TOTALLY REAL POINTS IN THE MULTIBROT SETS

ALESSIO CANGINI AND HANG FU

Abstract. We classify all totally real parabolic parameters in the multibrot sets, extending a theorem of Buff and Koch.

1. Introduction

Let $d \geq 2$ be an integer and consider the family of polynomials $f_c(z) = z^d + c \in \mathbb{C}[z]$. For $n \in \mathbb{N} = \{1, 2, \dots\}$, we denote by f_c^n the n-th iteration of f_c.

We focus on three subsets of the parameter space. The *multibrot set* M_d consists of all $c \in \mathbb{C}$ such that the orbit $\{f_c^n(0) \colon n \in \mathbb{N}\}$ is bounded. When $d = 2$, the multibrot set M_2 is the well-known *Mandelbrot set*. The set of *postcritically finite* parameters, denoted by PCF_d, consists of all $c \in \mathbb{C}$ such that the orbit $\{f_c^n(0) \colon n \in \mathbb{N}\}$ is finite. A point $z \in \mathbb{C}$ is *periodic* for f_c if there is $n \in \mathbb{N}$ such that $f_c^n(z) = z$. If $z \in \mathbb{C}$ is periodic and $n \in \mathbb{N}$ is the smallest period, then the *multiplier* of f_c at z is given by $(f_c^n)'(z)$. A periodic point is *parabolic* if its multiplier is a root of unity. A parameter $c \in \mathbb{C}$ is *parabolic* if f_c has a parabolic periodic point; the set of such parameters is denoted by Par_d.

Both PCF_d and Par_d are Galois invariant subsets of M_d. An algebraic number is *totally real* if all its Galois conjugates are real. The field of totally real numbers is denoted by $\mathbb{Q}^{\mathrm{tr}}$. Noytaptim and Petsche studied the arithmetic properties of PCF_d and classified $\mathrm{PCF}_d \cap \mathbb{Q}^{\mathrm{tr}}$ for all $d \geq 2$.

Theorem 1.1 ([NP24, Theorem 1])**.** *We have*

$$\begin{aligned} \mathrm{PCF}_2 \cap \mathbb{Q}^{\mathrm{tr}} &= \{0, -1, -2\}, \\ \mathrm{PCF}_d \cap \mathbb{Q}^{\mathrm{tr}} &= \{0, -1\} \text{ if } d \geq 4 \text{ is even}, \\ \mathrm{PCF}_d \cap \mathbb{Q}^{\mathrm{tr}} &= \{0\} \text{ if } d \geq 3 \text{ is odd}. \end{aligned}$$

In order to prove Theorem 1.1, Noytaptim and Petsche used capacity-theoretic tools to bound the degree of points in $\mathrm{PCF}_d \cap \mathbb{Q}^{\mathrm{tr}}$. Buff and Koch then realized

Date: October 1, 2025.

2020 *Mathematics Subject Classification.* Primary: 37P05, 37P15.

Key words and phrases. Arithmetic dynamics, multibrot sets, parabolic points, PCF points.

The authors acknowledge support by the Swiss National Science Foundation Grant "Rational points, arithmetic dynamics, and heights" no. 219397.

that, with the help of Kronecker's theorem, they were able to avoid the capacity-theoretic tools and substantially simplify the proof of Theorem 1.1 for the case $d = 2$. As a further application of their approach, they considered the totally real parabolic points and proved the following result.

Theorem 1.2 ([BK22, Proposition 2])**.** *We have*

$$\mathrm{Par}_2 \cap \mathbb{Q}^{\mathrm{tr}} = \{1/4, -3/4, -5/4, -7/4\}.$$

Given Theorems 1.1 and 1.2, it is natural to ask if we can also compute the set $\mathrm{Par}_d \cap \mathbb{Q}^{\mathrm{tr}}$ for $d \geq 3$. Our main theorem answers this question.

Theorem 1.3. *We have* $\mathrm{Par}_3 \cap \mathbb{Q}^{\mathrm{tr}} = \{\pm 2\sqrt{3}/9\}$ *and* $\mathrm{Par}_d \cap \mathbb{Q}^{\mathrm{tr}} = \emptyset$ *for* $d \geq 4$.

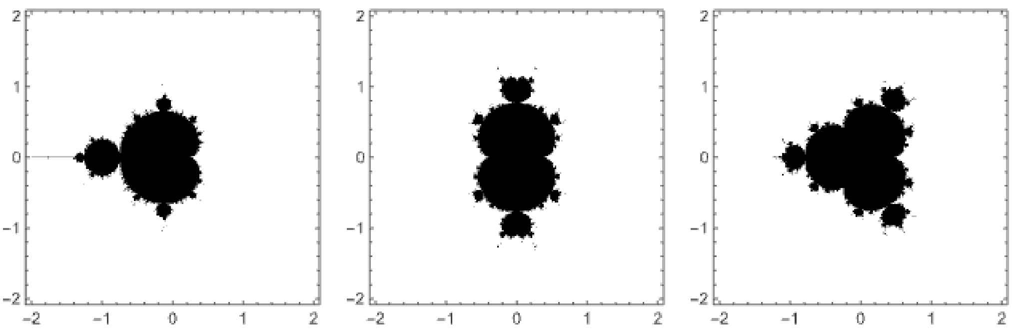

FIGURE 1. The multibrot sets M_2, M_3 and M_4. The images have been produced with MATHEMATICA.

We first note that the simplified proof of Theorem 1.1 for the case $d = 2$ given by Buff and Koch can be easily generalized to the cases $d \geq 3$ without too much extra effort. However, their proof of Theorem 1.2 relies on the crucial fact that if c is a totally real parabolic parameter, then $4c$ is a totally real algebraic integer, so that Kronecker's theorem can be applied. Due to the lack of this property, their idea does not seem to generalize to the cases $d \geq 3$. To tackle this difficulty, we employ the capacity-theoretic tools again.

Outline. In Section 2, we collect some basic properties of M_d and also review the capacity-theoretic tools mentioned above. In Section 3, we give the proof of Theorem 1.1, following the idea of Buff and Koch. In Section 4, we give the proof of Theorem 1.3.

Acknowledgements. We would like to thank Philipp Habegger for the helpful comments on the first draft of this paper.

2. Preliminary lemmas

In this section, we provide some general lemmas that will be used later. We do not assume any number is totally real.

2.1. **Real slices of multibrot sets.**

Since both $\mathrm{PCF}_d \cap \mathbb{Q}^{\mathrm{tr}}$ and $\mathrm{Par}_d \cap \mathbb{Q}^{\mathrm{tr}}$ are subsets of $\mathrm{M}_d \cap \mathbb{R}$, we first take a closer look at the real slices of multibrot sets.

We define the quantities

$$\alpha(d) = (d-1)d^{-d/(d-1)}, \qquad \beta(d) = -2^{1/(d-1)}, \qquad \gamma(d) = -(d+1)d^{-d/(d-1)},$$

where the $(d-1)$-th roots are always taken on the positive real axis. It is known that $\mathrm{M}_d \cap \mathbb{R} = [-\alpha(d), \alpha(d)]$ if d is odd [PR17] and $\mathrm{M}_d \cap \mathbb{R} = [\beta(d), \alpha(d)]$ if d is even [PRR17].

When d is odd, for any $c \in \mathrm{M}_d \cap \mathbb{R}$, the polynomial f_c always has a fixed point, either parabolic or attractive. When d is even, the dynamics of f_c with $c \in \mathrm{M}_d \cap \mathbb{R}$ is much more complicated. Fortunately, the following lemma enables us to locate the candidate totally real parabolic points in a small subinterval of $\mathrm{M}_d \cap \mathbb{R}$.

Lemma 2.1. *Let $d \geq 2$ be an integer. Then:*

(i) The polynomial $f_{\alpha(d)}$ has a parabolic fixed point.
(ii) If d is odd and $-\alpha(d) < c < \alpha(d)$, then f_c has an attractive fixed point.
(iii) If d is odd, then $f_{-\alpha(d)}$ has a parabolic fixed point.
(iv) If d is even and $\gamma(d) < c < \alpha(d)$, then f_c has an attractive fixed point.
(v) If d is even, then $f_{\gamma(d)}$ has a parabolic fixed point.
(vi) If d is even and $-1 < c < \gamma(d)$, then f_c has an attractive cycle of period 2.

We think this is a special case of the general phenomenon called *period-doubling bifurcation*, but we cannot find a good reference, so we provide an *ad hoc* proof.

Proof. Let $\delta(d) = d^{-1/(d-1)}$. We can easily check that

(i): the parabolic fixed point is $\delta(d)$;

(iii, v): the parabolic fixed point is $-\delta(d)$;

(ii, iv): the attractive fixed point is between $-\delta(d)$ and $\delta(d)$, by the intermediate value theorem.

For (vi), let z be a period 2 point of f_c but not a fixed point. Then z satisfies

$$\frac{f_c^2(z) - z}{f_c(z) - z} = \frac{(z^d + c)^d - z^d + z^d + c - z}{z^d + c - z} = 0,$$

namely,

$$\sum_{k=0}^{d-1} z^k (z^d + c)^{d-1-k} + 1 = 0.$$

Let

$$x = \frac{z^d + c}{z}. \tag{2.1}$$

Then we have

$$\sum_{k=0}^{d-1} x^k = -z^{-(d-1)}. \tag{2.2}$$

Let λ be the multiplier of z. Then we have

$$(f_c^2)'(z) = d^2 z^{d-1}(z^d + c)^{d-1} = \lambda,$$

and

$$\lambda^{-1} d^2 x^{d-1} = z^{-2(d-1)}. \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$g_\lambda(x) := \lambda \left(\sum_{k=0}^{d-1} x^k \right)^2 - d^2 x^{d-1} = 0. \tag{2.4}$$

Now suppose that $0 < \lambda < 1$. Then by Descartes' sign rule, the polynomial g_λ has at most two positive real roots. Since $g_\lambda(0) = \lambda > 0$, $g_\lambda(1) = d^2(\lambda - 1) < 0$, and $g_\lambda(+\infty) = +\infty$, we know that g_λ has a unique root $0 < x < 1$.

We claim that $\lambda \mapsto x$ is a bijection from $(0, 1)$ to $(0, 1)$. It is clearly injective by (2.4). It is also surjective because if $0 < x < 1$, then

$$0 < \frac{d^2 x^{d-1}}{\left(\sum_{k=0}^{d-1} x^k \right)^2} < \frac{d^2 x^{d-1}}{d^2 \left(\prod_{k=0}^{d-1} x^k \right)^{2/d}} = 1.$$

Combining (2.1) and (2.2), we get

$$c = z(x - z^{d-1}) = \frac{-\sum_{k=0}^{d} x^k}{\left(\sum_{k=0}^{d-1} x^k \right)^{d/(d-1)}}.$$

If $x = 0$, then $c = -1$. If $x = 1$, then $c = \gamma(d)$. Since the map $x \mapsto c$ is continuous, any $-1 < c < \gamma(d)$ is the image of some $0 < x < 1$, which is in turn the image of some $0 < \lambda < 1$. ■

The following lemma is a special case of [CG93, Theorems III.2.2 and III.2.3].

Lemma 2.2. *Any attractive or parabolic cycle of f_c attracts the orbit of the critical point 0. In particular:*

(i) The polynomial f_c has at most one attractive or parabolic cycle.

(ii) If f_c has an attractive or parabolic fixed point z_c, then the orbit of the critical point 0 converges to z_c.

2.2. **Arithmetic of parabolic parameters.** Milnor studied the arithmetic of unicritical polynomials and proved the following property of Par_d.

Lemma 2.3 ([Mil14, Theorem 1.1 and Remark 2.2])**.** *If $c \in \mathrm{Par}_d$, then*

$$d^{\frac{d}{d-1}} c \in \overline{\mathbb{Z}} \qquad \text{and} \qquad \left(d^{\frac{d}{d-1}} c\right) \overline{\mathbb{Z}} + d\overline{\mathbb{Z}} = \overline{\mathbb{Z}}.$$

This lemma gives a strong restriction on the elements of Par_d.

For any prime number p, there is a p-adic valuation $\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ with $\nu_p(p) = 1$. It can be extended to a valuation map $\nu_p : \overline{\mathbb{Q}} \to \mathbb{Q} \cup \{\infty\}$. Note that this extension is not unique. We fix one such extension.

Corollary 2.4. *If $c \in \mathrm{Par}_d$, then $\nu_p(c) = -\nu_p\left(d^{\frac{d}{d-1}}\right)$ if p divides d and $\nu_p(c) \geq 0$ otherwise.*

Proof. By Lemma 2.3, there are $x, y \in \overline{\mathbb{Z}}$ such that $d^{\frac{d}{d-1}} cx + dy = 1$. If p divides d, then $\nu_p\left(d^{\frac{d}{d-1}} c\right) = 0$. Indeed, if $\nu_p\left(d^{\frac{d}{d-1}} c\right) > 0$, then $0 < \nu_p\left(d^{\frac{d}{d-1}} cx + dy\right) = \nu_p(1) = 0$, which is a contradiction. If p does not divide d, then $0 \leq \nu_p\left(d^{\frac{d}{d-1}} c\right) = \nu_p\left(d^{\frac{d}{d-1}}\right) + \nu_p(c) = \nu_p(c)$. ∎

Corollary 2.5. *Let $c \in \mathrm{Par}_d$ and $P(T) = a_n T^n + \cdots + a_0 \in \mathbb{Z}[T]$ be the minimal polynomial of c with $a_n > 0$. Then we have $a_n = d^{\frac{nd}{d-1}} \in \mathbb{Z}$.*

Proof. Let $c_1 = c, c_2, \ldots, c_n$ be the Galois conjugates of c. They are also parabolic. For any $Q \in \overline{\mathbb{Q}}[T]$, we denote by $\nu_p(Q)$ the minimum of the p-adic valuation of its coefficients. By Gauss's lemma [BG06, Lemma 1.6.3], it follows that for any prime p, we have

$$0 = \nu_p(P) = \nu_p(a_n) + \sum_{i=1}^{n} \nu_p(T - c_i) = \nu_p(a_n) + \sum_{i=1}^{n} \min\{0, \nu_p(c_i)\}.$$

Thus, Corollary 2.4 implies that $\nu_p(a_n) = \frac{nd}{d-1}\nu_p(d)$ if p divides d and $\nu_p(a_n) = 0$ otherwise. Hence, we have $a_n = d^{\frac{nd}{d-1}} \in \mathbb{Z}$. ∎

2.3. **Capacity theory.** We recall a lemma from capacity theory, which was used by Noytaptim and Petsche.

Let $K \subseteq \mathbb{C}$ be a compact set. For any integer $n \geq 2$, we define the *n-th diameter* of K as

$$d_n(K) = \sup_{z_1, \ldots, z_n \in K} \prod_{1 \leq i < j \leq n} |z_i - z_j|^{\frac{2}{n(n-1)}}.$$

The sequence $\{d_n(K) \colon n \geq 2\}$ is monotone decreasing, by [Ran95, Theorem 5.5.2]. The *transfinite diameter* of K is defined as

$$d_\infty(K) = \lim_{n \to \infty} d_n(K).$$

The transfinite diameter of a real interval $[a,b] \subseteq \mathbb{R}$ is $d_\infty([a,b]) = (b-a)/4$, see [Ran95, Corollary 5.2.4]. The n-th diameters of $[a,b]$ were explicitly computed in [NP24], but it seems to us that they were already known by Stieltjes [Sti85] and Schur [Sch18].

Lemma 2.6 ([NP24, Theorem 2])**.** *For $n \geq 2$, the n-th diameter of a real interval $[a,b]$ is given by*

$$d_n([a,b]) = (b-a)D_n^{\frac{1}{n(n-1)}}$$

where $\{D_n \colon n \geq 2\}$ is defined recursively by $D_2 = 1$ and

$$D_n = \frac{n^n(n-2)^{n-2}}{2^{2n-2}(2n-3)^{2n-3}} D_{n-1}.$$

Actually, we do not need the full lemma. We only use the fact that $D_n^{\frac{1}{n(n-1)}}$ is decreasing and the first few values: $D_2 = 1$, $D_3 = 2^{-4}$, and $D_4 = 5^{-5}$.

3. Proof of Theorem 1.1

In this section, we provide an alternative proof of Theorem 1.1, following the idea of Buff and Koch.

The following lemma is extracted from their proof.

Lemma 3.1. *If $c \in \overline{\mathbb{Z}}$ and all its Galois conjugates are in the interval $[-2,0]$, then $c \in \{0,-1,-2\}$.*

Proof. Let $F(z) = z + z^{-1}$. Then F maps the unit circle onto the interval $[-2,2]$. In particular, F maps the arc $A = \{e^{i\theta} \colon \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$ onto the interval $[-2,0]$.

Let c satisfy the given condition and $a \in F^{-1}(c)$. Then $a \in \overline{\mathbb{Z}}$ and all its Galois conjugates are in A. By Kronecker's theorem [BG06, Theorem 1.5.9], a must be a root of unity. The condition $a \in A$ forces a to be -1, $\pm i$, or $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Hence, $c \in \{0,-1,-2\}$. ■

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. The inclusion $\supseteq$ is clear. Now let $c \in \mathrm{PCF}_d \cap \mathbb{Q}^{\mathrm{tr}}$. First we notice that by the definition, $c \in \overline{\mathbb{Z}}$ and all Galois conjugates of c are PCF. We distinguish two cases.

Case 1: d is odd. We have $-\alpha(d) \leq c \leq \alpha(d)$. In particular, by Lemmas 2.1 and 2.2, f_c has a fixed point z_c with $f_c^n(0) \to z_c$ as $n \to \infty$. Given that the orbit $\{f_c^n(0) \colon n \in \mathbb{N}\}$ is finite, we have $f_c^n(0) = z_c$ for large enough n. Since f_c is strictly increasing on $\mathbb{R}$, it follows that $z_c = 0$ and $c = 0$.

Case 2: d is even. We first assume that $0 \leq c \leq \alpha(d)$. As in Case 1, by Lemmas 2.1 and 2.2, f_c has a fixed point z_c with $f_c^n(0) \to z_c$ as $n \to \infty$. Given that the orbit $\{f_c^n(0)\colon n \in \mathbb{N}\}$ is finite, we have $f_c^n(0) = z_c$ for large enough n. Again, since f_c is strictly increasing on $\mathbb{R}_{\geq 0}$, we have $z_c = 0$ and $c = 0$. This argument works for all Galois conjugates of c. Therefore, we can assume that all Galois conjugates of c are contained in $[\beta(d), 0] \subseteq [-2, 0]$. By Lemma 3.1, we have $c \in \{0, -1, -2\}$ if $d = 2$ and $c \in \{0, -1\}$ if $d \geq 4$. ■

4. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. From now on, we assume that $c \in \mathrm{Par}_d \cap \mathbb{Q}^{\mathrm{tr}}$.

Let $P(T) = a_n T^n + \cdots + a_0 \in \mathbb{Z}[T]$ be the minimal polynomial of c with $a_n > 0$, let $c_1 = c, c_2, \ldots, c_n$ be the Galois conjugates of c, and let

$$\Delta(P) = a_n^{2(n-1)} \prod_{1 \leq i < j \leq n} (c_i - c_j)^2 \in \mathbb{Z}$$

be the discriminant of P. Since c is totally real, we have $\Delta(P) > 0$.

We first bound $\Delta(P)$ from below and above.

Lemma 4.1. *Assume that $c \in \mathrm{Par}_d \cap \mathbb{Q}^{\mathrm{tr}}$ and $d \geq 4$ is even. Then a_n^{n-1} divides $\Delta(P)$ and*

$$a_n^{n-1} \leq \Delta(P) \leq a_n^{2(n-1)} \left(2^{\frac{1}{d-1}} - 1\right)^{n(n-1)} D_n,$$

where D_n is defined in Lemma 2.6.

Proof. For any prime p that divides d, Corollaries 2.4 and 2.5 imply that

$$\begin{aligned}
\nu_p(\Delta(P)) &= 2(n-1)\nu_p(a_n) + 2 \sum_{1 \leq i < j \leq n} \nu_p(c_i - c_j) \\
&\geq 2(n-1)\frac{nd}{d-1}\nu_p(d) - n(n-1)\frac{d}{d-1}\nu_p(d) \\
&= n(n-1)\frac{d}{d-1}\nu_p(d) \\
&= (n-1)\nu_p(a_n).
\end{aligned}$$

Hence, it follows that a_n^{n-1} divides $\Delta(P)$ and that $a_n^{n-1} \leq \Delta(P)$.

Suppose that $d \geq 4$ is even. If $c \in \mathrm{Par}_d$ and $c > -1$, then $c = \alpha(d)$ or $c = \gamma(d)$ by Lemmas 2.1 and 2.2. However, neither of them is totally real. Therefore, we can assume that $c_1, \ldots, c_n \in [\beta(d), -1]$ and get the upper bound

$$\Delta(P) \leq a_n^{2(n-1)} d_n([\beta(d), -1])^{n(n-1)}.$$

By Lemma 2.6, we have

$$\Delta(P) \leq a_n^{2(n-1)} \left((-1 - \beta(d)) D_n^{\frac{1}{n(n-1)}}\right)^{n(n-1)} = a_n^{2(n-1)} \left(2^{\frac{1}{d-1}} - 1\right)^{n(n-1)} D_n.$$

This completes the proof. ■

Now we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. If d is odd, then the only real parabolic points are $\pm\alpha(d)$ by Lemmas 2.1 and 2.2. They are totally real only if $d = 3$.

If $d \geq 4$ is even, then by Corollary 2.5 and Lemma 4.1, we have

$$d^{\frac{d}{d-1}} \left(2^{\frac{1}{d-1}} - 1\right) D_n^{\frac{1}{n(n-1)}} \geq 1. \tag{4.1}$$

We can check that $\sigma(d) := d^{\frac{d}{d-1}} \left(2^{\frac{1}{d-1}} - 1\right)$ is decreasing and converges to $\log 2$ as $d \to \infty$. Moreover, we have seen that $\tau(n) := D_n^{\frac{1}{n(n-1)}}$ is decreasing and converges to $1/4$ as $n \to \infty$.

We want to show by contradiction that $\mathrm{Par}_d \cap \mathbb{Q}^{\mathrm{tr}} = \emptyset$ if $d \geq 4$ is even. We suppose to the contrary and distinguish four subcases.

Case 1: If $n \in \{1, 2\}$ and $d \geq 4$, then $a_n = d^{\frac{nd}{d-1}}$ is not an integer, contradicting Corollary 2.5.

Case 2: If $n \geq 3$ and $d \geq 6$, then

$$\sigma(d)\tau(n) \leq \sigma(6)\tau(3) = 6^{\frac{6}{5}}(2^{\frac{1}{5}} - 1)2^{-\frac{2}{3}} < 1,$$

contradicting (4.1).

Case 3: If $n \geq 4$ and $d \geq 4$, then

$$\sigma(d)\tau(n) \leq \sigma(4)\tau(4) = 4^{\frac{4}{3}}(2^{\frac{1}{3}} - 1)5^{-\frac{5}{12}} < 1,$$

contradicting (4.1).

Case 4: If $n = 3$ and $d = 4$, then

$$\begin{aligned} P(T) &= a_3(T - c_1)(T - c_2)(T - c_3) \\ &= a_3T^3 + a_2T^2 + a_1T + a_0, \text{ where } a_3 = 2^8, \end{aligned}$$

and

$$\Delta(P) = a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3.$$

By Lemma 4.1, $\Delta(P)$ is a multiple of $a_3^2 = 2^{16}$ and

$$\Delta(P) \leq a_n^{2(n-1)} \left(2^{\frac{1}{d-1}} - 1\right)^{n(n-1)} D_n = 2^{28}(2^{\frac{1}{3}} - 1)^6 < 2^{17}.$$

Therefore, we must have $\Delta(P) = 2^{16}$. Moreover, by Corollary 2.4, we have

$$\begin{aligned} \nu_2(a_0) &= \nu_2(2^8c_1c_2c_3) = 0, \\ \nu_2(a_1) &= \nu_2(2^8(c_1c_2 + c_2c_3 + c_3c_1)) \geq \lceil 8 - \tfrac{16}{3} \rceil = 3, && (4.2) \\ \nu_2(a_2) &= \nu_2(2^8(c_1 + c_2 + c_3)) \geq \lceil 8 - \tfrac{8}{3} \rceil = 6, && (4.3) \end{aligned}$$

$$\nu_2(a_3) = \nu_2(2^8) = 8.$$

Then

$$\begin{aligned}
\nu_2(a_1^2a_2^2) &\geq 3\cdot 2 + 6\cdot 2 = 18,\\
\nu_2(-4a_1^3a_3) &\geq 2 + 3\cdot 3 + 8 = 19,\\
\nu_2(-4a_0a_2^3) &\geq 2 + 0 + 6\cdot 3 = 20,\\
\nu_2(18a_0a_1a_2a_3) &\geq 1 + 0 + 3 + 6 + 8 = 18.
\end{aligned}$$

If either (4.2) or (4.3) is strict, then $\nu_2(a_1^2a_2^2), \nu_2(18a_0a_1a_2a_3) \geq 19$. Otherwise, $\nu_2(a_1^2a_2^2), \nu_2(18a_0a_1a_2a_3) = 18$ and $\nu_2(a_1^2a_2^2 + 18a_0a_1a_2a_3) \geq 19$. As a result,

$$2^{16} \equiv \Delta(P) \equiv -27a_0^2a_3^2 \equiv 5\cdot 2^{16} \mod 2^{19}.$$

This is a contradiction, so such P does not exist. ∎

References

[BG06] E. Bombieri and W. Gubler. *Heights in Diophantine geometry*. Vol. 4. New Mathematical Monographs. Cambridge University Press, Cambridge, 2006, pp. xvi+652. DOI (↑ 5, 6).

[BK22] X. Buff and S. Koch. *Totally real points in the Mandelbrot Set*. 2022. arXiv: `2211.15725 [math.DS]` (↑2).

[CG93] L. Carleson and T. W. Gamelin. *Complex dynamics*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993, pp. x+175. DOI (↑4).

[Mil14] J. Milnor. *Arithmetic of unicritical polynomial maps. Frontiers in complex dynamics*. Vol. 51. Princeton Math. Ser. Princeton Univ. Press, Princeton, NJ, 2014, pp. 15–24. DOI (↑5).

[NP24] C. Noytaptim and C. Petsche. Totally real algebraic integers in short intervals, Jacobi polynomials, and unicritical families in arithmetic dynamics. *Bull. Lond. Math. Soc.* 56.1 (2024), pp. 364–379. DOI (↑1, 6).

[PR17] P. Parisé and D. Rochon. Tricomplex dynamical systems generated by polynomials of odd degree. *Fractals* 25.3 (2017), pp. 1750026, 11. DOI (↑3).

[PRR17] P. Parisé, T. Ransford, and D. Rochon. Tricomplex dynamical systems generated by polynomials of even degree. *Chaotic Modeling and Simulation* 1 (2017), pp. 37–48 (↑ 3).

[Ran95] T. Ransford. *Potential theory in the complex plane*. Vol. 28. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1995, pp. x+232. DOI (↑5, 6).

[Sch18] I. Schur. Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z.* 1.4 (1918), pp. 377–402. DOI (↑6).

[Sti85] T. J. Stieltjes. Sur quelques théorèmes d'algèbre. *C. R. Acad. Sci. Paris* 100 (1885), pp. 439–440 (↑6).

Alessio Cangini, Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, 4051 Basel, Switzerland

Email address: alessio.cangini@unibas.ch

URL: https://sites.google.com/view/alessio-cangini

Hang Fu, Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, 4051 Basel, Switzerland

Email address: drfuhang@gmail.com

URL: https://sites.google.com/view/hangfu