced subgraph of G with $n' \geq 4$ vertices, where n' is even. Let v be any vertex of G', and let $B = (G, \beta)$ be any bicolored graph of G. Then there exists a string $w \in V(G')^*$ such that the following conditions hold:*

i. $|w| \leq 4n' - 4$,
ii. $B_w = B^{V(G')}$,
iii. *w starts with v.*

Lemma 9 states that if G' is a nontrivial connected induced subgraph of G with an odd number n' of vertices, then the vertices of G' can be color-reversed in G using at most $4n' - 3$ local inversions.

Lemma 9. *Let G be a graph and let G' be any connected induced subgraph of G with $n' \geq 5$ vertices, where n' is odd. Let $B = (G, \beta)$ be any bicolored graph of G. Then there exists a string $w \in V(G')^*$ such that the following conditions hold:*

i. $|w| \leq 4n' - 3$,
ii. $B_w = B^{V(G')}$.

Proof. Let a be any vertex of G' such that $G'' = G' - a$ is connected.

First, assume that a lies on a triangle, say abc in G'. Let $w_1 = abacbac$. Applying w_1 to B yields a bicolored graph B_1. By Proposition 3, B_1 has the same underlying graph G, and compared to B, only the color of a is flipped.

Since G'' is a connected induced subgraph of G with $n' - 1 \geq 4$ vertices (which is even), we can apply Lemma 8 to obtain a string w_2 starting with c that flips the colors of all vertices of G'' in B. Therefore, $B_{w_1 w_2}$ differs from B exactly in that all vertices of G' have their colors flipped, while the underlying graph remains G.

Let w be the string obtained from $w_1 w_2$ by removing the final c of w_1 and the initial c of w_2. By Proposition 1, $B_{w_1 w_2} = B_w$. Furthermore,

$$\begin{aligned} |w| &= |w_1| + |w_2| - 2 \\ &\leq 7 + (4(n' - 1) - 4) - 2 && \text{by Lemma 8} \\ &= 4n' - 3. \end{aligned}$$

Now assume that a does not lie on any triangle in G'. Since G'' is connected and has at least five vertices, there exist vertices $b, c \in V(G'')$ such that acb is an induced P_3. In this case, let $w_1 = cabacba$. By arguments analogous to the previous case, together with Lemma 2 and Lemma 7, we obtain the desired string w satisfying the two conditions. □

We are now ready to prove Theorem 1.

Theorem 1. *Let G be a connected graph with $n \geq 2$ vertices. If n is even, then $cr(G) \leq 4n-4$, and if n is odd, then $cr(G) \leq 4n-3$.*

Proof. If $n = 2$, then by Proposition 4, we have $cr(G) \leq 2 \leq 4 \cdot 2 - 4$. If $n = 3$, then by Proposition 4, we have $cr(G) \leq 9 \leq 4 \cdot 3 - 3$. If $n \geq 4$ and n is even, then the statement follows from Lemma 7. If $n \geq 5$ and n is odd, then the statement follows from Lemma 9. □

Corollary 1 follows immediately from Theorem 1, noting that each component of a graph with even order may itself have odd order.

Corollary 1. *Let G be a graph with n vertices and $t \geq 2$ components, and suppose G has no isolated vertices. Then $cr(G) \leq 4n - 3t$.*

We now state and prove our second main result.

Theorem 2. *Let G be a connected graph on $n \geq 2$ vertices. Let $B = (G, \beta)$ and $B' = (G, \beta')$ be two bicolored graphs on G. Then there exists a string w of length at most $\left\lfloor \frac{11n-3}{2} \right\rfloor$ such that $B_w = B'$.*

Proof. Let $V_0 = \{v \in V(G) : \beta(v) = \beta'(v)\}$ with $n_0 = |V_0|$, and let $V_1 = \{v \in V(G) : \beta(v) = -\beta'(v)\}$ with $n_1 = |V_1| = n - n_0$. If $n_1 = 0$, then $B = B'$ and we are done. If $n_0 = 0$, then $\beta = -\beta'$, and by Theorem 1 there is a string of length at most $4n - 3 \leq \frac{11n-3}{2}$ (since $n \geq 2$).

Assume henceforth $n_0, n_1 \geq 1$. Let $G_0 = G[V_0]$ and $G_1 = G[V_1]$. Let I_0 (resp. I_1) be the set of isolated vertices in G_0 (resp. G_1), and write $i_0 = |I_0|$, $i_1 = |I_1|$.

We consider two strategies and take the cheaper.

(1) Fix V_1 directly. Use Lemma 3 on each vertex of I_1 (cost 7 per isolate), then apply Theorem 1 on $G_1 - I_1$ (cost at most $4(n_1 - i_1) - 3$ if $n_1 - i_1 > 0$, else 0). Thus, the total cost satisfies

$$p \ \leq \ 7i_1 + \max\{\, 4(n_1 - i_1) - 3,\, 0 \,\} \ \leq \ 7n_1.$$

(2) Flip V_0, then flip all of G. First use Lemma 3 on I_0 (cost 7 per isolate), then apply Theorem 1 on $G_0 - I_0$ (cost at most $4(n_0 - i_0) - 3$ if $n_0 - i_0 > 0$, else 0), and finally apply Theorem 1 on G (cost at most $4n - 3$). Hence

$$q \ \leq \ 7i_0 + \max\{\, 4(n_0 - i_0) - 3,\, 0 \,\} + (4n - 3) \ \leq \ 7n_0 + 4n - 3.$$

For the given partition (n_0, n_1) we can realize B' from B with length at most $\min\{p, q\} \leq \min\{\, 7(n - n_0),\ 7n_0 + 4n - 3 \,\}$. Maximizing this upper bound over $0 \leq n_0 \leq n$ occurs when the two arguments are equal:

$$7(n - n_0) = 7n_0 + 4n - 3 \implies 14n_0 = 3n + 3,$$

which yields

$$\min\{p, q\} \ \leq \ 7\bigl(n - n_0\bigr) \ = \ \frac{14(n - n_0)}{2} \ = \ \frac{14n - 3n - 3}{2} \ = \ \frac{11n - 3}{2}.$$

Therefore, there exists a string w of length at most $\left\lfloor \frac{11n-3}{2} \right\rfloor$ such that $B_w = B'$. □

Corollary 2 follows immediately.

Corollary 2. *Let G be a graph without isolated vertices, and let $B = (G, \beta)$ and $B' = (G, \beta')$ be two bicolored graphs on G. Then there exists a string w of length at most $\left\lfloor \frac{11n-3t}{2} \right\rfloor$ such that $B_w = B'$, where t is the number of components of G.*

4 Stars and complete graphs

In this section, we obtain tighter bounds for stars and complete graphs.

Theorem 3. *Let S_n be the star graph on $n \geq 2$ vertices. Then $cr(S_n) \leq 3n$.*

Proof. Let c_0 be the center and $c_1, c_2, \ldots, c_{n-1}$ be the leaves of S_n.

Consider the bicolored graph $B = (S_n, \beta)$. Define

$$w_n = (c_1c_0c_1c_0c_1)(c_2c_0c_2)(c_3c_0c_3)\cdots(c_{n-1}c_0c_{n-1})(c_0).$$

We claim that $B_{w_n} = (S_n, -\beta)$.

For $2 \leq i \leq n$, let $A_i = \{c_0, c_1, \ldots, c_{i-1}\}$. We prove that $B_{w_i} = B^{A_i}$. For $i = 1$, $w_1 = c_1c_0c_1c_0c_1c_0$, and this case follows from Proposition 2. Assume that the statement holds for some $i = n-1$. We now prove it for $i = n$.

Observe that

$$\begin{aligned}
w &= (w_{n-1})(c_0c_{n-1}c_0c_{n-1}c_0c_{n-1})(c_{n-1}) && \text{assume}\\
&\sim (c_1c_0c_1c_0c_1)(c_2c_0c_2)(c_3c_0c_3)\cdots(c_{n-2}c_0c_{n-2})(c_0)(c_0c_{n-1}c_0c_{n-1}c_0c_{n-1})\\
&\quad (c_{n-1})\\
&= (c_1c_0c_1c_0c_1)(c_2c_0c_2)(c_3c_0c_3)\cdots(c_{n-2}c_0c_{n-2})(c_0c_0)(c_{n-1}c_0c_{n-1})(c_0)(c_{n-1}\\
&\quad c_{n-1})\\
&\sim (c_1c_0c_1c_0c_1)(c_2c_0c_2)(c_3c_0c_3)\cdots(c_{n-2}c_0c_{n-2})(c_{n-1}c_0c_{n-1})(c_0)\\
&\quad \text{by Proposition 1}\\
&= w_n.
\end{aligned}$$

Therefore, it is enough to show that $B_w = (S_n, -\beta)$.

By the induction hypothesis, $B_{w_{n-1}} = B^{A_{n-1}}$; that is, $B_{w_{n-1}}$ has underlying graph S_n, where the colors of all vertices except c_{n-1} are flipped. Let $w' = c_0c_{n-1}c_0c_{n-1}c_0c_{n-1}$. By Proposition 2, $(B_{w_{n-1}})_{w'}$ has the same underlying graph S_n, and the colors of c_0 and c_{n-1} are flipped relative to $B_{w_{n-1}}$. Thus, $B_{w_{n-1}w'}$ has all vertex-colors flipped except c_0, with respect to B. Finally, applying c_{n-1} to $(B_{w_{n-1}w'})$ preserves the underlying graph (since c_{n-1} has degree 1) and flips the color of c_0. Hence, all vertex colors are flipped, completing the proof. $\square$

Theorem 4. *For $n \geq 2$, we have $cr(K_n) \leq 3n$.*

Proof. Let $V(K_n) = \{c_0, c_1, \ldots, c_{n-1}\}$, and let $B = (K_n, \beta)$ be a bicolored graph. Define

$$w = (c_0)(w')(c_0),$$

where

$$w' = (c_1c_0c_1c_0c_1)(c_2c_0c_2)(c_3c_0c_3)\cdots(c_{n-1}c_0c_{n-1})(c_0),$$

which is the string used for the color-reversal of S_n in Theorem 3.

In B_{c_0}, the underlying graph is a star with center c_0, and the colors of all vertices except c_0 are flipped. By Theorem 3, $(B_{c_0})_{w'}$ has the same underlying graph (the star centered at c_0), and the colors of all vertices are flipped relative to B_{c_0}. Therefore, in $B_{c_0w'}$, only the color of c_0 is flipped relative to B. Finally, applying c_0 once more, $(B_{c_0w'})_{c_0}$ has underlying graph K_n, with all vertex colors flipped relative to B.

The claim follows since

$$\begin{aligned} w &= (c_0)(w')(c_0)\\ &= (c_0)(c_1c_0c_1c_0c_1)(c_2c_0c_2)(c_3c_0c_3)\cdots(c_{n-1}c_0c_{n-1})(c_0)(c_0)\\ &\sim (c_0)(c_1c_0c_1c_0c_1)(c_2c_0c_2)(c_3c_0c_3)\cdots(c_{n-1}c_0c_{n-1}) \qquad \text{by Proposition 1.} \end{aligned}$$

This completes the proof. □

5 Concluding remarks

We conclude with a few open problems. We have shown that any bicolored graph can be color-reversed in at most $4n-4$ local inversions when n is even, and in at most $4n-3$ inversions when n is odd. Although this bound is tight for some small graphs, such as P_3, K_3, and S_4, we do not believe it is tight for larger graphs. With the aid of a computer program, we determined $cr(G)$ for all graphs with at most 5 vertices and found that the value never exceeds $3n$.

Problem 1: Is it true that $cr(G) \le 3n$?

We proved that Problem 1 has an affirmative answer for stars and complete graphs. But is it tight for them?

Problem 2: Is it true that $cr(G) = 3n$ for all stars and complete graphs of at least 3 vertices?

As mentioned earlier, all our proofs are constructive, and the corresponding strings can be obtained in polynomial time. Nevertheless, the complexity of computing $cr(G)$ remains unknown.

Problem 3: Is the following problem solvable in polynomial time? Given a graph G and an integer k, decide whether $cr(G) \le k$.

Note that no lower bounds are currently known for $cr(G)$, which is another direction worth exploring. Some good lower bounds may help us to get a good approximation algorithm. In particular:

Problem 4: Does our algorithm (implied by Theorem 1) serve as an approximation algorithm with a good approximation factor for the optimization version of the problem?

The bound we obtained for the number of local inversions required to transform one bicolored graph into another (Theorem 2) is weaker than the color-reversal bound. However, it is unclear whether selective color-reversal inherently requires more local inversions than global color reversal.

Problem 5: Is it true that the minimum number of local inversions required to transform B into B' is at most $cr(G)$, where G is the underlying graph of B and B'?

A trivial brute-force algorithm for the decision version runs in $n^{O(k)}$ time, implying that the problem lies in XP.

Problem 6: Does the problem admit an FPT algorithm parameterized by k?

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