CUBIC FOURFOLDS WITH AN ORDER-7 AUTOMORPHISM

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ABSTRACT. We study smooth cubic fourfolds admitting an automorphism of order 7. It is known that the possible symplectic automorphism groups of such cubic fourfolds are precisely F_{21}, $\mathrm{PSL}(2,\mathbb{F}_7)$, and A_7. In this paper, we determine all possible full automorphism groups of smooth cubic fourfolds with an automorphism of order 7. We also investigate the moduli spaces of cubic fourfolds whose automorphism group is either F_{21} or $\mathrm{PSL}(2,\mathbb{F}_7)$, describing them both as GIT quotients and as locally symmetric varieties. In particular, we give an explicit description of the singular cubic fourfolds that appear in the boundary of the corresponding GIT quotients. For these two cases, we determine the commensurability classes of the monodromy groups by explicitly identifying certain arithmetic subgroups. As an interesting consequence, we prove that the period domain for cubic fourfolds equipped with an order-7 automorphism is isogenous to a Hilbert modular surface.

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Notation:

(1) For two groups G_1, G_2, we write $G_1 < G_2$ if G_1 is a subgroup of G_2.
(2) We use $L_2(7)$ to denote $\mathrm{PSL}_2(\mathbb{F}_7)$.
(3) For $\frac{1}{n}(m_1, \ldots, m_k)$, we mean the diagonal matrix $\mathrm{diag}(\zeta_n^{m_1}, \ldots, \zeta_n^{m_k})$, where $\zeta_n = \exp(\frac{2\pi\sqrt{-1}}{n})$ and $m_1, \ldots, m_k, n \in \mathbb{Z}$.
(4) For $g \in \mathrm{GL}(n, \mathbb{C})$, denote by $\overline{g}$ the image of g in $\mathrm{PGL}(n, \mathbb{C})$ under the natural map $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{PGL}(n, \mathbb{C})$ defined by quotient the center of $\mathrm{GL}(n, \mathbb{C})$.
(5) For permutation $\sigma \in S_n$, we also use σ to denote the matrix $A_\sigma \in \mathrm{GL}(n, \mathbb{C})$ such that $(e_{\sigma(1)}, .., e_{\sigma(n)}) = (e_1, .., e_n)A_\sigma$. For example, we use $(123) \in S_3$ to denote $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

1. Introduction

Cubic fourfolds play an important role in complex algebraic varieties thanks to their relation to hyperkähler geometry and rationality problems. What makes cubic fourfolds special is that their middle cohomology has Hodge numbers $0, 1, 21, 1, 0$, which is very similar to the middle cohomology of K3 surfaces. Therefore, the moduli space of cubic fourfolds shares very similar properties with the moduli space of K3 surfaces and they both have corresponding Torelli theorems. Historically, Voisin [Voi86, Voi08] proved a global torelli theorem for cubic fourfolds. An alternative proof was provided by Looijenga [Loo09]. Hassett [Has00] conjectured that the image of the period map is the complement is two specific Noether-Lefschetz divisors. This conjecture was independently confirmed by Looijenga [Loo09] and Laza [Laz10], who also identified the GIT compactification with the Looijenga compactification (with respect to one of the Noether-Lefschetz divisors) for the moduli space of cubic fourfolds.

In [YZ20], Yu–Zheng studied moduli space of cubic fourfolds with specified group action, and identified the GIT compactification of the moduli space to the Looijenga compactification of the corresponding arithmetic quotient of local period domain. In this paper, we follow the work of [YZ20], and focus on the study of smooth cubic fourfolds with an automorphism of order 7. According to [GAL11, Theorem 3.8] (see also [LZ22, Theorem 1.2 (7b)]), up to a linear coordinate transformation, any such cubic fourfold can be defined by an equation:

$$F_{a,b} = x_1^2x_2 + x_3^2x_4 + x_5^2x_6 + x_2^2x_3 + x_4^2x_5 + x_6^2x_1 + ax_1x_3x_5 + bx_2x_4x_6,$$

and the order-7 automorphism is given by $g_7 = \frac{1}{7}(1, 5, 4, 6, 2, 3)$. An automorphism of a smooth cubic fourfold is called symplectic if its action on $H^{3,1}$ is trivial. From the equation $F_{a,b}$, it is direct to see that any order-7 automorphism of a smooth cubic fourfold is symplectic.

In [LZ22], Laza–Zheng classified all symplectic automorphism groups of smooth cubic fourfolds. From [LZ22, Theorem 1.2, (7b), (8), (9)], we have

Theorem 1.1 (Laza–Zheng)**.** *If a smooth cubic fourfold X admits an automorphism of order 7, then $\mathrm{Aut}^s(X) = F_{21}, L_2(7)$ or A_7. Moreover, cubic fourfolds with $F_{21} < \mathrm{Aut}^s(X)$ form a 2-dimensional family. Those with $L_2(7) < \mathrm{Aut}^s(X)$ form a 1-dimensional subfamily of it and those with $A_7 < \mathrm{Aut}^s(X)$ are two isolated points.*

However, a full classification of automorphism groups of cubic fourfolds with order-7 actions has not been obtained yet. There is also a lack of detailed study of their moduli. In this paper, we give a comprehensive investigation of such cubic fourfolds, from both geometric and arithmetic perspectives.

Next we give the main theorem (see §4) of this paper. Note that we borrow many explicit equations from previous work including [GAL11], [Fu16], [HM19], [LZ22], [YYZ24], [Koi25].

Theorem 1.2. *If a smooth cubic fourfold X admits an automorphism of order 7, then $\mathrm{Aut}(X) = F_{21}, F_{21} \rtimes C_2, F_{21} \rtimes C_6, L_2(7), L_2(7) \rtimes C_2, A_7$ or S_7. More precisely, the pair $(\mathrm{Aut}^s(X), \mathrm{Aut}(X))$ equals to exactly one of the following cases:*

(1) (F_{21}, F_{21}). This is the generic case. And defining equation of X can be written as

$$x_1^2x_2 + x_3^2x_4 + x_5^2x_6 + x_2^2x_3 + x_4^2x_5 + x_6^2x_1 + ax_1x_3x_5 + bx_2x_4x_6,$$

with $a, b \in \mathbb{C}$ distinct.

(2) $(F_{21}, F_{21} \rtimes C_2)$. *This is a 1-dimensional family in* $\overline{\mathcal{F}}_{F_{21}}$. *In this case, defining equation of* X *can be written as*
$$x_1^2x_2 + x_3^2x_4 + x_5^2x_6 + x_2^2x_3 + x_4^2x_5 + x_6^2x_1 + a(x_1x_3x_5 + x_2x_4x_6),$$
where $a \in \mathbb{C}^\times$.

(3) $(F_{21}, F_{21} \rtimes C_6)$. *In this case,* X *is isomorphic to the Klein cubic fourfold, defined by the equation*
$$x_1^2x_2 + x_3^2x_4 + x_5^2x_6 + x_2^2x_3 + x_4^2x_5 + x_6^2x_1.$$
See also [YYZ24, Example 6.1 (9)].

(4) $(L_2(7), L_2(7))$. *It is the generic case for the family* $\overline{\mathcal{F}}_{L_2(7)}$, *which is a 1-dimensional subfamily in* $\overline{\mathcal{F}}_{F_{21}}$. *The defining equations are given by*
$$c_1(\sum_{i=1}^6 x_i^3 - (\sum_{i=1}^6 x_i)^3) + c_2(x_1x_2x_4 + x_2x_3x_5 + x_3x_4x_6 - (x_1x_3 + x_4x_5 + x_2x_6)(\sum_{i=1}^6 x_i)).$$

(5) $(L_2(7), L_2(7) \rtimes C_2)$. *There exists and only exists two* X *up to isomorphism. They are Galois conjugate to each other and the defining equations can be given by*
$$\sum_{i=1}^6 x_i^3 - (\sum_{i=1}^6 x_i)^3 \pm \frac{3\sqrt{2}}{2}(x_1x_2x_4 + x_2x_3x_5 + x_3x_4x_6 - (x_1x_3 + x_4x_5 + x_2x_6)(\sum_{i=1}^6 x_i)).$$
(*Equation of one of the examples for another coordinate was given by* [YYZ24, Example 6.1 (13)].)

(6) (A_7, A_7). *We only have 1 possibility for* X *up to isomorphism, explicit defining equations given by*
$$x_1^3 + x_2^3 + x_3^3 + \frac{12}{5}x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + x_3x_6^2 + \frac{4\sqrt{15}}{9}x_4x_5x_6.$$
This equation was first obtained by [YYZ24, Theorem 6.14]. *See* [Koi25, Page 16 (2.3)] *for an equation up to another coordinate.*

(7) (A_7, S_7). *We only have 1 possibility for* X *up to isomorphism, with defining equation*
$$\sum_{i=1}^6 x_i^3 - (\sum_{i=1}^6 x_i)^3.$$
see also [HM19, Table 11].

Along the proof of Theorem 1.2, we obtain a clear understanding of the stable and semistable locus, see Proposition 3.6 and Proposition 3.7 in §3.2.

We know from [YZ20] that there is an isomorphism $\mathcal{F} \cong (\mathbb{D}_T \setminus \mathcal{H}_s)/\Gamma_T$, where $\mathcal{F}$ is the moduli space of smooth cubic fourfolds with certain symmetries, $\mathbb{D}_T$ is the corresponding local period domain, $\mathcal{H}_s$ is a specific hyperplane arrangement in $\mathbb{D}_T$, and Γ_T is an arithmetic subgroup acting on $\mathbb{D}_T$. For details of these notations, see §2.

Using the a, b model $F_{a,b}$, we calculate the locus of singular cubic fourfolds and obtain the following results, see §3.3 and Proposition 3.18.

Proposition 1.3. *For cubic fourfolds inside the family* $F_{a,b}$, *there are three curves corresponding to singular cubic fourfolds and they only have isolated singularities of* A_1 *or* A_2

type. Moreover, there is an action of the cyclic group C_6 on the parameter space $\mathbb{A}^2$ for the space $F_{a,b}$ such that $C_6\backslash\mathbb{A}^2 \hookrightarrow \overline{\mathcal{F}}_{F_{21}}$ is a Zariski open subset. As a corollary, $\overline{\mathcal{F}}_{F_{21}}$ is a rational surface.

Proposition 1.4. *The moduli of smooth cubic fourfolds with $L_2(7) < \mathrm{Aut}^s(X)$, denoted by $\mathcal{F}_{L_2(7)}$, has compactification $\overline{\mathcal{F}}_{L_2(7)} \cong \mathbb{P}^1$. The boundary $\overline{\mathcal{F}}_{L_2(7)} \setminus \mathcal{F}_{L_2(7)}$ has exactly one determinantal cubic fourfold, one cubic fourfold with 7 nodes and one cubic fourfold with 14 nodes.*

Via period maps, the moduli spaces $\mathcal{F}_{F_{21}}$ and $\mathcal{F}_{L_2(7)}$ are Zariski open subsets of the corresponding arithmetic quotients of type IV domains. We investigate these two arithmetic quotients via tube domain models.

Let $\mathbb{D}_{T_1}$ be the local period domain of the $L_2(7)$-family and $\mathbb{D}_{T_2}$ be the local period domain of the F_{21}-family. Firstly, we compute the Gram matrices for T_1 and T_2 respectively; see §6.1 and §7.1.

The Gram matrix of T_1 is $\mathrm{diag}(\begin{bmatrix} -2 & 1 \\ 1 & 10 \end{bmatrix}, -28)$ up to an integral basis. By elementary number theory, one can check that this lattice has no nonzero isotropic vector. This implies the following:

Proposition 1.5 (= Proposition 6.11)**.** *The quotient curve $\Gamma_{T_1}\backslash\mathbb{D}_{T_1}$ is compact. In particular, it is not isogenous to $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$.*

The compactness of $\Gamma_{T_1}\backslash\mathbb{D}_{T_1}$ can also be deduced from Proposition 1.4 using period map, see Remark 6.12 for an argument.

In fact, we can determine what $\Gamma_{T_1}\backslash\mathbb{D}_{T_1}$ is isogenous to by computing Γ_{T_1} as follows. There is a natural surjection $\Phi_1\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}^+(1,2)$, see §6.3 for the detailed construction. Using this surjection, we have the following description of Γ_{T_1}:

Proposition 1.6 (= Proposition 6.5)**.** *Denote $\Gamma'' = \Phi_1^{-1}(\mathrm{O}^+(T_1) \cap \mathrm{SO}^+(1,2))$. We have $\Phi_1(\Gamma'') \cong \Gamma_{T_1}$ and the following description:*

$$\Gamma'' = \{ \begin{pmatrix} \frac{u\sqrt{u'}+v\sqrt{v'}}{2} & \frac{w\sqrt{w'}+x\sqrt{x'}}{2} \\ \frac{w\sqrt{w'}-x\sqrt{x'}}{2} & \frac{u\sqrt{u'}-v\sqrt{v'}}{2} \end{pmatrix} | u,v,w,x \in \mathbb{Z}, (u',v',w',x') = (1,21,6,14) \text{ or } (2,42,3,7)$$
$$\text{up to permutation by } \mathrm{V} < \mathrm{S}_4, \text{ and } u^2u' - v^2v' - w^2w' + x^2x' = 4\}$$

(Here we use V to denote the Klein 4-group generated by $(12)(34)$ and $(13)(24)$.)

We determine the type of Γ'' as an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$ up to commensurability and conjugacy and obtain the following (the notation follows [Mor15]):

Theorem 1.7 (= Theorem 6.13)**.** *$\Gamma_{T_1}\backslash\mathbb{D}_{T_1}$ is isogenous to $\mathrm{SL}(1,\mathbb{H}_{\mathbb{Z}}^{21,6})\backslash\mathbb{H}$.*

Similarly, we describe Γ_{T_2} up to a finite index. There is a natural surjection $\Phi_2\colon \mathrm{SL}(2,\mathbb{R})\times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}^+(2,2)$, see §6.3 for the detailed construction. Using this isomorphism, we have the following result:

Proposition 1.8 (= Proposition 7.4)**.** *We have that Γ_{T_2} is an index-2 subgroup of $\mathrm{O}^+(T_2)$ and $\mathrm{O}^+(T_2) = \langle -\mathrm{Id}, \Gamma_{T_2}\rangle$.*

On the other hand, $\mathrm{O}^+(T_2) = \langle \mathrm{SO}^+(T_2), P\rangle$, where P is an involution constructed in §7.1. Denote $\Gamma' = \Phi_2^{-1}(\mathrm{SO}^+(T_2))$, then we have $h \in \Gamma'$ if and only if

$$h = \left(\begin{pmatrix} \frac{\alpha_1\sqrt{\alpha}+\beta_1\sqrt{\beta}}{2} & \frac{1}{\sqrt{7}}\frac{\gamma_1\sqrt{\gamma}+\delta_1\sqrt{\delta}}{2} \\ \sqrt{7}\frac{\gamma_2\sqrt{\gamma}-\delta_2\sqrt{\delta}}{2} & \frac{\alpha_2\sqrt{\alpha}-\beta_2\sqrt{\beta}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\alpha_2\sqrt{\alpha}+\beta_2\sqrt{\beta}}{2} & \sqrt{7}\frac{\gamma_2\sqrt{\gamma}+\delta_2\sqrt{\delta}}{2} \\ \frac{1}{\sqrt{7}}\frac{\gamma_1\sqrt{\gamma}-\delta_1\sqrt{\delta}}{2} & \frac{\alpha_1\sqrt{\alpha}-\beta_1\sqrt{\beta}}{2} \end{pmatrix}\right)$$

where

- $(\alpha, \beta, \gamma, \delta) = (1, 21, 3, 7)$ *up to permutation by* $\mathrm{V} < \mathrm{S}_4$
- $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{Z}$ *such that* $\alpha_i - \beta_i, \gamma_i - \delta_i \in 2\mathbb{Z}$
- $\alpha_1\alpha_2\alpha - \beta_1\beta_2\beta - \gamma_1\gamma_2\gamma + \delta_1\delta_2\delta = 4$ *and* $\alpha_1\beta_2 - \alpha_2\beta_1 = \gamma_1\delta_2 - \gamma_2\delta_1$.

(The last condition is equivalent to saying that the determinants of h_1 and h_2 are 1.)

We will show that Γ' contains a Hilbert modular group as an index-4 subgroup, in particular, the modular surface admits a degree-4 covering from the Hilbert modular surface. This is stated as the following theorem.

Theorem 1.9. *There is a subgroup $H < \Gamma'$ such that $\Gamma'/H \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

There is a natural homomorphism $\psi\circ\varphi\colon \mathrm{SL}(2, F) \to \mathrm{SL}(2, \mathbb{R})\times\mathrm{SL}(2, \mathbb{R})$ mapping $\Gamma(\mathcal{O}_F \oplus \mathfrak{p})$ isomorphically onto H, where $F = \mathbb{Q}(\sqrt{21})$ and $\mathfrak{p} = (\frac{7+\sqrt{21}}{2})$, the prime ideal of $\mathcal{O}_F$ lying over 7.

In particular, there are natural morphisms

$$\Gamma(\mathcal{O}_F \oplus \mathfrak{p})\backslash\mathbb{H}^2 \to \Gamma'\backslash\mathbb{H}^2 \xrightarrow{\sim} \mathrm{SO}^+(T_2)\backslash\mathbb{D}_{T_2} \to \Gamma_{T_2}\backslash\mathbb{D}_{T_2} \cong \mathrm{O}^+(T_2)\backslash\mathbb{D}_{T_2}.$$

Each of them is of degree 2, except for the isomorphisms.

Remark 1.10. In [DFL18] and [FL21], the authors studied the geometry and monodromy group of the Wiman-Edge pencil. They proved that the monodromy group is commensurable to the Hilbert modular group $\mathrm{SL}(2, \mathbb{Z}[\sqrt{5}])$. In their case, the period map is defined via Jacobian of curves, and they used Picard–Lefschetz theory to calculate the monodromy group. In this paper (Theorem 1.9), by considering the Hodge structures on the middle cohomology of cubic fourfolds, we provide another monodromy group commensurable to Hilbert modular group.

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2. Preliminary: Cubic Fourfolds and Periods

2.1. **Periods for Cubic Fourfolds.** First we recall the global Torelli theorem for cubic fourfolds. It allows us to connect the GIT moduli space of cubic fourfolds to the moduli of their Hodge structures.

Let X be a smooth cubic fourfold, the middle cohomology $H^4(X,\mathbb{Z})$, with the natural intersection pairing, is the unimodular odd lattice Λ of signature $(21,2)$. Let $\eta_X \in H^4(X,\mathbb{Z})$ be the square of the hyperplane class of X, we can choose an isomorphism of lattices $\Lambda \cong \mathrm{E}_8^{\oplus 2} \oplus \mathrm{U}^{\oplus 2} \oplus \mathrm{I}_{3,0}$ such that $\eta_X = (1,1,1) \in I_{3,0}$. The primitive cohomology $H^4(X,\mathbb{Z})_{prim} = \langle \eta_X \rangle^{\perp}$ carries a Hodge structure of K3 type and let Λ_0 denote its lattice type, we have $\Lambda_0 \cong \mathrm{E}_8^{\oplus 2} \oplus \mathrm{U}^{\oplus 2} \oplus \mathrm{A}_2$. The local period domain for Hodge structures on Λ_0 is the 20-dimensional Type IV domain

$$\mathbb{D} = \{x \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) | (x,x) = 0, (x,\overline{x}) < 0\}^{+}$$

(where the subscript $+$ means taking one of the two connected components).

Let $\mathbb{P}(H^0(\mathbb{P}^5,\mathcal{O}(3)))^{sm}$ be the subset of $\mathbb{P}(H^0(\mathbb{P}^5,\mathcal{O}(3)))$ consisting of smooth cubic fourfolds. It is known that any isomorphism between two smooth cubic fourfolds can be lifted to a linear isomorphism. Therefore, the moduli of smooth cubic fourfolds is $\mathcal{M} = \mathrm{SL}(6,\mathbb{C})\backslash\mathbb{P}(H^0(\mathbb{P}^5,\mathcal{O}(3)))^{sm}$ with GIT compactification $\overline{\mathcal{M}} = \mathrm{SL}(6,\mathbb{C})\backslash\backslash\mathbb{P}(H^0(\mathbb{P}^5,\mathcal{O}(3)))$.

Let $\mathrm{O}(\Lambda_0)$ denote the automorphism group of the lattice Λ_0. For any $g \in \mathrm{O}(\Lambda_0)$, we use $\overline{g}$ to denote its image in the orthogonal group of the discriminant group $\mathrm{O}(A_{\Lambda_0})$. The global monodromy group for the family $\mathbb{P}(H^0(\mathbb{P}^5,\mathcal{O}(3)))^{sm}$ is (cf. [Bea86, Theorem 2])

$$\widehat{\Gamma} = \{g \in \mathrm{O}(\Lambda_0) | \overline{g} = Id_{A_{\Lambda_0}}, g(\mathbb{D}) = \mathbb{D}\}$$

Therefore, by associating a cubic fourfold with its Hodge structure, one obtains the global period map

$$\mathscr{P} \colon \mathcal{M} \to \widehat{\Gamma}\backslash\mathbb{D}$$

which is injective due to Voisin [Voi86, Voi08], see also Looijenga [Loo09]. Here the analytic variety $\widehat{\Gamma}\backslash\mathbb{D}$ is in fact a quasi-projective variety due to Baily-Borel compactification [BB66].

Definition 2.1.

- A norm 2 vector v in Λ_0 is called a short root. The orthogonal complements of each short root give a $\widehat{\Gamma}$-invariant hyperplane arrangements $\mathcal{H}_6$ in $\mathbb{D}$. The associated Heegner divisor is $\mathcal{C}_6 = \widehat{\Gamma}\backslash\mathcal{H}_6$.
- A norm 6 vector v in Λ_0 with divisibility 3 is called a long root. The orthogonal complements of each long root give a $\widehat{\Gamma}$-invariant hyperplane arrangements $\mathcal{H}_2$ in $\mathbb{D}$. The associated Heegner divisor is $\mathcal{C}_2 = \widehat{\Gamma}\backslash\mathcal{H}_2$.

Both $\mathcal{C}_6$ and $\mathcal{C}_2$ are irreducible divisors [Has00] in $\widehat{\Gamma}\backslash\mathbb{D}$. Let $\mathcal{M}^{\mathrm{ADE}}$ denote the moduli space of cubic fourfolds with at most ADE singularities, we state the following theorem due to [Loo09] and [Laz10].

Theorem 2.2. *The period map for cubic fourfolds gives an isomorphism of quasi-projective varieties*

$$\mathscr{P} \colon \mathcal{M} \to \widehat{\Gamma}\backslash(\mathbb{D} - (\mathcal{H}_2 \cup \mathcal{H}_6))$$

which extends to an isomorphism

$$\mathscr{P}\colon \mathcal{M}^{\mathrm{ADE}} \to \widehat{\Gamma}\backslash(\mathbb{D}-\mathcal{H}_2)$$

and further to an isomorphism

$$\mathscr{P}\colon \overline{\mathcal{M}} \to \overline{\widehat{\Gamma}\backslash\mathbb{D}}^{\mathcal{H}_2}$$

where $\overline{\widehat{\Gamma}\backslash\mathbb{D}}^{\mathcal{H}_2}$ *is the Looijenga compactification.*

2.2. **Moduli and Periods for Cubic Fourfolds with Symmetries.** Next we recall Yu–Zheng's work [YZ20] on moduli space of cubic fourfolds with a specified group action.

Let $\mathbb{C}^6$ be a complex vector space of dimension 6. The space $H^0(\mathbb{P}(\mathbb{C}^6), \mathcal{O}(3))$ is the space of degree 3 polynomials on $\mathbb{C}^6$ and we have a natural action of $\mathrm{SL}(6,\mathbb{C})$ on $H^0(\mathbb{P}(\mathbb{C}^6), \mathcal{O}(3))$. For each matrix $A \in \mathrm{SL}(6,\mathbb{C})$ and $F \in H^0(\mathbb{P}(\mathbb{C}^6), \mathcal{O}(3))$, we denote $(AF)(\vec{x}) = F(\vec{x}A)$.

The center of $\mathrm{SL}(6,\mathbb{C})$ is the group μ_6 consisting of 6-th roots of unity. Let $\widetilde{G}$ be a finite subgroup of $\mathrm{SL}(6,\mathbb{C})$ containing μ_6 and $G = \widetilde{G}/\mu_6$ its image in $\mathrm{PGL}(6,\mathbb{C})$. Let $\lambda\colon \widetilde{G} \to \mathbb{C}^\times$ be a character of $\widetilde{G}$ such that $\lambda|_{\mu_6}$ sends $\xi \in \mu_6$ to ξ^{-3} and let $\mathcal{V}_{\widetilde{G},\lambda}$ be the λ-eigenspace of $H^0(\mathbb{P}(\mathbb{C}^6), \mathcal{O}(3))$ with respect to G. Geometrically, $\mathcal{V}_{\widetilde{G},\lambda}$ contains some of the cubic fourfolds with automorphism group containing G. Now let

$$N = \{A \in \mathrm{SL}(6,\mathbb{C}) | A\widetilde{G}A^{-1} = \widetilde{G}, \lambda(AgA^{-1}) = \lambda(g), \forall g \in \widetilde{G}\}.$$

We know that N is a reductive subgroup of $\mathrm{SL}(6,\mathbb{C})$ and has an action on $\mathcal{V}_{\widetilde{G},\lambda}$. Let $\mathcal{V}^{sm}_{\widetilde{G},\lambda}$ be the subspace containing smooth cubic fourfolds and $\mathcal{V}^{ss}_{\widetilde{G},\lambda}$ the subspace containing semistable points with respect to the action of N. Respectively, we let $\mathcal{F} = N\backslash\backslash\mathcal{V}^{sm}_{\widetilde{G},\lambda}$ and $\overline{\mathcal{F}} = N\backslash\backslash\mathcal{V}^{ss}_{\widetilde{G},\lambda}$ be the corresponding GIT quotients. We borrow the following theorem form Yu–Zheng's work

Theorem 2.3. *There is a natural morphism* $j\colon \overline{\mathcal{F}} \to \overline{\mathcal{M}}$ *sending* $[F] \in \mathcal{F}$ *to* $[F] \in \mathcal{M}$ *for any* $F \in \mathcal{V}^{sm}_{\widetilde{G},\lambda}$*. This morphism is finite. Furthermore, if there is some point* $F \in \mathcal{V}^{sm}_{\widetilde{G},\lambda}$ *with automorphism group* $\mathrm{Aut}(Z(F)) = G$*, then this morphism is a normalization of its image. (Here we denote* $Z(F)$ *for its corresponding cubic fourfold).*

Now let $X = Z(F)$ be a generic point $F \in \mathcal{V}^{sm}_{\widetilde{G},\lambda}$, the action of G on X induces an action of G on $\Lambda = H^4(X,\mathbb{Z})$ and therefore on Λ_0 since it acts on η_X trivially. Meanwhile, G acts on $H^{3,1}(X)$ and this action is a character $\zeta\colon G \to \mathbb{C}^\times$. We only consider $\zeta = \overline{\zeta}$ here. Therefore, according to Yu–Zheng's work, we define $T = (\Lambda_0)_\zeta = \{x \in \Lambda_0 | gx = \zeta(g)x, \forall g \in G\}$ which is a lattice of signature $(n,2)$ with $n = \dim \overline{\mathcal{F}}$. And we define the local period domain $\mathbb{D}_T = \{x \in \mathbb{P}(T \otimes \mathbb{C}) | (x,x) = 0, (x,\overline{x}) < 0\}^+$. Let $\overline{G} < \widehat{\Gamma}$ denote the image of G in $\widehat{\Gamma}$, and let Γ_T be the normalizer of $\overline{G}$ in $\widehat{\Gamma}$. We borrow the following theorem from Yu–Zheng's work [YZ20]:

Theorem 2.4. *There is a global period map*

$$\mathscr{P}\colon \mathcal{F} \to \Gamma_T\backslash(\mathbb{D}_T - \mathcal{H}_s)$$

which is an algebraic isomorphism. Here $\mathcal{H}_s = \mathbb{D}_T \cap (\mathcal{H}_2 \cup \mathcal{H}_6)$*. Moreover, let* $\mathcal{F}^{\mathrm{ADE}}$ *denote the open subset parametrizing those with at most ADE singularities, then this period map* *extends to isomorphisms*

$$\mathscr{P}\colon \mathcal{F}^{\mathrm{ADE}} \to \Gamma_T\backslash(\mathbb{D}_T - \mathcal{H}_*)$$

$(\mathcal{H}_* = \mathbb{D}_T \cap \mathcal{H}_2)$ *and*

$$\mathscr{P}\colon \overline{\mathcal{F}} \to \overline{\Gamma_T\backslash\mathbb{D}_T}^{\mathcal{H}_*}$$

where $\overline{\Gamma_T\backslash\mathbb{D}_T}^{\mathcal{H}_*}$ *is the Looijenga compactification.*

3. GIT Analysis for Cubic Fourfolds Admitting Order-7 Action

In this section, we investigate the geometry of the GIT compactification of the moduli of cubic fourfolds with order-7 automorphism. We obtain a complete description of the singular cubic fourfolds inside it and their singularities.

3.1. **Some Preliminery Group-theoretic Analysis.**

By §2.1, to construct the GIT moduli of smooth cubic fourfolds with order-7 action, we need to construct $N\backslash\backslash\mathbb{P}\mathcal{V}_T$ for $T = (\widetilde{G}, \lambda)$. However, different projective representations of G and different characters λ may give different families.

The following lemma shows that there is no such ambiguity. More precisely, we only need to consider one projective representation for each $G = C_7, F_{21}$ or $L_2(7)$. In fact, we only need to deal with one linear lifting for this representation and the G-invariant cubic polynomials with respect to this linear lifting.

Fix generators g_3 and g_7 of F_{21} such that g_3 has order 3, g_7 has order 7 and $g_3g_7g_3^{-1} = g_7^2$. We identify C_7 with the subgroup $\langle g_7\rangle$ of F_{21}.

Consider 6-dimensional representations of C_7 and F_{21} determined by sending g_3 to $(e_i \mapsto e_{i+2})$, where $(e_1, e_2, \dots, e_6)$ represents the standard basis of $\mathbb{C}^6$, and sending g_7 to $\frac{1}{7}(1, 5, 4, 6, 2, 3)$.

Lemma 3.1. *For $G = C_7, F_{21}$ or $L_2(7)$, if there exists a smooth cubic fourfold X such that $G < \mathrm{Aut}^s(X)$ where G acts on X via some faithful projective representation $\rho\colon G \to$* $\mathrm{PGL}(6,\mathbb{C})$. *Then ρ is uniquely determined up to isomorphism and it can be lifted to a linear representation $\widetilde{\rho}\colon G \to$* $\mathrm{GL}(6,\mathbb{C})$.

For C_7 and F_{21}, $\widetilde{\rho}$ can be chosen as we describe above. For $L_2(7)$, see §3.4.

Moreover, for all smooth cubic fourfold X with $G < \mathrm{Aut}^s(X)$, there exists a coordinate change such that X is defined by a $\widetilde{\rho}(G)$-invariant polynomial.

Proof. (1) The case that $G = C_7$ directly follows from [Fu16, Theorem 1.1]. Any smooth cubic fourfold with C_7-action can be defined by the polynomial $x_1^2x_2 + x_3^2x_4 + x_5^2x_6 + x_2^2x_3 + x_4^2x_5 + x_6^2x_1 + ax_1x_3x_5 + bx_2x_4x_6$ with $a, b \in \mathbb{C}$ after a coordinate changing.

(2) For $G = F_{21}$, each projective representation ρ of G can be lifted to be a linear representation $\widetilde{\rho}$, see [Koi25, Page 4]. By [Fu16, Theorem 1.1] again, we may assume $\widetilde{\rho}$ is what we describe above. Such 6-dimensional representation of F_{21} is unique. We have $\det(\widetilde{\rho}(g_3)) = 1$. Suppose $\widetilde{\rho}(g_3)F = \lambda F$, then $\lambda^3 = 1$. By [Fu16, Lemma 3.2] or [YYZ24, Lemma 6.10], $1 = \det(\widetilde{\rho}(g_3)) = \lambda^2$. So $\lambda = 1$ and F is F_{21}-invariant for representation $\widetilde{\rho}$.

(3) For $G = L_2(7)$, if there is a smooth cubic fourfold X with $\mathrm{Aut}^s(X) \cong L_2(7)$ with $L_2(7)$ acting on X via some projective representation ρ, then ρ can be lifted to a linear representation $\widetilde{\rho}$, see [Koi25, Lemma 2.5]. By [Koi25, Page 5], let $g_4 \in L_2(7)$ be an order-4 element, then $\mathrm{Tr}(\widetilde{\rho}(g_4)) = 0$. There is only one 6-dimensional representation of $L_2(7)$

satisfying above condition, the irreducible one. Thus ρ is uniquely determined. And defining equation F of X must be $L_2(7)$-invariant for $\widetilde{\rho}$ since $L_2(7)$ is a simple group. $\square$

From now on, we denote $\mathcal{V}_{C_7}$ and $\mathcal{V}_{F_{21}}$ for C_7-invariant and F_{21}-invariant polynomials respectively.

By straightforward calculation we have

$$\begin{aligned}\mathcal{V}_{C_7} =&\mathrm{Span}_{\mathbb{C}}\{x_1^2x_2, x_3^2x_4, x_5^2x_6, x_2^2x_3, x_4^2x_5, x_6^2x_1, x_1x_3x_5, x_2x_4x_6\}\\ \mathcal{V}_{F_{21}} =&\mathrm{Span}_{\mathbb{C}}\{x_1^2x_2 + x_3^2x_4 + x_5^2x_6, x_2^2x_3 + x_4^2x_5 + x_6^2x_1, x_1x_3x_5, x_2x_4x_6\}\end{aligned}$$

Therefore, we define the following moduli spaces:

$$\begin{aligned}\mathcal{F}_{C_7} &= N_{\mathrm{PGL}(6,\mathbb{C})}(C_7)\backslash \mathbb{P}\mathcal{V}^{sm}_{C_7}\\ \mathcal{F}_{F_{21}} &= N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})\backslash \mathbb{P}\mathcal{V}^{sm}_{F_{21}}\end{aligned}$$

and their GIT compactifications $\overline{\mathcal{F}}_{C_7}$ and $\overline{\mathcal{F}}_{F_{21}}$ respectively.

Next we compute normalizer of C_7 and F_{21} in $\mathrm{PGL}(6,\mathbb{C})$.

Proposition 3.2. *The normalizer $N_{\mathrm{PGL}(6,\mathbb{C})}(C_7)$ of C_7 in $\mathrm{PGL}(6,\mathbb{C})$ as above is isomorphic to $(\mathbb{C}^*)^5 \rtimes C_6$.*

Here $(\mathbb{C}^*)^5$ is the subgroup of diagonal matrices in $\mathrm{PGL}(6,\mathbb{C})$. And C_6 is generated by the permutation matrix (123456).

Lemma 3.3. *For $g \in \mathrm{GL}(6,\mathbb{C})$, denote its image in $\mathrm{PGL}(6,\mathbb{C})$ by $\overline{g}$. If $\overline{gg_7g^{-1}} = \overline{g_7}$, then $gg_7g^{-1} = g_7$. In this case, g must be a diagonal matrix.*

Proof of Proposition 3.2. We have $gg_7g^{-1} = \lambda g_7$ for some $\lambda \in \mathbb{C}^\times$. Thus λg_7 is similar to g_7 and then $\lambda\zeta_7 = \zeta_7^k$ for some $k \in \mathbb{Z}$. Thus $\lambda = \zeta_7^{k-1}$. If $k \neq 1$, then λg_7 has eigenvalue 1, this is a contradiction since g_7 has no eigenvalue 1. So we have $gg_7g^{-1} = g_7$.

We have $gg_7 = g_7g$, hence g must be diagonal. $\square$

Now we can compute $N_{\mathrm{PGL}(6,\mathbb{C})}(C_7)$.

Proof. We consider the representation $\widetilde{\rho}$ of C_7 as above and consider the induced projective representation $\rho\colon C_7 \to \mathrm{PGL}(6,\mathbb{C})$. Consider an arbitrary $\overline{g} \in \mathrm{PGL}(6,\mathbb{C})$ such that $\overline{g}$ normalizes C_7 (here we denote C_7 by its image in $\mathrm{GL}(6,\mathbb{C})$ and $\mathrm{PGL}(6,\mathbb{C})$ again if there is no ambiguity), then $\overline{gg_7g^{-1}} = \overline{g_7}^k$ for some $k \in \mathbb{Z}$.

Let $g_\tau = (123456) \in \mathrm{GL}(6,\mathbb{C})$. Then g_τ stabilizes C_7 since $g_\tau g_7 g_\tau^{-1} = g_7^3$. Thus one can check $\overline{gg_7g^{-1}} = \overline{g_\tau^m g_7 g_\tau^{-m}}$ for some $m \in \mathbb{Z}/7\mathbb{Z}$ since 3 generates $(\mathbb{Z}/7\mathbb{Z})^*$. Thus after a left action of $\langle\overline{g_\tau}\rangle$, we may assume $\overline{gg_7g^{-1}} = \overline{g_7}$, that is $\overline{g_7}$ centralizes $\overline{g_7}$. Take a representative g of $\overline{g}$ in $\mathrm{GL}(6,\mathbb{C})$, then g is diagonal by Lemma 3.3.

In conclusion, for any element which normalizes C_7 in $\mathrm{PGL}(6,\mathbb{C})$, after a left multiplication of $\langle\overline{g_\tau}\rangle$, we may assume it is a centralizer of C_7. And it is easy to check that centralizer of C_7 is the group of diagonal matrices. We denote $(\mathbb{C}^\times)^5$ by all diagonal matrices in $\mathrm{PGL}(6,\mathbb{C})$. Conversely, the group generated by $(\mathbb{C}^\times)^5$ and $\langle\overline{g_\tau}\rangle$ inside $\mathrm{PGL}(6,\mathbb{C})$ is $(\mathbb{C}^\times)^5 \rtimes \langle\overline{g_\tau}\rangle \cong (\mathbb{C}^\times)^5 \rtimes C_6$ and it normalized C_7. Thus the normalizer of C_7 in $\mathrm{PGL}(6,\mathbb{C})$ is $(\mathbb{C}^\times)^5 \rtimes C_6$. $\square$

Lemma 3.4. *We have $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21}) < N_{\mathrm{PGL}(6,\mathbb{C})}(C_7)$, where $C_7 < F_{21}$ is the unique Sylow-7 subgroup.*

Proof. Note that $F_{21} \cong C_7 \rtimes C_3$, by Sylow's theorem, there is a unique order-7 subgroup, denoted by C_7. Any element in $\mathrm{PGL}(6,\mathbb{C})$ normalizing F_{21} must sending C_7 into another order-7 subgroup of F_{21}, which is forced to be C_7 itself. Thus we have $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21}) < N_{\mathrm{PGL}(6,\mathbb{C})}$, which is isomorphic to $(\mathbb{C}^\times)^5 \rtimes C_6$, see Lemma 3.2. $\square$

Proposition 3.5. *The normalizer $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$ of F_{21} in $\mathrm{PGL}(6,\mathbb{C})$ as above is isomorphic to $(\mathbb{C}^\times \times C_{21}) \rtimes C_6$.*

Here $\mathbb{C}^\times$ is the subgroup consisting of elements of the form $\mathrm{diag}(1,c,1,c,1,c)$, $c \in \mathbb{C}^\times$. The group C_{21} is generated by $\frac{1}{7}(1,5,4,6,2,3)$ and $\frac{1}{3}(0,0,1,1,2,2)$. And C_6 is generated by the permutation matrix (123456).

Proof. Now we consider the linear representation and induced projective representation of F_{21} as above.

Recall that $g_7 = \frac{1}{7}(1,5,4,6,2,3)$ and $g_3 = (135)(246)$.

Direct computation shows that $\overline{g_\tau} \in N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$, where $g_\tau = (123456)$. Take $\overline{g} \in N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$, then $\overline{g} \in N_{\mathrm{PGL}(6,\mathbb{C})}(C_7)$. Since we have $g_\tau g_7 g_\tau^{-1} = g_7^3$, after a left multiplication of $\langle \overline{g_\tau} \rangle$, we may assume $\overline{g}$ centralizes C_7. By Lemma 3.3, $\overline{g}$ is diagonal.

After all discussions above, the condition that $\overline{g}$ normalizes F_{21} is equivalent to that $\overline{g}$ conjugating g_3 to another element of order 3. All order-3 elements in F_{21} can be written as $g_7^m g_3^n$ for some $m \in \mathbb{Z}/7\mathbb{Z}$ and $n = 1$ or 2. And $g_7^m g_3^n = g_7^{m*(1-2n)^{-1}} g_3^n g_7^{-m*(1-2n)^{-1}}$. So after further a left multiplication of C_7 (which does not change the property of centralizing C_7), we may further assume that $\overline{g}$ conjugates $\overline{g_3}$ to $\overline{g_3}$ or $\overline{g_3}^2$. The latter case can never happen since $\overline{g}$ is diagonal. So $\overline{g}$ centralizes F_{21}.

Take a representative g of $\overline{g}$ in $\mathrm{GL}(6,\mathbb{C})$, then g is diagonal and $gg_3g^{-1} = \lambda g_3$ with $\lambda \in \mathbb{C}^\times$. Since λg_3 is similar to g_3, λ is 1, ω or ω^2, where $\omega = \exp(\frac{2\pi\sqrt{-1}}{3})$. Let $k := \frac{1}{3}(0,0,1,1,2,2)$. Note that $\omega^n g_3 = k^n g_3 k^{-n}$, $n = 0,1,2$ and that $gg_3g^{-1} = g_3$ with g diagonal is equivalent to that $g = \mathrm{diag}(a,b,a,b,a,b)$ for some $a,b \in \mathbb{C}^\times$. We know that $gg_3g^{-1} = \omega^n g_3$ is equivalent to that $g = k^n \mathrm{diag}(a,b,a,b,a,b)$ for $n = 0,1,2$ and $a,b \in \mathbb{C}^\times$.

In conclusion, after a left multiplication of $\langle \overline{g_\tau} \rangle$, C_7 and $\langle \overline{k} \rangle$, we may assume $\overline{g} \in N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$ is of the form $\overline{\mathrm{diag}(a,b,a,b,a,b)} = \overline{\mathrm{diag}(1,c,1,c,1,c)}$ with $n = 0,1,2$ and $a,b,c = \frac{b}{a} \in \mathbb{C}^\times$. So $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21}) < \langle \mathbb{C}^\times, \overline{g_7}, \overline{k}, \overline{g_\tau} \rangle \cong (\mathbb{C}^\times \times C_{21}) \rtimes C_6$. Conversely, it is clear that $(\mathbb{C}^\times \times C_{21}) \rtimes C_6 < N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$. So the conclusion follows. $\square$

3.2. **GIT Moduli of Cubic Fourfolds with F_{21}-Action.** By definition, there is a surjective morphism $\mathrm{SL}(6,\mathbb{C}) \to \mathrm{PGL}(6,\mathbb{C})$ and the normalizer of $G_1 = \mu_6 C_7$ or $G_2 = \mu_6 F_{21}$ in $\mathrm{SL}(6,\mathbb{C})$ is just the inverse image of the normalizer of C_7 or F_{21} in $\mathrm{PGL}(6,\mathbb{C})$.

Using these, we analyze the GIT stability and form the following

Proposition 3.6. *For $F = a_1x_1^2x_2 + a_2x_2^2x_3 + a_3x_3^2x_4 + a_4x_4^2x_5 + a_5x_5^2x_6 + a_6x_6^2x_1 + a_7x_1x_3x_5 + a_8x_2x_4x_6$, it is $N_{\mathrm{SL}(6,\mathbb{C})}(G_1)$-semistable if and only if one of the following holds:*

(1) $a_7, a_8 \neq 0$ *(2)* $a_2, a_4, a_6, a_7 \neq 0$ *(3)* $a_1, a_3, a_5, a_8 \neq 0$ *(4)* $a_1, a_2, .., a_6 \neq 0$.

It is $N_{\mathrm{SL}(6,\mathbb{C})}(G_1)$-stable if and only if $a_1, a_2, .., a_6 \neq 0$.

Proof. The connected component of $N_{\mathrm{SL}(6,\mathbb{C})}(G_1)$ containing the identity is just the subgroup of $\mathrm{SL}(6,\mathbb{C})$ containing all diagonal matrixes. Therefore, nontrivial 1-parameter subgroups of

$N_{\mathrm{SL}(6,\mathbb{C})}(G_1)$ are just $\lambda\colon \mathbb{C}^\times \to N_{\mathrm{SL}(6,\mathbb{C})}(G_1)$ with $\lambda(t) = \mathrm{diag}(t^{r_1}, \cdots, t^{r_6})$ such that $r_i \in \mathbb{Z}$, $\vec{r} \neq \vec{0}$ and $\sum_{i=1}^{6} r_i = 0$.

Let $\omega_{a_1} = 2r_1 + r_2, \cdots, \omega_{a_6} = 2r_6 + r_1, \omega_{a_7} = r_1 + r_3 + r_5, \omega_{a_8} = r_2 + r_4 + r_6$ and let $\omega_{\vec{r}} = \min\{\omega_{a_i} | a_i \neq 0\}$. By Hilbert-Mumford criterion [MFK94, Theorem 2.1], F is not semi-stable (resp. stable) if and only if there is some $\vec{r} \neq \vec{0}$, such that $\omega_{\vec{r}} > 0$(resp. $\omega_{\vec{r}} \geq 0$).

For sufficiency, first assume $a_7, a_8 \neq 0$, then there is no $\vec{r}$ such that $\omega_{a_7}, \omega_{a_8} > 0$ otherwise $\omega_{a_7} + \omega_{a_8} = \sum_{i=1}^{6} r_i > 0$. The others are similar. For necessity, suppose none of four conditions hold. Up to symmetry, we may assume $a_1, a_7 = 0$, take $\vec{r} = (-25, -1, 3, 1, -1, 23)$, we would have $\min\{\omega_{a_i} | i = 2, 3, 4, 5, 6, 8\} > 0$ and therefore F is not semistable. For stability, just assume $a_1 = 0$ and take $\vec{r} = (-8, -5, 10, 1, -2, 4)$ and repeat the same argument. $\square$

Proposition 3.7. *For $F = c_1(x_1^2x_2 + x_3^2x_4 + x_5^2x_6) + c_2(x_2^2x_3 + x_4^2x_5 + x_6^2x_1) + c_3x_1x_3x_5 + c_4x_2x_4x_6$, it is $N_{\mathrm{SL}(6,\mathbb{C})}(G_2)$-stable if and only if one of the following holds:*

(1) $c_1, c_2 \neq 0$ (2) $c_1, c_4 \neq 0$ (3) $c_2, c_3 \neq 0$ (4) $c_3, c_4 \neq 0$.

And F is $N_{\mathrm{SL}(6,\mathbb{C})}(G_2)$-semistable implies it is $N_{\mathrm{SL}(6,\mathbb{C})}(G_2)$-stable. In particular, the unstable locus is two projective lines $c_1 = c_3 = 0$ and $c_2 = c_4 = 0$.

Proof. The connected component of $N_{\mathrm{SL}(6,\mathbb{C})}(G_2)$ containing the identity is just

$$\mathbb{C}^\times = \mathrm{diag}(a, a^{-1}, a, a^{-1}, a, a^{-1}) < \mathrm{SL}(6, \mathbb{C}).$$

Therefore, all non-trivial 1-parameter subgroups are just

$$\lambda(t) = \mathrm{diag}(t^n, t^{-n}, t^n, t^{-n}, t^n, t^{-n}), n \in \mathbb{Z}/\{0\}.$$

Also by Hilbert-Mumford criterion, we obtain the conclusion. $\square$

3.3. **Singular Cubic Fourfolds with Order-7 Action.** In this section, we study singular cubic fourfolds which admit an order-7 action.

Proposition 3.8. *For a cubic fourfold X with an order-7 action, we write the defining equation of X as $F = a_1x_1^2x_2 + a_2x_2^2x_3 + a_3x_3^2x_4 + a_4x_4^2x_5 + a_5x_5^2x_6 + a_6x_6^2x_1 + a_7x_1x_3x_5 + a_8x_2x_4x_6$. If one of $a_i = 0$, $i = 1, \ldots, 6$, then X admits singularity which is not of type ADE.*

Proof. By [Laz09, Theorem 5.6], if X has singularity at worst of type ADE, then X is stable in the moduli of all cubic fourfolds. In particular, the stabilizer in $\mathrm{PGL}(6, \mathbb{C})$ of X is finite and $\mathrm{PGL}(6, \mathbb{C})$-orbit of X is closed in vector space of all cubic fourfolds. By Luna [Lun75, §3, Corollary 1], $N_{\mathrm{PGL}(6,\mathbb{C})}(C_7)$-orbit of X is closed. This forces X to be stable in $\mathcal{V}_{C_7}$, which contradicts Proposition 3.6. $\square$

Direct computation shows that:

Proposition 3.9. *For a cubic fourfold X admitting symplectic F_{21}-action, write defining equation of X as $F = c_1(x_1^2x_2 + x_3^2x_4 + x_5^2x_6) + c_2(x_2^2x_3 + x_4^2x_5 + x_6^2x_1) + c_3x_1x_3x_5 + c_4x_2x_4x_6$. If c_1 or c_2 equals 0, then singular locus of X has dimension at least 1.*

Thus for the remaining cases, we may assume X is defined by the equation $x_1^2x_2 + x_3^2x_4 + x_5^2x_6 + x_2^2x_3 + x_4^2x_5 + x_6^2x_1 + ax_1x_3x_5 + bx_2x_4x_6$ after suitable action of $N_{\mathrm{PGL}(6,\mathbb{C})}(C_7)$.

Lemma 3.10. *Suppose $F(x_1, \cdots, x_6) = x_1^2x_2 + x_2^2x_3 + \cdots + x_5^2x_6 + x_6^2x_1 + ax_1x_3x_5 + bx_2x_4x_6$ has a singularity $[t_1 : \cdots : t_6]$, then $t_i \neq 0$ for any i. Therefore, we can assume $t_1 = 1$ and set $t = t_2$. Then there exists some $0 \leq k \leq 2$ and $0 \leq l \leq 6$ such that the following equations hold*

$$\begin{cases} a + 2\zeta_7^l t + \omega^k(\zeta_7^l t)^2 = 0 \\ b + 2\omega^k(\zeta_7^l t)^{-1} + (\zeta_7^l t)^{-2} = 0 \end{cases}$$

(here $\zeta_7 = e^{\frac{2\pi i}{7}}$ and $\omega = e^{\frac{2\pi i}{3}}$)

And $[1 : t_2 : \cdots : t_6] = [1 : t : \omega^k\zeta_7^l : t\omega^k\zeta_7^{-2l} : \omega^{-k}\zeta_7^{-2l} : t\omega^{-k}\zeta_7^{-3l}]$. The converse also holds.

Proof. A singularity $[x_1 : x_2 : \cdots : x_6]$ in $Z(F)$ satisfies the following equations:

$$\begin{cases} x_6^2 + 2x_1x_2 + ax_3x_5 = 0, \quad x_2^2 + 2x_3x_4 + ax_5x_1 = 0, \quad x_4^2 + 2x_5x_6 + ax_1x_3 = 0 \\ x_1^2 + 2x_2x_3 + bx_4x_6 = 0, \quad x_3^2 + 2x_4x_5 + bx_6x_2 = 0, \quad x_5^2 + 2x_6x_1 + bx_2x_4 = 0 \end{cases}$$

If one of x_i is zero, we shall prove that all x_i are zero. Without loss of generality, we assume $x_1 = 0$. Then equations reduce to following:

$$\begin{cases} x_6^2 + ax_3x_5 = 0, \quad x_2^2 + 2x_3x_4 = 0, \quad x_4^2 + 2x_5x_6 = 0 \\ 2x_2x_3 + bx_4x_6 = 0, \quad x_3^2 + 2x_4x_5 + bx_6x_2 = 0, \quad x_5^2 + bx_2x_4 = 0 \end{cases}$$

If any of $x_2, \cdots, x_6$ is zero, then all $x_2, \cdots, x_6$ would be forced to be zero. Thus from now on, we only consider cases where $x_2, \cdots, x_6$ are all nonzero.

Then we actually just need to consider the 5 equations, from the second to the sixth. They gives

$$\begin{cases} -b = \frac{2x_2x_3}{x_4x_6} = \frac{x_3^2 + 2x_4x_5}{x_6x_2} = \frac{x_5^2}{x_2x_4} \\ x_2^2x_3 + 2x_3^2x_4 = 0 \\ x_4^2x_5 + 2x_5^2x_6 = 0 \end{cases}$$

Then the first equation reduced to

$$2x_2^2x_3 = x_3^2x_4 + 2x_4^2x_5 = x_5^2x_6$$

Let $e_i = x_i^2x_{i+1}$ we have the following:

$$\begin{cases} 2e_2 = e_3 + 2e_4 = e_5 \\ e_2 + 2e_3 = 0 \\ e_4 + 2e_5 = 0 \end{cases}$$

Plug the latter two equations into the first one we obtain that $-4e_3 = e_3 + 2e_4 = -\frac{1}{2}e_4$, we obtain $e_2, \cdots, e_5$ should be all zero.

Now, if none of the x_i is zero. Still denote $e_i = x_i^2x_{i+1}$, then multiply the six equations each with $x_1, x_2, x_3, x_4, x_5, x_6$ in order, the equations can be reduced to following:

$$\begin{cases} e_6 + 2e_1 + ax_1x_3x_5 = 0, \quad e_2 + 2e_3 + ax_1x_3x_5 = 0, \quad e_4 + 2e_5 + ax_1x_3x_5 = 0 \\ e_1 + 2e_2 + bx_2x_4x_6 = 0, \quad e_3 + 2e_4 + bx_2x_4x_6 = 0, \quad e_5 + 2e_6 + bx_2x_4x_6 = 0 \end{cases}$$

Hence we have

$$\begin{cases} e_6 + 2e_1 = e_2 + 2e_3 = e_4 + 2e_5 \\ e_1 + 2e_2 = e_3 + 2e_4 = e_5 + 2e_6 \end{cases}$$

Solving these linear equations of e_i, we have all $e_1 = e_3 = e_5$, and $e_2 = e_4 = e_6$. Then it's not hard to find out that $[x_1, x_2, \dots, x_6]$ can be written as the form in the proposition. Plug the results into the equations, we have singularities exists if and only if there exists some $0 \le k \le 2$ and $0 \le l \le 6$ such that the following equations hold

$$\begin{cases} a + 2\zeta_7^l t + \omega^k (\zeta_7^l t)^2 = 0 \\ b + 2\omega^k (\zeta_7^l t)^{-1} + (\zeta_7^l t)^{-2} = 0 \end{cases}$$

□

Proposition 3.11. *For a cubic fourfold $X = Z(F)$ with the following defining equation:*

$$x_1^2 x_2 + x_3^2 x_4 + x_5^2 x_6 + x_2^2 x_3 + x_4^2 x_5 + x_6^2 x_1 + a x_1 x_3 x_5 + b x_2 x_4 x_6$$

it is singular if and only if it lies in one of the following 3 curves inside $\mathbb{C}^2$: $Z(G)$, $\omega Z(G)$ and $\omega^2 Z(G)$ where $G = a^2b^2 - 6ab + 4a + 4b - 3$ and $\omega = e^{\frac{2\pi i}{3}}$. The cubic fourfold has exactly 7 isolated singularities if it lies on exactly one of the curves.

Proof. The equations about a, b in the last proposition can be rewritten as follow. Replace $\zeta_7 t$ by t, we have singularities exists if and only if there exists k and t s.t.

$$\begin{cases} a + 2t + \omega^k t^2 = 0 \\ b + 2\omega^k t^{-1} + t^{-2} = 0 \end{cases} \tag{3.1}$$

This gives 3 curves in $\mathbb{C}^2$. Let C be the curve where $k = 1$, then one sees that ωC, $\omega^2 C$ give exactly another 2 curves.

Then we only need to find the defining equations of C.

For $(a, b) \in C$, there exists $t \in \mathbb{C}$ s.t. $a + 2t + t^2 = 0$ and $b + 2t^{-1} + t^{-2} = 0$. Then $(t+1)^2 = 1 - a$, and $\frac{(t+1)^2}{t^2} = 1 - b$. If one of a, b is 1, then both a, b is 1. When $a, b \neq 1$, then $t^2 = \frac{1-a}{1-b}$. Then $a + 2t + t^2 = 0$ implies $4t^2 = (a + t^2)^2$, hence $4\frac{1-a}{1-b} = (a + \frac{1-a}{1-b})^2$. We have

$$4(1-a)(1-b) = (a - ab + 1 - a)^2$$

This can be reduced to

$$a^2b^2 - 6ab + 4a + 4b - 3 = 0$$

Notice that $a, b = 1$ is a solution of this equation. One can varifies for any a, b satisfying this equation, either $a, b = 1$, or $a, b \neq 1$. When $a, b = 1$, then $t = -1$ satisfies our need, thus $(1, 1) \in C$. When $a, b \neq 1$, let $t = -\frac{1}{2}\frac{1-ab}{1-b}$. Then $t^2 = \frac{1}{4}\frac{(1-ab)^2}{(1-b)^2} = \frac{1-a}{1-b}$, thus $a + 2t + t^2 = a - \frac{1-ab}{1-b} + \frac{1-a}{1-b} = 0$, and $b + 2\frac{1}{t} + \frac{1}{t^2} = b - 1 + \frac{(1+t)^2}{t^2} = b - 1 + (1-a)\frac{1-b}{1-a} = 0$. Hence the curve C has defining equation $G = a^2b^2 - 6ab + 4a + 4b - 3 = 0$. □

Proposition 3.12. *For singular cubic fourfolds $X = Z(F)$ where F is of the form*

$$F_{a,b} = x_1^2 x_2 + x_3^2 x_4 + x_5^2 x_6 + x_2^2 x_3 + x_4^2 x_5 + x_6^2 x_1 + a x_1 x_3 x_5 + b x_2 x_4 x_6,$$

generically, they only admits A_1 *singularities. The only exceptions are* $(a,b)=(1,1),(\omega,\omega)$ *or* (ω^2,ω^2)*, and they lie in the same orbit. The cubic fourfold* $Z(F_{1,1})$ *admits exactly* 7 *singularities of type* A_2*.*

Proof. Suppose $F_{a,b}$ has a singularity, by Lemma 3.10, we may assume the singularity is $[1:t:\omega^k\zeta_7^l:t\omega^k\zeta_7^{-2l}:\omega^{-k}\zeta_7^{-2l}:t\omega^{-k}\zeta_7^{-3l}]$, where t satisfies the equation in Lemma 3.10. For simplicity, let $c=\omega^k\zeta_7^l$, then the singularity is $[1:t:c:c^{19}t:c^5:c^{11}t]$. Meanwhile, $a=-2c^{15}t-c^{16}t^2, b=-2c^{13}t^{-1}-c^{12}t^{-2}$.

Consider the affine equation $F(1,x_2,..,x_6)$ and compute its Hessian matrix multiplied by $\frac{1}{2}$ at the point $(t,c,c^{19}t,c^5,c^{11}t)$, and compute its determinant, we obtain $-\frac{147}{16}c^{14}t(ct+1)$. Therefore, if it is a singularity not of type A_1, then we would have $t=-c^{-1}$. Put it back into the expressions of a,b, we get three possibilities $(a,b)=(1,1),(\omega,\omega)$ or (ω^2,ω^2). From the discussion above, these three cubic fourfolds lie in the same orbit and we only need to further analyze the singular locus of F with $(a,b)=(1,1)$.

Note that when $(a,b)=(1,1)$, it has 7 singularities of the form $[1:-\zeta_7^{6l}:\zeta_7^l:-\zeta_7^{4l}:\zeta_7^{5l}:-\zeta_7^{3l}]$ with $0\le l\le 6$. Since the action of $G_1=C_7$ acts transitively on these 7 singularities, we only need to consider $[1:-1:1:-1:1:-1]$. Consider the affine equation $F(1,x_2,..,x_6)$, and translate the point $(-1,1,-1,1,-1)$ to the origin, we obtain $f(x_2,..,x_6)=F(1,x_2-1,x_3+1,x_4-1,x_5+1,x_6-1)$. Take the matrix

$$\widehat{S}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 3/4 & 1/4 & 1 & 0 & 0\\ 1 & 1/2 & 1 & 1 & 0\\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

We obtain $\widehat{S}\cdot f=x_2^2-2x_3^2+\frac{7}{8}x_4^2-\frac{7}{4}x_5^2+x_6^3+\sum_{i=2}^{5}x_ig_i(x_2,..,x_6)$, $g_i\in\mathfrak{m}^2$ where $\mathfrak{m}=(x_2,...,x_6)$ is a maximal ideal of $\mathbb{C}[x_2,..,x_6]$.

We can further make coordinate changes and assume that $\widehat{S}\cdot f=x_2^2+x_3^2+x_4^2+x_5^2+x_6^3+\sum_{i=2}^{5}x_ig_i(x_2,..,x_6)$. And then make coordinate changes $x_i\mapsto x_i-\frac{1}{2}g_i$ to obtain $x_2^2+x_3^2+x_4^2+x_5^2+x_6^3+a_4x_6^4+\sum_{i=2}^{5}x_ih_i(x_2,..,x_6)$ with $h_i\in\mathfrak{m}^3$. Repeat in this way, we obtain $x_2^2+x_3^2+x_4^2+x_5^2+ux_6^3+g$ where u is a unit in $\mathbb{C}[[x_6]]$ and $g\in\mathfrak{m}^n$ for n sufficiently high. Using the finite determinacy theorem, see for example [GLS07], we know that it has the same singularity type as $x_2^2+x_3^2+x_4^2+x_5^2+ux_6^3$ and therefore $x_2^2+x_3^2+x_4^2+x_5^2+x_6^3$, which is an A_2 singularity. □

Note that $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$-orbit of generic $F_{a,b}$ intersects the (a,b)-plane at $F_{a,b}$, $F_{b,a}$, $F_{\omega a,\omega b}$, $F_{\omega b,\omega a}$, $F_{\omega^2 a,\omega^2 b}$, $F_{\omega^2 b,\omega^2 a}$, thus in the finite quotient $C_6\backslash\mathbb{C}^2$, the 3 curves $Z(G),\omega Z(G),\omega^2 Z(G)$ becomes $Z(G)$ quotient by a single relation $(a,b)\sim(b,a)$, i.e. a curve isomorphic to $Z(x^2-6x+4y-3=0)$ where $x=ab, y=a+b$. We have the following proposition.

Proposition 3.13. $\mathcal{F}_{F_{21}}^{\mathrm{ADE}}-\mathcal{F}_{F_{21}}\cong\Gamma_T\backslash(\mathcal{H}_*-\mathcal{H}_s)$ *is an irreducible curve.*

For a cubic fourfold X with exactly one node and a G-action, one can relate X to a degree-6 K3 surface S with G-action. Conversely, any degree-6 K3 surface S with G-action gives a one dimensional family of cubic fourfolds with G-action and the central fiber is a cubic fourfold with a node, see [LZ22, §5.5, Page 1493, Page 1494].

In our case, for a cubic fourfold X with an order-7 action admitting some nodes, one can similarly relates a surface by simulating the above construction. Specifically, we have the following conclusion:

Proposition 3.14. *For each cubic fourfold X with an order-7 action and with 7 nodes as in Proposition 3.11, around each node one can construct a degree 6 surface S with at least 6 singular points and with a $C_3 \cong F_{21}/C_7$-action.*

The C_3 action permutes these 6 singular points and they forms 2 C_3-orbits. And these 7 surfaces are isomorphic to each other with isomorphisms induced from the order-7 action.

Proof. For such a cubic fourfold $X = Z(F)$, we can make a coordinate change such that $F = x_6 q(x_1, .., x_5) + c(x_1, .., x_5)$ with q, c being degree 2 and 3 homogeneous polynomials respectively. Inside $\mathbb{P}^5$, we can form a complete intersection $S = \{q = c = 0\}$ and this surface parametrizes all lines through the node $p = [0 : 0 : 0 : 0 : 0 : 1]$ of $Z(F)$.

It is a classical construction that, if $[0 : \cdots : 0 : 1]$ is the only singularity of X and it is a node, then $\{q = c = 0\}$ would a K3 surface. However, here we have got six other nodes $p_1, .., p_6$. By Bézout's theorem, the line $\overline{pp_i}$ should lie in X, otherwise they should intersect X with multiplicity 3 but here p, p_i are two nodes and therefore with multiplicity at least 4.

Note that each of these lines $\overline{pp_i}$ corresponds to a singularity of S. Since p_i is a singularity of F, we have $\frac{\partial F}{\partial x_j}(p_i) = x_6 \frac{\partial q}{\partial x_j} + \frac{\partial c}{\partial x_j} = 0$ for $j = 1, .., 5$. Therefore, the Jacobian matrix $\frac{\partial(q,c)}{\partial(x_1,..,x_5)}$ has rank ≤ 1 at these lines and therefore they are singularities. The rest of the proposition is a direct application of Lemma 3.10. $\square$

Remark 3.15. As these surfaces are singular, the above proposition does not conflict with [LZ22, Theorem 5.4].

3.4. **GIT Moduli of Cubic Fourfolds with $L_2(7)$-Action.** In this section, we compute the moduli of cubic fourfold with $L_2(7)$-action.

It is a well-known fact that $L_2(7)$ can be realized as a subgroup of S_7 generated by permutations (12)(36) and (1234567). We denote them by g_2 and g_7 respectively.

Fix a $\mathbb{C}^7$ with natural basis $\{e_1, \ldots, e_7\}$. We identify $\mathrm{GL}(6, \mathbb{C})$ with subgroup of $\mathrm{GL}(7, \mathbb{C})$ fixing the subspace $\mathrm{Span}_{\mathbb{C}}\{6e_1 - e_2 - \cdots - e_7, 6e_2 - e_1 - e_3 - \cdots - e_7, \ldots, 6e_6 - e_1 - \cdots - e_5 - e_7\}$ and the element $e_1 + \cdots + e_7$. Explicitly, there is a homomorphism $\Psi\colon\ \mathrm{GL}(6, \mathbb{C}) \longrightarrow\ \ \mathrm{GL}(7, \mathbb{C})$

sending A to T $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} T^{-1}$, where

$$T = \begin{pmatrix} 6 & -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 6 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & 6 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 6 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 6 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 6 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

On the other hand, we consider the representation $\widetilde{\rho} : L_2(7) \to \mathrm{GL}(7, \mathbb{C})$ determined $g_2 \mapsto (12)(36) \in \mathrm{GL}(7, \mathbb{C})$ and $g_7 \mapsto (1234567) \in \mathrm{GL}(7, \mathbb{C})$.

We find that $\widetilde{\rho}$ factor through the subgroup $\Psi\colon \mathrm{GL}(6, \mathbb{C}) \hookrightarrow \mathrm{GL}(7, \mathbb{C})$ by $\widetilde{\rho} = \Psi \circ \rho$. Here ρ is the unique irreducible 6-dimensional representation of $L_2(7)$. We identify g_2 and g_7 with their image described as above.

Proposition 3.16. $N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7)) = L_2(7) \rtimes C_2 \cong \mathrm{PGL}(2, \mathbb{F}_7)$.

Proof. We have the following exact sequence:

$$1 \to C_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7)) \to N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7)) \to \mathrm{Aut}(L_2(7)),$$

where $C_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7))$ is the centralizer and $\mathrm{Aut}(L_2(7))$ is the automorphism group of $L_2(7)$.

We claim that $C_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7))$ is trivial and the rightmost arrow is surjective. So, $N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7)) \cong \mathrm{Aut}(L_2(7)) \cong L_2(7) \rtimes C_2 \cong \mathrm{PGL}(2, \mathbb{F}_7)$, see [Ste60, §3.4].

Let $\overline{g} \in C_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7))$, take a lifting $g \in \mathrm{GL}(6, \mathbb{C})$. Then $gg_2g^{-1} = \lambda g_2$ and $gg_7g^{-1} = \mu g_7$ for some $\lambda, \mu \in \mathbb{C}^\times$. By comparing eigenvalues of λg_2, μg_7 with g_2 and g_7 respectively, we know $\lambda = \mu = 1$ and $g \in C_{\mathrm{GL}(6,\mathbb{C})}(L_2(7))$. So g is an automorphism of $\mathbb{C}^6$ as an irreducible $L_2(7)$-module, see Lemma 3.1. By Schur's lemma, g is a scalar and thus $\overline{g}$ is trivial.

For surjectivity, consider

$$E = (243756) * (a(\mathrm{Id} + g_7^4 + g_7^6) + b(g_7 + g_7^2 + g_7^3 + g_7^5)),$$

where $a = \frac{2\sqrt{2}}{7} + \frac{1}{7}$ and $b = \frac{-3\sqrt{2}}{14} + \frac{1}{7}$. Direct computation shows that $E \in \mathrm{GL}(6, \mathbb{C}) < \mathrm{GL}(7, \mathbb{C})$ described above, $E^2 = (162)(457) \in L_2(7)$ and $\overline{E} \in N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7))$. And $\overline{E} \notin L_2(7)$ since E is not a permutation matrix. $\square$

From [Koi25, §7.2], we know that cubic fourfolds with faithful $L_2(7)$-action are given as

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 0, \\ c_1(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3) \\ \; + c_2(x_1x_2x_4 + x_2x_3x_5 + x_3x_4x_6 + x_1x_5x_6 + x_1x_3x_7 + x_4x_5x_7 + x_2x_6x_7) = 0, \end{cases}$$

and they are invariant under the action of the representation $\widetilde{\rho}\colon L_2(7) \to \mathrm{GL}(7, \mathbb{C})$ above.

Let $F_1 = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3$ and $F_2 = x_1x_2x_4 + x_2x_3x_5 + x_3x_4x_6 + x_1x_5x_6 + x_1x_3x_7 + x_4x_5x_7 + x_2x_6x_7$, then $f_1 = F_1((x_1, ..., x_6, 0)7 * T^{-1})$ and $f_2 = F_2((x_1, ..., x_6, 0)7 * T^{-1})$ are invariant polynomials over $\mathbb{C}^6$ under the action of the representation $\rho\colon L_2(7) \to \mathrm{SL}(6, \mathbb{C})$

above. Note that

$$7 * T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore, f_1 and f_2 are obtained from replacing x_7 by $-(x_1 + \cdots + x_6)$ in F_1 and F_2 respectively.

From [LZ22, Theorem 1.2] and [YZ20, Proposition 4.10], we know that the family of cubic fourfolds with $L_2(7)$-action form a one-dimensional family. Since $N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7))$ is 0-dimensional and f_1, f_2 are clearly linearly independent, we know that the space $\mathcal{V}_{L_2(7)}$ is just the subspace generated by f_1, f_2.

Now let $E' = \Psi^{-1}(E)$, we have $E'(f_1) = \frac{3\sqrt{2}}{2} f_2$, $E'(f_2) = \frac{\sqrt{2}}{3} f_1$, this makes the action of $N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7))/L_2(7) \cong C_2$ on $\mathbb{P}(\mathcal{V}_{L_2(7)})$ clear. Therefore, we have the following proposition.

Proposition 3.17. *Denote $\mathcal{F}_{L_2(7)}$ by $N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7))\backslash \mathbb{P}\mathcal{V}^{sm}_{L_2(7)}$, and $\overline{\mathcal{F}}_{L_2(7)}$ by its GIT compactification, then we have $\overline{\mathcal{F}}_{L_2(7)} \cong C_2\backslash\mathbb{P}^1 \cong \mathbb{P}^1$.*

By intersecting the family $\mathbb{P}\mathcal{V}_{L_2(7)} \cong \mathbb{P}^1$ with the singular locus described in Proposition 3.11 in $\mathbb{P}\mathcal{F}_{G_1}$, we can characterize the singular cubic fourfolds with $L_2(7)$-action.

Proposition 3.18. *Inside the $\mathbb{P}\mathcal{V}_{L_2(7)} \cong \mathbb{P}^1$, there are exactly 6 points corresponding to 6 singular cubic fourfolds with $L_2(7)$-action. Two of them have 7 nodes, another two have 14 nodes, and remaining two cubic fourfolds have singular locus as a smooth quadratic surface, and they are known as determinantal cubic fourfolds. Each pair of cubic fourfolds as above contributes 1 point in $\overline{\mathcal{F}}_3$. The equations for those six singular cubic fourfolds are given in Table 1.*

Proof. Since we have already made clear the singular cubic fourfolds in the F_{21}-family, we make some coordinate change to let the $\mathcal{V}_{L_2(7)}$ lie in $\mathcal{V}_{F_{21}}$.

Explicitly, we first find some copy of F_{21} inside $\widetilde{\rho}(L_2(7))$, which is generated by $g_7'' = (1234567)$ and $g_3'' = (235)(476)$. Also, let $g_7' = \Psi^{-1}(g_7'')$ and $g_3' = \Psi^{-1}(g_3'')$. To conjugate this F_{21} generated by g_7' and g_3' to the one we are already familiar with, which is the one generated by $g_3 \colon x_i \mapsto x_{i+2}$ and $g_7 = \frac{1}{7}(1, 5, 4, 6, 2, 3)$, we need to find some isomorphism of representation.

Here we use a trick and let $S = \sum_{i=0}^{2}\sum_{j=0}^{6}(g_3^i g_7^j)^{-1}(g_3')^i(g_7')^j$. Therefore, we would have $Sg_7'S^{-1} = g_7$ and $Sg_3'S^{-1} = g_3$. The invariant polynomials with respect to $S\rho(L_2(7))S^{-1}$ would be linear combinations of $S \cdot f_1$ and $S \cdot f_2$, generating an $\mathbb{P}^1$ inside $\mathbb{P}(\mathcal{V}_{F_{21}})$.

Recall that cubic fourfolds with non-isolated singular locus in $\mathbb{P}(\mathcal{V}_{F_{21}})$ give two projective planes, and the $\mathbb{P}^1$ above intersects them in exactly 2 points. It is easy to check that they are in the same orbit as the determinantal cubic fourfolds.

Meanwhile, it can be checked that $\frac{1}{21}S\cdot f_1+\frac{t}{7}S\cdot f_2$ and $[1:1:2-\frac{7ct}{(t+c)^2}:2+\frac{7(\frac{1}{2}+c)t}{(t-\frac{1}{2}-c)^2}]$ in $\mathbb{P}(\mathcal{V}_{F_{21}})$ are in the same orbit, where $c=-\frac{1}{2}\zeta_7^4-\frac{1}{2}\zeta_7^2-\frac{1}{2}\zeta_7-\frac{1}{2}$ (For example, using the action of $\mathrm{diag}(1,s,1,s,1,s)$). Using Proposition 3.11, we obtain explicit equations of singular cubic fourfolds in the $L_2(7)$-family, see Table 1. Note that two equations in each row can be permuted to each other using the action of E' above. $\square$

Equations		Singular Type
$f_1+3\frac{1+\sqrt{-7}}{4}f_2$	$f_1+3\frac{1-\sqrt{-7}}{4}f_2$	Determinantal cubic fourfolds
$f_1-\frac{3}{10}f_2$	f_1-15f_2	7 nodes
$f_1-\frac{3}{2}f_2$	f_1-3f_2	14 nodes

Table 1: Singular cubic fourfolds inside $\mathbb{P}(\mathcal{V}_{L_2(7)})$

4. Full Automorphism Groups of Cubic Fourfolds Admitting Order-7 Action

In this section we determine all possible full automorphism groups of cubic fourfolds admitting an order-7 action. Precisely, we prove Theorem 1.2.

For any smooth cubic fourfold X, the symplectic automorphism group $\mathrm{Aut}^s(X)$ is a normal subgroup of the automorphism group $\mathrm{Aut}(X)$. Meanwhile, there is a natural embedding $\mathrm{Aut}(X)\to \mathrm{PGL}(6,\mathbb{C})$ according to [MM63]. So any element of $\mathrm{Aut}(X)$ lies in the normalizer of $\mathrm{Aut}^s(X)$ in $\mathrm{PGL}(6,\mathbb{C})$. Therefore we have the following lemma:

Lemma 4.1. *For $X\in\mathbb{P}\mathcal{V}_G$ a smooth cubic fourfold with $\mathrm{Aut}^s(X)=G$, we have $\mathrm{Aut}(X)=\mathrm{Stab}_{N_{\mathrm{PGL}(6,\mathbb{C})}(G)}(X)$. Here $\mathrm{Stab}_{N_{\mathrm{PGL}(6,\mathbb{C})}(G)}(X)$ is the stabilizer of X under the action of $N_{\mathrm{PGL}(6,\mathbb{C})}(G)$.*

4.1. **Case F_{21}.** For F_{21} case, recall that $\mathbb{P}\mathcal{V}_{F_{21}}\cong\mathbb{P}^3$. We will use $[a,b,c,d]\in\mathbb{P}^3$ to represent a cubic fourfold $X\in\mathbb{P}\mathcal{V}_{F_{21}}$ with defining equation $a(x_1^2x_2+x_3^2x_4+x_5^2x_6)+b(x_2^2x_3+x_4^2x_5+x_6^2x_1)+cx_1x_3x_5+dx_2x_4x_6$.

The action of $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$ on $\mathbb{P}\mathcal{V}_{F_{21}}$ with respect to the coordinates described as above determines a projective representation $R\colon N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})\to\mathrm{PGL}(4,\mathbb{C})$.

Directly compute all possible stabilizers of $X\in\mathbb{P}\mathcal{V}_{F_{21}}$ for the action of $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$. Orbit-stabilizer theorem shows that orbits which contain some smooth cubic fourfolds are disjoint union of some affine curves. More precisely, we will obtain the following list for such orbits:

cubic fourfold	stabilizer	number of orbit components	orbit degree
$[1,a,b,c]$, $a,b,c\neq 0$, $c\neq a^3b$	F_{21}	6	3
$[1,a,0,b]$ or $[1,a,b,0]$, $a,b,c\neq 0$	F_{21}	6	2
$[1,a,b,a^3b]$, $a,b\neq 0$	$F_{21}\rtimes C_2$	3	3
$[1,a,0,0]$, $a\neq 0$	$F_{21}\rtimes C_6$	1	1

Table 2: Stabilizers and Orbits

For smooth cubic fourfold $X = [1, a, b, c]$ with $a \neq 0$ and $c \neq a^3 b$, generically X is smooth with $\mathrm{Aut}^s(X) = F_{21}$. In this case, $\mathrm{Aut}(X) = F_{21}$ by Lemma 4.1.

For cubic fourfold $X = [1, a, 0, 0]$, X is isomorphic to the Klein fourfold $[1, 1, 0, 0]$. We have X is smooth and $\mathrm{Aut}^s(X) = F_{21}$. So $\mathrm{Aut}(X) = F_{21} \rtimes C_6$.

We claim that for a generic $X = [1, a, b, a^3 b]$ with $a, b \neq 0$, we have X is smooth and $\mathrm{Aut}^s(X) = F_{21}$. Thus $\mathrm{Aut}(X) = F_{21} \rtimes C_2$ once $\mathrm{Aut}^s(X) = F_{21}$.

Consider the sub-family $\mathbb{A}^2 \cong \{[1, a, b, a^3 b]\} \subset \mathbb{P}\mathcal{V}_{F_{21}}$. Note that $[1, 1, 0, 0] \in \mathbb{A}^2$ and $X = [1, 1, 0, 0]$ is smooth and $\mathrm{Aut}^s(X) = F_{21}$. The claim follows since smooth cubic fourfolds $X \in \mathbb{P}\mathcal{V}_{F_{21}}$ with $\mathrm{Aut}^s(X) = F_{21}$ forms a dense subset containing an open subset of $\mathbb{P}\mathcal{V}_{F_{21}}$.

Combining Lemma 3.1, we have actually taken into consideration all smooth cubic fourfold X with $\mathrm{Aut}^s(X) = F_{21}$.

4.2. **Case $L_2(7)$ and A_7.** For $G = L_2(7)$, recall that $\mathbb{P}\mathcal{V}_{L_2(7)} \cong \mathbb{P}^1$. We use $[a, b]$ to represent a cubic fourfold $X \in \mathbb{P}\mathcal{V}_{L_2(7)}$ with defining equation $af_1 + bf_2$, see §3.4 for expression of f_1 and f_2.

Note that $N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7)) = L_2(7) \rtimes C_2$ acts on $\mathbb{P}\mathcal{V}_{L_2(7)}$ factor through the quotient C_2. And C_2 acts on $\mathbb{P}\mathcal{V}_{L_2(7)}$ by sending $[a, b]$ to $[\frac{3\sqrt{2}}{2}b, \frac{\sqrt{2}}{3}a]$.

So for $X \neq [1, \frac{3}{2}\sqrt{2}], [1, -\frac{3}{2}\sqrt{2}]$ with X smooth and $\mathrm{Aut}^s(X) = L_2(7)$, we have $\mathrm{Aut}(X) = L_2(7)$.

For $X = [1, \frac{3}{2}\sqrt{2}]$ or $[1, -\frac{3}{2}\sqrt{2}]$, by proposition 3.18, we have X is smooth. We know $\mathrm{Aut}^s(X) \neq A_7$. Otherwise, by [LZ22, Theorem 1.8 (2)], $L_2(7) \rtimes C_2 < A_7$ or S_7. However, $N_{S_7}(L_2(7)) = N_{A_7}(L_2(7)) = L_2(7)$. So $\mathrm{Aut}^s(X) = L_2(7)$. By Lemma 4.1 again, $\mathrm{Aut}(X) = L_2(7) \rtimes C_2$.

Denote them by X_1 and X_2 respectively. Above argument shows that they are Galois conjugate to each other with $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}/\mathbb{Q}))$. The following proposition shows that they are not isomorphic to each other.

Proposition 4.2. *Let $X_1 \neq X_2 \in \mathbb{P}\mathcal{V}_{L_2(7)}$ with automorphism group $L_2(7) \rtimes C_2$. Then X_1 is not isomorphic to X_2.*

Proof. We first prove $N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7) \rtimes C_2) = N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7))$. Take $\overline{g} \in N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7) \rtimes C_2)$. Denote $\overline{g} L_2(7) \overline{g}^{-1}$ by G. Then G is a normal subgroup of $L_2(7) \rtimes C_2$ and thus $G \cap L_2(7)$ is a normal subgroup of $L_2(7)$. Since $L_2(7)$ is a simple group, $G \cap L_2(7)$ must be $L_2(7)$ itself or $\{\mathrm{id}\}$. On the other hand, $G/G \cap L_2(7) = 1$ or C_2, which forces $G = L_2(7)$.

Recall that $\mathrm{Aut}(X_1)$ and $\mathrm{Aut}(X_2)$ are exactly the same subgroup of $\mathrm{PGL}(6, \mathbb{C})$, isomorphic to $L_2 \rtimes C_2$. If $X_1 \cong X_2$, then one can find $\overline{g} \in \mathrm{PGL}(6, \mathbb{C})$ such that $\overline{g} X_1 = X_2$ by [MM63]. Then $\overline{g}\mathrm{Aut}(X_1)\overline{g}^{-1} < \mathrm{Aut}(X_2)$. So $\overline{g} \in N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7) \rtimes C_2) = L_2(7) \rtimes C_2$. We know that $\overline{g} \in L_2(7) \rtimes C_2 = \mathrm{Aut}(X_1)$. So $X_1 = X_2$, this is a contradiction. $\square$

Remark 4.3. When embedding X_1 and X_2 into $\mathbb{P}\mathcal{V}_{F_{21}}$, they are of the type $[1, a, b, a^3 b]$ since there is an extra order-2 element in $N_{\mathrm{PGL}(6,\mathbb{C})}(L_2(7)) = \mathrm{Aut}(X_i)$ normalizing F_{21}, $i = 1, 2$.

Act them to the plane $\mathbb{A}^2 = \{[1, 1, a, b]\} \subset \mathbb{P}\mathcal{V}_{F_{21}}$, then they are in the diagonal.

Remark 4.4. We have $E' \cdot F_{X_1} = F_{X_1}$ and $E' \cdot F_{X_2} = -F_{X_2}$ where F_{X_i} is the defining equation of X_i, $i = 1, 2$.

So X_1 and X_2 are corresponds to different symmetric type for cubic fourfolds with $L_2(7) \rtimes C_2$-action, see [YZ20, §2.2].

For $G = A_7$, [LZ22, Theorem 1.8 (2)] shows that there are 2 smooth cubic fourfolds with symplectic automorphism group equals A_7. By [LZ22, Theorem 1.8 (2)], X defined by f_1 is smooth, satisfying $\mathrm{Aut}^s(X) = A_7$ and $\mathrm{Aut}(X) = S_7$. And [YYZ24, Theorem 6.14], [Koi25, Page 16 (2.3)] give explicit expressions of the other smooth X with $\mathrm{Aut}^s(X) = \mathrm{Aut}(X) = A_7$.

In conclusion, we have proved Theorem 1.2.

5. More Geometric Consequences

In this section, we apply the previous explicit analysis to obtain more geometric consequences.

5.1. Moduli Spaces $\mathcal{F}_{C_7}$ and $\mathcal{F}_{F_{21}}$.

We first show that $\overline{\mathcal{F}}_{F_{21}}$ is a rational surface. From the proof, we can also determine $\mathcal{F}_{F_{21}}^{\mathrm{ADE}}$.

Proposition 5.1. *There is a C_6 action on $\mathbb{A}^2$ which gives an open embedding $C_6\backslash\mathbb{A}^2 \hookrightarrow \overline{\mathcal{F}}_{F_{21}}$, which identifies $C_6\backslash\mathbb{A}^2$ with $\mathcal{F}_{F_{21}}^{\mathrm{ADE}}$.*

Moreover, $C_6\backslash\mathbb{A}^2$ is rational and therefore $\overline{\mathcal{F}}_{F_{21}}$ is rational.

Proof. Consider $\mathbb{A}^2 = \{[1, 1, a, b]|\ a, b \in \mathbb{C}\} \subset \mathbb{P}\mathcal{V}_{F_{21}}$. Consider that C_6 acts on $\mathbb{A}^2 = \{[1, 1, a, b]|\ a, b \in \mathbb{C}\ \}$ by sending $[1, 1, a, b]$ to $[1, 1, \omega b, \omega a]$. Computing the intersection of $\mathbb{A}^2$ with $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$-orbits inside $\mathbb{P}(\mathcal{V}_{F_{21}})$, we can show that the natural embedding $\mathbb{A}^2 \to \mathbb{P}\mathcal{V}_{F_{21}}$ induces an injective morphism

$$C_6\backslash\mathbb{A}^2 \to \overline{\mathcal{F}}_{F_{21}}$$

which maps onto an open subvariety. Since $\overline{\mathcal{F}}_{F_{21}}$ is normal, so is this open subvariety. Therefore, applying Zariski's main theorem, we know that $C_6\backslash\mathbb{A}^2$ is isomorphic to its image and therefore $C_6\backslash\mathbb{A}^2 \hookrightarrow \overline{\mathcal{F}}_{F_{21}}$ is an open embedding.

The property that $\mathcal{F}_{F_{21}}^{\mathrm{ADE}} \cong C_6\backslash\mathbb{A}^2$ is a direct result from Proposition 3.8 and Proposition 3.12.

Therefore we only need to show that $C_6\backslash\mathbb{A}^2$ is rational. By computing invariants, we have

$$C_6\backslash\mathbb{A}^2 \cong \operatorname{Spec}\mathbb{C}[(\alpha+\beta)^3, (\alpha\beta)^3, \alpha\beta(\alpha+\beta)] \cong \operatorname{Spec}\mathbb{C}[x, y, z]/(xy - z^3).$$

It is easy to show that this affine cubic surface is rational by considering projection to the first and third coordinates. $\square$

Moreover, we have the following proposition, showing that it is equivalent to study $\overline{\mathcal{F}}_{C_7}$ and $\overline{\mathcal{F}}_{F_{21}}$.

Proposition 5.2. *We have $\overline{\mathcal{F}}_{C_7} \cong \overline{\mathcal{F}}_{F_{21}}$ as algebraic varieties.*

Proof. Using Theorem 2.3, we know that there is a natural finite morphism $\overline{\mathcal{F}}_{F_{21}} \to \overline{\mathcal{M}}$ which is in fact a normalization of its image since generically, cubic fourfolds with F_{21}-action has exactly automorphism group F_{21}.

Meanwhile, there is a natural morphism $\mathbb{P}(\mathcal{V}_{F_{21}}) \to \mathbb{P}(\mathcal{V}_{C_7})$ sending $[a:b:c:d]$ to $[a:a:a:b:b:b:c:d]$ with respect to the basis given at the beginning. This morphism restricts to the semistable locus and gives $\mathbb{P}(\mathcal{V}_{F_{21}}^{ss}) \to \mathbb{P}(\mathcal{V}_{C_7}^{ss}) \to N_{\mathrm{SL}(6,\mathbb{C})}(C_7)\backslash\backslash\mathbb{P}(\mathcal{V}_{C_7}^{ss})$. Note that since C_7 is the only sylow-7 subgroup of F_{21}, we have $N_{\mathrm{SL}(6,\mathbb{C})}(F_{21}) < N_{\mathrm{SL}(6,\mathbb{C})}(C_7)$. Therefore, the above morphism is equivariant under the action of $N_{\mathrm{SL}(6,\mathbb{C})}(F_{21})$ and descends to a morphism $\overline{\mathcal{F}}_{F_{21}} \to \overline{\mathcal{F}}_{C_7}$. This morphism forces the natural morphism $\overline{\mathcal{F}}_{C_7} \to \overline{\mathcal{M}}$ to be generically 1 to 1 and hence a normalization of its image.

On the other hand, it is easy to check that each element in $\mathbb{P}(\mathcal{V}_{C_7})$ can become of the form $[a:a:a:b:b:b:c:d]$ after some action from $N_{\mathrm{SL}(6,\mathbb{C})}(C_7)$. Therefore, $\overline{\mathcal{F}}_{C_7} \to \overline{\mathcal{M}}$ is mapped into the image of $\overline{\mathcal{F}}_{F_{21}} \to \overline{\mathcal{M}}$. Using the universal property of normalization, we obtain a natural morphism $\overline{\mathcal{F}}_{C_7} \to \overline{\mathcal{F}}_{F_{21}}$.

The two morphisms $\overline{\mathcal{F}}_{C_7} \to \overline{\mathcal{F}}_{F_{21}}$ and $\overline{\mathcal{F}}_{F_{21}} \to \overline{\mathcal{F}}_{C_7}$ are clearly birational morphisms. It is also clear that they are inverse to each other and we are done. $\square$

5.2. **Relation to Degree-Two K3.** As [LZ22, Theorem 1.2 (8)] mentioned, there is a relation between cubic fourfold with $L_2(7)$ action with degree-2 K3-surface. We give a more precise description in this section.

By proposition 3.18, there exist exactly 2 determinantal cubic fourfolds in family $\mathbb{P}\mathcal{V}_{L_2(7)}$. We denote them by X_1 and X_2 respectively. The singularity locus of each of X_i with $i = 1, 2$ is a quadratic smooth rational surface V_i with $i = 1, 2$, which is given by Veronese embedding from $\mathbb{P}^2$ to $\mathbb{P}^5$. And V_i naturally inherent the $L_2(7)-$action from X_i, $i = 1, 2$. For a generic cubic fourfold X with a G-action with $C_7 < G < L_2(7)$, the intersection of X with each of V_i gives a sextic curve $C_{i,X}$ admitting a G-action.

There is a sextic curve C with $G < \mathrm{PGL}(3,\mathbb{C})-$action is related to a degree-2 K3-surface S with a G-action. Here S is constructed by considering degree-2 covering of $\mathbb{P}^2$ branched over C. Explicitly, suppose C has the defining equation $f(z_1, z_2, z_3)$, a homogeneous polynomial with degree 6. Then $S \subset \mathbb{P}(1, 1, 1, 3)$ (the weighted projective space) is defined by the equation $y^2 = f(z_1, z_2, z_3)$, with G acting on the coordinates z_1, z_2, z_3 the same as G acts on $\mathbb{P}^2$ and G acting on y trivially.

For $G = L_2(7)$, there is explicit expression of sextic curve with $L_2(7)$-action in [Smi07, Page 16] and [BB25, Proposition 8.1]. We provide an alternative approach with more details as following:

Proposition 5.3. *For any cubic fourfold $X \in \mathbb{P}\mathcal{V}_{L_2(7)}$ with $X \neq X_2$, $X \cap V_1$ is the same smooth sextic curve C_1. Thus the above construction relates the whole family $\mathbb{P}\mathcal{V}_{L_2(7)} \setminus \{X_2\}$ to one degree-2 K3 surface S with an $L_2(7)$-action. Symmetrically, we have the same result for X_2 and V_2.*

Proof. We have singular locus V_1 of X_1 is $L_2(7)-$invariant. So $X \cap V_1$ has a $L_2(7)-$action for $X \in \mathbb{P}\mathcal{V}_{L_2(7)}$. Combining argument above, we only need to check $X \cap V_1$ is a smooth sextic curve.

We have a determinantal cubic fourfold $X_1' = [1, 0, -2, -1] \in \mathbb{P}\mathcal{V}_{F_{21}}$. The singular locus of X_1' is a surface $V_1' = \{[z_2z_3, z_1^2, z_1z_3, z_2^2, z_1z_3, z_3^2] |\ [z_1, z_2, z_3] \in \mathbb{P}^1\ \} \cong \mathbb{P}^2$. Let $s = -2\zeta_7^5 - 2\zeta_7^3 - \zeta_7 - 1$, where $\zeta_7 = \exp(\frac{2\pi\sqrt{-1}}{7})$. There is

$$g = \mathrm{diag}(1, s^{-1}, 1, s^{-1}, 1, s^{-1}) \in N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$$

such that $gX_1 = X_1'$. Then singular locus V_1 of X_1 is $g^{-1}V_1'$. For any $X \in \mathbb{P}\mathcal{V}_{L_2(7)}$, $X \cap V_1 = g^{-1}(gX \cap V_1')$. So we only need to compute $gX \cap V_1'$.

For X defined by $F(x_1, \ldots, x_6)$, $X \cap V_1' \subset \mathbb{P}^2$ is defined by $F(z_2z_3, z_1^2, z_1z_3, z_2^2, z_1z_3, z_3^2)$.

Direct computation shows that, up to scalar in $\mathbb{C}^\times$,

$$gSf_1(z_2z_3, z_1^2, z_1z_3, z_2^2, z_1z_3, z_3^2) = gSf_2(z_2z_3, z_1^2, z_1z_3, z_2^2, z_1z_3, z_3^2) = z_1^5z_3+z_2^5z_1+z_3^5z_2-5z_1^2z_2^2z_3^2.$$

So $gSf_1 \cap V_1' = gSf_2 \cap V_1' = \{[z_1, z_2, z_3]|\ z_1^5z_3 + z_2^5z_1 + z_3^5z_2 - 5z_1^2z_2^2z_3^2 = 0\}$, which is a smooth sextic curve in $\mathbb{P}^2$. For $X \in \mathbb{P}\mathcal{V}_{L_2(7)}$ defined by $aSf_1 + bSf_2$ for some $a, b \in \mathbb{C}$, $gX \cap V_1'$ is defined by $(agSf_1 + bgSf_2)(z_2z_3, z_1^2, z_1z_3, z_2^2, z_1z_3, z_3^2)$, giving either the same sextic curve as above or the whole $\mathbb{P}^2$. The latter case can happen only when $gX = X_1'$, which is equivalent to $X = X_1$. $\square$

5.3. **Boundary of $\overline{\mathcal{F}}_{F_{21}}$.** Now we study the boundary of the moduli space $\overline{\mathcal{F}}_{F_{21}} \setminus \mathcal{F}_{F_{21}}$. We have obtained the following stratification:

$$\overline{\mathcal{F}}_{F_{21}} \supset \mathcal{F}_{F_{21}}^{\mathrm{ADE}} = \mathbb{A}^2/C_6 \supset \mathcal{F}_{F_{21}}.$$

Recall that $\mathcal{F}_{F_{21}}^{\mathrm{ADE}} - \mathcal{F}_{F_{21}}$ is an irreducible curve described in Proposition 3.13. So, it remains to determine $\overline{\mathcal{F}}_{F_{21}} - \mathcal{F}_{F_{21}}^{\mathrm{ADE}}$.

By Proposition 3.7 and Proposition 5.1, we know that a cubic fourfold in $\overline{\mathcal{F}}_{F_{21}} - \mathcal{F}_{F_{21}}^{\mathrm{ADE}}$ can be written as the following types: $[1, 0, a, b]$, $[0, 1, b, a]$ and $[0, 0, 1, b]$ with $b \neq 0$.

Up to an $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$-action, the remaining types are $[1, 0, t, -1]$ and $[0, 0, 1, 1]$ with $t \in \mathbb{C}$. Moreover, two different cubic fourfolds $[1, 0, t, -1]$ and $[1, 0, t', -1]$ are in the same $N_{\mathrm{PGL}(6,\mathbb{C})}(F_{21})$-orbit if and only if $t = -t'$.

So $\overline{\mathcal{F}}_{F_{21}} - \mathcal{F}_{F_{21}}^{\mathrm{ADE}}$ consists of two parts: image of $C_2 \backslash \mathbb{A}^1$ induced by the natural injection $t \mapsto [1, 0, t, -1]$, which is a rational irreducible curve, and a point represented by $[0, 0, 1, 1]$.

In conclusion, we obtain the following description of the boundary of $\overline{\mathcal{F}}_{F_{21}}$.

Proposition 5.4. *The boundary $\overline{\mathcal{F}}_{F_{21}} - \mathcal{F}_{F_{21}}$ of moduli space of cubic fourfold with F_{21}-action consists of the following 3 parts: 2 irreducible curves and 1 point.*

One of the irreducible curves consists of singular cubic fourfolds with singularities of ADE type and the other irreducible curve consists of cubic fourfolds of type $[1, 0, t, -1]$. The point is given by the cubic fourfold $[0, 0, 1, 1]$.

From [YZ20], we have the following commutative diagram with horizontal morphisms being isomorphisms. The complements of vertical left embeddings were described in Proposition 5.4.

$$\begin{tikzcd}
\mathcal{F}_{F_{21}} \arrow[r, "\cong"] \arrow[d, hook] & \Gamma_T\backslash(\mathbb{D} - \mathcal{H}_s) \arrow[d, hook] \\
C_6\backslash\mathbb{A}^2 \cong \mathcal{F}_{F_{21}}^{ADE} \arrow[r, "\cong"] \arrow[d, hook] & \Gamma_T\backslash(\mathbb{D} - \mathcal{H}_*) \arrow[d, hook] \\
\overline{\mathcal{F}}_{F_{21}} \arrow[r, "\cong"] & \Gamma_T\backslash\overline{\mathbb{D}}^{\mathcal{H}_*}
\end{tikzcd}$$

Since horizontal maps are isomorphisms, we also determine the boundary of quotient of Type IV domain $(\overline{\Gamma_T\backslash\mathbb{D}}^{\mathcal{H}_*}) - (\Gamma_T\backslash(\mathbb{D}-\mathcal{H}_s))$.

Note that the determinantal cubic fourfold $[1,0,-2,1]$ lies in the boundary of the moduli spaces $\overline{\mathcal{F}}_{F_{21}}$, so there is a corresponding point $p \in \overline{\Gamma_T\backslash\mathbb{D}}^{\mathcal{H}_*}$. This implies $\mathcal{H}_*$ is nonempty. So the Looijenga compactification $\overline{\Gamma_T\backslash\mathbb{D}}^{\mathcal{H}_*}$ is different to the Baily-Borel compactification $\overline{\Gamma_T\backslash\mathbb{D}}^{\mathrm{BB}}$.

6. Periods of the $L_2(7)$-family

In this section, we focus on the $L_2(7)$-family and determine its monodromy and period domain. In the end, we characterize the quotient curve $\Gamma_{T_1}\backslash\mathbb{D}_{T_1}$ in an explicit way. For convenience, we use notation T instead of T_1, Γ instead of Γ_{T_1} and Φ instead of Φ_1 through this section. The calculation is inspired by [Dol96].

6.1. **Calculation of the Lattice T.** We compute the lattice corresponding to the local period domain of $L_2(7)$-family. We mainly use Conway-Sloane's notation [CS13, Chapter 15] and the classification result of Höhn–Mason [HM16, §4]. The reader can also refer to [LZ22, Appendix A] for a brief introduction to the Conway-Sloane symbol.

Definition 6.1. A pair (G,S) is called a Leech pair, if S is a positive definite even lattice with no vectors of norm two, and G is a finite group acting faithfully on S such that the induced action of G on the discriminant group A_S is trivial, and the invariant sublattice S^G is trivial.

By [GHV12, Appendix B.2] (see also [MM25, Corollary 4.19] and [Zhe25, Lemma 1.1]), there is the following useful lemma:

Lemma 6.2. *Suppose (G,S) is a Leech pair, and* $\operatorname{rank}(S)+l(A_S)\leq 24$. *Then there exists a primitive embedding of S into the Leech lattice $\mathbb{L}$.*

Now let $G = L_2(7)$, and S the coinvariant lattice of the action of G on Λ_0. By [LZ22, Lemma 4.2], (G,S) is a Leech pair satisfying the condition in Lemma 6.2. Then there is a primitive embedding $S \hookrightarrow \mathbb{L}$. Thus the $L_2(7)$-action on S can be extended to $\mathbb{L}$, such that S is the coinvariant lattice in $\mathbb{L}$. It is known from [LZ22] that the invariant sublattice $\mathbb{L}^G$ corresponds to [HM16, Table 1, item 77], which has discriminant form $q_S = 4_1^{+1}7^{+2}$. And S is the orthogonal complement of $\mathbb{L}^G$ in $\mathbb{L}$, so S has discriminant form $q_S = 4_7^{+1}7^{+2}$. The lattice T is the orthogonal complement of S in Λ_0, and thus T has signature $(1,2)$. We have the following embeddings:

$$S \oplus E_6 \hookrightarrow \Lambda_0 \oplus E_6 \hookrightarrow \mathrm{II}_{26,2}.$$

The lattice $\mathrm{II}_{26,2}$ is the even unimodular lattice of signature $(26,2)$, known as the Borcherds lattice. The lattice T is the orthogonal complement of $S \oplus E_6$ in $\mathrm{II}_{26,2}$. Then $q_T = 4_1^{+1}7^{+2}3^{-1}$. By [Nik80, Theorem 1.13.2], such T is unique. One can directly construct a Gram matrix for T with respect to certain basis as below:

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & 10 & 0 \\ 0 & 0 & -28 \end{pmatrix}. \tag{6.1}$$

6.2. **Characterization of Γ.** By abuse of notation, we denote the above matrix by T.

According to [LZ22, §4.1.2], $(L_2(7), S)$ is a saturated Leech pair and therefore $L_2(7) = \{h \in \widehat{\Gamma} | h|_T = \mathrm{Id}_T\}$. Now let N denote the normalizer of $L_2(7)$ inside $\widehat{\Gamma}$. For each element $f \in N$ and $h \in L_2(7)$, $f^{-1}hf = h'$ for some $h' \in L_2(7)$. Therefore, for any $t \in T$, we have $h(f(t)) = f(t)$, i.e. $f(t)$ is $L_2(7)$-invariant. By definition, we have $f(t) \in T$. Therefore, we can restrict f on T. When forming quotients, there is no difference between considering N or its image in $\mathrm{O}^+(T)$ via restriction. Therefore, we denote $\Gamma = \mathrm{Im}(N \to \mathrm{O}^+(T))$. We have the following characterization of Γ.

Proposition 6.3. *Recall that the discriminant group of T is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/7\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z}$. An element $f \in \mathrm{O}^+(T)$ is in Γ if and only if $\overline{f}|_{\mathbb{Z}/3\mathbb{Z}}$ is trivial.*

Proof. From [HM16, §4], the discriminant group A_S of S is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/7\mathbb{Z})^2$. For lattice Λ_0, we have $A_{\Lambda_0} \cong \mathbb{Z}/3\mathbb{Z}$. Let A_T denote the discriminant group of T. From their discriminant groups we see that $\Lambda_0/(S \oplus T)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/7\mathbb{Z})^2$. Let $H_S = A_S$ and H_T be the subgroup $\mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/7\mathbb{Z})^2$ of A_T. From [Nik80, Proposition 1.5.1], we see that the primitive embedding $T \hookrightarrow \Lambda_0$ with orthogonal complement S is determined by an isomorphism $\gamma \colon H_T \xrightarrow{\sim} H_S$ with $q_S \circ \gamma = -q_T$ (where q_S and q_T are their corresponding discriminant-quadratic forms). Also note that the projection $\Lambda_0^*/(S \oplus T) \to T^*/T = A_T$ induces an embedding $A_{\Lambda_0} \to A_T$ to the copy of $\mathbb{Z}/3\mathbb{Z}$.

Now suppose $f \in \mathrm{O}^+(T)$ satisfies that $\overline{f}|_{\mathbb{Z}/3\mathbb{Z}}$ is trivial. Note that from [HM16, §4] we have $\mathrm{O}(S) \to A_S = H_S$ being surjective, thus we can choose some $g \in \mathrm{O}(S)$ such that $\overline{g} = \gamma \circ \overline{f} \circ \gamma^{-1}$. Therefore, also from [Nik80, Corollary 1.5.2], we can extend f to some $\widetilde{f} \in \mathrm{O}^+(\Lambda_0)$ such that $\widetilde{f}|_T = f, \widetilde{f}|_S = g$. Note that $\widetilde{f} \in \widehat{\Gamma}$ since $\overline{f}|_{\mathbb{Z}/3\mathbb{Z}}$ is trivial. Meanwhile, $\widetilde{f} L_2(7) \widetilde{f}^{-1} = L_2(7)$ is easy to check, therefore, $f \in \Gamma$.

Conversely, any $f \in \Gamma$ is the image of some element in $\widehat{\Gamma}$, then $f|_{\mathbb{Z}/3\mathbb{Z}}$ is trivial. The proposition follows. ∎

For each element $f \in \mathrm{O}^+(T)$, if $\overline{f}|_{\mathbb{Z}/3\mathbb{Z}}$ is not trivial, then it sends 1 to -1 in $\mathbb{Z}/3\mathbb{Z}$, thus $-f$ acts trivially on $\mathbb{Z}/3\mathbb{Z}$. Meanwhile, it is obvious that $-\mathrm{Id} \notin \Gamma$, therefore we have the following

Proposition 6.4. *As a group, $\mathrm{O}^+(T)$ is generated by Γ and $-\mathrm{Id}$. And Γ is an index 2 subgroup of $\mathrm{O}^+(T)$.*

Let $\mathrm{O}(1,2)$ be the orthogonal group of $T \otimes \mathbb{R}$, and identify $\mathrm{O}^+(T) < \mathrm{O}(1,2)$. Multiplying those elements in Γ with determinant -1 by $-\mathrm{Id}$, we obtain an isomorphism from Γ to $\Gamma' = \mathrm{O}^+(T) \cap \mathrm{SO}(1,2)$. The quotients of $\mathbb{D}_T$ by Γ or Γ' are obviously the same, therefore we are reduced to considering $\Gamma' = \mathrm{O}^+(T) \cap \mathrm{SO}(1,2)$.

6.3. **Tube Domain Model for the $L_2(7)$-family.** Let $e_1, e_2, e_3 \in T$ be the integral basis for which the Gram matrix is (6.1). Then $(e_1)^2 = -2, (e_2)^2 = 10, (e_1, e_2) = 1, (e_3)^2 = -28$.

Let $f, e, g \in T \otimes \mathbb{R}$ be an $\mathbb{R}$-basis such that if we denote

$$B = \begin{pmatrix} -\frac{\sqrt{21}}{42} - \frac{1}{2} & 0 & -\frac{\sqrt{21}}{42} + \frac{1}{2} \\ -\frac{\sqrt{21}}{21} & 0 & -\frac{\sqrt{21}}{21} \\ 0 & \frac{\sqrt{14}}{14} & 0 \end{pmatrix},$$

we have $\begin{pmatrix} f & e & g \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} B$.

Note that

$$B^T T B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Therefore, f, g are isotropic and $(f, g) = 1, (e)^2 = -2, (f, e) = 0, (g, e) = 0$.

With this basis, we can identify $T \otimes \mathbb{R}$ with the quadratic space of binary forms over $\mathbb{R}$, sending $\alpha f + \beta e + \gamma g$ to $\alpha x^2 + 2\beta xy + \gamma y^2$, where the quadratic form on this space is $Q(\alpha, \beta, \gamma) = -2\beta^2 + 2\alpha\gamma$. Note that $\beta^2 - \alpha\gamma$ is the discriminant of the binary form $\alpha x^2 + 2\beta xy + \gamma y^2$. Each $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ can be seen as an isometry on this space via the action

$$h \cdot [\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}] = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix}$$

With respect to the basis f, e, g, the action of h corresponds to a matrix $A(h) \in \mathrm{SO}^+(1, 2) = \mathrm{SO}^+(1, 2; \mathbb{R})$. Straightforward calculation shows that

$$A(h) = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}.$$

Let us denote $\Phi\colon \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SO}^+(1, 2)$ be the morphism given by sending h to $BA(h)B^{-1}$ (i.e. considering the basis e_1, e_2, e_3). This gives a surjective morphism $\mathrm{SL}(2, \mathbb{R}) \to \mathrm{SO}^+(1, 2)$ of degree two, which induces a group isomorphism $\mathrm{PSL}(2, \mathbb{R}) \xrightarrow{\sim} \mathrm{SO}^+(1, 2)$.

Let $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ be the complex upper plane, with $\mathrm{SL}(2, \mathbb{R})$ acting on it by $h \cdot \frac{\omega_1}{\omega_2} = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}$. it is easy to check that there is a biholomorphic morphism $\mathbb{H} \to \mathbb{D}_T$ sending z to $z^2 f + ze + g$, or equivalently sending $\frac{\omega_1}{\omega_2}$ to $\omega_1^2 f + \omega_1\omega_2 e + \omega_2^2 g$, which is the classic tube domain model. As binary forms, we can write $\omega_1^2 f + \omega_1\omega_2 e + \omega_2^2 g$ as $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Therefore, we have the following equivariant diagram

$$\begin{array}{ccccc} \mathrm{SL}(2, \mathbb{R}) & \xrightarrow{\Phi} & \mathrm{SO}^+(1, 2) & \hookrightarrow & \mathrm{O}(1, 2) \\ \circlearrowright & & \circlearrowright & & \circlearrowright \\ \mathbb{H} & \xrightarrow{\cong} & \mathbb{D}_T & \hookrightarrow & \mathbb{P}(T \otimes \mathbb{C}) \end{array}$$

Next we characterize the curve $\mathbb{D}_T/\Gamma$ as an arithmetic quotient of $\mathbb{H}$. For this, we compute the inverse image $\Phi^{-1}(\Gamma') = \Phi^{-1}(\mathrm{O}^+(T) \cap \mathrm{SO}^+(1, 2))$. By definition, they are those $h \in \mathrm{SL}(2, \mathbb{R})$ such that $\Phi(h)$ has integer coefficients.

Proposition 6.5. *Denote* $\Gamma'' = \Phi^{-1}(\mathrm{O}^+(T) \cap \mathrm{SO}^+(1,2))$*. We have*

$$\Gamma'' = \{ \begin{pmatrix} \frac{u\sqrt{u'}+v\sqrt{v'}}{2} & \frac{w\sqrt{w'}+x\sqrt{x'}}{2} \\ \frac{w\sqrt{w'}-x\sqrt{x'}}{2} & \frac{u\sqrt{u'}-v\sqrt{v'}}{2} \end{pmatrix} | u,v,w,x \in \mathbb{Z}, (u',v',w',x') = (1,21,6,14) \text{ or } (2,42,3,7)$$
$$\text{up to permutation by } \mathrm{V} < \mathrm{S}_4, \text{ and } u^2u' - v^2v' - w^2w' + x^2x' = 4\}$$

(Here we use $\mathrm{V} = \{1, (12)(34), (13)(24), (14)(23)\}$ *to denote the Klein* 4*-group).*

Proof. For any $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ such that $\Phi(h)$ has integer coefficients. Set $\Phi(h) = N = \begin{pmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \\ n_7 & n_8 & n_9 \end{pmatrix}$ with $n_i \in \mathbb{Z}$. Writing out $A(h) = B^{-1}NB$, we obtain this matrix.

$$\begin{pmatrix} \frac{21(n_1+n_5)+(n_1+2n_2+10n_4-n_5)\sqrt{21}}{42} & -\frac{\sqrt{14}}{14}n_3+(-\frac{\sqrt{6}}{4}+\frac{\sqrt{14}}{28})n_6 & \frac{21(-n_1+n_4+n_5)+(n_1+2n_2-11n_4-n_5)\sqrt{21}}{42} \\ (-\frac{\sqrt{6}}{6}-\frac{\sqrt{14}}{2})n_7-\frac{\sqrt{6}}{3}n_8 & n_9 & (-\frac{\sqrt{6}}{6}+\frac{\sqrt{14}}{2})n_7-\frac{\sqrt{6}}{3}n_8 \\ \frac{21(-n_1+n_4+n_5)-(n_1+2n_2-11n_4-n_5)\sqrt{21}}{42} & \frac{\sqrt{14}}{14}n_3+(-\frac{\sqrt{6}}{4}-\frac{\sqrt{14}}{28})n_6 & \frac{21(n_1+n_5)-(n_1+2n_2+10n_4-n_5)\sqrt{21}}{42} \end{pmatrix}$$

Now let $u = a+d, v = a-d, w = b+c, x = b-c$, direct computation shows that $u^2, v^2, w^2, x^2 \in \mathbb{Z}$ and $\sqrt{21}uv, \sqrt{21}wx, 2\sqrt{6}uw \in \mathbb{Z}$ (and therefore $\sqrt{21}uv, \sqrt{21}wx, \sqrt{6}uw \in \mathbb{Z}$). Therefore, we can set $u = u_1\sqrt{u_2}, .., x = x_1\sqrt{x_2}$ with $u_1, .., x_1 \in \mathbb{Z}$, $u_2, .., x_2 \in \mathbb{Z}^+$ and $u_2, .., x_2$ square-free. Since $\sqrt{21}uv \in \mathbb{Z}$, we have $u_2v_2 = 21y_1^2$ for some $y_1 \in \mathbb{Z}^+$, y_1 square-free and cannot be divided by 3 or 7. This gives four possibilities, which are $u_2 = 21y_1$, $7y_1$, $3y_1$ or y_1. Similarly for w_2, x_2, we have $w_2x_2 = 21y_2^2, u_2w_2 = 6y_3^2$ with $y_2, y_3 \in \mathbb{Z}^+$, square-free, and not divided by $3, 7$ and $2, 3$ respectively.

Up to symmetry, we only need to consider $u_2 = 6y_3 = 6w_2$ or $u_2 = 3y_3, w_2 = 2y_3$. In the first case, we must have w_2 not divided by $2, 3$ and thus $w_2 = y_2$ or $7y_2$. Further discussion of the relation of u_2 and v_2 shows that $(u_2, v_2, w_2, x_2) = (6y_2, 14y_2, y_2, 21y_2)$ or $(42y_2, 2y_2, 7y_2, 3y_2)$. The "determinant = 1" condition gives $u_1^2u_2 - v_1^2v_2 - w_1^2w_2 + x_1^2x_2 = 4$ and therefore 4 is divided by y_2. However, $v_2 = 14y_2$ or $2y_2$ shows that y_2 is not divided by 2. The only possibility is that $y_2 = 1$.

Similarily, in the second case, $u_2 = 3y_3, w_2 = 2y_3$ we will have $(u_2, v_2, w_2, x_2) = (21y_1, y_1, 14y_1, 6y_1)$ or $(3y_1, 7y_1, 2y_1, 42y_1)$ and we must have $y_1 = 1$. Therefore, $(u_2, v_2, w_2, x_2) = (1, 21, 6, 14)$ or $(2, 42, 3, 7)$ up to permutation by $\mathrm{V} < \mathrm{S}_4$.

Conversely, for any element h in the group in the statement of the theorem, we already have $\Phi(h)$ having integer coefficients, which can be seen applying elementary number theory. For example, when $h = \begin{pmatrix} \frac{u+v\sqrt{21}}{2} & \frac{w\sqrt{6}+x\sqrt{14}}{2} \\ \frac{w\sqrt{6}-x\sqrt{14}}{2} & \frac{u-v\sqrt{21}}{2} \end{pmatrix}$, we have $u^2 - 21v^2 - 6w^2 + 14x^2 = 4$. Observing the coefficients of $\Phi(h)$, we see that $\Phi(h)$ has integer coefficients if and only if $u - v, w - x \in 2\mathbb{Z}$. (This is the same as u is even if and only v is even, w is even if and only x is even). They all can be deduced from $u^2 - 21v^2 - 6w^2 + 14x^2 = 4$. For example, if we modulo 2, we obtain $u^2 - v^2 \equiv 0$ and thus $u - v \in 2\mathbb{Z}$ or $u + v \in 2\mathbb{Z}$. However, $u - v \in 2\mathbb{Z}$ is the same as $u + v \in 2\mathbb{Z}$. Similar argument applies to w, x if we modulo 4. Therefore, the theorem is proven. $\square$

We investigate more information about the group Γ'' as below. For convenience, we call elements of the form $\begin{pmatrix} \frac{u\sqrt{u'}+v\sqrt{v'}}{2} & \frac{w\sqrt{w'}+x\sqrt{x'}}{2} \\ \frac{w\sqrt{w'}-x\sqrt{x'}}{2} & \frac{u\sqrt{u'}-v\sqrt{v'}}{2} \end{pmatrix}$ in Γ'' have type (u', v', w', x'). There are eight types of elements here.

Proposition 6.6. *Elements of type* $(1, 21, 6, 14)$ *form a normal subgroup* H *of index* 8 *in* Γ''*, with each type of elements form a coset in* Γ'' *with repect to this subgroup* H*. Moreover,* $\Gamma''/H \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$.

Proof. Direct computation shows that H is a subgroup. For elements of type (u', v', w', x'), we can choose some element r of this type (for example, see below). Also by direct computation, we can show that left multiplication by r and r^{-1} will give a bijection between elements of type $(1, 21, 6, 14)$ and elements of type (u', v', w', x'). Therefore, the set of elements of type (u', v', w', x') is just the left coset rH. Right multiplication will give the same conclusion, thus H is a normal subgroup. $\square$

The following table will make the structure clear.

Type	Representative	Preserves $\mathbb{Z}/3\mathbb{Z}$ in A_T?	As elements in $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$
$(1, 21, 6, 14)$	Id	Yes	$(0, 0, 0)$
$(21, 1, 14, 6)$	$\begin{pmatrix} \frac{\sqrt{21}+3}{2} & \frac{\sqrt{14}+\sqrt{6}}{2} \\ \frac{\sqrt{14}-\sqrt{6}}{2} & \frac{\sqrt{21}-3}{2} \end{pmatrix}$	No	$(1, 0, 0)$
$(6, 14, 1, 21)$	$\begin{pmatrix} \frac{3\sqrt{6}+\sqrt{14}}{2} & 3 \\ 3 & \frac{3\sqrt{6}-\sqrt{14}}{2} \end{pmatrix}$	No	$(0, 1, 0)$
$(14, 6, 21, 1)$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	Yes	$(1, 1, 0)$
$(2, 42, 3, 7)$	$\begin{pmatrix} 0 & \frac{\sqrt{3}+\sqrt{7}}{2} \\ \frac{\sqrt{3}-\sqrt{7}}{2} & 0 \end{pmatrix}$	Yes	$(0, 0, 1)$
$(42, 2, 7, 3)$	$\begin{pmatrix} \sqrt{2} & \frac{3\sqrt{7}+5\sqrt{3}}{2} \\ \frac{3\sqrt{7}-5\sqrt{3}}{2} & -\sqrt{2} \end{pmatrix}$	No	$(0, 1, 1)$
$(3, 7, 2, 42)$	$\begin{pmatrix} \sqrt{3} & \sqrt{2} \\ \sqrt{2} & \sqrt{3} \end{pmatrix}$	No	$(0, 1, 1)$
$(7, 3, 42, 2)$	$\begin{pmatrix} \frac{\sqrt{7}+\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{7}-\sqrt{3}}{2} \end{pmatrix}$	Yes	$(1, 1, 1)$

Table 3: Elements of the Group Γ''/H

6.4. **Commensurability Class of Γ.** Recall that arithmetic subgroups of $\mathrm{SL}(2, \mathbb{R})$ up to commensurability and conjugacy have been completely classified, see for example [Mor15, Chapter 6]. For reader's convenience, we gather these facts as below

Definition 6.7. A closed subgroup $\Delta < \mathrm{SL}(2, \mathbb{R})$ is cocompact if $\mathrm{SL}(2, \mathbb{R})/\Delta$ is compact.

We borrow the following proposition [Mor15, Chapter 6, Proposition 6.1.5]

Proposition 6.8. $\mathrm{SL}(2,\mathbb{Z})$ *is the only noncocompact, arithmetic subgroup of* $\mathrm{SL}(2,\mathbb{R})$ *up to commensurability and conjugacy.*

What is left is those cocompact arithmetic subgroups. We first recall what cocompact arithmetic subgroups can appear according to [Mor15, Proposition 6.2.4, Propsition 6.2.5 and Proposition 6.2.6].

Definition 6.9. (1) For any field F, and any nonzero $a,b \in F$, the corresponding quaternion algebra over F is the ring

$$\mathbb{H}_F^{a,b} = \{p + qi + rj + sk | p,q,r,s \in F\},$$

where

- addition is defined in the obvious way, and
- multiplication is determined by the relations

$$i^2 = a, j^2 = b, ij = k = -ji$$

together with the requirement that every element of F is in the center of $\mathbb{H}_F^{a,b}$.

(2) The reduced norm of $x = p + qi + rj + sk \in \mathbb{H}_F^{a,b}$ is

$$\mathrm{N}_{red}(x) = x\overline{x} = p^2 - aq^2 - br^2 + abs^2 \in F$$

where $\overline{x} = p - qj - rj - sk$ is the conjugate of x.

For $a,b \in \mathbb{R}^+$, there is an isomorphism of rings $\psi\colon \mathbb{H}_{\mathbb{R}}^{a,b} \xrightarrow{\sim} \mathrm{M}_{2\times 2}(\mathbb{R})$ sending i to $\begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}$, j to $\begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$ and k to $\begin{pmatrix} 0 & \sqrt{a} \\ -b\sqrt{a} & 0 \end{pmatrix}$ such that $\mathrm{N}_{red}(x) = \det(\psi(x))$. Let $\mathrm{SL}(1,\mathbb{H}_F^{a,b}) \coloneqq \{g \in \mathbb{H}_F^{a,b} | \mathrm{N}_{red}(x) = 1\}$, then under the identification above, we have $\mathrm{SL}(2,\mathbb{R}) = \mathrm{SL}(1,\mathbb{H}_{\mathbb{R}}^{a,b})$. We recall the following fact according to [Mor15, Proposition 6.2.4, Proposition 6.2.5 and Proposition 6.2.6]

Proposition 6.10. *Every cocompact and arithmetic subgroup of* $\mathrm{SL}(2,\mathbb{R})$ *up to commensurability and conjugacy is of the following form*

- $\mathrm{SL}(1,\mathbb{H}_{\mathbb{Z}}^{a,b}) < \mathrm{SL}(1,\mathbb{H}_{\mathbb{R}}^{a,b}) \cong \mathrm{SL}(2,\mathbb{R})$, *where* $a,b \in \mathbb{Z}^+$ *and* $(0,0,0,0)$ *is the only integer solution* (p,q,r,s) *of the Diophantine equation*

$$w^2 - ax^2 - by^2 + abz^2 = 0$$

- $\mathrm{SL}(1,\mathbb{H}_{\mathcal{O}}^{a,b}) < \mathrm{SL}(1,\mathbb{H}_{\mathbb{R}}^{a,b}) \cong \mathrm{SL}(2,\mathbb{R})$, *where* $\mathcal{O}$ *is the ring of integers of a totally real algebraic number field* $F \neq \mathbb{Q}$ *and* $a,b \in \mathcal{O}$ *are positive such that* $\sigma(a)$ *and* $\sigma(b)$ *are negative for every place* $\sigma \neq \mathrm{Id}$.

Next we show Γ is a cocompact arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$.

Proposition 6.11. *The curve* $\mathbb{D}_T/\Gamma$ *is compact, and therefore not isogenous to* $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$.

Proof. By Proposition 6.8, we only need to show that $\mathbb{D}_T/\Gamma$ is compact. This can be shown by considering the Baily-Borel compactification. In fact, the boundaries of Baily-Borel compactification are the isotropic subspaces of $T \otimes \mathbb{Q}$ [BB66]. However, there is no isotropic vector. The claim is equivalent to that the equation $-x^2 + xy + 5y^2 = 14z^2$ has no coprime integer solution.

If there is an integer solution x, y, z with $\gcd(x, y, z) = 1$, then x, y must be even integers. So 4 divides $14z^2$ and thus z is forced to be an even integer, a contradiction to the coprime assumption. $\square$

Remark 6.12. The compactness of $\Gamma\backslash\mathbb{D}_T$ can also be deduced from the GIT analysis in Proposition 3.18.

More precisely, $\overline{\mathcal{F}}_{L_2(7)} \cong \mathbb{P}^1$ can be obtained from $\mathcal{F}^{\mathrm{ADE}}_{L_2(7)}$ by adding one point that corresponds to the determinantal cubic fourfold. And note that the point of determinantal cubic fourfold is of codimension 1. By Theorem 2.4, the period map:

$$\mathcal{F}^{\mathrm{ADE}}_{L_2(7)} \to \Gamma_T\backslash(\mathbb{D} - \mathcal{H}_*)$$

extends to

$$\overline{\mathcal{F}}_{L_2(7)} \cong \overline{\Gamma_T\backslash\mathbb{D}}^{\mathcal{H}_*} = \overline{\Gamma_T\backslash\mathbb{D}}^{BB}. \tag{6.2}$$

From analysis of singular cubic fourfolds in $\overline{\mathcal{F}}_{L_2(7)}$, the image of map 6.2 should be $\Gamma_T\backslash\mathbb{D}$. Therefore, $\Gamma_T\backslash\mathbb{D}$ is itself compact.

In fact, we can further determine the type of $\mathbb{H}$ as an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{R})$

Theorem 6.13. *The curve $\mathbb{D}_T/\Gamma$ is isogenous to $\mathbb{H}/\mathrm{SL}(1,\mathbb{H}^{21,6}_{\mathbb{Z}})$.*

Proof. We show $\mathrm{SL}(1,\mathbb{H}^{21,6}_{\mathbb{Z}}) < H$ up to a conjugation.

Consider $A = \begin{pmatrix} 0 & 6^{-\frac{1}{4}} \\ -6^{\frac{1}{4}} & 0 \end{pmatrix}$, then

$$AHA^{-1} = \{\begin{pmatrix} \frac{u+v\sqrt{21}}{2} & \frac{3w-x\sqrt{21}}{6} \\ 3w + x\sqrt{21} & \frac{u-v\sqrt{21}}{2} \end{pmatrix} | u, v, w, x \in \mathbb{Z}, u^2 - 21v^2 - 6w^2 + 14x^2 = 4\}.$$

For equation $u^2 - 21v^2 - 6w^2 + 14x^2 = 4$, we analyze modulo 3 and obtain $u^2 + 2x^2 \equiv 1 \pmod 3$. We must have $3|x$, otherwise $u^2 \equiv -1 \pmod 3$, which is impossible. Replace x by $3x$, and one obtains

$$AHA^{-1} = \{\begin{pmatrix} \frac{u+v\sqrt{21}}{2} & \frac{w-x\sqrt{21}}{2} \\ 6(\frac{w+x\sqrt{21}}{2}) & \frac{u-v\sqrt{21}}{2} \end{pmatrix} | u, v, w, x \in \mathbb{Z}, u^2 - 21v^2 - 6w^2 + 126x^2 = 4\}.$$

And $\mathrm{SL}(1,\mathbb{H}^{21,6}_{\mathbb{Z}})$ is a subgroup of AHA^{-1} consisting of matrices where u, v, w, x are even integers.

Therefore, $\mathrm{SL}(1,\mathbb{H}^{21,6}_{\mathbb{Z}}) < \Gamma''$ and thus they are commensurable with each other, since both are arithmetic subgroups of $\mathrm{SL}(2,\mathbb{R})$.

Modulo 7 and using infinite descent, one can show that equation $u^2 - 21v^2 - 6w^2 + 126x^2 = 0$ does not have non-trivial integer solution. Therefore, $\mathrm{SL}(1,\mathbb{H}^{21,6}_{\mathbb{Z}})$ satisfies the conditions in Proposition 6.10. $\square$

7. Periods of the F_{21}-family

For convenience, we use notation T instead of T_2, Γ instead of Γ_{T_2} and Φ instead of Φ_2 inside this section.

7.1. Calculation of the Lattice T. The lattice T for smooth cubic fourfolds with F_{21}-action was calculated by [BGM25, Table 1]. We review the calculation for reader's convenience.

Now let $G = F_{21}$. Let S be the coinvariant lattice for the action of F_{21} on Λ_0. By [LZ22, Lemma 4.2], (G, S) is a Leech pair satisfying the condition in Lemma 6.2. Then by Lemma 6.2, there is a primitive embedding $S \hookrightarrow \mathbb{L}$. Thus the F_{21}-action on S can be extended to $\mathbb{L}$, such that S is the coinvariant lattice in $\mathbb{L}$. It is known from [LZ22] that the invariant sublattice $\mathbb{L}^G$ corresponds to [HM16, Table 1, item 52], hence $\mathbb{L}^G$ has discriminant form 7^{+3}.

Similar as §6.1, T has discriminant form $q_T = 7^{+3}3^{-1}$ and such T is unique by [Nik80, Theorem 1.13.2]. By direct construction, we know there exists an integral basis for T, such that the Gram matrix is:

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & 10 & 0 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 7 & 0 \end{pmatrix} \tag{7.1}$$

7.2. Characterization of Γ. By abuse of notation, we still denote the above matrix by T. Also from [LZ22, §4.1.2], we know that (F_{21}, S) is a saturated Leech pair and therefore $F_{21} = \{h \in \widehat{\Gamma} | h|_T = \mathrm{Id}_T\}$. Also let N be the normalizer of F_{21} inside $\widehat{\Gamma}$. The same as the discussion in the last section, we have a well-defined restriction map $N \to \mathrm{O}^+(T)$. Therefore, we may just assume $\Gamma = \mathrm{Im}(N \to \mathrm{O}^+(T))$.

Recall that from the above calculation, the discriminant group A_{Λ_0}, A_S and A_T are isomorphic to $\mathbb{Z}/3\mathbb{Z}$, $(\mathbb{Z}/7\mathbb{Z})^3$ and $(\mathbb{Z}/7\mathbb{Z})^3 \oplus \mathbb{Z}/3\mathbb{Z}$ respectively. Also from [HM16, §4], we know that $\mathrm{O}(S) \to A_S$ is surjective. Therefore, we can completely copy the proof of Proposition 6.2 and obtain the following

Proposition 7.1. *An element $f \in \mathrm{O}^+(T)$ is in Γ if and only if $\overline{f}|_{\mathbb{Z}/3\mathbb{Z}}$ is trivial.*

Up to a multiplication by $-\mathrm{Id}$, we may just consider $\mathrm{O}^+(T)$ itself. More precisely

Proposition 7.2. *As a group, $\mathrm{O}^+(T)$ is generated by Γ and $-\mathrm{Id}$. Γ is an index 2 subgroup of $\mathrm{O}^+(T)$.*

An important thing to point out is that unlike the $L_2(7)$ case, $-\mathrm{Id}$ in $\mathrm{O}^+(T)$ has determinant 1. Therefore, we cannot find the copy of Γ to be $\mathrm{O}^+(T) \cap \mathrm{SO}(2,2)$. Instead, we first find an element

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$\in \mathrm{O}^+(T)$ which is of determinant -1, and thus form the following result.

Proposition 7.3. *The group* $\mathrm{SO}^+(T)$ *is a normal subgroup of* $\mathrm{O}^+(T)$ *with index two, and the quotient group* $\mathrm{O}^+(T)/\mathrm{SO}^+(T)$ *is generated by* P *above.*

We will first compute $\mathrm{SO}^+(T)$ and give a modular interpretation of $\mathbb{D}_T/\mathrm{SO}^+(T)$. The latter will be a 2 to 1 cover of the F_{21}-family.

7.3. **Tube Domain Model for the F_{21}-family.** Now let $e_1, e_2, e_3, e_4 \in T$ be the integral basis for which we have the above Gram matrix (7.1). Then we have $(e_1)^2 = -2, (e_2)^2 = 10, (e_1, e_2) = 1, (e_3, e_4) = 7$. Now take $f_1, f_2, g_1, g_2 \in T \otimes \mathbb{R}$ such that if we denote

$$B = \begin{pmatrix} -\frac{\sqrt{21}}{42} - \frac{1}{2} & 0 & 0 & -\frac{\sqrt{21}}{42} + \frac{1}{2} \\ -\frac{\sqrt{21}}{21} & 0 & 0 & -\frac{\sqrt{21}}{21} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & 0 \end{pmatrix}, \tag{7.2}$$

we have $\begin{pmatrix} f_1 & f_2 & g_2 & g_1 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \end{pmatrix} B$

Therefore, the Gram matrix with respect to f_1, f_2, g_2, g_1 is just

$$B^T T B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

With respect to this basis, there is an accidental isometry from $T \otimes \mathbb{C}$ to $\mathrm{M}_2(\mathbb{C})$, sending $z_{11} f_1 + z_{12} f_2 + z_{21} g_2 + z_{22} g_1$ to $\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$. Here $\mathrm{M}_2(\mathbb{C})$ is endowed with the quadratic form $2(z_{11}z_{22} - z_{12}z_{21})$.

Now consider the group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Each element $(h_1, h_2) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ can be seen as an isometry of the space $\mathrm{M}_2(\mathbb{C})$ via the action $(h_1, h_2) \cdot M = h_1 M h_2^T$. In this way, we obtain a surjective morphism $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}^+(2,2;\mathbb{R})$ with kernel $\pm(\mathrm{I},\mathrm{I})$. When pass to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSO}^+(2,2;\mathbb{R})$ we have kernel being $(\pm\mathrm{I}, \pm\mathrm{I})$. More explicitely, set $h = (h_1, h_2) = (\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})$ and let $A(h)$ be the matrix of the action of h with respect to the basis f_1, f_2, g_2, g_1, then we have

$$A(h) = \begin{pmatrix} a_1 h_2 & b_1 h_2 \\ c_1 h_2 & d_1 h_2 \end{pmatrix} = h_1 \otimes h_2.$$

Now denote by $\Phi\colon \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}^+(2,2)$ the morphism sending h to $BA(h)B^{-1}$, the matrix of the action of h with respect to the basis e_1, e_2, e_3, e_4.

Now let $\mathbb{H}^2 = \{(z_1, z_2) \in \mathbb{C}^2 | \Im(z_1) > 0, \Im(z_2) > 0\}$ be the product of upper half planes. The group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ via $h \cdot (z_1, z_2) = (\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2})$. There is a classical tube domain model identification of $\mathbb{D}_T$ as $\mathbb{H}^2$. Precisely, there is a biholomorphic morphism $\mathbb{H}^2 \to \mathbb{D}_T$, sending (z_1, z_2) to $z_1 z_2 f_1 + z_1 f_2 + z_2 g_2 + g_1$. The latter is $\begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix} = \begin{pmatrix} z_1 \\ 1 \end{pmatrix} \begin{pmatrix} z_2 & 1 \end{pmatrix}$

when identified as a matrix. Therefore, we obtain an equivariant diagram:

$$\begin{array}{ccccc}
\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\Phi} & \mathrm{SO}^+(2,2) & \hookrightarrow & \mathrm{O}(2,2) \\
\circlearrowright & & \circlearrowright & & \circlearrowright \\
\mathbb{H}^2 & \xrightarrow{\cong} & \mathbb{D}_T & \hookrightarrow & \mathbb{P}(T\otimes\mathbb{C})
\end{array}$$

First we find the inverse image $\Phi^{-1}(\mathrm{SO}^+(T))$. Note that $h\in\Phi^{-1}(\mathrm{SO}^+(T))$ if and only if $\Phi(h)$ has integer coefficients.

Proposition 7.4. *Denote $\Gamma'=\Phi^{-1}(\mathrm{SO}^+(T))$, then we have $h\in\Gamma'$ if and only if*

$$h=(\begin{pmatrix}\frac{\alpha_1\sqrt{\alpha}+\beta_1\sqrt{\beta}}{2} & \frac{1}{\sqrt{7}}\frac{\gamma_1\sqrt{\gamma}+\delta_1\sqrt{\delta}}{2}\\ \sqrt{7}\frac{\gamma_2\sqrt{\gamma}-\delta_2\sqrt{\delta}}{2} & \frac{\alpha_2\sqrt{\alpha}-\beta_2\sqrt{\beta}}{2}\end{pmatrix},\begin{pmatrix}\frac{\alpha_2\sqrt{\alpha}+\beta_2\sqrt{\beta}}{2} & \sqrt{7}\frac{\gamma_2\sqrt{\gamma}+\delta_2\sqrt{\delta}}{2}\\ \frac{1}{\sqrt{7}}\frac{\gamma_1\sqrt{\gamma}-\delta_1\sqrt{\delta}}{2} & \frac{\alpha_1\sqrt{\alpha}-\beta_1\sqrt{\beta}}{2}\end{pmatrix})$$

where

- $(\alpha,\beta,\gamma,\delta)=(1,21,3,7)$ *up to permutation by* $\mathrm{V}=\langle(13)(24),(12)(34)\rangle<\mathrm{S}_4$*,*
- $\alpha_i,\beta_i,\gamma_i,\delta_i\in\mathbb{Z}$ *such that* $\alpha_i-\beta_i,\gamma_i-\delta_i\in 2\mathbb{Z}$*,*
- $\alpha_1\alpha_2\alpha-\beta_1\beta_2\beta-\gamma_1\gamma_2\gamma+\delta_1\delta_2\delta=4$ *and* $\alpha_1\beta_2-\alpha_2\beta_1=\gamma_1\delta_2-\gamma_2\delta_1$*.*

(The last condition is equivalent to saying that the determinants of h_1 and h_2 are 1.)

Proof. Set $h=(h_1,h_2)=(\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix},\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix})$ and suppose

$$\Phi(h)=\begin{pmatrix}n_1 & n_2 & n_3 & n_4\\ n_5 & n_6 & n_7 & n_8\\ n_9 & n_{10} & n_{11} & n_{12}\\ n_{13} & n_{14} & n_{15} & n_{16}\end{pmatrix}=BA(h)B^{-1}$$

where $n_i\in\mathbb{Z}$ and the expression of B is given in 7.2. Writing it out, we see that

$$A(h)=\begin{pmatrix}a_1h_2 & b_1h_2\\ c_1h_2 & d_1h_2\end{pmatrix}=\begin{pmatrix}a_1a_2 & a_1b_2 & b_1a_2 & b_1b_2\\ a_1c_2 & a_1d_2 & b_1c_2 & b_1d_2\\ c_1a_2 & c_1b_2 & d_1a_2 & d_1b_2\\ c_1c_2 & c_1d_2 & d_1c_2 & d_1d_2\end{pmatrix}$$

equals to .

$$\begin{pmatrix}\frac{21(n_1+n_6)+(n_1+n_2+5n_5-n_6)\sqrt{21}}{42} & \frac{n_7-2n_3-n_7\sqrt{21}}{2} & \frac{2n_4-n_8+n_8\sqrt{21}}{14} & \frac{21(-n_1+n_5+n_6)+(n_1+2n_2-11n_5-n_6)\sqrt{21}}{42}\\ \frac{-21n_9-(n_9+2n_{10})\sqrt{21}}{42} & n_{11} & -\frac{n_{12}}{7} & \frac{21n_9-(n_9+2n_{10})\sqrt{21}}{42}\\ \frac{21n_{13}+(n_{13}+2n_{14})\sqrt{21}}{6} & -7n_{15} & n_{16} & \frac{-21n_{13}+(n_{13}+2n_{14})\sqrt{21}}{6}\\ \frac{21(-n_1+n_5+n_6)-(n_1+2n_2-11n_5-n_6)\sqrt{21}}{42} & \frac{-n_7+2n_3-n_7\sqrt{21}}{2} & \frac{-2n_4+n_8+n_8\sqrt{21}}{14} & \frac{21(n_1+n_6)-(n_1+n_2+5n_5-n_6)\sqrt{21}}{42}\end{pmatrix}$$

Now if we set $u_1=a_1+d_2, v_1=a_1-d_2, w_1=\sqrt{7}(b_1+c_2), x_1=\sqrt{7}(b_1-c_2)$ and $u_2=a_2+d_1, v_2=a_2-d_1, w_2=\frac{1}{\sqrt{7}}(b_2+c_1), x_2=\frac{1}{\sqrt{7}}(b_2-c_1)$, then using information from the above matrix, we can prove various properties of these numbers.

First, we claim that $u_i^2,v_i^2,w_i^2,x_i^2\in\mathbb{Z}$. For example, $u_1^2=a_1^2+2a_1d_2+d_2^2$, the term $a_1d_2=n_{11}$ is an integer. The terms $a_1^2=\det(a_1h_2)$ and $d_2^2=\det(d_2h_1)$ can be expressed in terms of n_i and we can obtain

$$u_1^2 = n_1 n_{11} - n_3 n_9 + n_6 n_{11} - n_7 n_{10} + 2n_{11}.$$

Therefore, $u_1^2 \in \mathbb{Z}$. The others are similar.

Second, note that $u_1u_2 = a_1a_2 + d_1d_2 + a_1d_1 + a_2d_2, v_1v_2 = a_1a_2 + d_1d_2 - (a_1d_1 + a_2d_2), w_1w_2 = b_1b_2 + c_1c_2 + b_1c_1 + b_2c_2, x_1x_2 = b_1b_2 + c_1c_2 - (b_1c_1 + b_2c_2)$. Among them, $a_1a_2+d_1d_2, b_1b_2+c_1c_2$ are easily seen to be integers. Taking the $(2,3)$ rows and $(2,3)$ columns, the corresponding submatrix has determinant $a_1d_1a_2d_2 - b_1c_1b_2c_2 = (a_1d_1 - b_1c_1)a_2d_2 + b_1c_1(a_2d_2 - b_2c_2) = a_2d_2 + b_1c_1 = a_2d_2 + a_1d_1 - 1 = b_1c_1 + b_2c_2 + 1$, which is an integer. Therefore, $u_1u_2, v_1v_2, w_1w_2, x_1x_2 \in \mathbb{Z}$.

Combining the two observations above, we can set $u_i = \alpha_i\sqrt{\alpha}$, $v_i = \beta_i\sqrt{\beta}$, $w_i = \gamma_i\sqrt{\gamma}$, $x_i = \delta_i\sqrt{\delta}$, where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are integers and $\alpha, \beta, \gamma, \delta$ are positive square-free integers. Next we want to determine the possibilities of α, β, γ, δ. Using the same tricks before, we obtain $\sqrt{21}u_2v_2, \sqrt{21}w_2x_2, \sqrt{3}u_2w_2 \in \mathbb{Z}$. Therefore, we can set $\alpha\beta = 21y_1^2, \gamma\delta = 21y_2^2, \alpha\gamma = 3y_3^2$ with $y_1, y_2, y_3 \in \mathbb{Z}^+$, square free, y_1, y_2 not divided by $3, 7$ and y_3 not divided by 3. Note that $\alpha\gamma = 3y_3^2$ gives two possibilities. One is $\alpha = 3\gamma$ and the other is $\gamma = 3\alpha$. Suppose $\alpha = 3\gamma$, then γ is not divided by 3. Therefore $\gamma = 7y_2$ or $\gamma = y_2$. Meanwhile, since α is divided by 3, we obtain $\alpha = 21y_1$ or $3y_1$. Suppose $\gamma = 7y_2$, then $\alpha \neq 3y_1$, otherwise $\alpha = 3\gamma = 21y_2$ will imply that y_1 is divided by 7. Similarily, when $\gamma = y_2$, we do not have $\alpha = 21y_1$. The same argument can run again for $\gamma = 3\alpha$. In summary, we obtain that $(\alpha, \beta, \gamma, \delta) = (y_2, 21y_2, 3y_2, 7y_2)$ up to permutation by $\mathrm{V} < \mathrm{S}_4$.

Determinant of h_1 or h_2 being 1 tells you that $\alpha_1\alpha_2\alpha - \beta_1\beta_2\beta - \gamma_1\gamma_2\gamma + \delta_1\delta_2\delta = 4$ and $\alpha_1\beta_2 - \alpha_2\beta_1 = \gamma_1\delta_2 - \gamma_2\delta_1$. From the first equation above, we note that 4 is divided by y_2 and therefore $y_2 = 1$ or 2.

Now we already know h must be of the form in the statement of this lemma, but only assuming $(\alpha, \beta, \gamma, \delta) = (1, 21, 3, 7)$ or $(2, 42, 6, 14)$ up to permutation by $\mathrm{V} < \mathrm{S}_4$ and having determinants 1. By writing out the matrix $\Phi(h) = BA(h)B^{-1}$, an elementary case-by-case observation shows that $\Phi(h)$ has integer coefficients if and only if $\frac{\alpha_i^2-\beta_i^2}{4}, \frac{\gamma_i^2-\delta_i^2}{4} \in \mathbb{Z}$ when $(\alpha, \beta, \gamma, \delta) = (1, 21, 3, 7)$ up to permutation and $\frac{\alpha_i^2-\beta_i^2}{2}, \frac{\gamma_i^2-\delta_i^2}{2} \in \mathbb{Z}$ when $(\alpha, \beta, \gamma, \delta) = (2, 42, 6, 14)$ up to permutation. They are all the same as $\alpha_i - \beta_i, \gamma_i - \delta_i \in 2\mathbb{Z}$.

Finally we show $(\alpha, \beta, \gamma, \delta) = (2, 42, 6, 14)$ (up to permutation) is impossible. For example, when $(\alpha, \beta, \gamma, \delta) = (2, 42, 6, 14)$, we would have $\alpha_1\alpha_2 - 21\beta_1\beta_2 - 3\gamma_1\gamma_2 + 7\delta_1\delta_2 = 2$. Mod out 4 and using $\alpha_1\beta_2 - \alpha_2\beta_1 = \gamma_1\delta_2 - \gamma_2\delta_1$, this gives us $(\alpha_1-\beta_1)(\alpha_2-\beta_2) + (\gamma_1-\delta_1)(\gamma_2-\delta_2) \equiv 2 \pmod 4$ but not 0. Therefore, the whole lemma has been proven. $\square$

Let us denote the subgroup of elements of type $(1, 21, 3, 7)$ as H. Similar as Proposition 6.6, we obtain the following result.

Proposition 7.5. *H is an index 4 normal subgroup of Γ' above. With $\Gamma'/H \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

We also make a table to make its structure clear

Type	Representative
$(1, 21, 3, 7)$	$(\mathrm{Id}, \mathrm{Id})$
$(21, 1, 7, 3)$	$(\mathrm{Id}, -\mathrm{Id})$

$(3,7,1,21)$	$(\begin{pmatrix} 0 & \frac{1}{\sqrt{7}} \\ -\sqrt{7} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sqrt{7} \\ \frac{1}{\sqrt{7}} & 0 \end{pmatrix})$
$(7,3,21,1)$	$(\begin{pmatrix} 0 & \frac{1}{\sqrt{7}} \\ -\sqrt{7} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{7} \\ -\frac{1}{\sqrt{7}} & 0 \end{pmatrix})$

Table 4: Elements of Γ'/H

Note that $(\mathrm{Id}, -\mathrm{Id})$ has no effect on $\mathbb{H}^2$. Therefore we can view $\mathbb{D}_T/\Gamma$ as $\mathbb{H}^2$ quotient by H and an involution

$$\widehat{F} = (\begin{pmatrix} 0 & \frac{1}{\sqrt{7}} \\ -\sqrt{7} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{7} \\ -\frac{1}{\sqrt{7}} & 0 \end{pmatrix}).$$

Moreover, H is isomorphic to a Hilbert modular group. To make it precise, let $F = \mathbb{Q}(\sqrt{21})$ and $\mathcal{O}_F = \mathbb{Z}[\frac{1+\sqrt{21}}{2}]$ be its ring of integers. It is well known that $\mathcal{O}_F$ is a PID. Let $\mathfrak{p} = (\frac{7+\sqrt{21}}{2})$ be its only prime ideal lying over $7 \in \mathbb{Z}$. The Hilbert modular group corresponding to $\mathfrak{p}$ is

$$\Gamma(\mathcal{O}_F \oplus \mathfrak{p}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,F) | a, d \in \mathcal{O}_F, b \in \mathfrak{p}^{-1}, c \in \mathfrak{p}\}.$$

It is a subgroup of $\mathrm{SL}(2,F)$.

Now let $\varphi\colon \mathrm{SL}(2,F) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ be a group homomorphism, sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix})$. Here we use a' to denote the Galois conjugate of a inside F. Meanwhile, let $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and ψ_C be an automorphism of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ sending $((h_1, h_2)$ to (h_1, Ch_2C^{-1}). We will have

Proposition 7.6. *The morphism $\psi \circ \varphi\colon \mathrm{SL}(2,F) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ maps $\Gamma(\mathcal{O}_F \oplus \mathfrak{p})$ isomorphically onto H.*

Proof. According to Lemma 7.4, we know that $h \in H$ if and only if

$$\begin{aligned} h &= (\begin{pmatrix} \frac{\alpha_1+\beta_1\sqrt{21}}{2} & \frac{1}{7}\frac{\gamma_1\sqrt{21}+7\delta_1}{2} \\ \frac{\gamma_2\sqrt{21}-7\delta_2}{2} & \frac{\alpha_2-\beta_2\sqrt{21}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\alpha_2+\beta_2\sqrt{21}}{2} & \frac{\gamma_2\sqrt{21}+7\delta_2}{2} \\ \frac{1}{7}\frac{\gamma_1\sqrt{21}-7\delta_1}{2} & \frac{\alpha_1-\beta_1\sqrt{21}}{2} \end{pmatrix}) \\ &= (\begin{pmatrix} \frac{\alpha_1+\beta_1\sqrt{21}}{2} & \frac{1}{7}\frac{\gamma_1\sqrt{21}+7\delta_1}{2} \\ \frac{\gamma_2\sqrt{21}-7\delta_2}{2} & \frac{\alpha_2-\beta_2\sqrt{21}}{2} \end{pmatrix}, C\begin{pmatrix} \frac{\alpha_1-\beta_1\sqrt{21}}{2} & \frac{1}{7}\frac{-\gamma_1\sqrt{21}+7\delta_1}{2} \\ \frac{-\gamma_2\sqrt{21}-7\delta_2}{2} & \frac{\alpha_2+\beta_2\sqrt{21}}{2} \end{pmatrix}C^{-1}) \\ &= \psi \circ \varphi(\begin{pmatrix} \frac{\alpha_1+\beta_1\sqrt{21}}{2} & \frac{1}{7}\frac{\gamma_1\sqrt{21}+7\delta_1}{2} \\ \frac{\gamma_2\sqrt{21}-7\delta_2}{2} & \frac{\alpha_2-\beta_2\sqrt{21}}{2} \end{pmatrix}) \end{aligned}$$

such that $\alpha_i - \beta_i, \gamma_i - \delta_i \in 2\mathbb{Z}$ and satisfies the "determinant $= 1$" condition.

Therefore, H lies in the image of $\psi \circ \varphi$. Meanwhile, $\psi \circ \varphi$ is clearly injective, thus we only need to prove that the inverse image of H is just $\Gamma(\mathcal{O}_F \oplus \mathfrak{p})$.

Note that $\frac{\alpha_1+\beta_1\sqrt{21}}{2} = \frac{\alpha_1-\beta_1}{2} + \beta_1\frac{1+\sqrt{21}}{2} \in \mathcal{O}_F$, the same apllies to $\frac{\alpha_2-\beta_2\sqrt{21}}{2}$. Meanwhile, $\frac{-7\delta_2+\gamma_2\sqrt{21}}{2} = (-(\gamma_2+2\delta_2)+\frac{\gamma_2+\delta_2}{2}\frac{1+\sqrt{21}}{2})\frac{7+\sqrt{21}}{2} \in \mathfrak{p}$ and almost the same applies to $\frac{1}{7}\frac{\gamma_1\sqrt{21}+7\delta_1}{2}$. Therefore, the inverse image of H lies in $\Gamma(\mathcal{O}_F \oplus \mathfrak{p})$. Conversely, each element in $\mathcal{O}_F$ can be written as $\frac{\alpha_1+\beta_1\sqrt{21}}{2}$ with $\alpha_1 - \beta_1 \in 2\mathbb{Z}$ and each element in $\mathfrak{p}$ can be written as $\frac{-7\delta_2+\gamma_2\sqrt{21}}{2}$ with $\delta_2 - \gamma_2 \in 2\mathbb{Z}$. The same also applies to $\mathfrak{p}^{-1}$. Therefore, the proposition is proven. ☐

Recall that in [vdG12, Chapter 2, §7], we can call $\Gamma(\mathcal{O}_F \oplus \mathfrak{p})\backslash\mathbb{H}^2$ a Hilbert modular surface. As a summary of above description of Γ, we have the following theorem:

Theorem 7.7. *There are natural morphisms*

$$\Gamma(\mathcal{O}_F \oplus \mathfrak{p})\backslash\mathbb{H}^2 \to \Gamma'\backslash\mathbb{H}^2 \xrightarrow{\sim} \mathrm{SO}^+(T)\backslash\mathbb{D}_T \to \Gamma\backslash\mathbb{D}_T \cong \mathrm{O}^+(T)\backslash\mathbb{D}_T.$$

Each of them is of degree 2 except the isomorphisms. The first degree 2 morphism is a quotient by

$$(\begin{pmatrix} 0 & \frac{1}{\sqrt{7}} \\ -\sqrt{7} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{7} \\ -\frac{1}{\sqrt{7}} & 0 \end{pmatrix})$$

and the second is a quotient by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Therefore, the period domain $\Gamma\backslash\mathbb{D}_T$ (for cubic fourfolds with order-7 action) is commensurable to the Hilbert modular surface $\Gamma(\mathcal{O}_F \oplus \mathfrak{p})\backslash\mathbb{H}^2$.

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