ON MINIMAL PRODUCT-ONE SEQUENCES OF MAXIMAL LENGTH OVER THE NON-ABELIAN GROUP OF ORDER pq

D.V. AVELAR[1], F.E. BROCHERO MARTÍNEZ[2], AND S. RIBAS[3]

Abstract. Let G be a finite group. A sequence over G is a finite multiset of elements of G, and it is called product-one if its terms can be ordered so that their product is the identity of G. The large Davenport constant $\mathsf{D}(G)$ is the maximal length of a minimal product-one sequence, that is, a product-one sequence that cannot be partitioned into two nontrivial product-one subsequences. Let p, q be odd prime numbers with $p \mid q-1$ and let $C_q \rtimes C_p$ denote the non-abelian group of order pq. It is known that $\mathsf{D}(C_q \rtimes C_p) = 2q$. In this paper, we describe all minimal product-one sequences of length $2q$ over $C_q \rtimes C_p$. As an application, we further investigate the k-th elasticity (and, consequently, the union of sets containing k) of the monoid of product-one sequences over these groups.

1. Introduction

Let G be a finite group written multiplicatively. A sequence S over G is a finite multiset of elements from G. The *zero-sum problems* investigate the conditions under which a given sequence over G contains a subsequence whose product is the identity of G. Such subsequences are called *product-one*. The *small Davenport constant* $\mathsf{d}(G)$ is defined as the maximal length of a sequence over G that do not have product-one subsequences. Moreover, the *large Davenport constant* $\mathsf{D}(G)$ is defined as the maximal length of a minimal product-one sequence, that is, a product-one sequence that cannot be decomposed into two nontrivial product-one subsequences. This invariant appears in the pioneering works due to Rogers [31], van Emde Boas and Kruyswijk [8] and Olson [25, 26]. In particular, Rogers showed that $\mathsf{D}(G)$ represents how many prime ideal factors a prime element can have in an algebraic number field with class group isomorphic to G. This is a crucial relation between zero-sum problems and factorization theory in Krull monoids (see [14]).

By definition, it follows that $\mathsf{d}(G) + 1 \le \mathsf{D}(G)$, with equality for abelian groups. Furthermore, a simple application of Pigeonhole Principle yields $\mathsf{D}(G) \le |G|$, with equality for cyclic groups. Let C_n be the cyclic group of order n. By the Fundamental Theorem of Abelian Groups, if G is a nontrivial finite abelian group, then there exist unique $1 < n_1 \mid n_2 \mid \ldots \mid n_r$ such that $G \cong C_{n_1} \otimes \ldots \otimes C_{n_r}$, where r is the rank of G and n_r is the exponent of G. Set $\mathsf{D}^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1)$. Some routine arguments yield $\mathsf{D}(G) \ge \mathsf{D}^*(G)$, with equality for abelian groups of rank at most 2 and for p-groups [25, 26]. However, there exist infinitely many groups for which the inequality is strict [15].

Although the main focus of the zero-sum problems has been on abelian groups due to the relation with factorization theory, it has been extended to non-abelian groups in the 80s. This explains the use multiplicative notation and the term *product-one* instead of additive notation and the term *zero-sum*. Nevertheless, for non-abelian groups, the small Davenport constant does not admit an interpretation in terms of monoid factorizations.

Date: October 2, 2025.
2010 *Mathematics Subject Classification.* 11P70 (primary), 11B75, 13F15 (secondary).
Key words and phrases. Zero-sum problems, inverse zero-sum problems, large Davenport constant, groups of order pq, k-th elasticity, union of sets of lengths.
[1]partially supported by FAPERJ grant E-26/210.315/2024 and FAPEMIG grant APQ-04712-25.
[2]partially supported by FAPEMIG grant RED-00133-21, and CNPq grants 316843/2023-7 and 420721/2025-8.
[3]partially supported by FAPEMIG grants RED-00133-21, APQ-01712-23 and APQ-04712-25, and CNPq grant 420721/2025-8.

Concerning the large Davenport constant over non-abelian groups, we refer the reader to [11, 18]. In particular, Geroldinger and Grynkiewicz [11] fully determined the large Davenport constant for groups having a cyclic subgroup of index 2, and Grynkiewicz [18] proved that if $G \cong C_q \rtimes C_p$ is the non-abelian group of order pq, where p, q are odd primes with $p \mid q-1$, then $\mathsf{D}(G) = 2q$. He also obtained several upper bounds for $\mathsf{D}(G)$ for general groups G. Among others, we highlight $\mathsf{D}(G) \le \mathsf{d}(G) + 2|G'| - 1$, where G' is the commutator subgroup of G, and $\mathsf{D}(G) \le 2|G|/p$, where G is non-cyclic and p is the smallest prime divisor of $|G|$.

The *direct problem* associated to $\mathsf{D}(G)$ asks for the exact value of $\mathsf{D}(G)$, while the associated *inverse problem* seeks to describe the exceptional sequences of length exactly $\mathsf{D}(G)$ that cannot be partitioned into two nontrivial product-one subsequences (see also [1, 2, 3, 4, 5, 6, 19, 20, 23, 24, 28, 29, 30, 33] for other recent developments on the direct and inverse problems over non-abelian groups).

The goal of this paper is to establish, by arguments similar to those in [18], the inverse problem associated to the large Davenport constant of $C_q \rtimes C_p$, that is, a complete characterization of minimal product-one sequences of length $2q$ over $C_q \rtimes C_p$. This is in the same idea as [24], where Oh and Zhong solved the inverse problem for $\mathsf{D}(G)$ over dihedral and dicyclic groups.

The minimal product-one sequences over G are precisely the atoms, that is, irreducible elements of the monoid $\mathcal{B}(G)$ of the product-one sequences over G. Building on our characterization of minimal product-one sequences of length $\mathsf{D}(G)$, we investigate the unions of sets of lengths in $\mathcal{B}(G)$ by studying the k-th elasticity of $\mathcal{B}(G)$. This approach is also similar to that employed by Oh and Zhong in [24] for dihedral and dicyclic groups.

The main result of this paper is the following.

Theorem 1.1. Let S be a minimal product-one sequence over $C_q \rtimes C_p$ of length $|S| = \mathsf{D}(C_q \rtimes C_p) = 2q$. Then there exist $x, y \in C_q \rtimes C_p$ for which $C_q \rtimes C_p = \langle x, y \colon x^p = y^q = 1, yx = xy^s, \mathrm{ord}_q(s) = p \rangle$ and

$$S = y^{[q-1]} \boldsymbol{\cdot} x \boldsymbol{\cdot} y^{[q-1]} \boldsymbol{\cdot} x^{p-1} y^{s^{p-1}+1}. \tag{1}$$

The paper is organized as follows. In Section 2, we present the prerequisite notation and definitions that will be used througout the paper, as well as some properties of the group $C_q \rtimes C_p$. In Section 3, we present several auxiliary results required for the proof of Theorem 1.1. In Section 4, we prove two results that lead to the proof of Theorem 1.1. In Section 5, we apply our description of the atoms of $C_q \rtimes C_p$ to the study of the k-th elasticity, which in turn yields information on the union of sets of lengths containing k.

2. Notation and preliminaries

We use the standard notation from group theory. In particular, for a finite group G,

- if $A, B \subset G$, then the *product-set* of A and B is the set $AB = \{ab \colon a \in A, b \in B\}$. For a singleton $A = \{a\}$, we denote $aB = \{ab \colon b \in B\}$;
- if $A \subset G$, then $\langle A \rangle \le G$ denotes the subgroup generated by A;
- $Z(G) = \{g \in G \colon gh = hg \text{ for every } h \in G\} \trianglelefteq G$ is the *centre* of G;
- $[g, h] = g^{-1}h^{-1}gh \in G$ is the *commutator* of $g, h \in G$;
- $G' = \langle [g, h] \colon g, h \in G \rangle \trianglelefteq G$ is the *commutator subgroup* of G;
- $C_G(g) = \{h \in G \colon gh = hg\} \le G$ is the *centralizer* of $g \in G$; and
- for $A, B \subset G$ and $g \in G$, set the conjugations $A^g = \{g^{-1}ag \colon a \in A\}$ and $A^B = \{b^{-1}ab \colon a \in A, b \in B\}$.

In what follows, we present the necessary definitions concerning sequences, ordered sequences, and the group under consideration in this paper. The notation in this paper is consistent with [18, 24].

2.1. **Sequences over groups.** Let G be a finite group written multiplicatively and let $\mathcal{F}(G)$ be the free abelian monoid with basis G with operation denoted by the bold dot $\boldsymbol{\cdot}$. A sequence S over G is an element of $\mathcal{F}(G)$,

meaning that S is a finite multiset of elements of G (allowing repetition and the order is disregarded). In particular, if $S \in \mathcal{F}(G)$, then

$$S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_k = \prod_{i=1}^{k} g_i = \prod_{g \in G} g^{[\mathsf{v}_g(S)]},$$

where $\mathsf{v}_g(S)$ is the *multiplicity* of g in S and $|S| = k = \sum_{g \in G} \mathsf{v}_g(S)$ is the *length* of S. We observe that $g_1 \cdot g_2 = g_1 g_2 \in G$ denotes the product of g_1 and g_2, while $g_1 \boldsymbol{\cdot} g_2 \in \mathcal{F}(G)$ denotes a two-term sequence. A *subsequence* of S is a divisor $T \mid S$ in $\mathcal{F}(G)$. In other words, $T \mid S$ if and only if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for every $g \in G$. In this case, we write $S \boldsymbol{\cdot} T^{[-1]} = \prod_{g \in G} g^{[\mathsf{v}_g(S) - \mathsf{v}_g(T)]}$. For a subset $K \subset G$, we denote $\mathsf{v}_K(S) = \sum_{g \in K} \mathsf{v}_g(S)$. Moreover, the *support* of S is the set $\mathrm{supp}(S) = \{g \in G \colon \mathsf{v}_g(S) > 0\}$.

The *set of products* and the *set of subproducts* of S are

$$\pi(S) = \left\{ \prod_{i=1}^{|S|} g_{\sigma(i)} \in G \colon \sigma \text{ is a permutation of } [1, |S|] \right\} \quad \text{and} \quad \Pi(S) = \bigcup_{\substack{T \mid S \\ |T| \ge 1}} \pi(T),$$

respectively. The sequence $S \in \mathcal{F}(G)$ is called

(i) *trivial* if $|S| = 0$ (in this case, S is the identity of $\mathcal{F}(G)$);
(ii) *product-one* if $1 \in \pi(S)$;
(iii) *product-one free* if $1 \notin \Pi(S)$; and
(iv) *minimal product-one* if $1 \in \pi(S)$ and $S \neq T_1 \boldsymbol{\cdot} T_2$ for every T_1, T_2 nontrivial product-one sequences.

Let

$$\mathcal{B}(G) = \{S \in \mathcal{F}(G) \colon 1 \in \pi(S)\}$$

denote the *set of product-one sequences over* G, and let

$$\mathcal{A}(G) = \{S \in \mathcal{B}(G) \colon S \text{ is minimal product-one}\}.$$

We observe that $\mathcal{B}(G)$ is a submonoid of $\mathcal{F}(G)$ and $\mathcal{A}(G)$ is the set of *atoms* (or irreducible elements) of $\mathcal{B}(G)$. With this notation, the *large Davenport constant* is

$$\mathsf{D}(G) = \sup\{|S| \colon S \in \mathcal{A}(G)\},$$

and the *small Davenport constant* is

$$\mathsf{d}(G) = \sup\{|S| \colon S \in \mathcal{F}(G) \text{ and } 1 \notin \Pi(S)\}.$$

Moreover, we observe that if $S \in \mathcal{F}(G)$, then $\pi(S)$ is contained in a G'-coset, that is, $\pi(S) = Ag$ for some $A \subset G'$ and some $g \in G$. This implies that if $S_1, \ldots, S_t \in \mathcal{F}(G)$, then, for each $j \in [1, t]$, $\pi(S_j) = A_j g_j$ for some $A_j \subset G'$ and $g_j \in G$. Since G' is a normal subgroup of G, we have, for each $i \in [1, t]$, that $A_i' = (A_i)^{(g_1 \cdots g_{i-1})^{-1}}$ for some $A_i' \subset G'$. It follows that $\pi(S_1) \ldots \pi(S_t) = (A_1 g_1) \ldots (A_t g_t) = A_1' \ldots A_t' (g_1 \ldots g_t)$. In the special case where $G' \cong C_q$ with q prime, then classical results on product-set cardinalities in C_q, such as the Cauchy-Davenport Theorem (Lemma 3.1), may be applied to bound the cardinality of the product-set $\pi(S_1) \ldots \pi(S_t)$. Throughout the paper, this will be done without further reference to the intermediate sets A_i'.

2.2. **Ordered sequences over groups.** Let $\mathcal{F}^*(G)$ denote the free non-abelian monoid with basis G, that is, $\mathcal{F}^*(G)$ is the semigroup of words over the alphabet G. The elements of $\mathcal{F}^*$, called *ordered sequences* over G, are written as

$$S^* = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_k = \prod_{j=1}^{k} g_j.$$

By disregarding the order of the elements in $\mathcal{F}^*(G)$, we obtain a natural map $[\cdot] : \mathcal{F}^*(G) \to \mathcal{F}(G)$. An ordered sequence $S^* \in \mathcal{F}^*(G)$ with $[S^*] = S$ is called an *ordering* of the sequence $S \in \mathcal{F}(G)$. Furthermore, if

$S = [S^*]$, then we set $\mathrm{supp}(S^*) = \mathrm{supp}(S)$, $|S^*| = |S|$, and $\mathsf{v}_g(S^*) = \mathsf{v}_g(S)$ for every $g \in G$ to be the *support* of S^*, the *length* of S^*, and the *multiplicity* of $g \in G$ in S^*.

Let $S^* = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_k \in \mathcal{F}^*(G)$. For any subset $J \subset [1,k]$, set $S^*(J) = \prod_{j \in J} g_j$, where the order is taken in increasing order of the indices in J. We say that $S^*(J)$ is an *ordered subsequence* of S^*. For integers $0 \le i \le j$, we abbreviate $S^*(i,j) = S^*([i,j])$ and $S^*(j) = S^*(\{j\})$; the former is called a *consecutive subsequence*, while the latter denotes the *j-th term* of S^*. Moreover, $\pi : \mathcal{F}^*(G) \to G$ denotes the *product* of S^* in the order the terms appear, that is, $\pi(g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_k) = g_1 \ldots g_k$. If $S = [S^*]$, then it is clear that $\pi(S^*) \in \pi(S)$. A *factorization* of $S^* \in \mathcal{F}^*(G)$ (of length t) is a t-tuple $(S_1^*, \ldots, S_t^*)$ of nontrivial consecutive subsequences $S_i^* \mid S^*$ such that $S^* = S_1^* \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} S_t^*$.

2.3. **On the group $C_q \rtimes C_p$.** We consider the groups G of order pq, where $p \le q$ are prime numbers. If $p = q$, then either $G \cong C_{p^2}$ or $G \cong C_p^2$, whence G is abelian. Suppose now that $p < q$. If $p \nmid q-1$, then an immediate consequence of Sylow's Theorem is that $G \cong C_{pq}$ is cyclic. It remains to analyse the case $p \mid q-1$. Another application of Sylow's Theorem yields, up to isomorphism, exactly two groups of order pq: the cyclic group C_{pq}, and only one non-abelian group, which can be written as the semidirect product $C_q \rtimes C_p$. The particular case $p = 2$ corresponds to the dihedral group of order $2q$, which has been extensively studied (see [3, 4, 11, 23, 27]), therefore we will assume that p, q are both odd prime numbers with $p \mid q-1$. From now on, we denote

$$\mathcal{G} = C_q \rtimes C_p \cong \langle \alpha, \tau : \alpha^q = \tau^p = 1, \alpha\tau = \tau\alpha^s \rangle, \tag{2}$$

where s has order p modulo q. The commutator subgroup of $\mathcal{G}$ is $\mathcal{G}' = \langle \alpha \rangle$ and its center is $Z(\mathcal{G}) = \{1\}$. The centralizer of $g \in \mathcal{G} \backslash \{1\}$ is $C_{\mathcal{G}}(g) = \langle g \rangle$. Moreover, $\mathrm{ord}(g) = q$ for every $g \in \mathcal{G}' \backslash \{1\}$ and $\mathrm{ord}(g) = p$ for every $g \in \mathcal{G} \backslash \mathcal{G}'$. Since p, q are odd and $p \mid q-1$, it follows that $q \ge 2p+1$.

It is worth mentioning that the direct and inverse problems over $\mathcal{G}$ associated to other invariants are already known, such as $\mathsf{d}(\mathcal{G}) = p+q-2$ [3, Lemma 14] (see also [5] for the inverse problem), and the Erdős-Ginzburg-Ziv constant [3, Theorem 15] (see also [29] for the inverse problem).

3. Preliminary results

The proof of Theorem 1.1 closely follows the approach used by Grynkiewicz [18] in his solution of the direct problem. Several lemmas from his paper and also from [11] are employed, some of them in an adapted form. We begin by stating a few general lemmas.

Lemma 3.1 (Cauchy-Davenport Theorem [17, Theorem 6.2])**.** Let $G \cong C_q$, where q is a prime number, and let $A, B \subset G$ be non-empty subsets. Then

$$|AB| \ge \min\{q, |A| + |B| - 1\}.$$

Lemma 3.2 ([11, Lemma 2.1])**.** Let G be a group, let $U^* \in \mathcal{F}^*(G)$ be an ordered sequence with $\pi(U^*) = 1$ and let $[U^*] \in \mathcal{A}(G)$ an atom. Then there are no consecutive product-one subsequences of U^* that are proper and nontrivial.

Lemma 3.3 ([11, Lemma 2.2])**.** Let G be a group and let $S \in \mathcal{F}(G)$ be a product-one sequence. If $T \mid S$ is a subsequence with $\pi(T) \subseteq G'$, then $\pi(S \boldsymbol{\cdot} T^{[-1]}) \subseteq G'$. In particular, if $T \mid S$ is a product-one subsequence, then $\pi(S \boldsymbol{\cdot} T^{[-1]}) \subseteq G'$.

Lemma 3.4 ([11, Lemma 2.4.1])**.** Let G be a finite group. Then every ordered sequence $S \in \mathcal{F}^*(G)$ of length $|S| \ge |G|$ has a consecutive, product-one subsequence that is nontrivial. In particular, we have $\mathsf{d}(G) + 1 \le \mathsf{D}(G) \le |G|$.

Lemma 3.5 ([13, Theorem 5.4.5.2], see also [9, Theorem 2.1]). Let $G \cong C_n$, where $n \ge 3$, and let $S \in \mathcal{F}(G)$ be a product-one free sequence of length $|S| \ge \frac{n+1}{2}$. Then there exists $g \in \mathrm{supp}(S)$ such that $\mathsf{v}_g(S) \ge 2|S| - n + 1$. In particular, $\mathsf{D}(G) = n$ and if $|S| = n - 1$, then $S = g^{[n-1]}$.

The following lemma is a crucial technical tool that will be invoked repeatedly in the proof of the main theorem. It embodies a simple yet effective algorithm, and for further details on the underlying idea we refer to the discussion after Lemma 3.2 in [18].

Lemma 3.6 ([18, Lemma 3.3]). Let G be a non-abelian finite group, let $S^* \in \mathcal{F}^*(G)$ be an ordered sequence, let $H \le G$ be an abelian subgroup, let

$$\omega \ge 1, \quad \omega_H \in \mathbb{Z}, \quad \text{and} \quad \omega_0 \in \{0\} \cup [2, |S^*|] \quad \text{with} \quad \omega_0 \le \omega,$$

and suppose that $|\pi(S_0)| \ge |S_0| = \omega_0$ and $\pi(S_0) \cap (G \backslash Z(G)) \ne \varnothing$ (if $\omega_0 > 0$), where $S_0 = [S^*(1, \omega_0)]$, and that there are at least ω_H terms of $S \boldsymbol{\cdot} S_0^{[-1]}$ from H.

Then there exists an ordered sequence $S'^* \in \mathcal{F}(G)$ with

$$[S'^*] = [S^*] \quad \text{and} \quad \pi(S'^*) \in \pi(S^*)^G$$

having a factorization

$$S'^* = T_1^* \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_{r-1}^* \boldsymbol{\cdot} T_r^* \boldsymbol{\cdot} R^*,$$

where $T_1^*, \ldots, T_r^*, R^* \in \mathcal{F}^*(G)$ and $r \ge 0$, such that, letting $R = [R^*]$ and $T_i = [T_i^*]$ for $i \in [1, r]$, we have $S_0 \mid T_1$ (if $\omega_0 > 0$),

$$\pi(T_i) \cap (G \backslash Z(G)) \ne \varnothing \quad \text{and} \quad |\pi(T_i)| \ge |T_i| \ge 2 \quad \text{for } i \in [1, r], \quad \pi(T_i)^G = \pi(T_i) \quad \text{for } i \in [1, r-1],$$

and either

(i) $\sum_{i=1}^r |T_i| \le \omega - 1$ and $\langle \mathrm{supp}(R) \rangle < G$ is a proper subgroup, or
(ii) $\omega \le \sum_{i=1}^r |T_i| \le \omega + 1$, with the upper bound only possible if $|T_r| = 2$ and $\sum_{i=1}^{r-1} |T_i| = \omega - 1$, and there are at least ω_H terms of R from H, or
(iii) $\sum_{i=1}^r |T_i| \le \omega - 1$ and there are precisely ω_H terms of R from H.

From now on, all results are restricted to the group $\mathcal{G}$ defined in Equation (2).

Lemma 3.7 ([18, Lemma 5.3]). Let $S \in \mathcal{F}(\mathcal{G} \backslash \{1\})$ and $g \in \mathcal{G} \backslash \mathcal{G}'$. Then $|\pi(g \boldsymbol{\cdot} S)| \ge \min\{q, |g \boldsymbol{\cdot} S|\}$.

Lemma 3.8 ([18, Lemma 5.4]). Let $S \in \mathcal{F}(\mathcal{G}' \backslash \{1\})$ and $g_1, g_2 \in \mathcal{G} \backslash \mathcal{G}'$. Suposse $g_1 g_2 \notin \mathcal{G}'$. Then

$$|\pi(g_1 \boldsymbol{\cdot} g_2 \boldsymbol{\cdot} S)| \ge \min\{q, 2|S| + 1\}.$$

Lemma 3.9 ([18, Lemma 5.5]). Let $S \in \mathcal{F}(\mathcal{G} \backslash \{1\})$. If $\langle \mathrm{supp}(S) \rangle = \mathcal{G}$, then $|\pi(S)| \ge \min\{p, |S|\}$.

Lemma 3.10 ([18, Lemma 5.8]). Let $S \in \mathcal{F}(\mathcal{G})$. If $|S| \ge q + 2p - 3$, then there is a nontrivial, product-one subsequence $T \mid S$ with $|T| \le q$.

Lemma 3.11 ([18, Lemma 5.11]). Let $T_1, \ldots, T_r \in \mathcal{F}(\mathcal{G})$ be sequences for which

$$\pi(T_i) \cap (\mathcal{G} \backslash Z(\mathcal{G})) \ne \varnothing \quad \text{and} \quad |\pi(T_i)| \ge |T_i| \ge 2 \quad \text{for } i \in [1, r], \quad \pi(T_i)^{\mathcal{G}} = \pi(T_i) \quad \text{for } i \in [1, r-1].$$

Then the following hold:

(1) $|\pi(T_1) \ldots \pi(T_r)| \ge \min\{q - 1, \sum_{i=1}^r |\pi(T_i)|\} \ge \min\{q - 1, \sum_{i=1}^r |T_i|\}$;
(2) if $\sum_{i=1}^r |T_i| \ge q + 1$, then $|\pi(T_1) \ldots \pi(T_r)| = q$.

The following result is essentially contained in the proof of [3, Lemma 14] (see also [20, Lemma 4]), and we reproduce it here for convenience.

Lemma 3.12. If $T \in \mathcal{F}(\mathcal{G})$ such that $|T| \geq q$ and $\pi(T) \cap \mathcal{G}' \neq \varnothing$, then T contains a product-one subsequence.

Proof. Since $\mathcal{G}/\mathcal{G}' \cong C_p$, which is an abelian group, we obtain that $\pi(T) \subseteq \mathcal{G}'$. Let us factorize T as

$$T = T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_k,$$

where $\pi(T_i) \subseteq \mathcal{G}'$ and each T_i is minimal with respect to this property. Since $|T| \geq q > p = \mathsf{d}(\mathcal{G}/\mathcal{G}') + 1$ and $2p + 1 \leq q$, we obtain that $k \geq 3$. Let us denote $A_1 = \pi(T_1)$ and $A_i = \pi(T_i) \cup \{1\}$ for all $i \in [2, k]$.

We claim that either $1 \in \pi(T_i)$ for some $i \in [1, k]$, in which case we conclude the proof, or $1 \notin \pi(T_i)$ and $|\pi(T_i)| \geq |T_i|$ for all $i \in [1, k]$. In the latter case, it follows from the Cauchy–Davenport Theorem (Lemma 3.1) that

$$|A_1 \ldots A_k| \geq \min\left\{q, \sum_{i=1}^{k} |A_i| - (k-1)\right\} = \min\left\{q, \sum_{i=1}^{k} |\pi(T_i)| + (k-1) - (k-1)\right\} \geq \min\left\{q, \sum_{i=1}^{k} |T_i|\right\} = q.$$

Since $A_1 \ldots A_k \subseteq \Pi(T)$, we conclude that T has a product-one subsequence.

We remark that if $g_1, g_2 \in \mathcal{G}$ are such that $g_1 g_2 = g_2 g_1 \in \mathcal{G}' \backslash \{1\}$, then $g_1, g_2 \in \mathcal{G}' \backslash \{1\}$. Let $g_1 = \tau^{a_1} \alpha^{b_1}$ and $g_2 = \tau^{a_2} \alpha^{b_2}$. Hence,

$$\tau^{a_1 + a_2} \alpha^{b_1 + b_2 s^{a_1}} = \tau^{a_1 + a_2} \alpha^{b_2 + b_1 s^{a_2}} = \alpha^u \in \mathcal{G}' \backslash \{1\},$$

and we have $a_1 + a_2 \equiv 0 \pmod{p}$ and $b_1 + b_2 s^{a_1} \equiv b_2 + b_1 s^{a_2} \equiv u \pmod{q}$. Then

$$u \equiv b_1 + b_2 s^{a_1} \equiv s^{a_1}(b_1 s^{-a_1} + b_2) \equiv s^{a_1}(b_1 s^{a_2} + b_2) \equiv u s^{a_1} \pmod{q}.$$

Since $u \not\equiv 0 \pmod{q}$, we obtain that $a_1 \equiv 0 \equiv a_2 \pmod{p}$, and thus $g_1, g_2 \in \mathcal{G}' \backslash \{1\}$.

Now we finish the proof by showing that if $1 \notin \pi(T_i)$, then $|\pi(T_i)| \geq |T_i|$. Let $|T_i| = t_i$, write $T_i = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_{t_i}$, and define $\pi_j(T_i) = g_j \ldots g_{t_i} g_1 \ldots g_{j-1}$ for $j \in [1, t_i]$. Observe that $|\{\pi_j(T_i)\}| \leq |\pi(T_i)|$. We will show that $|\{\pi_j(T_i)\}| = t_i$, that is, $|T_i| \leq |\pi(T_i)|$. If $|\{\pi_j(T_i)\}| < t_i$, then there exist $1 \leq k < \ell \leq t_i$ such that $\pi_k(T_i) = \pi_\ell(T_i)$. Then

$$g_k \ldots g_{t_i} g_1 \ldots g_{k-1} = g_\ell \ldots g_{t_i} g_1 \ldots g_{\ell-1}.$$

Let $g = g_k \ldots g_{\ell-1}$ and $g' = g_\ell \ldots g_{t_i} g_1 \ldots g_{k-1}$. Hence, $gg' = g'g \in \mathcal{G}' \backslash \{1\}$, since $1 \notin \pi(T_i)$. But it follows from the remark above that $g, g' \in \mathcal{G}' \backslash \{1\}$, which contradicts the minimality of T_i. Therefore, $|\{\pi_j(T_i)\}| = t_i$, and we conclude the proof.

□

The following results are adapted from [18]. Their proofs are similar.

Lemma 3.13 (Adapted from [18, Lemma 5.9])**.** Let $S \in \mathcal{A}(\mathcal{G})$ with $\mathsf{v}_{\mathcal{G} \backslash \mathcal{G}'}(S) \geq 3$. If $|S| \geq 2q$, then

$$\mathsf{v}_{\mathcal{G}'}(S) \leq \frac{q-3}{2}.$$

Proof. At first, we claim that there exist $g_1, g_2 \in \operatorname{supp}(S) \cap (\mathcal{G} \backslash \mathcal{G}')$ such that $g_1 g_2 \notin \mathcal{G}'$. In fact, since $\mathsf{v}_{\mathcal{G} \backslash \mathcal{G}'}(S) \geq 3$, let $x, y, z \in \operatorname{supp}(S) \cap (\mathcal{G} \backslash \mathcal{G}')$ and assume that $xy, xz, yx \in \mathcal{G}'$. Let $\phi_{\mathcal{G}'}(g) = g\mathcal{G}'$ be the canonical homomorphism. Then $\mathcal{G}' = \phi_{\mathcal{G}'}(xy) = \phi_{\mathcal{G}'}(xz)$ and, hence, $\phi_{\mathcal{G}'}(y) = \phi_{\mathcal{G}'}(z)$. This implies that $\mathcal{G}' = \phi_{\mathcal{G}'}(yz) = \phi_{\mathcal{G}'}(y)^2$. Since $|\mathcal{G}/\mathcal{G}'| = p$ is an odd prime number, we obtain that $y \in \mathcal{G}'$, which is a contradiction.

Now, we will show that $\mathsf{v}_{\mathcal{G}'}(S) \leq \frac{q-3}{2}$ and, to this end, let us assume otherwise, that is, assume that $\mathsf{v}_{\mathcal{G}'}(S) \geq \frac{q-1}{2}$. Let $T \mid S$ be a subsequence such that $\operatorname{supp}(T) \subseteq \mathcal{G}'$ and $|T| = \frac{q-1}{2}$. Then

$$|S \boldsymbol{\cdot} (g_1 \boldsymbol{\cdot} g_2 \boldsymbol{\cdot} T)^{[-1]}| = |S| - |T| - 2 \geq 2q - \frac{q-1}{2} - 2 = q + \frac{q-1}{2} - 1 \geq q + p - 1 = \mathsf{d}(\mathcal{G}) + 1.$$

Let $R \mid S$ be a nontrivial product-one subsequence such that $g_1 \boldsymbol{\cdot} g_2 \boldsymbol{\cdot} T \mid S \boldsymbol{\cdot} R^{[-1]}$. It follows from Lemma 3.3 that $\pi(S \boldsymbol{\cdot} R^{[-1]}) \subseteq \mathcal{G}'$. However, since $\mathrm{supp}(T) \subseteq \mathrm{supp}(S) \subseteq \mathcal{G} \backslash \{1\}$, it follows from Lemma 3.8 that

$$|\pi(g_1 \boldsymbol{\cdot} g_2 \boldsymbol{\cdot} T)| \ge \min\{q, 2|T|+1\} = q,$$

which implies that $1 \in \pi(S \boldsymbol{\cdot} R^{[-1]})$. Therefore, $R \boldsymbol{\cdot} (S \boldsymbol{\cdot} R^{[-1]})$ is a nontrivial factorization of S into two product-one subsequences, contradicting the fact that S is an atom. □

Lemma 3.14 (Adapted from [18, Lemma 5.12])**.** Let $S \in \mathcal{A}(\mathcal{G})$. If $|S| \ge 2q$, then

$$\mathsf{v}_H(S) \le q-1 \quad \text{for every subgroup} \quad H \le \mathcal{G} \quad \text{with} \quad |H| = p.$$

Proof. Let $S^* \in \mathcal{F}(\mathcal{G})$ be such that $\pi(S^*) = 1$ and $S = [S^*]$, and let us assume that there exists a subgroup $H \le \mathcal{G}$ with $|H| = p$ such that $\mathsf{v}_H(S) \ge q$. We will apply Lemma 3.6 to S^* using H with

$$\omega = q+1, \quad \omega_H = p+1, \quad \text{and} \quad \omega_0 = 0.$$

Let

$$S'^* = T_1^* \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r^* \boldsymbol{\cdot} R^*$$

be the factorization obtained in Lemma 3.6, and we will analyze the three cases. First, we remark that since $\pi(S'^*) \in \pi(S^*)^{\mathcal{G}} = \{1\}$, it follows that $\pi(S'^*) = 1$.

Case 1. $\sum_{i=1}^r |T_i| \le \omega - 1 = q$ and $K = \langle \mathrm{supp}(R) \rangle < \mathcal{G}$ is a proper subgroup.

Since $\mathsf{v}_H(S) \ge q$ and $\sum_{i=1}^r |T_i| \le q$, we have two possibilities: either there is at least one element of $H \backslash \{1\}$ in K, or there is none different from 1.

Assume first that there is a element of H in K and, since K is a proper subgroup, we must have $H = K$. However, since $|R| = |S| - \sum_{i=1}^r |T_i| \ge 2q - q = q > p$, Lemma 3.4 guarantees a nontrivial and proper product-one consecutive subsequence of R^*, contradicting Lemma 3.2 and the fact that S is an atom.

Now, assume that K contains no element of H other than 1. In this case, we have $\mathsf{v}_H(S) = q$, $\sum_{i=1}^r |T_i| = q$, and $|R| \ge q$. Let us denote $K = \langle \tau^a \alpha^b \rangle$. If $a \not\equiv 0 \pmod p$, then $|K| = p < q \le |R|$. On the other hand, if $a \equiv 0 \pmod p$, that is, $K = \mathcal{G}'$, then $\mathsf{d}(\mathcal{G}') = q-1 < q \le |R|$. In both cases, we also obtain a contradiction by Lemma 3.4, Lemma 3.2, and the fact that S is an atom.

Case 2. $q+1 = \omega \le \sum_{i=1}^r |T_i| \le \omega + 1 = q+2$ and there are at least $\omega_H = p+1$ elements of R from H.

Since $\mathsf{d}(H) = p-1$, there exists a nontrivial product-one subsequence $R' \mid R$. As a consequence of Lemma 3.3, we have that $\pi(S \boldsymbol{\cdot} R'^{[-1]}) \subseteq \mathcal{G}'$. Now, observe that $T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r \mid S \boldsymbol{\cdot} R'^{[-1]}$ and, by Lemma 3.11.(2), $|\pi(T_1) \ldots \pi(T_r)| = q$. Then $\pi(S \boldsymbol{\cdot} R'^{[-1]}) = \mathcal{G}'$. Therefore, $S = R' \boldsymbol{\cdot} (S \boldsymbol{\cdot} R'^{[-1]})$ is a nontrivial factorization of S into two product-one subsequences, contradicting the fact that S is an atom.

Case 3. $\sum_{i=1}^r |T_i| \le \omega - 1 = q$ and $\mathsf{v}_H(R) = \omega_H = p+1$.

Since $\mathsf{v}_H(R) = \omega_H = p+1 < q \le \mathsf{v}_H(S)$, we obtain that

$$\mathsf{v}_H(T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r) = \mathsf{v}_H(S) - \mathsf{v}_H(R) \ge q - p - 1.$$

Since H is abelian and $|\pi(T_i)| \ge |T_i| \ge 2$, each T_i has terms from $\mathcal{G} \backslash H$. Then

$$\sum_{i=1}^r |T_i| \ge q - p - 1 + r \ge q - p,$$

where $r \ge 1$. Notice that if $\sum_{i=1}^r |T_i| = q-p$, then $r = 1$ and $|R| = q+p$. Now, since $\sum_{i=1}^r |T_i| \le q$, we obtain that

$$\mathsf{v}_{\mathcal{G} \backslash H}(R) = \begin{cases} |S| - |T_1| - \mathsf{v}_H(R) = 2q - q + p - p - 1 = q-1, & \text{if } r = 1, \\ |S| - \sum_{i=1}^r |T_i| - \mathsf{v}_H(R) \ge 2q - q - p - 1 = q - p - 1 > p - 1, & \text{if } r \ge 2. \end{cases}$$

Then there exists $R' \mid R$ such that

$$|R'| = \begin{cases} q-1, & \text{if } r = 1, \\ p-1, & \text{if } r \geq 2, \end{cases}$$

and $\text{supp}(R') \cap H = \varnothing$. Let $g \in \text{supp}(R) \cap H$. Then $\langle \text{supp}(g \boldsymbol{\cdot} R') \rangle = \mathcal{G}$, and Lemma 3.9 guarantees that

$$|\pi(g \boldsymbol{\cdot} R')| \geq \min\{p, |g \boldsymbol{\cdot} R'|\} = p.$$

Since $\pi(g \boldsymbol{\cdot} R')$ is contained in a $\mathcal{G}'$-coset, we can apply the Cauchy-Davenport Theorem (Lemma 3.1) and, together with Lemma 3.11.(1), we obtain that

$$\begin{aligned} |\pi(T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r)\pi(g \boldsymbol{\cdot} R')| &\geq \min\{q, |\pi(T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r)| + |\pi(g \boldsymbol{\cdot} R')| - 1\} \\ &\geq \min\left\{q, \min\{q-1, \sum_{i=1}^{r} |T_i|\} + |\pi(g \boldsymbol{\cdot} R')| - 1\right\} \end{aligned}$$

and, since $q \geq 2p+1$,

$$\min\left\{q, \min\left\{q-1, \sum_{i=1}^{r} |T_i|\right\} + |\pi(g \boldsymbol{\cdot} R')| - 1\right\} \geq \begin{cases} \min\{q, \min\{q-1, q-p\} + q - 1\} = q & \text{if } r = 1, \\ \min\{q, \min\{q-1, q-p+1\} + p - 1\} = q & \text{if } r \geq 2. \end{cases}$$

In both cases, since $\mathsf{v}_H(R) = p+1$ and $\mathsf{v}_H(g \boldsymbol{\cdot} R') = 1$, we still have $p = |H|$ terms of $R \boldsymbol{\cdot} (g \boldsymbol{\cdot} R')^{[-1]}$ from H. Since $\mathsf{d}(H) = p-1$, there exists a nontrivial product-one subsequence $R'' \mid R \boldsymbol{\cdot} (g \boldsymbol{\cdot} R')^{[-1]}$. It follows from Lemma 3.3 that $\pi(S \boldsymbol{\cdot} R''^{[-1]}) \subseteq \mathcal{G}'$. However, since $T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r \boldsymbol{\cdot} g \boldsymbol{\cdot} R' \mid S \boldsymbol{\cdot} R''^{[-1]}$, it follows that $\pi(S \boldsymbol{\cdot} R''^{[-1]}) = \mathcal{G}'$. Therefore, $S = R'' \boldsymbol{\cdot} (S \boldsymbol{\cdot} R''^{[-1]})$ is a nontrivial factorization of S into two product-one subsequences.

Summing up, we have that $\mathsf{v}_H(S) \leq q-1$ for every subgroup $H \leq \mathcal{G}$ with $|H| = p$.

□

4. Proof of Theorem 1.1

The proof of Theorem 1.1 follows from the next two results. The first states that any atom of length $\mathsf{D}(\mathcal{G})$ must have at most two terms from $\mathcal{G} \backslash \mathcal{G}'$.

Theorem 4.1. Let $S \in \mathcal{F}(\mathcal{G})$ such that $|S| = 2q$. If $\mathsf{v}_{\mathcal{G} \backslash \mathcal{G}'}(S) \geq 3$, then $S \notin \mathcal{A}(\mathcal{G})$.

Proof. Let us assume that $S \in \mathcal{A}(\mathcal{G})$ and let $S^* \in \mathcal{F}(\mathcal{G})$ be such that $\pi(S^*) = 1$ and $S = [S^*]$. It follows from Lemma 3.13 that $\mathsf{v}_{\mathcal{G}'}(S) \leq \frac{q-3}{2}$. In light of Lemma 3.10, since $|S| = 2q > q + 2p - 3$, there exists $U \mid S$ such that $|U| \leq q$ and $1 \in \pi(U)$. Let U be minimal with respect to this property, that is, U is the shortest product-one subsequence of S. We will split the proof into three cases.

Case 1. $|U| = q$.

Let $V = S \boldsymbol{\cdot} U^{[-1]}$. It follows from Lemma 3.3 that $\pi(V) \subseteq \mathcal{G}'$. Let us assume that $1 \notin \pi(V)$; otherwise, $S = U \boldsymbol{\cdot} V$ is a factorization of S into two product-one subsequences, contradicting the fact that $S \in \mathcal{A}(\mathcal{G})$. As a consequence of Lemma 3.12, V contains a product-one subsequence of lenght at most $q-1$, contradicting the minimality of U.

Case 2. $|U| \leq q - p$.

We first claim that U can be taken as a nontrivial product-one subsequence with $|U| \leq p$ and $|\langle \text{supp}(U) \rangle| = p$. Let $W = S \boldsymbol{\cdot} U^{[-1]}$. If $\mathsf{v}_{\mathcal{G}'}(W) = 0$, let W_0 be the trivial sequence. Otherwise, since

$$|W| = |S| - |U| \geq 2q - q + p = q + p > \frac{q-3}{2} \geq \mathsf{v}_{\mathcal{G}'}(S),$$

let $W_0 \mid W$ be the subsequence containing all terms of W from $\mathcal{G}'$ and exactly one term from $\mathcal{G} \backslash \mathcal{G}'$. Moreover, if W_0 is nontrivial, Lemma 3.7 guarantees that $|\pi(W_0)| \geq |W_0| \geq 2$ and, hence, $\pi(W_0) \cap (\mathcal{G} \backslash \{1\}) \neq \varnothing$. Let us fix

W^* as any ordering of W such that $[W^*(1, |W_0|)] = W_0$. Now, we apply Lemma 3.6 to W by taking $H = \{1\}$, $\omega = q + 1$, $\omega_H = -1$, and $\omega_0 = |W_0| \le \frac{q-1}{2}$. Let $W'^* = T_1^* \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r^* \boldsymbol{\cdot} R^*$ be the factorization obtained in Lemma 3.6. We note that since $\omega_H = -1$, the third case of this lemma does not occur. If the second case holds, that is, $q + 1 \le \sum_{i=1}^r |T_i|$, then Lemma 3.11.(2) implies that $|\pi(T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r)| \ge |\pi(T_1) \ldots \pi(T_r)| = q$. However, since $1 \in \pi(U)$, then $\pi(S \boldsymbol{\cdot} U^{[-1]}) = \pi(W) \subseteq \mathcal{G}'$ by Lemma 3.3. Thus, $\pi(W) = \mathcal{G}'$, and hence, $S = W \boldsymbol{\cdot} U$ is a nontrivial factorization of S into two product-one subsequences, contradicting the fact that S is an atom. Finally, assume that the first case of Lemma 3.6 holds, that is, $\sum_{i=1}^r |T_i| \le \omega - 1 = q$ and $K = \langle \mathrm{supp}(R) \rangle$ is a proper subgroup of $\mathcal{G}$. In this case,

$$|R| = |W| - \sum_{i=1}^r |T_i| \ge q + p - q = p.$$

Since $W_0 \mid T_1$, there is no term in R from $\mathcal{G}'$ and thus $|K| = p$. Then $|R| \ge p = \mathsf{d}(K) + 1$, and there exists a nontrivial product-one subsequence of R with at most p elements. This proves that there exists a nontrivial product-one subsequence U of S with $|U| \le p$ and $|\langle \mathrm{supp}(U) \rangle| = p$. From now on, let U be this product-one subsequence of S.

Let us define $W = S \boldsymbol{\cdot} U^{[-1]}$, W_0 and W^* as done above. We will use Lemma 3.6 on W again, using the same parameters: $H = \{1\}$, $\omega = q + 1$, $\omega_H = -1$, and $\omega_0 = |W_0| \le \frac{q-1}{2}$. Statements (2) and (3) of Lemma 3.6 do not hold, as argued in the paragraph above. Then let us assume that $\sum_{i=1}^r |T_i| \le \omega - 1 = q$ and that $K = \langle \mathrm{supp}(R) \rangle$ is a proper subgroup of $\mathcal{G}$. Then

$$|R| = |S| - |U| - \sum_{i=1}^r |T_i| \ge 2q - p - q = q - p \ge p + 1.$$

Let $K' = \langle \mathrm{supp}(U) \rangle$. If $K = K'$, then all terms of $R \boldsymbol{\cdot} U$ belong to the same subgroup of order p, and $|R \boldsymbol{\cdot} U| = |S| - \sum_{i=1}^r |T_i| \ge 2q - q = q$, which contradicts Lemma 3.14. Therefore, $K \ne K'$. Since $|R| > \mathsf{d}(K)$, there exists a nontrivial product-one subsequence $L \mid R$ with $\langle \mathrm{supp}(L) \rangle = K$.

Let us define $V = S \boldsymbol{\cdot} L^{[-1]} = W \boldsymbol{\cdot} U \boldsymbol{\cdot} L^{[-1]}$ and $Z = R \boldsymbol{\cdot} U \boldsymbol{\cdot} L^{[-1]} \mid V$. First, we observe that since $1 \in \pi(L)$, then $\pi(V) \subseteq \mathcal{G}'$ by Lemma 3.3. Moreover, there are terms of Z from both K and K'. Since $1 \notin \mathrm{supp}(S)$, there exist elements $g \in K \backslash \{1\}$ and $g' \in K' \backslash \{1\}$ such that $g, g' \in \mathrm{supp}(Z)$. Since $K \ne K'$, we obtain that $gg' \ne g'g$. If $q - 1 \le \sum_{i=1}^r |T_i| \le q$, since $\pi(g \boldsymbol{\cdot} g')$ is contained in a $\mathcal{G}'$-coset, we apply the Cauchy-Davenport Theorem (Lemma 3.1) and, together with Lemma 3.11.(1), we obtain that

$$\begin{aligned} |\pi(T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r)\pi(g \boldsymbol{\cdot} g')| &\ge \min\{q, |\pi(T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r)| + |\pi(g \boldsymbol{\cdot} g')| - 1\} \\ &\ge \min\left\{q, \min\left\{q - 1, \sum_{i=1}^r |T_i|\right\} + |\pi(g \boldsymbol{\cdot} g')| - 1\right\} \\ &\ge \min\{q, \min\{q - 1, q - 1\} + 2 - 1\} = q. \end{aligned}$$

Since $T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r \boldsymbol{\cdot} g \boldsymbol{\cdot} g' \mid V$, we obtain that $\pi(V) = \mathcal{G}'$, and hence, $S = V \boldsymbol{\cdot} L$ is a nontrivial factorization of S into two product-one subsequences. Therefore, $\sum_{i=1}^r |T_i| \le q - 2$.

Let us fix $V_0 = T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r$ and V^* as any ordering of V such that $[V^*(1, |V_0|)] = V_0$. Then $|V_0| \le q - 2$, and as a consequence of Lemma 3.11.(1),

$$|\pi(V_0)| = |\pi(T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r)| \ge |\pi(T_1) \ldots \pi(T_r)| \ge \min\left\{q - 1, \sum_{i=1}^r |T_i|\right\} = |V_0|.$$

Let us apply Lemma 3.6 again but with the parameters $H = \{1\}$, $\omega = q + 1$, $\omega_H = -1$, and $\omega_0 = |V_0| \le q - 2$. Again, the statements (2) and (3) from Lemma 3.6 do not hold. Let $V'^* = T_1'^* \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r'^* \boldsymbol{\cdot} R'$ be the factorization obtained from Lemma 3.6. Since $V_0 \mid T_1'$, then $T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_r \mid T_1'$ and $R' \mid Z$. Now, $\mathrm{supp}(Z) \subseteq K \cup K'$ with $\mathsf{v}_{K'}(Z) = \mathsf{v}_{K'}(U) = |U| \le p$. Then at most p terms of R' are from K', and the remaining are from K. However,

$\langle \mathrm{supp}(R') \rangle$ is a proper subgroup of $\mathcal{G}$ and

$$|R'| = |V'| - \sum_{i=1}^{r} |T_i'| = |S| - |L| - \sum_{i=1}^{r} |T_i'| \geq 2q - p - q = q - p \geq p + 1.$$

Thus, all terms of R' are from K. But, since $\mathrm{supp}(R' \cdot L) \subseteq K$ and

$$|R' \cdot L| = |S| - \sum_{i=1}^{r} |T_i'| \geq 2q - q = q,$$

we obtain a contradiction by Lemma 3.14. Therefore, this case cannot occur.

Case 3. $q - p < |U| \leq q - 1$.

Since $|S| - |U| \geq q + 1 > \frac{q-3}{2} + p$, there exist $g_1, g_2 \in \mathrm{supp}(S \cdot U^{[-1]}) \cap (\mathcal{G} \backslash \mathcal{G}')$ with $\langle g_1, g_2 \rangle = \mathcal{G}$. Let us fix $W = S \cdot (U \cdot g_1 \cdot g_2)^{[-1]}$. Since $|W| = |S| - |U| - 2 \geq q - 1 > \mathsf{v}_{\mathcal{G}'}(S)$, let us proceed as in case 2. and let W_0 be the trivial sequence if $\mathsf{v}_{\mathcal{G}'}(W) = 0$, and let W_0 be the sequence consisting of all terms of W from $\mathcal{G}'$ and one term from $\mathcal{G} \backslash \mathcal{G}'$. In the latter case, Lemma 3.7 guarantees that $|\pi(W_0)| \geq |W_0| \geq 2$ and, hence, $\pi(W_0) \cap (\mathcal{G} \backslash \{1\}) \neq \varnothing$. Let $W^* \in \mathcal{F}(\mathcal{G})$ be any ordering of W such that $[W^*(1, |W_0|)] = W_0$, and let us apply Lemma 3.6 with the parameters $H = \{1\}$, $\omega = q - p - 1$, $\omega_H = -1$, and $\omega_0 = |W_0| \leq \frac{q-1}{2} \leq \omega$. Let $W'^* = T_1^* \cdot \ldots \cdot T_r^* \cdot R^*$ be the factorization obtained in Lemma 3.6. Since $\omega_H = -1$, statement (3) of this lemma does not occur. Let us analyze the other two statements.

Subcase 3.1. Assume that the second statement of Lemma 3.6 holds, that is,

$$\omega = q - p - 1 \leq \sum_{i=1}^{r} |T_i| \leq \omega + 1 = q - p < q - 1 \leq |W|.$$

As a consequence of Lemma 3.11.(1): if $\sum_{i=1}^{r} |T_i| = q - p - 1$, then

$$|R| = |W| - \sum_{i=1}^{r} |T_i| \geq (q - 1) - (q - p - 1) = p$$

and

$$|\pi(T_1) \ldots \pi(T_r)| \geq \min \left\{ q - 1, \sum_{i=1}^{r} |T_i| \right\} \geq \min \{q - 1, q - p - 1\} = q - p - 1;$$

if $\sum_{i=1}^{r} |T_i| = q - p$, then

$$|R| = |W| - \sum_{i=1}^{r} |T_i| \geq (q - 1) - (q - p) = p - 1$$

and

$$|\pi(T_1) \ldots \pi(T_r)| \geq \min \left\{ q - 1, \sum_{i=1}^{r} |T_i| \right\} \geq \min \{q - 1, q - p\} = q - p.$$

That is,

$$|R| \geq p - 1 + \epsilon$$

and

$$|\pi(T_1) \ldots \pi(T_r)| \geq q - p - \epsilon,$$

where $\epsilon \in \{0, 1\}$. As a consequence of Lemma 3.7,

$$|\pi(R \cdot g_1 \cdot g_2)| \geq \min \{q, |R| + 2\} = p + 1 + \epsilon.$$

Now, using the Cauchy-Davenport Theorem (Lemma 3.1), we obtain that

$$|(\pi(T_1) \ldots \pi(T_t))(\pi(R \cdot g_1 \cdot g_2))| \geq \min \{q, |\pi(T_1) \ldots \pi(T_t)| + |\pi(R \cdot g_1 \cdot g_2)| - 1\} = q.$$

Then $1 \in \pi(S \cdot U^{[-1]}) = \pi(W \cdot g_1 \cdot g_2)$ and $S \notin \mathcal{A}(\mathcal{G})$.

Subcase 3.2. Assume now that the first statement of Lemma 3.6 holds, that is, $\sum_{i=1}^{r} |T_i| \le \omega - 1 = q - p - 2$ and $\langle \mathrm{supp}(R) \rangle < \mathcal{G}$ is a proper subgroup. Then

$$|R| = |W| - \sum_{i=1}^{r} |T_i| \ge (q-1) - (q-p-2) = p+1.$$

Since $\mathsf{v}_{\mathcal{G}'}(R) = 0$, $\langle \mathrm{supp}(R) \rangle$ must have order p. Since $|R| > \mathsf{d}(\langle \mathrm{supp}(R) \rangle) + 1 = p$, there exists $L \mid R$ such that $1 \in \pi(L)$ and $|L| \le p$. But $|L| \le p < q - p < |U|$, which contradicts the minimality of the length of U, and this case cannot hold. This completes the proof.

□

In order to describe the minimal product-one sequences of maximal length $\mathsf{D}(\mathcal{G})$, we need to study product-one sequences S for which $\mathsf{v}_{\mathcal{G}\setminus\mathcal{G}'}(S) = 2$.

Proposition 4.2. Let $S \in \mathcal{A}(\mathcal{G})$ with $|S| = 2q$ and $|S_{\mathcal{G}\setminus\mathcal{G}'}| = 2$. Then there exist $x, y \in \mathcal{G}$ and $s \in \mathbb{Z}_q^*$ for which $\mathcal{G} = \langle x, y \mid x^p = y^q = 1, yx = xy^s, \mathrm{ord}_q(s) = p \rangle$ and

$$S = y^{[q-1]} \boldsymbol{\cdot} x \boldsymbol{\cdot} y^{[q-1]} \boldsymbol{\cdot} x^{p-1} y^{s^{p-1}+1}. \tag{3}$$

Proof. It is easy to verify that the sequence S given by Eq. (3) is a product-one sequence, since

$$y^{q-1} \cdot x \cdot y^{q-1} \cdot x^{p-1} y^{s^{p-1}+1} = y^{-1} \cdot x \cdot x^{p-1} \cdot y^{-s^{p-1}} \cdot y^{s^{p-1}+1} = 1.$$

We claim that S is minimal. Indeed, if S is not minimal, then $S = S_1 \boldsymbol{\cdot} S_2$ with $S_1, S_2 \in \mathcal{F}(\mathcal{G})$ both nontrivial product-one sequences. It is clear that either $S_1 = y^{[q]}$ or $S_2 = y^{[q]}$. Say $S_1 = y^{[q]}$. This implies that $S_2 = y^{[q-2]} \boldsymbol{\cdot} x \boldsymbol{\cdot} x^{p-1} y^{s^{p-1}+1}$. Since S_2 has product-one, it follows that

$$1 = y^t \cdot x \cdot y^{q-2-t} \cdot x^{p-1} y^{s^{p-1}+1} = y^t \cdot x \cdot x^{p-1} \cdot y^{(-2-t)s^{p-1}+s^{p-1}+1} = y^{t+(-2-t)s^{p-1}+s^{p-1}+1}$$

for some $t \in [0, q-2]$. Therefore $s^{p-1}(t+1) - (t+1) \equiv 0 \pmod q$. Since $1 \le t+1 \le q-1$, we have $\gcd(t+1, q) = 1$ and hence $s^{p-1} \equiv 1 \pmod q$, a contradiction since $\mathrm{ord}_q(s) = p$.

On the other hand, let S be a minimal product-one sequence of length $2q$ for which $|S_{\mathcal{G}\setminus\mathcal{G}'}| = 2$. We may write

$$S = x^a y^{b_1} \boldsymbol{\cdot} x^{p-a} y^{b_2} \boldsymbol{\cdot} \prod_{i=1}^{2q-2} y^{c_i}, \quad a \in [1, p-1], \ \ b_1, b_2 \in [0, q-1], \ \ c_i \in [1, q-1].$$

We may assume that $a = 1$ and $b_1 = 0$. Indeed,

$$\begin{cases} \mathrm{ord}_q(s^a) = p, \\ (x^a y^{b_1})^p = x^{ap} y^{b_1 + b_1 s^a + b_1 s^{2a} + \ldots + b_1 s^{(p-1)a}} = y^{b_1 \left(\frac{s^{ap}-1}{s^a-1}\right)} = 1, \\ y \cdot x^a y^{b_1} = x^a y^{s^a + b_1} = x^a y^{b_1} \cdot y^{s^a}, \end{cases}$$

and this means that $\{x^a y^{b_1}, y\}$ and $\{x, y\}$ generate non-abelian groups of order pq, which is unique up to isomorphism. Therefore both $\{x^a y^{b_1}, y\}$ and $\{x, y\}$ generate isomorphic groups.

Since S is a product-one sequence, it follows that

$$\pi^*(T_1) \cdot x \cdot \pi^*(T_2) \cdot x^{p-1} y^{b_2} = 1, \quad T_1, T_2 \in \mathcal{F}(\mathcal{G}').$$

If either T_1 or T_2 is a product-one sequence, then S is not an atom. By Lemma 3.5, we must have $T_j = (y^{c_j})^{[q-1]}$ for some $c_j \in [1, q-1]$, $j = 1, 2$. Similarly to the previous paragraph, we may assume that $c_1 = 1$ since $\{x, y^{c_1}\}$ and $\{x, y\}$ generate isomorphic groups. Therefore

$$S = x \boldsymbol{\cdot} x^{p-1} y^{b_2} \boldsymbol{\cdot} y^{[q-1]} \boldsymbol{\cdot} (y^{c_2})^{[q-1]},$$

so that

$$x^{p-1}y^{b_2}\cdot y^{q-1}\cdot x\cdot y^{c_2(q-1)} = y^{(b_2-1)s-c_2} = 1$$

if and only if

$$(b_2-1)s \equiv c_2 \pmod q.$$

In this case, if $c_2 \neq 1$, then, by Lemma 3.5, $y^{[c_2]}\boldsymbol{\cdot}(y^{c_2})^{[q-2]} \mid T_1\boldsymbol{\cdot}T_2$ has a nontrivial ordered product-one subsequence $T_0^* = y^{[\ell]}\boldsymbol{\cdot}(y^{c_2})^{[k]}$, with $1 \le \ell \le c_2$, $1 \le k \le q-2$ and $\ell + c_2k \equiv 0 \pmod q$, which can be chosen to be consecutive, so that

$$1 = y^{(b_2-1)s-c_2} = x^{p-1}y^{b_2}\cdot y^{c_2}\cdot y^{q-1-c_2}\cdot x\cdot (y^{c_2})^{q-2-k}\cdot y^{c_2-\ell} \in \pi(S\boldsymbol{\cdot}T_0^{[-1]}).$$

Therefore $T_0\boldsymbol{\cdot}(S\boldsymbol{\cdot}T_0^{[-1]})$ is a decomposition of S into nontrivial product-one subsequences. Thus $c_2 = 1$ and this completes the proof.

□

Now we are able to prove the main theorem of this paper.

Proof of Theorem 1.1. Let $S \in \mathcal{A}(\mathcal{G})$ with $|S| = 2q$. By Theorem 4.1, $\mathsf{v}_{\mathcal{G}\setminus\mathcal{G}'}(S) \le 2$. If $\mathsf{v}_{\mathcal{G}\setminus\mathcal{G}'}(S) = 0$, then $S \in \mathcal{F}(\mathcal{G}')$. Since $\mathcal{G}' \cong C_q$, it follows that $\mathsf{D}(\mathcal{G}') = q$, therefore S is not an atom. If $\mathsf{v}_{\mathcal{G}\setminus\mathcal{G}'}(S) = 1$, then $\pi(S)\cap\mathcal{G}' = \varnothing$, therefore S is not a product-one sequence. This implies that $\mathsf{v}_{\mathcal{G}\setminus\mathcal{G}'}(S) = 2$. By Proposition 4.2, it follows that S is of the form (1), and we are done. □

5. The union of sets of lengths containing k and the k-th elasticity of $\mathcal{B}(C_q \rtimes C_p)$

In this section, a *monoid* is a commutative cancelative semigroup with unit element. Suppose that $\mathbb{M}$ is an atomic monoid, that is, every non-unit element can be written as a finite product of atoms, and let $\mathcal{A}(\mathbb{M})$ denote the set of atoms (irreducible elements) of $\mathbb{M}$. In this sense, if $a \in \mathbb{M}$, then there exist $u_1, \dots, u_k \in \mathcal{A}(\mathbb{M})$ such that $a = u_1 \dots u_k$. This k is called the *length of the factorization* of a, and we define the *set of lengths* of a as

$$\mathsf{L}(a) = \{k \in \mathbb{N}\colon a \text{ has a factorization of length } k\}.$$

The *system of sets of lengths* of $\mathbb{M}$ is

$$\mathcal{L}(\mathbb{M}) = \{\mathsf{L}(a)\colon a \in \mathbb{M}\}.$$

If not every element of $\mathbb{M}$ is invertible, then, for $k \in \mathbb{N}$, the *union of sets of lengths containing* k is

$$\mathcal{U}_k(\mathbb{M}) = \bigcup_{\substack{L\in\mathcal{L}(\mathbb{M})\\ k\in L}} L.$$

Let $\rho_k(\mathbb{M}) = \sup\mathcal{U}_k(\mathbb{M})$ be *k-th elasticity of* $\mathbb{M}$, and let $\lambda_k(\mathbb{M}) = \inf\mathcal{U}_k(\mathbb{M})$. For a subset $L \subset \mathbb{N}$, let $\rho(L) = \frac{\sup L}{\min L} \in \mathbb{Q}_{\ge 1}\cup\{\infty\}$ be the *elasticity of* L. The *elasticity of* $\mathbb{M}$ is defined as $\rho(\mathbb{M}) = \sup\{\rho(L)\colon L \in \mathcal{L}(\mathbb{M})\}$. It is possible to show that

$$\rho(\mathbb{M}) = \sup\left\{\frac{\rho_k(\mathbb{M})}{k}\colon k\in\mathbb{N}\right\} = \lim_k \frac{\rho_k(\mathbb{M})}{k} \quad \text{and} \quad \frac{1}{\rho(\mathbb{M})} = \inf\left\{\frac{\lambda_k(\mathbb{M})}{k}\colon k\in\mathbb{N}\right\} = \lim_k \frac{\lambda_k(\mathbb{M})}{k}$$

(see [10, Proposition 2.4]). We have that

$$\mathcal{U}_k(\mathbb{M}) = \{\ell \in \mathbb{N}\colon \text{ there exist } u_1,\dots,u_k,v_1,\dots,v_\ell \in \mathcal{A}(\mathbb{M}) \text{ such that } u_1\dots u_k = v_1\dots v_\ell\}.$$

From this, it is clear that $k \in \mathcal{U}_k(\mathbb{M})$ for every $k \in \mathbb{N}$. Furthermore, $\mathcal{U}_k(\mathbb{M}) + \mathcal{U}_\ell(\mathbb{M}) \subset \mathcal{U}_{k+\ell}(\mathbb{M})$, but the converse is not necessarily true. Moreover, $\ell \in \mathcal{U}_k(\mathbb{M})$ if and only if $k \in \mathcal{U}_\ell(\mathbb{M})$, and $1 \in \mathcal{U}_k(\mathbb{M})$ if and only if $k = 1$, which is also equivalent to $\mathcal{U}_k(\mathbb{M}) = \{1\}$.

In zero-sum theory over a finite group G, the monoid $\mathcal{B}(G)$ of product-one sequences is atomic, being Krull precisely when G is abelian [21, Proposition 3.4]. In this case, $\mathcal{B}(G)$ is a natural model for studying the

arithmetic of Krull monoids, and has been extensively investigated (see [10, 32]). For non-abelian groups, $\mathcal{B}(G)$ is no longer Krull but remains a C-monoid [7, Theorem 3.2], hence still enjoying finiteness properties for arithmetical invariants [13, 16] (see also [21, 22]).

For brevity, we write $*(\mathcal{B}(G)) = *(G)$, where $* \in \{\rho, \rho_k, \lambda_k, \mathcal{U}_k, \mathcal{L}, \mathcal{A}, \dots\}$. It is known that $\mathcal{U}_k(G)$ is the singleton $\{k\}$ if and only if $|G| \le 2$, and in this case we obtain that $\mathcal{B}(G)$ is half-factorial (see [10, Proposition 3.3.2]). Thus it is convenient to assume that $|G| \ge 3$. We have the following results.

Theorem 5.1 ([21, Theorem 5.5.1])**.** Let G be a finite group with $|G| \ge 3$. Then for every $k \in \mathbb{N}$, $\mathcal{U}_k(G) = [\lambda_k(G), \rho_k(G)]$ is a finite interval.

Theorem 5.2 ([24, Proposition 5.3])**.** Let G be a finite group with $|G| \ge 3$. For every $\ell \in \mathbb{N}_0$, we have

$$\lambda_{\ell\,\mathsf{D}(G)+j}(G) = \begin{cases} 2\ell & \text{for } j = 0, \\ 2\ell + 1 & \text{for } j \in [1, \rho_{2\ell+1}(G) - \ell\,\mathsf{D}(G)], \\ 2\ell + 2 & \text{for } j \in [\rho_{2\ell+1}(G) - \ell\,\mathsf{D}(G) + 1, \mathsf{D}(G) - 1], \end{cases}$$

provided that $\ell\,\mathsf{D}(G) + j > 0$.

It is worth mentioning that if G is infinite, then $\mathcal{U}_k(G) = \mathbb{N}_{\ge 2}$ [13, Theorem 7.4.1]. In the context of the preceding theorems, $\rho_k(G)$ becomes a central invariant in the study of the interplay between zero-sum problems and factorization theory. In this direction, the following bounds hold.

Proposition 5.3 ([12, Lemma 1])**.** Let G be a finite group with $|G| \ge 3$.

(i) $k + \ell \le \rho_k(G) + \rho_\ell(G) \le \rho_{k+\ell}(G)$;

(ii) $\rho_{2k}(G) = k\mathsf{D}(G)$ and

$$k\mathsf{D}(G) + 1 \le \rho_{2k+1}(G) \le k\mathsf{D}(G) + \left\lfloor \frac{\mathsf{D}(G)}{2} \right\rfloor. \tag{4}$$

In particular, $\rho(G) = \dfrac{\mathsf{D}(G)}{2}$.

(iii) If $\rho_{2k+1}(G) \ge m$ for some $m \in \mathbb{N}$ and $\ell \ge k$, then $\rho_{2\ell+1}(G) \ge m + (\ell - k)\mathsf{D}(G)$.

We observe that $\rho_k(G)$ is fully determined in terms of $\mathsf{D}(G)$ when k is even. Nevertheless, for odd k, the lower bound on Inequality (4) is attained for cyclic groups (see [12, Corollary 1 and Proposition 6]), while the upper bound is conjectured to be eventually attained for non-cyclic abelian groups.

Conjecture 5.4 ([12, Conjecture 1])**.** Let G be a finite non-cyclic abelian group with $\mathsf{D}(G) \ge 4$. Then there exists $k_0 \in \mathbb{N}$ such that

$$\rho_{2k+1}(G) = k\mathsf{D}(G) + \left\lfloor \frac{\mathsf{D}(G)}{2} \right\rfloor$$

for each $k \ge k_0$.

By item (iii) of previous proposition, if this conjecture holds for some k_0, then it also holds for every $k \ge k_0$. Oh and Zhong investigated this problem for dihedral and dicyclic groups. In particular, they proved that the upper bound in Inequality (4) is attained when G is the dihedral group of order $2n$ with n odd (see [24, Theorem 5.4]). On the other hand, for dihedral groups of order $2n$ with n even, as well as for dicyclic groups of order $4m$, $m \ge 2$, they showed that for every $k \ge 2$, ρ_k attains neither the lower nor the upper bound in Inequality (4) (see [24, Theorem 5.5]).

For the group $C_q \rtimes C_p$, in this section we show that neither the lower nor the upper bound in Inequality (4) is attained, a phenomenon similar to [24, Theorem 5.5]. This occurs because the extremal sequences described in Theorem 1.1 somehow resemble those obtained for dihedral groups of order $2n$ with n even and for dicyclic groups. The main result of this section is stated as follows.

Theorem 5.5. Let p, q be odd prime numbers with $p \mid q-1$ and let $\mathcal{G} \cong C_q \rtimes C_p$. For every $k \in \mathbb{N}$, we have that

$$k\mathsf{D}(\mathcal{G}) + 2 \le \rho_{2k+1}(\mathcal{G}) \le k\mathsf{D}(\mathcal{G}) + \frac{\mathsf{D}(\mathcal{G})}{2} - 1.$$

Proof. Recall that $\mathsf{D}(\mathcal{G}) = 2q$. From Proposition 5.3(iii), in order to prove the first inequality it suffices to show that $\rho_3(\mathcal{G}) \ge 2q+2$. As a consequence of Theorem 4.2, we consider the minimal product-one sequences

$$S_1 = y^{[2q-2]} \boldsymbol{\cdot} x \boldsymbol{\cdot} x^{-1} y^{s^{p-1}+1}, \; S_2 = (y^{-1})^{[2q-2]} \boldsymbol{\cdot} x^{-1} y^{-1} \boldsymbol{\cdot} xy^{-1} \in \mathcal{A}(\mathcal{G}).$$

Moreover, since the products

$$x^{-1} \cdot xy^{-s-1} = y^{-s-1}, \quad x^{-1} \cdot xy^s = y^s, \quad x^{-1} y^{s^{p-1}} \cdot xy^{-s-1} = y^{-s}, \quad x^{-1} y^{s^{p-1}} \cdot xy^s = y^{s+1}$$

are all different from 1, it follows that

$$S_3 = x^{-1} \boldsymbol{\cdot} xy^{-s-1} \boldsymbol{\cdot} xy^s \boldsymbol{\cdot} x^{-1} y^{s^{p-1}} \in \mathcal{A}(\mathcal{G})$$

is a minimal product-one sequence as well. We obtain a distinct factorization

$$S_1 \boldsymbol{\cdot} S_2 \boldsymbol{\cdot} S_3 = U_1 \boldsymbol{\cdot} U_2 \boldsymbol{\cdot} U_3 \boldsymbol{\cdot} U_4 \boldsymbol{\cdot} U_5^{[2q-2]},$$

where

$$U_1 = x \boldsymbol{\cdot} x^{-1}, \; U_2 = x^{-1} y^{s^{p-1}+1} \boldsymbol{\cdot} xy^{-s-1}, \; U_3 = x^{-1} y^{-1} \boldsymbol{\cdot} xy^s, \; U_4 = x^{-1} y^{s^{p-1}} \boldsymbol{\cdot} xy^{-1}, \; U_5 = y \boldsymbol{\cdot} y^{-1} \in \mathcal{A}(\mathcal{G})$$

are minimal product-one sequences. This implies that $2q+2 \in \mathcal{U}_3(\mathcal{G})$, whence $\rho_3(\mathcal{G}) = \sup \mathcal{U}_3(\mathcal{G}) \ge 2q+2$.

For the upper bound, we assume that $\rho = \rho_{2k+1}(\mathcal{G}) = q(2k+1)$ for some $k \in \mathbb{N}$. Suppose in addition that k is minimal with this property. By assumption, there exist minimal product-one sequences $V_1, \ldots, V_{2k+1} \in \mathcal{A}(G)$ such that

$$\rho \in \mathsf{L}(V_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} V_{2k+1}).$$

By definition, there exist minimal product-one sequences $W_1, \ldots, W_\rho \in \mathcal{A}(G)$ such that

$$T = V_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} V_{2k+1} = W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_\rho.$$

If $1^{[2]} \mid T$, then the sequence $T \boldsymbol{\cdot} (1^{[2]})^{[-1]}$ contradicts the minimality of k. If $1 \mid T$ but $1^{[2]} \nmid T$, say $V_{2k+1} = W_\rho = 1$, then we consider $T \boldsymbol{\cdot} 1^{[-1]} = V_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} V_{2k} = W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{\rho-1}$. Since $\mathsf{D}(\mathcal{G}) = 2q$ and $1^{[2]} \nmid T$, it follows that $|V_i| \le 2q$ and $|W_j| \ge 2$ for every $i \in [1, 2k]$ and $j \in [1, \rho-1]$. Thus

$$4qk \ge |V_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} V_{2k}| = |W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{\rho-1}| \ge 2(\rho-1) = 4qk + 2q - 2,$$

a contradiction since $q \ge 7$. Hence $1 \nmid T$ and this implies that $|W_j| \ge 2$ for every $j \in [1, \rho]$. Since $|V_i| \le 2q$ for every $i \in [1, 2k+1]$, it follows that

$$2q(2k+1) \ge |V_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} V_{2k+1}| = |W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_\rho| \ge 2q(2k+1),$$

whence $|V_i| = 2q = \mathsf{D}(\mathcal{G})$ and $|W_j| = 2$ for every $i \in [1, 2k+1]$ and $j \in [1, \rho]$. By Theorem 1.1, each V_i has an unique element $g_i \in \mathcal{G}$ with $\mathrm{ord}(g_i) = q$ such that $g_i \in \mathrm{supp}(V_i)$. More precisely, $g_i^{[2q-2]} \mid V_i$. On the other hand, since $|W_j| = 2$, if $g^{[2q-2]} \mid V_i$ for some $i \in [1, 2k+1]$, then there exists ℓ such that $(g^{-1})^{[2q-2]} \mid V_\ell$. Since $2k+1$ is odd, there exists V_i such that none of the terms of order q can be paired with its inverse to form W_j for some j. This leads to the desired contradiction, thereby completing the proof of the theorem.

□

References

[1] D.V. Avelar, F.E. Brochero Martínez, S. Ribas; *A note on Bass' conjecture.* J. Number Theory **249** (2023), 462–469.

[2] D.V. Avelar, F.E. Brochero Martínez, S. Ribas; *On the direct and inverse zero-sum problems over $C_n \rtimes_s C_2$.* J. Combin. Theory Ser. A **197** (2023), 105751.

[3] J. Bass; *Improving the Erdős-Ginzburg-Ziv theorem for some non-abelian groups.* J. Number Theory **126** (2007), 217–236.

[4] F.E. Brochero Martínez, S. Ribas; *Extremal product-one free sequences in Dihedral and Dicyclic Groups.* Discrete Math. **341** (2018), 570-578.

[5] F.E. Brochero Martínez, S. Ribas; *Extremal product-one free sequences in $C_q \rtimes_s C_p$.* J. Number Theory **204** (2019), 334–353.

[6] F.E. Brochero Martínez, S. Ribas; *Extremal product-one free sequences over $C_n \rtimes_s C_2$.* Discrete Math. **345** (2022), 113062.

[7] K. Cziszter, M. Domokos, A. Geroldinger; *The interplay of invariant theory with multiplicative ideal theory and with arithmetic combinatorics.* In: Multiplicative ideal theory and factorization theory, 43–95, Springer Proc. Math. Stat. 170, Springer (2016).

[8] P. van Emde Boas, Kruyswijk; *A combinatorial problem on finite abelian groups III.* Z.W. Math. Centrum, Amsterdam (1969).

[9] W. Gao, A. Geroldinger, W.A. Schmid; *Inverse zero-sum problems.* Acta Arith. **128.3** (2007), 245–279.

[10] A. Geroldinger, *Sets of lengths.* Amer. Math. Monthly **123**(10) (2016), 960–988.

[11] A. Geroldinger, D.J. Grynkiewicz; *The large Davenport constant I: Groups with a cyclic, index 2 subgroup.* J. Pure Appl. Algebra **217** (2013), 863–885.

[12] A. Geroldinger, D.J. Grynkiewicz, P. Yuan; *On products of k atoms II.* Moscow J. Comb. Number Theory **5**(3) (2015), 3–58.

[13] A. Geroldinger, F. Halter-Koch; *Non-Unique Factorizations: Algebraic, Combinatorial and Analytic Theory.* Pure and Applied Mathematics 278. Chapman & Hall/CRC (2006).

[14] A. Geroldinger, I.Z. Ruzsa; *Combinatorial Number Theory and Additive Group Theory.* Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser Basel, 1st ed. (2009).

[15] A. Geroldinger, R. Schneider; *On Davenport's constant.* J. Combin. Theory Series A **61** (1992), 147–152.

[16] A. Geroldinger, Q. Zhong; *A characterization of seminormal C-monoids.* Boll. Unione Mat. Ital. **12** (2019), 583–597.

[17] D.J. Grynkiewicz; *Structural Additive Theory.* Developments in Mathematics 30, Springer (2013).

[18] D.J. Grynkiewicz; *The large Davenport constant II: General upper bounds.* J. Pure Appl. Algebra **217** (2013), 2221–2246.

[19] D. Han, H. Zhang; *Erdős-Ginzburg-Ziv theorem and Noether number for $C_m \ltimes_\varphi C_{mn}$.* J. Number Theory **198** (2019), 159–175.

[20] C.G. Karthick Babu, R. Ranjanbera, M. Ghosh, B. Sury; *Davenport constant and its variants for some non-abelian groups.* Preprint (2024). Available at `https://arxiv.org/pdf/2406.09210`.

[21] J.S. Oh; *On the algebraic and arithmetic structure of the monoid of product-one sequences.* J. Commut. Algebra **12**(3) (2020), 409–433.

[22] J.S. Oh; *On the algebraic and arithmetic structure of the monoid of product-one sequences II.* Period. Math. Hungar. **78** (2019), 203–230.

[23] J.S. Oh, Q. Zhong; *On Erdős-Ginzburg-Ziv inverse theorems for dihedral and dicyclic groups.* Israel J. Math. **238** (2020), 715–743.

[24] J.S. Oh, Q. Zhong; *On minimal product-one sequences of maximal length over dihedral and dicyclic groups.* Commun. Korean Math. Soc. **35** (2020), 83–116.

[25] J.E. Olson; *A combinatorial problem on finite Abelian groups I.* J. Number Theory **1** (1969), 8–10.

[26] J.E. Olson; *A combinatorial problem on finite Abelian groups II.* J. Number Theory **1** (1969), 195–199.

[27] J.E. Olson, E.T. White; *Sums from a sequence of group elements.* In: H. Zassenhaus (Ed.), Number Theory and Algebra, Academic Press, New York (1977), 215–222.

[28] Y. Qu, Y. Li; *On a conjecture of Zhuang and Gao.* Colloq. Math. **171** (2023), 113–126.

[29] Y. Qu, Y. Li; *Extremal product-one free sequences and $|G|$-product-one free sequences of a metacyclic group.* Discrete Math. **345**(8) (2022), 112938.

[30] S. Ribas; *Some zero-sum problems over $\langle x, y \mid x^2 = y^{n/2}, y^n = 1, yx = xy^s \rangle$.* Bull. Braz. Math. Soc. (N.S.) **56** (12) (2025), Article 12.

[31] K. Rogers; *A Combinatorial problem in Abelian groups.* Proc. Cambridge Phil. Soc. **59** (1963), 559–562.

[32] W.A. Schmid; *Some recent results and open problems on sets of lengths of Krull monoids with finite class group.* In: Multiplicative ideal theory and factorization theory, 323–352, Springer Proc. Math. Stat. 170, Springer (2016).

[33] J. Yang, X. Zhang, L. Feng; *On the direct and inverse zero-sum problems over non-split metacyclic groups.* Discrete Math. **347** (12) (2024), 114213.

Departamento de Análise, Universidade Federal Fluminense (UFF), Niterói, RJ, 24210-201, Brazil,
Email address: daniloavelar@id.uff.br

Departamento de Matemática, Universidade Federal de Minas Gerais (UFMG), Belo Horizonte, MG, 31270-901, Brazil,
Email address: fbrocher@mat.ufmg.br

Departamento de Matemática, Universidade Federal de Ouro Preto, Ouro Preto, MG, 35402-136, Brazil
Email address: savio.ribas@ufop.edu.br