Joint Inference for the Regression Discontinuity Effect and Its External Validity

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Abstract

The external validity of regression discontinuity (RD) designs is essential for informing policy and remains an active research area in econometrics and statistics. However, we document that only a limited number of empirical studies explicitly address the external validity of standard RD effects. To advance empirical practice, we propose a simple joint inference procedure for the RD effect and its local external validity, building on Calonico, Cattaneo, and Titiunik (2014, *Econometrica*) and Dong and Lewbel (2015, *Review of Economics and Statistics*). We further introduce a locally linear treatment effects assumption, which enhances the interpretability of the treatment effect derivative proposed by Dong and Lewbel. Under this assumption, we establish identification and derive a uniform confidence band for the extrapolated treatment effects. Our approaches require no additional covariates or design features, making them applicable to virtually all RD settings and thereby enhancing the policy relevance of many empirical RD studies. The usefulness of the method is demonstrated through an empirical application, highlighting its complementarity to existing approaches.

Keywords: Extrapolation, regression discontinuity designs, robust bias correction

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1 Introduction

In causal analysis, internal validity and external validity are two central concerns. Internal validity is a necessary condition for drawing causal conclusions, but external validity is equally essential for informing policy. Thus, internally valid causal analysis alone is not sufficient, as emphasized in Duflo et al. (2007, Chapter 8) and Vivalt (2020). Nevertheless, external validity is often overlooked, as Peters et al. (2016) report for randomized controlled trials, suggesting that empirical studies rarely address this issue explicitly.

Regression discontinuity (RD) designs, one of the most popular and credible empirical strategies, face a similar, if not more severe, challenge. In the RD framework, treatment status changes discontinuously at a cutoff point, enabling identification of the treatment effect at the threshold under a mild smoothness assumption (Hahn et al., 2001). This attractive feature, however, comes at the cost of uncertain external validity. Standard RD designs provide no information about treatment effects away from the cutoff, thereby limiting the generalizability of the RD estimates regardless of their statistical significance. For example, Figure 1 illustrates two scenarios in which the RD effects at the cutoff are identical, yet their policy implications differ. In Figure 1a, the RD effect plausibly extends beyond the cutoff, whereas in Figure 1b it may be questionable whether one can conclude with confidence that the treatment has a positive effect for everyone. Both types of (observable) regression functions are well-documented across empirical studies.

Despite this crucial limitation, we find only 26 papers that discuss external validity among 70 empirical RD studies published in leading economic journals between 2020 and 2024 (details are provided in Section 2). Furthermore, only four studies cite recent theoretical proposals for extrapolation. These results may suggest not only that researchers devote limited attention to external validity, but also that extrapolation strategies proposed in theoretical econometrics are not necessarily well aligned with every empirical context, because they typically require stronger assumptions, informative covariates, or design features such as multiple cutoff points (Angrist and Rokkanen, 2015; Bertanha and

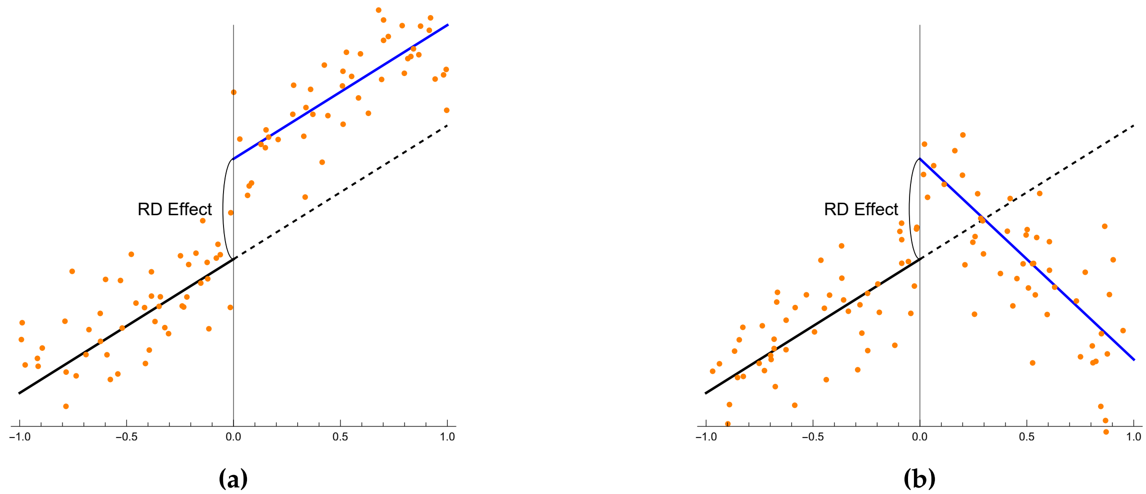


Figure 1: Same RD Effect, Different Policy Implication

Imbens, 2020; Cattaneo et al., 2021; Deaner and Kwon, 2025; Okamoto and Ozaki, 2025). This observation motivates us to develop a simple, practically accessible econometric tool.

To this end, this article develops simple joint inference procedures by incorporating the local extrapolation method of Dong and Lewbel (2015)—reviewed in the following paragraph—into the widely used robust bias correction (RBC) methods of Calonico et al. (2014). The Dong and Lewbel approach is particularly attractive because it does not require additional covariates or design features and imposes only a weak differentiability assumption; in other words, it can be applied to most empirical settings. Embedding it within the RBC framework ensures compatibility with standard practice: since our confidence region is a natural extension of methods already familiar to empirical researchers, it requires no changes to the research design (including bandwidth choice), and hence, remains directly comparable to existing approaches.

Now, after introducing the Dong and Lewbel strategy, we describe our proposal more concretely. Let $\tau(x)$ denote the treatment effect function with $x = 0$ as the cutoff. Whereas most RD studies focus on identifying and inferring $\tau(0)$, Dong and Lewbel (2015) proposes examining $\tau'(0)$, since this derivative contains information about the local extrapolatability of the RD effect $\tau(0)$. For instance, if $\tau'(0) = 0$, the estimated treatment effect plausibly extends beyond the cutoff, as illustrated in Figure 1a. By contrast, if $\tau'(0) < 0$, the treatment effect may decline at other values of the running variable, raising concerns about the generalizability of the RD effect, as shown in Figure 1b.

In this respect, it is evident that the policy implications of RD studies become clearer when both $\tau(0)$ and $\tau'(0)$ are reported. Unfortunately, however, none of the surveyed studies report the derivative. With the aim of encouraging broader adoption of Dong and Lewbel (2015), this article develops a joint inference procedure for the pair $(\tau(0), \tau'(0))$. In particular, we propose constructing a joint confidence region by deriving the joint asymptotic distribution under the RBC framework of Calonico et al. (2014). This RBC-based confidence region should be familiar and accessible to applied researchers and is therefore well-positioned for widespread adoption in empirical work.

Although it is an essential first step, the local external validity indicator $\tau'(0)$ can sometimes be difficult to interpret, particularly when it is non-zero, compared with a more comprehensive extrapolation analysis such as Cattaneo et al. (2021). For instance, what does $\tau'(0) = 0.5$ imply about treatment effects away from the cutoff point? To address this limitation, we further introduce a *locally linear treatment effects assumption*, which complements the contribution of Dong and Lewbel (2015) and makes their identified treatment effect derivative $\tau'(0)$ easier to interpret in practice. Under this additional assumption, RD effects away from the cutoff are characterized by the linear function, $\tau(x) = \tau(0) + \tau'(0)x$. This locally linear assumption and resulting identification are *more general* than simply extending the standard RD effect, which effectively restricts $\tau'(0)$ to be zero. Moreover, the limit distribution used to derive the joint confidence region can also be employed to construct a uniform confidence band for the extrapolated RD effects, thereby enabling researchers to conduct a more informative extrapolation analysis that goes beyond merely assessing external validity.

Our proposed approaches require no additional covariates or design features, making them applicable to virtually all RD settings and thereby enhancing the policy relevance of many empirical studies. We illustrate the usefulness by applying them to the educational context originally studied by Londoño-Vélez et al. (2020). Consistent with alternative approaches based on multiple cutoff points (Cattaneo et al., 2021; Okamoto and Ozaki, 2025), we find evidence of external validity and positive treatment effects even away from the cutoff. We further demonstrate that the locally linear treatment effects assumption underlying our procedure yields estimates comparable to those obtained by Cattaneo et al.'s method in this application. Moreover, our method complements existing approaches based on multiple cutoffs by enabling extrapolation over regions where those approaches cannot identify treatment effects.

Plan of the Article

Section 2 reviews recent empirical studies to examine the current state of internal and external validity analysis in RD designs. Section 3 presents the main results of the paper, focusing on the sharp RD design. After introducing the setup and notation, Section 3.2 develops a robust local linear confidence region, and Section 3.3 proposes an extrapolation method. Section 4 reports an empirical application, and Section 5 concludes. All proofs, along with the generalization to local polynomial confidence regions, are provided in Appendix A, while the extension to the fuzzy RD design is included in the Online Appendix.

2 Empirical Practices in Recent RD Studies

2.1 Sample and Procedure

We begin by reviewing empirical RD studies published between 2020 and 2024 in leading economics journals: *American Economic Journal: Applied Economics* (AEJ: Applied), *American Economic Journal: Economic Policy* (AEJ: EP), *American Economic Review* (AER), *American Economic Review: Insights* (AERI), *Journal of Human Resources* (JHR), and *Journal of Labor Economics* (JOLE). We identified articles containing the phrase "regression discontinuity" using the EBSCOhost database. After excluding theoretical contributions, we obtain a sample of 70 empirical studies.

To systematically assess whether these studies discuss the external and internal validity of their RD design—and to avoid subjective judgment—we apply the following criteria. External validity is identified by (i) the presence of keywords such as "external validity," "extrapolat*," or "transportability" (where the asterisk (*) denotes a wildcard that allows for variations of the root word, e.g., *extrapolate*, *extrapolation*), and (ii) citations to leading contributions including Angrist and Rokkanen (2015), Bertanha and Imbens (2020), Cattaneo et al. (2021), and Dong and Lewbel (2015). Internal validity is assessed using a parallel strategy. However, discussions of internal validity seldom appear under the explicit phrase "internal validity." Therefore, in addition to this phrase, we also search for related keywords—such as "continuity," "manipulat*," and "placebo"—that suggest a discussion of identification assumptions. We further record citations to Calonico et al. (2014) and McCrary (2008), which provide valid inference procedures and a diagnostic test that are essential for internally valid RD analysis. In what follows, we conduct text analysis using R (R Core Team, 2023), instead of relying on manual inspection, to enhance reproducibility.

2.2 Results

The survey results are summarized in Table 1. Almost all RD studies carefully address the internal validity of their analysis. By contrast, far fewer articles explicitly discuss external validity. In total, only 26 out of 70 articles contain keywords related to external validity, and only four cite the leading contributions in this area.

This pattern may suggest two possible implications. First, the relatively small number of citations compared to keyword mentions indicates that most existing proposals are not necessarily directly applicable to many empirical RD settings, so that discussions of external validity are often conducted on a case-by-case basis. Second, compared to internal

Journal	# of Articles	Internal Validity		External Validity		TED
		Keywords	Citation	Keywords	Citation	
AEJ: Applied	16	16	13	7	1	0
AEJ: EP	16	15	15	4	1	0
AER/AERI	10	8	8	4	0	0
JHR	22	22	15	9	1	0
JOLE	6	6	6	2	1	0
Total	70	67	57	26	4	0

Table 1: Reporting of Internal and External Validity in Empirical RD Studies

Note: The table reports, for each journal, the number of empirical RD studies published between 2020 and 2024 (column 2). Columns 3-4 show the number of articles that mention at least one of the internal validity keywords or cite at least one of the designated references (Calonico et al., 2014; McCrary, 2008). Columns 5-6 report the corresponding counts for external validity, based on the presence of keywords or citations to leading contributions (Angrist and Rokkanen, 2015; Bertanha and Imbens, 2020; Cattaneo et al., 2021; Dong and Lewbel, 2015). The last column reports the number of studies that explicitly present the treatment effect derivative (TED) following Dong and Lewbel (2015).

validity, researchers devote relatively little attention to external validity. This is evident from the fact that the procedure of Dong and Lewbel (2015) has not been widely adopted in empirical practice, despite its broad applicability.

3 External Validity and Extrapolation under Sharp RD

To advance empirical practice, this section proposes a simple inference procedure, building on Dong and Lewbel (2015). To encourage wider adoption of their useful strategy, we incorporate it into the popular RBC confidence intervals of Calonico et al. (2014). This integration makes our procedure highly compatible with standard practice. Because it is a straightforward extension of methods researchers are already familiar with, it requires no changes to the research design and remains directly comparable to existing studies based on the same estimator.

Extending this idea, we also introduce an additional layer of assumptions that enables researchers to conduct a more comprehensive extrapolation analysis, rather than only assessing external validity. The proposed strategy again does not require additional covariates or design features, and is therefore applicable to a wide range of RD studies.

3.1 Setup

We focus on the sharp RD design, while the extension to the fuzzy RD case is provided in the Online Appendix. We observe a random sample $(Y_i(0), Y_i(1), X_i) : i = 1, \ldots, n$. Here, $Y_i(1)$ denotes the potential outcome under treatment, and $Y_i(0)$ the potential outcome under control. X_i is the running variable with density $f(x)$. Unit i receives the treatment if $X_i \geq 0$ and is untreated otherwise. Hence, the observed outcome is $Y_i = Y_i(0)\mathbf{1}\{X_i < 0\} + Y_i(1)\mathbf{1}\{X_i \geq 0\}$.

Define the treatment effect function as $\tau(x) := \mathbb{E}[Y_i(1) - Y_i(0)|X_i = x]$. Then, the sharp RD effect is given by $\tau_{\text{SRD}} := \tau(0)$. To assess a local external validity, Dong and Lewbel (2015) introduces the sharp RD treatment effect derivative (TED),[1] which is given by $\tau'_{\text{SRD}} := \tau'(0)$. Under a weak differentiability assumption, these are identified as

$$\tau_{\text{SRD}} = \mu_+ - \mu_-, \ \mu_+ := \lim_{x \downarrow 0} \mu(x), \ \mu_- := \lim_{x \uparrow 0} \mu(x),$$
$$\tau'_{\text{SRD}} = \mu'_+ - \mu'_-, \ \mu'_+ := \lim_{x \downarrow 0} \mu'(x), \ \mu'_- := \lim_{x \uparrow 0} \mu'(x),$$

where $\mu(x) := \mathbb{E}[Y_i|X_i = x]$. Dong and Lewbel (2015) shows that both τ_{SRD} and τ'_{SRD} can be estimated from standard local linear regression, but their analysis does not extend to a joint inference procedure for the pair $(\tau_{\text{SRD}}, \tau'_{\text{SRD}})$. Our first goal is to perform valid inference on τ_{SRD} and τ'_{SRD} jointly.

3.2 Robust Local Polynomial Confidence Region

We begin by introducing maintained assumptions, which are fairly general and standard in the literature (Calonico et al., 2014):

[1] See also DiNardo and Lee (2011, Section 3.3.2) and Cerulli et al. (2017) for related contributions on derivative-based external validity or extrapolation analysis.

Assumption 1. *In a neighbourhood* $(-\kappa_0, \kappa_0)$ *around the cutoff:*

(i) $\mathbb{E}\left[Y_i^4 | X_i = x\right]$ *is bounded, and* $f(x)$ *is continuous and bounded away from zero.*

(ii) $\mu_+(x) := \mathbb{E}\left[Y_i(1)|X_i = x\right]$ *and* $\mu_-(x) := \mathbb{E}\left[Y_i(0)|X_i = x\right]$ *are S-times continuously differentiable.*

(iii) $\sigma_+^2(x) := \mathbb{V}\left[Y_i(1)|X_i = x\right]$ *and* $\sigma_-^2(x) := \mathbb{V}\left[Y_i(0)|X_i = x\right]$ *are continuous and bounded away from zero.*

Assumption 2. *The kernel function K is a symmetric, nonnegative, bounded, compactly supported on* $[-1, 1]$*, and continuous second-order kernel.*

Following the RD literature (Calonico et al., 2014; Hahn et al., 2001), we employ the local linear regression, while a more general treatment is given in Appendix A. Specifically, we estimate $\hat{\tau}_{\mathtt{SRD}} = \hat{\mu}_+ - \hat{\mu}_-$ and $\hat{\tau}'_{\mathtt{SRD}} = \hat{\mu}'_+ - \hat{\mu}'_-$, where

$$\begin{aligned}
\left(\hat{\mu}_+, \hat{\mu}'_+\right)^\top &= \underset{(b_0,b_1)^\top \in \mathbb{R}^2}{\arg\min} \sum_{i=1}^n \mathbf{1}\left\{X_i \geq 0\right\} \left(Y_i - b_0 - b_1 X_i\right)^2 \frac{1}{h} K\left(\frac{X_i}{h}\right), \\
\left(\hat{\mu}_-, \hat{\mu}'_-\right)^\top &= \underset{(b_0,b_1)^\top \in \mathbb{R}^2}{\arg\min} \sum_{i=1}^n \mathbf{1}\left\{X_i < 0\right\} \left(Y_i - b_0 - b_1 X_i\right)^2 \frac{1}{h} K\left(\frac{X_i}{h}\right).
\end{aligned}$$

We here note that Calonico et al. (2014) propose the use of local *quadratic* regression for estimating the first derivative in the context of regression kink designs. In a similar vein, one could estimate derivatives by running a *separate* local quadratic regression. However, we prefer *simultaneous* estimation of the mean and first-derivative functions via local linear regression, for several reasons. First, it aligns with Dong and Lewbel (2015), which proposes a simultaneous estimation using local linear regression. Second, at the boundary point, the variance inflation by choosing higher-order polynomials is severe compared to interior points, so the finite sample performance may not improve by increasing the polynomial order (see Gelman and Imbens, 2019; Pei et al., 2022; and Ruppert and Wand, 1994, Remark 4 for related discussions). Finally, and most importantly, the leading biases of $\hat{\mu}_+$ and $\hat{\mu}'_+$ depend on the same quantity $\mu_+^{(2)}$ when simultaneous smoothing is employed. This feature is attractive because only a single quantity needs to be estimated for bias correction.

It is well-known that the conditional biases of these estimators are given by

$$\mathbb{E}\left[\hat{\tau}_{\mathtt{SRD}}|X_1,\ldots,X_n\right]-\tau_{\mathtt{SRD}}=h^2\mathrm{B}_{\mathtt{SRD}}\left(\mu_+^{(2)}-\mu_-^{(2)}\right)\left\{1+o_p(1)\right\},$$
$$\mathbb{E}\left[\hat{\tau}'_{\mathtt{SRD}}|X_1,\ldots,X_n\right]-\tau'_{\mathtt{SRD}}=h\mathrm{B}'_{\mathtt{SRD}}\left(\mu_+^{(2)}-\mu_-^{(2)}\right)\left\{1+o_p(1)\right\},$$

where $\mathrm{B}_{\mathtt{SRD}}$ and $\mathrm{B}'_{\mathtt{SRD}}$ are constants depending only on observables. We can estimate the unknown components, $\mu_+^{(2)}$ and $\mu_-^{(2)}$, using the local quadratic estimation with the same kernel K and a possibly different bandwidth b. With these local quadratic estimators, we can estimate the leading bias terms by $h^2\mathrm{B}_{\mathtt{SRD}}(\hat{\mu}_+^{(2)}-\hat{\mu}_-^{(2)})$ and $h\mathrm{B}'_{\mathtt{SRD}}(\hat{\mu}_+^{(2)}-\hat{\mu}_-^{(2)})$. Then, our bias-corrected estimators for $(\tau_{\mathtt{SRD}},\tau'_{\mathtt{SRD}})$ are defined as

$$\tilde{\tau}_{\mathtt{SRD}}:=\hat{\tau}_{\mathtt{SRD}}-h^2\mathrm{B}_{\mathtt{SRD}}\left(\hat{\mu}_+^{(2)}-\hat{\mu}_-^{(2)}\right),\quad\text{and}\quad\tilde{\tau}'_{\mathtt{SRD}}:=\hat{\tau}'_{\mathtt{SRD}}-h\mathrm{B}'_{\mathtt{SRD}}\left(\hat{\mu}_+^{(2)}-\hat{\mu}_-^{(2)}\right).$$

The following lemma derives the joint limit of these two estimators:

Lemma 1. *Suppose Assumptions 1-2 hold. If $S\geq 3$, $\max\{h,b\}<\kappa_0$, $n\min\{h^5,b^5\}\times\max\{h^2,b^2\}\to 0$, $n\min\{h,b\}\to\infty$, and Ω is invertible, then*

$$\Omega^{-1/2}\mathrm{diag}(1,h)\tilde{\Delta}(\tau_{\mathtt{SRD}},\tau'_{\mathtt{SRD}})\to_d\mathcal{N}\left((0,0)^\top,\mathrm{diag}(1,1)\right),$$

where

$$\tilde{\Delta}(t,t'):=\begin{pmatrix}\tilde{\tau}_{\mathtt{SRD}}-t\\ \tilde{\tau}'_{\mathtt{SRD}}-t'\end{pmatrix},\ \Omega:=\begin{pmatrix}\mathrm{V}_{\mathtt{SRD}} & \mathrm{C}_{\mathtt{SRD}}\\ \mathrm{C}_{\mathtt{SRD}} & \mathrm{V}'_{\mathtt{SRD}}\end{pmatrix},$$

$\mathrm{V}_{\mathtt{SRD}}$, $\mathrm{V}'_{\mathtt{SRD}}$, *and* $\mathrm{C}_{\mathtt{SRD}}$ *are provided in Appendix A.* ♣

This is just a straightforward extension of the main theorem of Calonico et al. (2014). Importantly, there is *no* additional unknown components in Ω compared to the asymptotic variance of Calonico et al. (2014). Hence, the same variance estimation procedures proposed in Calonico et al. (2014) are also valid in our case.

Let $\hat{\Omega}$ be a consistent estimator, and define $\hat{\Omega}_h := \mathrm{diag}(1, h^{-1})\hat{\Omega}\mathrm{diag}(1, h^{-1})$. Due to Lemma 1, we obtain that $\tilde{\Delta}(\tau_{\mathtt{SRD}}, \tau'_{\mathtt{SRD}})^\top \hat{\Omega}_h^{-1} \tilde{\Delta}(\tau_{\mathtt{SRD}}, \tau'_{\mathtt{SRD}}) \to_d \chi^2_2$. Hence, we can construct an asymptotically valid $(1-\alpha)\%$ confidence region for $(\tau_{\mathtt{SRD}}, \tau'_{\mathtt{SRD}})$ as follows:

Proposition 1. *Let $\hat{\Omega}$ be invertible and $\hat{\Omega} \to_p \Omega$. Under the same assumptions in Lemma 1, an asymptotic $1-\alpha$ confidence region of $(\tau_{\mathtt{SRD}}, \tau'_{\mathtt{SRD}})$ is given by*

$$\mathcal{R}_{1-\alpha} := \left\{ (t, t')^\top \in \mathbb{R}^2 : \tilde{\Delta}(t, t')^\top \hat{\Omega}_h^{-1} \tilde{\Delta}(t, t') \leq c_{1-\alpha} \right\},$$

where $c_{1-\alpha}$ is the $100(1-\alpha)$-percentile of χ^2_2 distribution. ♣

This confidence region can be represented graphically as an ellipse; see Section 4 for example. The proposed construction has a practically appealing feature. The rate conditions on the bandwidths in Lemma 1 accommodate the popular MSE-optimal choice. Hence, we can still use the usual MSE-optimal bandwidth for the treatment effect $\tau_{\mathtt{SRD}}$ as h, and that for $\mu_+^{(2)} - \mu_-^{(2)}$ as b, following Calonico et al. (2014, Section 4). This approach is most closely aligned with common practice and has the advantage that additional bandwidth selection is not needed.

3.3 Inference under a Locally Linear Treatment Effects Assumption

The confidence region $\mathcal{R}_{1-\alpha}$ is itself a useful object for assessing the external validity of standard RD estimates. However, in many empirical contexts, a more comprehensive extrapolation analysis, such as that of Cattaneo et al. (2021), may be of more direct relevance than merely verifying external validity. To this end, we introduce an additional assumption, which we term the *locally linear treatment effects assumption*:

Assumption 3 (Locally Linear Treatment Effects). *For some $\delta_1, \delta_2 > 0$, the treatment effect function $\tau(x)$ is a linear function of x over the region $[-\delta_1, \delta_2]$.*

Before discussing this assumption, we provide the following identification result of the extrapolated treatment effects:

Lemma 2. *Suppose that $\mu_+(x)$ and $\mu_-(x)$ are continuously differentiable. Then, under Assumption 3, $\tau(x)$ is identified over $[-\delta_1, \delta_2]$ as $\tau(x) = \tau_{\mathtt{SRD}} + \tau'_{\mathtt{SRD}}x$.* ♣

Assumption 3 is admittedly restrictive and may not hold in all empirical settings.[2] Even so, it provides a natural and tractable starting point for extrapolation analysis, reflecting the common belief that treatment effects are likely to evolve smoothly. Moreover, this extrapolation result is *more general* than simply extending the standard RD effect $\tau_{\mathtt{SRD}}$. Hence, it provides a more encompassing perspective than a sole focus on the treatment effect at the cutoff point. (In cases where additional information such as multiple cutoffs is available, this assumption can also be tested. We do this in our empirical application.)

Further, with the help of the additional assumption, Lemma 2 provides a way to intuitively and graphically interpret Dong and Lewbel's TED. Of course, the TED is a more general quantity and should not be restricted to our interpretation in every scenario. Nevertheless, our framework offers a natural lens through which their TED can be visually understood in practice, and facilitates a more informative extrapolation analysis.

Under the locally linear treatment effects assumption, we can further obtain a useful uncertainty quantification method:

Proposition 2. *In addition to the assumptions made in Proposition 1, suppose that Assumption*

[2] Ghosh et al. (2025) recently proposed a similar (and stronger) linear treatment effects assumption in the sharp RD setting (see their Assumption 2). However, they introduced it to improve inference on the standard RD estimand $\tau_{\mathtt{SRD}}$ and do not consider external validity or extrapolation issues.

3 *also holds. Then,* $\lim_{n\to\infty} \mathbb{P}\left[\tau(x) \in \mathcal{U}_{1-\alpha}, \forall x \in [-\delta_1, \delta_2]\right] = 1-\alpha$*, where*

$$\mathcal{U}_{1-\alpha} := \Bigg[\tilde{\tau}_{\mathtt{SRD}} + \tilde{\tau}'_{\mathtt{SRD}} x - c^{\star}_{1-\alpha}\sqrt{(1,x)\hat{\Omega}_h(1,x)^{\top}}, \\ \tilde{\tau}_{\mathtt{SRD}} + \tilde{\tau}'_{\mathtt{SRD}} x + c^{\star}_{1-\alpha}\sqrt{(1,x)\hat{\Omega}_h(1,x)^{\top}}\Bigg],$$

and $c^{\star}_{1-\alpha}$ *is defined by* $P(c^{\star}_{1-\alpha}) = 1-\alpha$*, where*

$$P(s) := \frac{\ell}{\pi}\left[1 - \exp\left(-\frac{s^2}{2}\right)\right] + \frac{2}{\pi}\int_0^{\pi/2-\ell/2} 1 - \exp\left(-\frac{s^2}{2\cos(u)^2}\right) du.$$

Here, $\ell \in [0,\pi]$ *denotes the smallest angle between the directions* $\hat{v}(-\delta_1)$ *and* $\hat{v}(\delta_2)$*, where* $\hat{v}(x) := \hat{\Omega}_h^{1/2}(1,x)^{\top}/||\hat{\Omega}_h^{1/2}(1,x)^{\top}||$. ♣

Although it may appear obscure at first glance, the confidence band $\mathcal{U}_{1-\alpha}$ is related to the joint confidence region $\mathcal{R}_{1-\alpha}$. Noting that $1-\exp(-s^2/2)$ is the distribution function of the standard Rayleigh distribution, it follows that $c^{\star}_{1-\alpha} \to \sqrt{c_{1-\alpha}}$ as $\min\{\delta_1,\delta_2\} \to \infty$, i.e., $\ell \to \pi$. In this sense, the confidence band $\mathcal{U}_{1-\alpha}$ admits an alternative interpretation: as the extrapolation interval expands, $\mathcal{U}_{1-\alpha}$ converges to the *projection envelope* $\mathcal{E}_{1-\alpha}$ of the ellipse $\mathcal{R}_{1-\alpha}$ onto the linear functions, where

$$\mathcal{E}_{1-\alpha} := \Bigg[\tilde{\tau}_{\mathtt{SRD}} + \tilde{\tau}'_{\mathtt{SRD}} x - \sqrt{c_{1-\alpha}}\sqrt{(1,x)\hat{\Omega}_h(1,x)^{\top}}, \\ \tilde{\tau}_{\mathtt{SRD}} + \tilde{\tau}'_{\mathtt{SRD}} x + \sqrt{c_{1-\alpha}}\sqrt{(1,x)\hat{\Omega}_h(1,x)^{\top}}\Bigg],$$

which can be regarded as an envelope of the *implied* linear extrapolations under $\mathcal{R}_{1-\alpha}$.

Moreover, we can show that $c^{\star}_{1-\alpha} \to \Psi^{-1}(1-\alpha/2)$ as $\max\{\delta_1,\delta_2\} \to 0$, where Ψ^{-1} denotes the quantile function of the standard normal distribution. Thus, $\mathcal{U}_{1-\alpha}$ collapses to the usual RBC confidence interval for $\tau_{\mathtt{SRD}}$ of Calonico et al. (2014). That is, $\mathcal{U}_{1-\alpha}$ can be viewed as a natural extension of the RBC confidence intervals to linear extrapolation analysis.

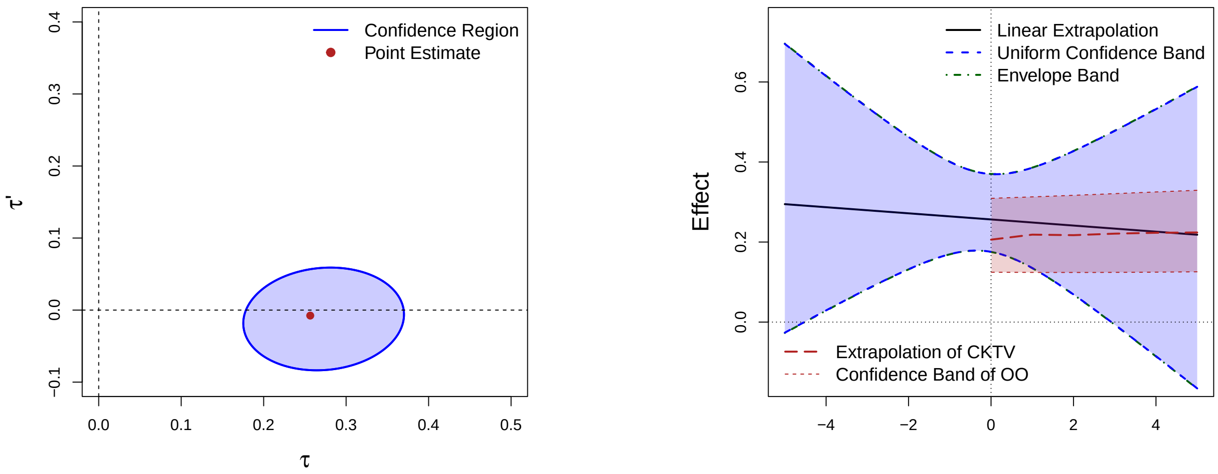
4 Empirical Illustration

We illustrate the potential usefulness of our proposal by applying it to the study of Londoño-Vélez et al. (2020). Their research examines the effect of Ser Pilo Paga (SPP), a financial aid program introduced in Colombia in 2014, on students' probability of enrolling in higher education, using an RD design. Making use of the multiple cutoff points in the dataset, Okamoto and Ozaki (2025) evaluates its external validity through their bounding approach, together with the extrapolation method proposed by Cattaneo et al. (2021).

In what follows, and in line with Okamoto and Ozaki (2025), we restrict attention to a specific subgroup of the full sample. In our subsample of interest ($n = 19738$), eligibility for the SPP scholarship is determined by a family wealth index. After normalization so that the cutoff value is zero, the wealth index ranges from -25.94 to 56.86, with the first quartile at -3.76 and the third quartile at 22.78. See Okamoto and Ozaki (2025) for a concise summary. Further details can be found in Londoño-Vélez et al. (2020).

We begin by assessing the external validity of the RD treatment effect $\tau_{\mathtt{SRD}}$ using the robust confidence region. For bandwidth selection and standard error estimation, we follow `rdrobust` package (Calonico et al., 2015). Figure 2a reports the 95% confidence region $\mathcal{R}_{0.95}$, where the horizontal axis represents the treatment effect and the vertical axis corresponds to the TED. The figure shows that the SPP has a sizable and statistically significant effect on enrolment probability. At the same time, the confidence interval along the TED dimension is relatively tight and centred close to zero, indicating that $\tau'_{\mathtt{SRD}}$ is small and statistically indistinguishable from zero.

This finding provides direct evidence of the local external validity of the RD effect of the SPP program. Importantly, such an implication could not be drawn from the treatment effect $\tau_{\mathtt{SRD}}$ alone; it emerges only when the derivative is taken into account. Hence, as emphasized by Dong and Lewbel (2015), examining $\tau'_{\mathtt{SRD}}$ alongside $\tau_{\mathtt{SRD}}$ is crucial for clarifying the policy relevance of RD estimates. Our joint inference procedure operationalizes this insight by providing a formal statistical framework that allows both quan-



(a) 95% Robust Confidence Region

(b) Linear Extrapolation and Confidence Band

Figure 2: External Validity and Extrapolation of Treatment Effects

Note: In Panel (b), Linear Extrapolation indicates the point estimates $\hat{\tau}_{\text{SRD}} + \hat{\tau}'_{\text{SRD}}x$, Uniform Confidence Band shows $\mathcal{U}_{0.95}$, and Envelope Band presents $\mathcal{E}_{0.95}$. For clarity of presentation, we report only the point estimates of Cattaneo et al. (2021) and the uniform confidence band for the bounds proposed by Okamoto and Ozaki (2025).

tities to be assessed simultaneously.

While suggestive, the confidence region $\mathcal{R}_{0.95}$ in Figure 2a provides only local information and is less intuitive than a full extrapolation analysis. To address this limitation, we impose the locally linear treatment effects assumption over $[-5, 5]$ (i.e. $\delta_1 = \delta_2 = 5$), with its plausibility briefly discussed at the end of this section. Figure 2b displays the uniform confidence band $\mathcal{U}_{0.95}$ together with the envelope $\mathcal{E}_{0.95}$; in our case, the two regions nearly coincide.

Figure 2b delivers clearer implications. Consistent with the robust confidence region in Figure 2a, the uniform confidence band for the linearly extrapolated effects indicates positive treatment effects over $[-4, 3]$, although the effectiveness of SPP at more distant scores remains uncertain. This evidence of positive treatment effects away from the cutoff is consistent with findings from alternative approaches that exploit multiple cutoff points, such as Cattaneo et al. (2021) and Okamoto and Ozaki (2025), represented by the red dashed lines and shaded region in Figure 2b. This comparison highlights the effectiveness of our proposed method, particularly in settings where additional information or design features are not available. However, we can also observe that the confidence band for the multi-cutoff method is much tighter than our proposed one, suggesting that additional external information can enhance the precision of the RD studies.

One important limitation of their approaches based on multiple cutoffs is that they do not apply to the left-hand region $[-5, 0)$. In contrast, our method enables the extrapolation for the left. Our results in Figure 2b suggest that the extrapolated effects are also positive in this range. These results indicate that our procedure can complement existing proposals and provide informative additional insights, even when extra design features are available.

Finally, we note that the linearly extrapolated line on the right-hand side of the cutoff lies within the confidence band of Okamoto and Ozaki (2025) and closely resembles the extrapolated curve of Cattaneo et al. (2021). This concordance supports the plausibility of the locally linear treatment effects assumption in this application.

5 Conclusion

Motivated by our survey, which suggests that recent empirical RD studies pay little attention to external validity and that more accessible econometric tools are needed, this article proposes a simple procedure for jointly inferring the RD effect and its local external validity. Building on the insight of Calonico et al. (2014) and Dong and Lewbel (2015), we formalised inference on both the treatment effect and its derivative, and developed a joint confidence region. We then introduced a locally linear treatment effects assumption, which complements the contribution of Dong and Lewbel (2015) by rendering their treatment effect derivative indicator more easily interpretable. Under this assumption, we further derived a uniform confidence band for linearly extrapolated RD effects.

Our approach requires no additional covariates or design features, making it applicable to virtually all RD settings and thereby enhancing the policy relevance of many empirical studies. Its usefulness is illustrated through an application to the Ser Pilo Paga program in Colombia (Londoño-Vélez et al., 2020). Our results underscore the broad applicability of our proposals and the informativeness of the Dong and Lewbel treatment effect derivative, particularly when combined with our proposed local linearity assumption.

A Proofs and Generalization

A.1 Proofs of Lemma 1 and Proposition 1

This subsection provides proofs of Lemma 1 and Proposition 1 in a more general setting, where we employ $p(\geq 1)$-th order local polynomial regression for the main estimation and use local polynomial regression of order $q \geq p+1$ for the bias estimation. As the matrix notation is more convenient, we begin by introducing some additional notations. Our p-th order local polynomial estimators are given by $\hat{\mu}_+ = e_0^\top \hat{\beta}_{+,p}$ and $\hat{\mu}'_+ = e_1^\top \hat{\beta}_{+,p}$, where $e_0 := (1, 0, \ldots, 0)^\top \in \mathbb{R}^{p+1}$, $e_1 := (0, 1, 0, \ldots, 0)^\top \in \mathbb{R}^{p+1}$,

$$\hat{\beta}_{+,p} := \underset{\beta \in \mathbb{R}^{p+1}}{\arg\min} \sum_{i=1}^{n} \mathbf{1}\left\{X_i \geq 0\right\} \left(Y_i - r_p(X_i)^\top \beta\right)^2 \frac{1}{h} K\left(\frac{X_i}{h}\right),$$

and $r_p(u) := (1, u, \ldots, u^p)$. $\hat{\mu}_-$ and $\hat{\mu}'_-$ is defined similarly. Note that $\hat{\mu}_+$ and $\hat{\mu}'_+$ have the following closed form expression:

$$\hat{\mu}_+ = \frac{1}{n} e_0^\top \Gamma_+^{-1} R^\top W_+ Y, \; \hat{\mu}'_+ = \frac{1}{n}\frac{1}{h} e_1^\top \Gamma_+^{-1} R^\top W_+ Y,$$

where $Y := (Y_1, \ldots, Y_n)^\top$, $W_+ := \mathrm{diag}(\mathbf{1}\{X_1 \geq 0\} K(X_1/h)/h, \ldots, \mathbf{1}\{X_n \geq 0\} K(X_n/h)/h)$, $R := (r_p(X_1/h), \ldots, r_p(X_n/h))^\top$, and $\Gamma_+ := R^\top W_+ R/n$. The counterparts replacing $+$

with $-$ are defined similarly. We will focus on the $+$ case as long as the $-$ case can be treated analogously below.

Now, we assume $n\min\{h,b\}\to\infty$, $n\min\{h^{2(p+1)+1},b^{2(p+1)+1}\}\max\{h^2,b^{2(q-p)}\}\to 0$, and $\max\{h,b\}<\kappa_0$ throughout. We also assume Assumptions 1-2 hold with $S\geq q+1\geq p+2$.

The usual Taylor expansion argument suggests that

$$\mathbb{E}\left[\hat{\mu}_+|X_1,\ldots,X_n\right]-\mu_+\approx h^{p+1}\frac{\mu_+^{(p+1)}}{(p+1)!}e_0^\top\Gamma_+^{-1}\vartheta_+ =: h^{p+1}\frac{\mu_+^{(p+1)}}{(p+1)!}\mathcal{B}_+$$
$$\mathbb{E}\left[\hat{\mu}_+'|X_1,\ldots,X_n\right]-\mu_+'\approx h^{p}\frac{\mu_+^{(p+1)}}{(p+1)!}e_1^\top\Gamma_+^{-1}\vartheta_+ =: h^{p}\frac{\mu_+^{(p+1)}}{(p+1)!}\mathcal{B}_+',$$

where $\vartheta_+ := R^\top W_+ S/n$ and $S := ((X_1/h)^{p+1},\ldots,(X_n/h)^{p+1})^\top$. This motivates our bias-corrected estimators:

$$\tilde{\mu}_+ := \hat{\mu} - h^{p+1}\frac{\hat{\mu}_+^{(p+1)}}{(p+1)!}\mathcal{B}_+,\ \text{and}\ \tilde{\mu}_+' := \hat{\mu}' - h^{p}\frac{\hat{\mu}_+^{(p+1)}}{(p+1)!}\mathcal{B}_+',$$

where $\hat{\mu}_+^{(p+1)}$ are constructed using the q-th order local polynomial regression:

$$\hat{\mu}_+^{(p+1)} := (p+1)!\frac{1}{b^{p+1}}e_{p+1}^\top\check{\Gamma}_+^{-1}\check{R}^\top\check{W}_+Y/n,$$

where $e_{p+1}\in\mathbb{R}^{q+1}$ is the $(p+2)$-th standard basis vector, and the checked quantities (e.g., $\check{\Gamma}_+$) are defined by replacing p with q and h with b. Then, Lemma S.A.4 of Calonico et al. (2014) tells us that $\mathbb{V}\left[\tilde{\mu}_+|x_i,\ldots,X_n\right] = V_+$ and $\mathbb{V}\left[h\tilde{\mu}'_+|x_i,\ldots,X_n\right] = V'_+$, where

$$
\begin{aligned}
V_+ &:= \frac{1}{n} e_0^\top \Gamma_+^{-1}(R^\top W_+ \Sigma_Y W_+ R/n)\Gamma_+^{-1} e_0 \\
&\quad + h^{2(p+1)} \frac{1}{nb^{2(p+1)}}(p+1)!^2 e_{p+1}^\top \check{\Gamma}_+^{-1}(\check{R}^\top \check{W}_+ \Sigma_Y \check{W}_+ \check{R}/n)\check{\Gamma}_+^{-1} e_{p+1} \frac{\mathcal{B}_+^2}{(p+1)!^2} \\
&\quad - 2h^{p+1} \frac{1}{nb^{p+1}}(p+1)! e_0^\top \Gamma_+^{-1}(RW_+ \Sigma_Y \check{W}_+ \check{R}/n)\check{\Gamma}_+^{-1} e_{p+1} \frac{\mathcal{B}_+}{(p+1)!} \\
&= \frac{1}{n} e_0^\top \Gamma_+^{-1}(R^\top W_+ \Sigma_Y W_+ R/n)\Gamma_+^{-1} e_0 \\
&\quad + \frac{h^{2(p+1)}}{nb^{2(p+1)}} e_{p+1}^\top \check{\Gamma}_+^{-1}(\check{R}^\top \check{W}_+ \Sigma_Y \check{W}_+ \check{R}/n)\check{\Gamma}_+^{-1} e_{p+1} \mathcal{B}_+^2 \\
&\quad - 2\frac{h^{p+1}}{nb^{p+1}} e_0^\top \Gamma_+^{-1}(RW_+ \Sigma_Y \check{W}_+ \check{R}/n)\check{\Gamma}_+^{-1} e_{p+1} \mathcal{B}_+, \\
V'_+ &:= \frac{1}{n} e_1^\top \Gamma_+^{-1}(R^\top W_+ \Sigma_Y W_+ R/n)\Gamma_+^{-1} e_1 \\
&\quad + \frac{h^{2(p+1)}}{nb^{2(p+1)}} e_{p+1}^\top \check{\Gamma}_+^{-1}(\check{R}^\top \check{W}_+ \Sigma_Y \check{W}_+ \check{R}/n)\check{\Gamma}_+^{-1} e_{p+1} (\mathcal{B}'_+)^2 \\
&\quad - 2\frac{h^{p+1}}{nb^{p+1}} e_1^\top \Gamma_+^{-1}(RW_+ \Sigma_Y \check{W}_+ \check{R}/n)\check{\Gamma}_+^{-1} e_{p+1} \mathcal{B}'_+,
\end{aligned}
$$

and $\Sigma_Y := \operatorname{diag}(\mathbb{V}\left[Y_i|X_1\right],\ldots,\mathbb{V}\left[Y_i|X_n\right])$. Note that the only unknown quantity is Σ_Y.

We can derive a similar representation for the covariance. Note that

$$
\begin{aligned}
&\mathbb{C}\mathrm{ov}\left[\tilde{\mu}_+, h\tilde{\mu}'_+|X_1,\ldots,X_n\right] = \mathbb{C}\mathrm{ov}\left[\hat{\mu}_+ - h^{p+1}\hat{B}_+, h\left(\hat{\mu}'_+ - h^p \hat{B}'_+\right)|X_1,\ldots,X_n\right] \\
&= h\mathbb{C}\mathrm{ov}\left[\hat{\mu}_+, \hat{\mu}'_+|X_1,\ldots,X_n\right] - h^{p+1}\mathbb{C}\mathrm{ov}\left[\hat{\mu}_+, \hat{B}'_+|X_1,\ldots,X_n\right] \\
&\quad - h^{p+2}\mathbb{C}\mathrm{ov}\left[\hat{B}_+, \hat{\mu}'_+|X_1,\ldots,X_n\right] + h^{2(p+1)}\mathbb{C}\mathrm{ov}\left[\hat{B}_+, \hat{B}'_+|X_1,\ldots,X_n\right].
\end{aligned}
$$

Then we can show that $\mathbb{C}\text{ov}\left[\tilde{\mu}_+, h\tilde{\mu}'_+ | X_1, \ldots, X_n\right] = C_+(\Sigma_Y)$, where

$$
\begin{aligned}
C_+(\Sigma) := &\frac{1}{n} e_0^\top \Gamma_+^{-1} (R^\top W_+ \Sigma W_+ R / n) \Gamma_+^{-1} e_1 \\
&- \frac{h^{p+1}}{n b^{p+1}} e_0^\top \Gamma_+^{-1} (R W_+ \Sigma \check{W}_+ \check{R} / n) \check{\Gamma}_+^{-1} e_{p+1} \mathcal{B}'_+ \\
&- \frac{h^{p+1}}{n b^{p+1}} e_1^\top \Gamma_+^{-1} (R W_+ \Sigma \check{W}_+ \check{R} / n) \check{\Gamma}_+^{-1} e_{p+1} \mathcal{B}_+ \\
&+ \frac{h^{2(p+1)}}{n b^{2(p+1)}} e_{p+1}^\top \check{\Gamma}_+^{-1} (\check{R}^\top \check{W}_+ \Sigma \check{W}_+ \check{R} / n) \check{\Gamma}_+^{-1} e_{p+1} \mathcal{B}_+ \mathcal{B}'_+ .
\end{aligned}
$$

Then, by the Cramer-Wold device, it suffices to show that

$$
\frac{a_0 \left(\tilde{\mu}_+ - \mu_+\right) + a_1 h \left(\tilde{\mu}'_+ - \mu'_+\right)}{\sqrt{a_0^2 V_+ + 2 a_0 a_1 C_+ + a_1^2 V'_+}} \to_d \mathcal{N}(0, 1)
$$

for any $(a_0, a_1)^\top \in \mathbb{R}^2$. First, the same manipulation as Lemma S.A.4 of Calonico et al. (2014, p. 16 of Supplement) shows that

$$
\begin{aligned}
&\frac{a_0 \left(\tilde{\mu}_+ - \mu_+\right) + a_1 h \left(\tilde{\mu}'_+ - \mu'_+\right)}{\sqrt{a_0^2 V_+ + 2 a_0 a_1 C_+ + a_1^2 V'_+}} \\
&\quad = \underbrace{\frac{a_0 \left(\tilde{\mu}_+ - \mathbb{E}\left[\tilde{\mu}_+ | X_1, \ldots, X_n\right]\right) + a_1 h \left(\tilde{\mu}'_+ - \mathbb{E}\left[\tilde{\mu}'_+ | X_1, \ldots, X_n\right]\right)}{\sqrt{a_0^2 V_+ + 2 a_0 a_1 C_+ + a_1^2 V'_+}}}_{=:\xi} + o_p(1).
\end{aligned}
$$

Second, we will show below that $\xi = \xi_* + o_p(1)$, where $\xi_* := \sum_{i=1}^n \omega_{n,i} \varepsilon_i$, $\omega_{n,i} := (a_0 \omega_{0,n,i} + a_1 h \omega_{1,n,i}) / \sqrt{\omega_{2,n}}$, $\varepsilon_i = Y_i - \mu(X_i)$, $\mu(X) := \mathbb{E}\left[Y | X\right]$,

$$
\begin{aligned}
\omega_{0,n,i} &:= e_0^\top \Gamma_{+,\star}^{-1} \frac{\mathbf{1}_i}{nh} K\left(\frac{X_i}{h}\right) r_p\left(\frac{X_i}{h}\right) - \frac{h^{p+1}}{b^{p+1}} e_0^\top \Gamma_{+,\star}^{-1} \vartheta_{+,\star} e_{p+1}^\top \check{\Gamma}_{+,\star}^{-1} \frac{\mathbf{1}_i}{nb} K\left(\frac{X_i}{b}\right) r_q\left(\frac{X_i}{b}\right), \\
\omega_{1,n,i} &:= e_1^\top \Gamma_{+,\star}^{-1} \frac{\mathbf{1}_i}{nh^2} K\left(\frac{X_i}{h}\right) r_p\left(\frac{X_i}{h}\right) - \frac{h^p}{b^{p+1}} e_1^\top \Gamma_{+,\star}^{-1} \vartheta_{+,\star} e_{p+1}^\top \check{\Gamma}_{+,\star}^{-1} \frac{\mathbf{1}_i}{nb} K\left(\frac{X_i}{b}\right) r_q\left(\frac{X_i}{b}\right),
\end{aligned}
$$

$\mathbf{1}_i := \mathbf{1}\{X_i \geq 0\}$, $\Gamma_{+,\star} := \int_0^1 K(u) r_p(u) r_p(u)^\top f(uh)\, du$, $\check{\Gamma}_{+,\star} := \int_0^1 K(u) r_q(u) r_q(u)^\top f(ub)\, du$, and $\vartheta_{+,\star} := \int_0^1 K(u) r_p(u) u^{p+1} f(uh)\, du$; and

$$\omega_{2,n} = \sum_{i=1}^{n} \mathbb{E}\left[(a_0 \omega_{0,n,i} + a_1 h \omega_{1,n,i})^2 \varepsilon_i^2\right].$$

To see this, first observe that $R^\top W_+(\varepsilon_1, \ldots, \varepsilon_n)^\top = \sum_{i=1}^n \mathbf{1}_i K(X_i/h)/h\varepsilon_i \times r_p(X_i/h)$ and the analogous equality holds for $\check{R}^\top \check{W}_+(\varepsilon_1, \ldots, \varepsilon_n)^\top$. Then, by Lemma S.A.1 (a.1) and (a.3) of Calonico et al. (2014) and the definition of $\mathcal{B}_+, \mathcal{B}'_+$, we can obtain that

$$a_0 \left(\tilde{\mu}_+ - \mathbb{E}\left[\tilde{\mu}_+ | X_1, \ldots, X_n\right]\right) + a_1 h \left(\tilde{\mu}'_+ - \mathbb{E}\left[\tilde{\mu}'_+ | X_1, \ldots, X_n\right]\right) = \sum_{i=1}^{n} \omega_{n,i} \varepsilon_i + o_p(1).$$

Further, by the forms of definitions of V_+, V'_+, C_+, $\omega_{0,n,i}$, and $\omega_{1,n,i}$, it is straightforward to show that

$$\sqrt{a_0^2 V_+ + 2 a_0 a_1 C_+ + a_1^2 V'_+} = \sum_{i=1}^{n} \mathbb{E}\left[(a_0 \omega_{0,n,i} + a_1 h \omega_{1,n,i})^2 \varepsilon_i^2\right] + o_p(1).$$

Hence, we obtain that $\xi = \xi_* + o_p(1)$. Now, note that the convergence rates of $\mathbb{E}[\omega_{0,n,i}^2 \varepsilon_i^2]$, $\mathbb{E}[\omega_{1,n,i}^2 \varepsilon_i^2]$, and $\mathbb{E}\left[(a_0 \omega_{0,n,i} + a_1 h \omega_{1,n,i})^2 \varepsilon_i^2\right]$ are balanced. Therefore, combined with Lemma S.A.4 of Calonico et al. (2014, p. 17), we obtain that $\sum_{i=1}^n \mathbb{E}\left[|\omega_{0,n,i}\varepsilon_i/\sqrt{\omega_{2,n}}|^4\right] = o(1)$ and $\sum_{i=1}^n \mathbb{E}\left[|h\omega_{1,n,i}\varepsilon_i/\sqrt{\omega_{2,n}}|^4\right] = o(1)$. Hence, using the elementary inequality $|x+y|^4 \leq 8(|x|^4 + |y|^4)$, we obtain that $\sum_{i=1}^n \mathbb{E}\left[|\omega_{n,i}\varepsilon_i|^4\right] = o(1)$. The Liapunov central limit theorem shows $\xi_* \to_d \mathcal{N}(0,1)$. By the random sampling assumption, we obtain the desired result with $\mathrm{V}_{\mathtt{SRD}} := V_+ + V_-$, $\mathrm{V}'_{\mathtt{SRD}} := V'_+ + V'_-$, and $\mathrm{C}_{\mathtt{SRD}} := \mathrm{C}_{\mathtt{SRD}}(\Sigma_Y) = C_+(\Sigma_Y) + C_-(\Sigma_Y)$.

Proof of Lemma 1. This lemma is a special case of the argument above. □

Proof of Proposition 1. This proposition is a corollary to the previous lemma. □

A.2 Proofs of Lemma 2 and Proposition 2

Proof of Lemma 2. Under the differentiability condition, τ_{SRD} and τ'_{SRD} are both identified (Dong and Lewbel, 2015, Theorem 1). Then, a linear function over the interval including $x = 0$ is identified. Hence, Assumption 3 implies the identification. $\square$

Proof of Proposition 2. Lemma 1 have shown that $\tilde{\Delta}(\tau_{\text{SRD}}, \tau'_{\text{SRD}}) \to_d \mathcal{N}((0,0)^\top, \Omega_h)$. Then, we have that

$$\frac{(1,x)\tilde{\Delta}(\tau_{\text{SRD}}, \tau'_{\text{SRD}})}{\sqrt{(1,x)\Omega_h(1,x)^\top}} \to_d v(x)^\top Z, \text{ where } Z \sim \mathcal{N}\left((0,0)^\top, \text{diag}(1,1)\right)$$

and $v(x) := \Omega_h^{1/2}(1,x)^\top / ||\Omega_h^{1/2}(1,x)^\top||$. We are interested in the limiting behavior of $\sup_{x\in[-\delta_1,\delta_2]} (1,x)\tilde{\Delta}(\tau_{\text{SRD}}, \tau'_{\text{SRD}}) / \sqrt{(1,x)\Omega_h(1,x)^\top}$. By the continuous mapping, its limit is $S := \sup_{x\in[-\delta_1,\delta_2]} |v(x)^\top Z|$. By transforming to polar coordinates, we can decompose the normal variable Z as $Z = R(\cos\phi, \sin\phi)^\top$, where R follows the standard Rayleigh distribution, $\phi \sim \text{Uniform}[0, 2\pi)$, and $R \perp\!\!\!\perp \phi$. Hence, we have $\sup_{x\in[-\delta_1,\delta_2]} |v(x)^\top Z| = R \cdot \sup_{x\in[-\delta_1,\delta_2]} |v(x)^\top (\cos\phi, \sin\phi)^\top|$.

Now, recalling that the definition of $v(x)$, it is on the unit circle. To study the direction of $v(x)$, we define the non-normalized version $u(x) := \Omega_h^{1/2}(1,x)^\top$, and identify $\mathbb{R}^2$ with the complex plane so that we examine the argument of the complex number $u_1(x) + iu_2(x)$, where $u(x) = (u_1(x), u_2(x))^\top$. By construction, there exist real constants a_1, a_2, b_1, b_2 such that $u(x) = (a_1 + b_1 x, a_2 + b_2 x)^\top$. Then we compute:

$$\frac{\partial}{\partial x} \arg\left(u_1(x) + iu_2(x)\right) = \frac{a_1 b_2 - a_2 b_1}{(a_1 + b_1 x)^2 + (a_2 + b_2 x)^2} > 0,$$

since $a_1 b_2 - a_2 b_1 = \det(\Omega_h^{1/2}) > 0$ by the regularity of Ω_h. Hence, the argument of $u(x)$ is strictly increasing in x, and so is $v(x)$. Therefore, as x ranges over $[-\delta_1, \delta_2]$, $v(x)$ traces out an arc of the unit circle. Let $\theta(x)$ denote this argument, and it is easy to see that there exists some closed interval $[\theta_l, \theta_u]$ such that $\theta(x) \in [\theta_l, \theta_u]$, $\theta_u - \theta_l \leq \pi$, and the boundary points are attained at $x = -\delta_1$ or $x = \delta_2$. Then, we can rewrite that

$$\begin{aligned} S &= R \cdot \sup_{x \in [-\delta_1, \delta_2]} |v(x)^\top (\cos\phi, \sin\phi)^\top| \\ &= R \cdot \sup_{x \in [-\delta_1, \delta_2]} |\cos(\theta(x) - \phi)| \\ &= R \cdot \sup_{\theta \in [\theta_l, \theta_u]} |\cos(\theta - \phi)|, \end{aligned}$$

where the second equality uses the definition of the inner product. The distribution of the last term is equal to that of $R \cdot \sup_{\theta \in [0, \theta_u - \theta_l]} |\cos(\theta - \phi)|$, whose distribution is again equivalent to the distribution of $R \cdot \sup_{\theta \in [0, \theta_u - \theta_l]} |\cos(\theta - \varphi)|$, where $\varphi \sim \text{Uniform}[0, \pi]$. Then, we can further rewrite this as

$$R \cdot \begin{cases} 1 & \text{if } \varphi \in [0, \theta_u - \theta_l] =: \mathrm{I}_1 \\ |\cos((\theta_u - \theta_l) - \varphi)| & \text{if } \varphi \in [\theta_u - \theta_l, \pi/2 + (\theta_u - \theta_l)/2] =: \mathrm{I}_2 \\ |\cos(-\varphi)| & \text{if } \varphi \in [\pi/2 + (\theta_u - \theta_l)/2, \pi] =: \mathrm{I}_3. \end{cases}$$

Figure 3 must be instructive to derive this. Again, by observing Figure 3, we finally obtain that

$$\begin{aligned} &S = R \cdot \begin{cases} 1 & \text{with probability } (\theta_u - \theta_l)/\pi \\ \cos(U) & \text{with probability } 1 - (\theta_u - \theta_l)/\pi, \end{cases} \\ &U \sim \text{Uniform}[0, \pi/2 - (\theta_u - \theta_l)/2], \ R \perp\!\!\!\perp U. \end{aligned}$$

Hence, the distribution function of S is given by

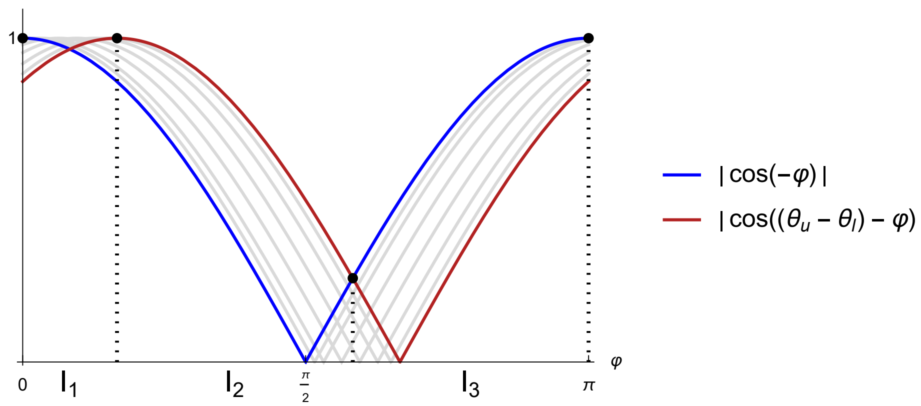


Figure 3: Illustration of I_1, I_2, and I_3

$$
\begin{aligned}
&\mathbb{P}\left[S \leq s\right] \\
&= \frac{\theta_u - \theta_l}{\pi}\mathbb{P}\left[R \leq s\right] + \left(1 - \frac{\theta_u - \theta_l}{\pi}\right)\int_0^{\pi/2-(\theta_u-\theta_l)/2} \mathbb{P}\left[R \leq \frac{s}{\cos(u)}\right]\frac{2}{\pi - (\theta_u - \theta_l)}\,du \\
&= \frac{\theta_u - \theta_l}{\pi}\left[1 - \exp\left(-\frac{s^2}{2}\right)\right] + \frac{2}{\pi}\int_0^{\pi/2-(\theta_u-\theta_l)/2} 1 - \exp\left(-\frac{s^2}{2\cos(u)^2}\right)\,du.
\end{aligned}
$$

Thus, combined with the consistency of $\hat{\Omega}$, the critical value defined in the proposition provides asymptotically correct coverage. □

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Online Appendix for "Joint Inference for the Regression Discontinuity Effect and Its External Validity"

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S1 Fuzzy RD Design

Extension of our proposals to the fuzzy RD case is straightforward. Suppose a random sample $(Y_i(0), Y_i(1), T_i(0), T_i(1), X_i)$. Here, $T(0)$ and $T(1)$ denote the potential treatment indicator, and we observe $T_i = T_i(1)\mathbf{1}\{X_i \geq 0\} + T_i(0)\mathbf{1}\{X_i < 0\}$. In fuzzy designs, our target parameters are $\tau_{\texttt{FRD}} := \mathbb{E}\left[Y_i(1) - Y_i(0) | X_i = 0, T_i(1) > T_i(0)\right]$ and its derivative with respect to x, $\tau'_{\texttt{FRD}}$. Under Assumption A3 of Dong and Lewbel (2015), these are identified as

$$\tau_{\texttt{FRD}} := \frac{\tau_Y}{\tau_T} := \frac{\mu_{Y,+} - \mu_{Y,-}}{\mu_{T,+} - \mu_{T,-}}, \text{ and } \tau'_{\texttt{FRD}} := \frac{\tau'_Y}{\tau'_T} := \frac{\mu'_{Y,+} - \mu'_{Y,-}}{\mu'_{T,+} - \mu'_{T,-}},$$

where $\mu_{A,+} := \lim_{x \downarrow 0} \mu_A(x)$, $\mu_{A,-} := \lim_{x \uparrow 0} \mu_A(x)$, and $\mu_A(x) := \mathbb{E}\left[A_i | X_i = x\right]$, where A equals either Y or T. Below, We sometimes write $\tau^{(0)}_{\texttt{FRD}} = \tau_{\texttt{FRD}}$ and $\tau^{(1)}_{\texttt{FRD}} = \tau'_{\texttt{FRD}}$.

In addition to the assumptions made in Lemma 1, we assume the following:

Assumption 4. $h \to 0$. *Furthermore, in a neighbourhood $(-\kappa_0, \kappa_0)$ around the cutoff:*

(i) $\mu_{T,+}(x) := \mathbb{E}\left[T_i(1) | X_i = x\right]$ and $\mu_{T,-}(x) := \mathbb{E}\left[T_i(0) | X_i = x\right]$ are S-times continuously differentiable.

(i) $\sigma^2_{T,+}(x) := \mathbb{V}\left[T_i(1) | X_i = x\right]$ and $\sigma^2_{T,-}(x) := \mathbb{V}\left[T_i(0) | X_i = x\right]$ are continuous and bounded away from zero.

Below, we suppose that the assumptions made in Lemma 1 and Assumption 4 are satisfied. Let $\hat{\tau}_{\texttt{FRD}}$ and $\hat{\tau}'_{\texttt{FRD}}$ be the ratio of the p-th order local polynomial "sharp RD" estimators using the same bandwidth h for both denominator and numerator. The bias-corrected estimators are given by

$$\tilde{\tau}_{\mathtt{FRD}} = \hat{\tau}_{\mathtt{FRD}} - h^{p+1}\left[\left\{\frac{1}{\hat{\tau}_T}\frac{\hat{\mu}^{(p+1)}_{Y,+}}{(p+1)!} - \frac{\hat{\tau}_Y}{\hat{\tau}_T^2}\frac{\hat{\mu}^{(p+1)}_{T,+}}{(p+1)!}\right\}\mathcal{B}_+ - \left\{\frac{1}{\hat{\tau}_T}\frac{\hat{\mu}^{(p+1)}_{Y,-}}{(p+1)!} - \frac{\hat{\tau}_Y}{\hat{\tau}_T^2}\frac{\hat{\mu}^{(p+1)}_{T,-}}{(p+1)!}\right\}\mathcal{B}_-\right],$$
$$\tilde{\tau}'_{\mathtt{FRD}} = \hat{\tau}'_{\mathtt{FRD}} - h^{p}\left[\left\{\frac{1}{\hat{\tau}'_T}\frac{\hat{\mu}^{(p+1)}_{Y,+}}{(p+1)!} - \frac{\hat{\tau}'_Y}{(\hat{\tau}'_T)^2}\frac{\hat{\mu}^{(p+1)}_{T,+}}{(p+1)!}\right\}\mathcal{B}_+ - \left\{\frac{1}{\hat{\tau}'_T}\frac{\hat{\mu}^{(p+1)}_{Y,-}}{(p+1)!} - \frac{\hat{\tau}'_Y}{(\hat{\tau}'_T)^2}\frac{\hat{\mu}^{(p+1)}_{T,-}}{(p+1)!}\right\}\mathcal{B}_-\right].$$

It is useful to rewrite (Calonico et al., 2014, Supplement, p. 24):

$$\tilde{\tau}^{(\nu)}_{\mathtt{FRD}} = \underbrace{\frac{1}{\tau^{(\nu)}_T}\left(\tilde{\tau}^{(\nu)}_Y - \tau^{(\nu)}_Y\right) - \frac{\tau^{(\nu)}_Y}{(\tau^{(\nu)}_T)^2}\left(\tilde{\tau}^{(\nu)}_T - \tau^{(\nu)}_T\right)}_{=:\mathring{\tau}^{(\nu)}_{\mathtt{FRD}}} + R_n - R_n^{\mathtt{bc}},$$

where R_n and $R_n^{\mathtt{bc}}$ are given in Calonico et al. (2014, Supplement, p. 24). The conditional variance of $\mathring{\tau}_{\mathtt{FRD}}$ and $h\mathring{\tau}'_{\mathtt{FRD}}$ is derived by Calonico et al. (2014, Theorem A.2). Let $\mathrm{V}_{\mathtt{FRD}}$ and $\mathrm{V}'_{\mathtt{FRD}}$ denote them. Observing that $\mathring{\tau}^{(\nu)}_{\mathtt{FRD}}$ is a (weighted) difference of the two bias-corrected sharp RD estimands, the covariance can be computed as $\mathbb{C}\mathrm{ov}\left[\mathring{\tau}_{\mathtt{FRD}}, h\mathring{\tau}'_{\mathtt{FRD}} | X_1, \dots, X_n\right] = \mathrm{C}_{\mathtt{FRD}}$, where

$$\mathrm{C}_{\mathtt{FRD}} := \frac{1}{\tau_T\tau'_T}\mathrm{C}_{\mathtt{SRD}}(\Sigma_Y) - \frac{\tau_Y}{\tau_T^2\tau'_T}\mathrm{C}_{\mathtt{SRD}}(\Sigma_{Y,T}) - \frac{\tau'_Y}{\tau_T(\tau'_T)^2}\mathrm{C}_{\mathtt{SRD}}(\Sigma_{Y,T}) + \frac{\tau_Y\tau'_Y}{\tau_T^2(\tau'_T)^2}\mathrm{C}_{\mathtt{SRD}}(\Sigma_T),$$

where $\Sigma_{Y,T} := \mathrm{diag}(\mathbb{C}\mathrm{ov}\left[Y_i, T_i | X_i\right], \dots, \mathbb{C}\mathrm{ov}\left[Y_i, T_i | X_n\right])$ and Σ_T is defined analogously to Σ_Y. Then, a similar argument, combined with Theorem A.2 of Calonico et al. (2014), proves the joint asymptotic normality: $\Omega_f^{-1/2}\mathrm{diag}(1,h)\tilde{\Delta}_f(\tau_{\mathtt{FRD}}, \tau'_{\mathtt{FRD}}) \to_d \mathcal{N}((0,0)^\top, \mathrm{diag}(1,1))$, where

$$\tilde{\Delta}_f(t,t') := \begin{pmatrix} \tilde{\tau}_{\mathtt{FRD}} - t \\ \tilde{\tau}'_{\mathtt{FRD}} - t' \end{pmatrix}, \ \Omega_f := \begin{pmatrix} \mathrm{V}_{\mathtt{FRD}} & \mathrm{C}_{\mathtt{FRD}} \\ \mathrm{C}_{\mathtt{FRD}} & \mathrm{V}'_{\mathtt{FRD}} \end{pmatrix}.$$

Given this result, a similar confidence region to that in Proposition 1 is readily constructed. Instead of Assumption 3, we can assume the local linearity of $\mathbb{E}[Y_i(1) - Y_i(0) | X_i = x, T_i(1) > T_i(0)]$ in $x \in [-\delta_1, \delta_2]$, thereby obtaining analogous confidence bands to Proposition 2.