Ramsey numbers of long even cycles versus books

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Abstract

For any positive integers k and n, let $B_n^{(k)}$ be the book graph consisting of n copies of the complete graph K_{k+1} sharing a common K_k. Let C_m be a cycle of length m. Prior work by Allen, Łuczak, Polcyn, and Zhang (2023) established the Ramsey number $R(C_m, B_n^{(1)})$ for all sufficiently large even integer $m = \Omega(n^{9/10})$. Recently, Hu, Lin, Łuczak, Ning, and Peng (2025) obtained the exact value of $R(C_m, B_n^{(2)})$ under the same asymptotic conditions. A natural problem is to determine the exact value of $R(C_m, B_n^{(k)})$ for each fixed $k \geq 3$ under similar conditions. This paper provides a complete solution to this problem. The lower bound is proved by an explicit construction, while the tight upper bound is established by analyzing the corresponding Ramsey graph using semi-random ideas.

Keywords: Ramsey number; Book; Cycle

1 Introduction

Let Γ be a graph on N vertices. For any two graphs G and H, let $\Gamma \to (G, H)$ denote that either Γ contains a copy of G or whose complement contains a copy of H. The Ramsey number $R(G, H)$ is the minimum N such that there exists a graph Γ on N vertices satisfying $\Gamma \to (G, H)$. We say Γ is a (G, H)-Ramsey graph if $\Gamma \nrightarrow (G, H)$. The existence of the Ramsey number $R(G, H)$ follows from [15].

Let $B_n^{(k)}$ be the book graph consisting of n copies of the complete graph K_{k+1} all sharing a common K_k. Books were deeply involved in several recent breakthroughs, such as the exponential improvements [5, 10] on the diagonal Ramsey numbers, which built upon the book algorithm.

For a graph H and an integer N, the Turán number $ex(N, H)$ is the maximum number of edges in a graph on N vertices that is H-free (i.e., it contains no copy of H). Let C_m be an cycle of length m. For the even cycle, we only know the correct order of $ex(N, C_m)$ for $m = 4, 6, 10$, we refer the reader to the survey [8] and references therein. It is widely open of $ex(N, C_m)$ for each even m of the remaining cases. In general, for even integer m,

$$N^{1+(4+o(1))/3m} \leq ex(N, C_m) \leq O(\sqrt{m} \ln m N^{1+2/m} + m^2 N),$$

where the upper bound is due to Bukh and Jiang [4] who improved the classical result due to Bondy and Simonovits [3], while the lower bound by Lubotzky, Phillips, and Sarnak [14] follows

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from the construction of regular graphs of large girth. As observed by Allen, Łuczak, Polcyn, and Zhang [1] together with a simple induction, we know that for each fixed $k \geq 1$ and even $m \geq 4$,

$$n + \Omega(n^{(4+o(1))/3m}) \leq R(C_m, B_n^{(k)}) \leq n + O(\sqrt{m} \ln mn^{2/m} + m^2). \tag{1}$$

There exists a gap between the upper and lower bounds of $R(C_m, B_n^{(k)})$.

In this paper, we will consider $R(C_m, B_n^{(k)})$, particularly when m is even and sufficiently large. For convenience, we adopt the following notation for the remainder of this paper: for each fixed $k \geq 1$, let

$$g_k = R(C_m, B_n^{(k)}).$$

It is known from Dirac [6] that a graph on N vertices with minimum degree at least $N/2$ is hamiltonian; and if its minimum degree is larger than $N/2$, then it is pancyclic by Bondy [2]. Therefore, for even $m \geq 2n \geq 4$, $g_1 = m$. Zhang, Broersma, and Chen [20] determined the value of g_1 when $m \geq 4$ is even and $3n/4 + 1 \leq m \leq 2n$.

Recently, Allen et al. [1] established the following result.

Theorem 1.1 (Allen, Łuczak, Polcyn, and Zhang [1]) *For every $t \geq 2$, an even integer $m \geq ct^9$ where $c > 0$ is a constant, and n satisfying $(t-1)(m-1) \leq n-1 < t(m-1)$,*

$$g_1 = \max\{t(m-1)+1, n + \lfloor (n-1)/t \rfloor + 1\}.$$

By Theorem 1.1, the value of g_1 is known for all sufficiently large even integers

$$m \geq \Omega(n^{9/10}).$$

On the other hand, as observed in [1], Theorem 1.1 fails if $m \leq O(\log n/\log\log n)$ from (1).

For the case $k = 2$, we denote this number simply by $R(C_m, B_n)$. As noted by Faudree, Rousseau, and Sheehan [7]: we know practically nothing about $R(C_m, B_n)$ when m is even and greater than four. Only recently have we gained some ground on this problem. We refer the reader to [11, 13, 17] and references therein. In particular, extending Theorem 1.1, we obtained the value of g_2 for all sufficiently large even $m \geq \Omega(n^{9/10})$.

Theorem 1.2 (Hu, Lin, Łuczak, Ning, and Peng [11]) *For every integer $t \geq 2$, an even integer m sufficiently large, and n satisfying $(t-1)(m-1) \leq n-1 < t(m-1)$,*

$$g_2 = \begin{cases} \max\{(t+1)(m-1)+1, n+2\lfloor (n-1)/t \rfloor + 2\} & \text{if } \lfloor (n-1)/t \rfloor + 1 < m-1, \\ n + 2\lfloor (n-1)/t \rfloor + 1 & \text{if } \lfloor (n-1)/t \rfloor + 1 = m-1. \end{cases}$$

In general, the behavior of g_k for $k \geq 3$ would be more difficult to characterize. We completely determine its value for each $k \geq 3$ under similar conditions, specifically establishing the value for all sufficiently large even m with $m \geq \Omega(n^{9/10})$.

Theorem 1.3 *For all integers $t \geq 2$, $k \geq 3$, an even integer $m \geq c(t+k)^9$ where $c > 0$ is a constant, and n satisfying $(t-1)(m-1) \leq n-1 < t(m-1)$, the following hold:*

(i) If $\lfloor \frac{n-1}{t} \rfloor < m-k$, then $g_k = \max\{(t+k-1)(m-1)+1, n+k\lfloor (n-1)/t \rfloor + k\}$.

(ii) If $\lfloor \frac{n-1}{t} \rfloor = m-\sigma$ where $\lceil \frac{k}{2} \rceil + 1 \leq \sigma \leq k$, then $g_k = n + k\lfloor (n-1)/t \rfloor + \sigma - 1$.

(iii) If $\lfloor \frac{n-1}{t} \rfloor = m - \sigma$ *where* $2 \le \sigma \le \lceil \frac{k}{2} \rceil$*, and if*

$$g_{k-1} = n + (k-1)\lfloor (n-1)/t \rfloor + \ell \ \textit{ where } \sigma - 1 \le \ell \le \lceil \tfrac{k-1}{2} \rceil,$$

then, given that $k - 1 = \mu_\ell \ell + \alpha_\ell$ *with* $0 \le \alpha_\ell < \ell$*, we have*

$$g_k = \begin{cases} n + k\lfloor (n-1)/t \rfloor + \ell + 1 & \textit{if } r_k \le r, \\ n + k\lfloor (n-1)/t \rfloor + \ell & \textit{if } r_k > r, \end{cases}$$

where $r_k = t\lfloor \frac{n-1}{t} \rfloor + t + k - \ell - n$*, and* $r = \frac{\mu_\ell - 1}{\mu_\ell}(t + k - \alpha_\ell) + \alpha_\ell \mu_\ell$*.*

Remark. Upon initial inspection, the seemingly peculiar condition $r_k \le r$ (or $r_k > r$) is, in fact, inherent, as can be seen from the remark at the end of the proof of Theorem 1.3.

From Theorem 1.1, Theorem 1.2, and Theorem 1.3(i) for $t = 2$, we have the following result.

Corollary 1.1 *For each integer* $k \ge 1$*, there exists a positive integer* $n_0 = n_0(k)$ *such that for all even* $n \ge n_0$*,* $R(C_n, B_n^{(k)}) = (k+1)(n-1) + 1$*.*

Notation. Let $G = (V, E)$ be a graph. For any subset $U \subseteq V$, we use $G[U]$ to denote the subgraph induced by U, and use $G - U$ to denote the subgraph obtained from G by deleting vertices of U. For any vertex $v \in V$, let $N_G(v)$ and $d_G(v)$ be the neighborhood and degree of v, respectively. Write $N_G(v, U) = N_G(v) \cap U$ and $d_G(v, U) = |N_G(v, U)|$. Let $[n] = \{1, 2, \ldots, n\}$, and $[m, n] = \{m, m+1, \ldots, n\}$.

This paper is organized as follows. In Section 2, we will prove lower bounds for Theorem 1.3. In Section 3, we will collect several previous useful lemmas; and in Section 4, we have some properties for $(C_m, B_n^{(k)})$-Ramsey graphs, including a crucial lemma for the proof of the upper bound of Theorem 1.3(iii). Upper bounds for Theorem 1.3 will be proven in Section 5. In the concluding remarks, a slightly improved bound for m in Theorem 1.3 will be presented, together with a discussion of some interesting problems.

2 Constructions of the lower bounds

For convenience, let

$$p = \lfloor (n-1)/t \rfloor, \ \text{ and } \ n - 1 = tp + q \ \text{ where } \ 0 \le q \le t - 1.$$

(i) Suppose $p < m - k$. We will show that $g_k > \max\{(t+k-1)(m-1), n + kp + k - 1\}$. The construction is similar to that in [1, 11]. Let F be the disjoint union of $t + k - 1$ cliques of size $m - 1$. One can check that F is C_m-free, and its complement is $B_n^{(k)}$-free since $(t-1)(m-1) < n$ from the assumption. Thus we have $g_k > |V(F)| = (t+k-1)(m-1)$.

It remains to show that $g_k > n + kp + k - 1$. Let $v_1, v_2, \ldots, v_{t+k-1-q}$ be distinct vertices from a K_{p+k}. Let F_1 be a graph consisting of the K_{p+k} and extra $t + k - 1 - q$ vertex-disjoint K_{p+1}'s such that the ith K_{p+1} shares v_i with the K_{p+k} for $1 \le i \le t + k - 1 - q$. Let F_2 be a graph obtained from F_1 by adding q vertex-disjoint copies of K_{p+1}, see Figure 1. Note that F_2 is C_m-free as $p + k < m$ from the assumption, and

$$|V(F_2)| = (t+k-1)(p+1) + (p+k) - (t+k-1-q) = n + k(p+1) - 1,$$

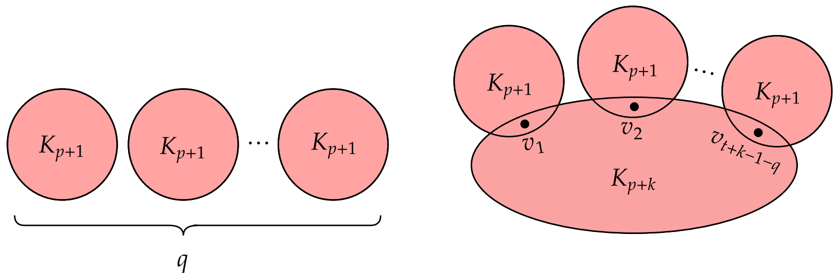

Figure 1: The graph F_2

by noting $n-1 = tp+q$.

Observe that if $u_1, u_2, \ldots, u_k$ form an independent set in F_2, then $|\cup_{1\le i\le k} N_{F_2}(u_i)| \ge kp$. Thus $u_1, u_2, \ldots, u_k$ have at most $|V(F_2) \setminus \cup_{1\le i\le k} N_{F_2}(u_i)| - k = n-1$ common non-neighbors in F_2, and so the complement of F_2 does not contain $B_n^{(k)}$ as a subgraph. The lower bound follows.

(ii) We assume that $p = m-\sigma$, where (for the convenience of proving (iii)) we extend the range of σ to $2 \le \sigma \le k$, and we will show that $g_k > n+kp+\sigma-2$.

Similar to the above, let $v_1, v_2, \ldots, v_{t+k-1-q}$ be distinct vertices from a $K_{p+\sigma-1}$. Let F_1 be a graph consisting of the $K_{p+\sigma-1}$ and extra $t+k-1-q$ vertex-disjoint K_{p+1}'s such that the ith K_{p+1} shares v_i with the $K_{p+\sigma-1}$ for $1 \le i \le t+k-1-q$. Let F_2 be a graph obtained from F_1 by adding q vertex-disjoint copies of K_{p+1}. Note that F_2 is C_m-free as $p+\sigma-1 = m-1$, and

$$\begin{aligned}|V(F_2)| &= (t+k-1)(p+1) + (p+\sigma-1) - (t+k-1-q)\\ &= (t+k)p + \sigma - 1 + q\\ &= n+kp+\sigma-2.\end{aligned}$$

Observe that if $u_1, u_2, \ldots, u_k$ form an independent set in F_2, then

$$|\cup_{1\le i\le k} N_{F_2}(u_i)| \ge (k-1)p + (p+\sigma-2) - (k-1) = k(p-1)+\sigma-1,$$

where the minimum will be achieved if some vertex u_i was chosen from the $K_{p+\sigma-1}$ other than $v_1, v_2, \ldots, v_{t+k-1-q}$. Therefore, $u_1, u_2, \ldots, u_k$ have at most $n-1$ common non-neighbors in F_2, and so the complement of F_2 contains no $B_n^{(k)}$. Thus we obtain $g_k > n+kp+\sigma-2$.

(iii) For this case, we assume that

$$p = m-\sigma, \quad \text{where } 2 \le \sigma \le \lceil \tfrac{k}{2} \rceil.$$

Furthermore, we assume that for $k \ge 3$, $g_{k-1} \ge n+(k-1)p+\ell$, where $\sigma - 1 \le \ell \le \lceil \frac{k-1}{2} \rceil$. Given that $k-1 = \mu_\ell \ell + \alpha_\ell$ where $0 \le \alpha_\ell < \ell$, we will show that

$$g_k \ge \begin{cases} n+kp+\ell+1 & \text{if } r_k \le r,\\ n+kp+\ell & \text{if } r_k > r,\end{cases} \tag{2}$$

where $r_k = tp+t+k-\ell-n$, and $r = \frac{\mu_\ell - 1}{\mu_\ell}(t+k-\alpha_\ell) + \alpha_\ell \mu_\ell$.

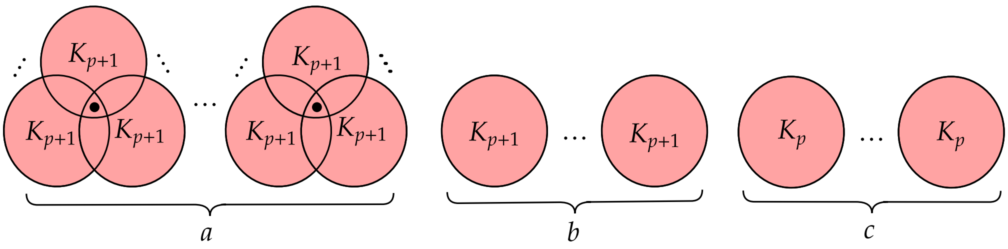

Figure 2: The graph $\Gamma_{a,b,c}$

First we will show (2) holds for $k = 3$. For this case, we know $\sigma = 2$, and so $p + 1 = m - 1$. By Theorem 1.2, $g_2 = n + 2p + 1$. Thus $\ell = 1$, $\mu_1 = 2$ and $\alpha_1 = 0$. Also, $r_3 = tp + t + 2 - n$.

Assume $r_3 \leq \frac{t+3}{2}$. We will prove $g_3 > n+3p+1$. Let Λ be the graph consisting of $2r_3$ copies of K_{p+1}'s paired up in twos where each such pair of K_{p+1}'s shares a common vertex, and let Γ_3 be obtained from Λ by adding extra $t + 3 - 2r_3$ vertex-disjoint K_{p+1}'s. Note that Γ_3 is C_m-free as $p + 1 = m - 1$ from the assumption. Moreover, $|V(\Gamma_3)| = (t + 3)(p + 1) - r_3 = n + 3p + 1$. Observe that if u_1, u_2 and u_3 form an independent set in Γ_3, then $|\cup_{1\leq i\leq 3} N_{\Gamma_3}(u_i)| \geq 3p - 1$. Therefore, u_1, u_2 and u_3 have at most $n - 1$ common neighbors in the complement of Γ_3, and so the complement of Γ_3 is $B_n^{(3)}$-free. The lower bound follows for this case.

Now assume $r_3 > \frac{t+3}{2}$, and we will show that $g_3 > n + 3p$. Let Γ_3' be the graph obtained from q vertex-disjoint K_{p+1}'s by adding extra $t + 3 - q$ copies of K_{p+1}'s with a common vertex. Clearly Γ_3' is C_m-free as $p + 1 = m - 1$. Moreover, recall $q = n - 1 - tp$, so we have that

$$|V(\Gamma_3')| = (t + 3)(p + 1) - (t + 2 - q) = tp + q + 3(p + 1) - 2 = n + 3p.$$

Note that if u_1, u_2 and u_3 form an independent set in Γ_3', then $|\cup_{1\leq i\leq 3} N_{\Gamma_3'}(u_i)| \geq 3p - 2$. Therefore, u_1, u_2 and u_3 have at most $n - 1$ common neighbors in the complement of Γ_3', and so the complement of Γ_3' is $B_n^{(3)}$-free. So the assertion holds for $k = 3$.

In the following, we assume $k \geq 4$ and suppose that (2) holds for smaller k. We will show that (2) also holds for k. We first define a family of graphs $\Gamma_{a,b,c}$, see Figure 2.

Definition 2.1 *Let a, b, c be non-negative integers. The graph $\Gamma_{a,b,c}$ is constructed as follows:*

(1) It contains a clusters, where each cluster is formed by some copies of K_{p+1} with one common vertex.

(2) It has b pairwise vertex-disjoint copies of K_{p+1} not intersecting any cluster.

(3) It has c pairwise vertex-disjoint copies of K_p not intersecting any cluster or K_{p+1}.

Case 1. $r_k \leq r$.

Observe that $r_k = tp + t + k - \ell - n \geq k - \ell$. Moreover, we have $k - 1 = \mu_\ell \ell + \alpha_\ell$ with $0 \leq \alpha_\ell < \ell$, and $\ell \leq \lceil \frac{k-1}{2} \rceil$. So we can express $r_k - \alpha_\ell$ as $(\mu_\ell - 1)\kappa_\ell + \gamma_\ell$, where $\kappa_\ell \geq \ell > \alpha_\ell$ and $0 \leq \gamma_\ell \leq \mu_\ell - 2$. Let G_1 be a graph consisting of $\mu_\ell + 1$ copies of K_{p+1} with a common vertex, G_2 a graph consisting of μ_ℓ copies of K_{p+1} with a common vertex, and G_3 a graph consisting of

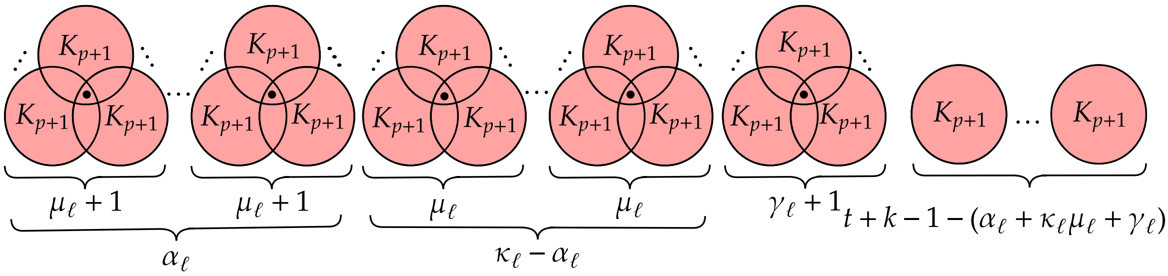

Figure 3: The graph Γ_k

$\gamma_\ell+1$ copies of K_{p+1} with a common vertex. Let Γ_k be the union graph consisting of α_ℓ disjoint copies of G_1, $(\kappa_\ell-\alpha_\ell)$ disjoint copies of G_2, one copy of G_3, and extra $t+k-1-(\alpha_\ell+\kappa_\ell\mu_\ell+\gamma_\ell)$ copies of K_{p+1}, and all of these G_1's, G_2's, G_3's, and K_{p+1}'s are vertex-disjoint from each other, see Figure 3. Clearly, Γ_k is C_m-free since $p+1\le m-1$. We will verify that the complement of Γ_k is $B_n^{(k)}$-free. Note that the total number of vertices in these G_i's for $1\le i\le 3$ is counted

$$\begin{aligned}
&\alpha_\ell((\mu_\ell+1)(p+1)-\mu_\ell)+(\kappa_\ell-\alpha_\ell)(\mu_\ell(p+1)-(\mu_\ell-1))+(\gamma_\ell+1)(p+1)-\gamma_\ell\\
&=(p+1)(\alpha_\ell(\mu_\ell+1)+(\kappa_\ell-\alpha_\ell)\mu_\ell+\gamma_\ell+1)-\alpha_\ell\mu_\ell-(\kappa_\ell-\alpha_\ell)(\mu_\ell-1)-\gamma_\ell\\
&=(p+1)(\alpha_\ell+\kappa_\ell\mu_\ell+\gamma_\ell+1)-(\alpha_\ell+\kappa_\ell(\mu_\ell-1)+\gamma_\ell)\\
&=(p+1)(\alpha_\ell+\kappa_\ell\mu_\ell+\gamma_\ell+1)-r_k.
\end{aligned}$$

Thus $|V(\Gamma_k)|=(t+k)(p+1)-r_k=k(p+1)+(n-k+\ell)=n+kp+\ell$ by noting $r_k=tp+t+k-\ell-n$. We aim to compute, for an independent set $u_1,\ldots,u_k$ in Γ_k, the minimum possible size of $|\cup_{i=1}^k N_{\Gamma_k}(u_i)|$. This minimum is achieved in the worst case, when none of the u_i is a center vertex of any G_j $(1\le j\le 3)$. Recall $k-1=\mu_\ell\ell+\alpha_\ell$ with $0\le\alpha_\ell<\ell$. Therefore,

$$\begin{aligned}
|\cup_{1\le i\le k}N_{\Gamma_k}(v_i)|&\ge\alpha_\ell((\mu_\ell+1)p+1)+(\ell-\alpha_\ell)(\mu_\ell p+1)+p+1-k\\
&=(\mu_\ell\ell+\alpha_\ell)p+\ell+p+1-k\\
&=k(p-1)+\ell+1.
\end{aligned}$$

It follows that $u_1,u_2,\ldots,u_k$ have at most $n-1$ common neighbors in the complement of Γ_k, and so the complement of Γ_k is $B_n^{(k)}$-free.

Case 2. $r_k>r$.

We aim to show $g_k>n+kp+\ell-1$. If $\ell=\sigma-1$, then we can apply F_2 as in (ii) (which is not of type $\Gamma_{a,b,c}$) to obtain a lower bound for g_k, i.e., $g_k>n+kp+\sigma-2=n+kp+\ell-1$. So we may assume that $\sigma\le\ell\le\lceil\frac{k-1}{2}\rceil$.

Claim 2.1 *If Γ_{k-1} is of type $\Gamma_{a,b,c}$ on $n+(k-1)p+\ell-1$ vertices and $\Gamma_{k-1}\nrightarrow(C_m,B_n^{(k-1)})$, and Γ_k is obtained from Γ_{k-1} by adding a K_p, then Γ_k is of type $\Gamma_{a,b,c}$ with $n+kp+\ell-1$ vertices satisfying $\Gamma_k\nrightarrow(C_m,B_n^{(k)})$.*

Proof. From the construction, Γ_k is of type $\Gamma_{a,b,c}$ with $n+kp+\ell-1$ vertices. Moreover, Γ_k is C_m-free. It remains to show that the complement of Γ_k is $B_n^{(k)}$-free. Since there is no $B_n^{(k-1)}$ in the complement of Γ_{k-1}, we have that for any $k-1$ vertices $u_1, u_2, \ldots, u_{k-1}$ that form an independent set in Γ_{k-1}, $|\cup_{1\le i\le k-1} N_{\Gamma_{k-1}}(u_i)| \ge |V(\Gamma_{k-1})|-(k-1)-(n-1) = (k-1)(p-1)+\ell$. Since Γ_k is of type $\Gamma_{a,b,c}$, any vertices $u_1, u_2, \ldots, u_k$ that form an independent set in Γ_k must satisfy that $|\cup_{1\le i\le k} N_{\Gamma_k}(v_i)| \ge (k-1)(p-1)+\ell+(p-1) = k(p-1)+\ell$. Therefore, $u_1, u_2, \ldots, u_k$ have at most $n-1$ common neighbors, and so the complement of Γ_k is $B_n^{(k)}$-free as desired. □

Note that we have proved (2) for $k=3$. Specifically, $\sigma=2$ when $k=3$. For $r_3 > \frac{t+3}{2}$, we know $g_3 > n+3p$, which corresponds to $\ell=\sigma-1=1$. For $r_3 \le \frac{t+3}{2}$, there exists a Γ_3 of type $\Gamma_{a,b,c}$ on $n+3p+1$ vertices such that $\Gamma_3 \nrightarrow (C_m, B_n^{(3)})$. In the following, suppose $k\ge 4$ and that for $2\le\sigma\le\lceil\frac{k-1}{2}\rceil$, there exists a Γ_{k-1} of type $\Gamma_{a,b,c}$ on $n+(k-1)p+\ell-1$ vertices (except when $\ell=\sigma-1$) with $\Gamma_{k-1} \nrightarrow (C_m, B_n^{(k-1)})$. We will show this holds for k.

If k is even, then $2\le\sigma\le\lceil\frac{k}{2}\rceil=\lceil\frac{k-1}{2}\rceil$. Moreover, if $r_k\le r$, then by Case 1, there exists a Γ_k of type $\Gamma_{a,b,c}$ on $n+kp+\ell$ vertices such that $\Gamma_k \nrightarrow (C_m, B_n^{(k)})$. If $r_k > r$, by the induction hypothesis and Claim 2.1, there exists a Γ_k of type $\Gamma_{a,b,c}$ on $n+kp+\ell-1$ vertices (except when $\ell=\sigma-1$) that satisfies $\Gamma_k \nrightarrow (C_m, B_n^{(k)})$.

If k is odd, we first consider the case where $\sigma=\lceil\frac{k}{2}\rceil=\frac{k+1}{2}=\lceil\frac{k-1}{2}\rceil+1$. By the lower bound of Theorem 1.3(ii), we know $g_{k-1}\ge n+(k-1)p+(\sigma-1)$. We may assume $\ell=\sigma-1$, as assuming a larger value for ℓ would only strengthen the lower bound. Now if $r_k\le r$, then applying the result from Case 1 we get a Γ_k of type $\Gamma_{a,b,c}$ on $n+kp+(\sigma-1)$ vertices satisfying $\Gamma_k \nrightarrow (C_m, B_n^{(k)})$. If $r_k>r$, then as observed at the beginning of Case 2, we have $g_k > n+kp+\sigma-2$.

Now we may assume $2\le\sigma\le\lceil\frac{k-1}{2}\rceil$. If $r_k\le r$, then by Case 1 there exists a Γ_k of type $\Gamma_{a,b,c}$ with $n+kp+\ell$ vertices satisfying that $\Gamma_k \nrightarrow (C_m, B_n^{(k)})$. If $r_k>r$, then by induction and Claim 2.1 there exists a Γ_k of type $\Gamma_{a,b,c}$ on $n+kp+\ell-1$ vertices (except when $\ell=\sigma-1$) satisfying $\Gamma_k \nrightarrow (C_m, B_n^{(k)})$.

We complete the proof of the lower bounds of Theorem 1.3. □

3 Useful lemmas

In the following, we will focus on the upper bounds of Theorem 1.3. The proofs proceed by induction on $k\ge 2$, analyzing the $(C_m, B_n^{(k)})$-Ramsey graph where m is even, i.e., R contains no even cycle C_m and whose complement is $B_n^{(k)}$-free. The strategy is twofold. First, if the graph has sufficiently large minimum degree, it must contain consecutive even cycles across a wide range, ensuring the existence of a C_m. This approach has been successfully applied in [1, 11, 13] and even to a problem in spectral graph theory [12]. Second, we utilize the extremal properties of such Ramsey graphs (see Section 4) to derive a contradiction when the graph's order is large enough.

A graph G is Hamiltonian if it contains a cycle that visits every vertex exactly once; such a cycle is called a Hamiltonian cycle. Let $\delta(G)$ be the minimum degree in G, and let $c(G)$ and $ec(G)$ denote the length of a longest cycle and a longest even cycle in G, respectively. We say that a graph G is pancyclic if it contains cycles of every length from 3 to $|V(G)|$. We begin by reviewing the following classical results.

Lemma 3.1 (Dirac [6]) *If an n-vertex graph G is 2-connected, then $c(G)\ge\min\{2\delta(G), n\}$.*

Bondy obtained a stronger result if graph G has minimum degree $\delta(G) \geq |V(G)|/2$.

Lemma 3.2 (Bondy [2]) *If an n-vertex graph G has minimum degree $\delta(G) \geq n/2$, then either G is pancyclic or G is the complete bipartite graph $K_{n/2,n/2}$.*

The following result guarantees long even cycles in graphs.

Lemma 3.3 (Voss and Zuluaga [18]) *If G is a 2-connected graph on n vertices, then we have $ec(G) \geq \min\{2\delta(G), n-1\}$.*

We also need the following result on the existence of consecutive even cycles.

Lemma 3.4 (Gould, Haxell, and Scott [9]) *Let $\varepsilon > 0$, $K = 75 \cdot 10^4/\varepsilon^5$, and let G be a graph with $n \geq \frac{45K}{\varepsilon^4}$ vertices and minimum degree at least εn. Then G contains a cycle of length t for every even integer $t \in [4, ec(G) - K]$.*

The following result shows that if the minimum degree of a graph is large enough, then for any pair of vertices there are paths of consecutive lengths.

Lemma 3.5 (Williamson [19]) *Let G be a graph with n vertices. If $\delta(G) \geq \frac{n}{2} + 1$, then for any two vertices x, y and any $2 \leq \ell \leq n-1$, there is an (x, y)-path of length ℓ in G.*

The following lemma states that if the minimum degree of a graph G is large enough, then we can find a small set W such that $G - W$ is a union of some vertex-disjoint 2-connected subgraphs.

Lemma 3.6 (Allen, Łuczak, Polcyn, and Zhang [1]) *Let $n \geq \ell \geq 2$. For each graph G with n vertices and minimum degree $\delta(G) \geq n/\ell + \ell$, there exists an $s < \ell$ and a subset $W \subset V(G)$ with $|W| \leq s - 1$, such that $G - W$ is a union of s vertex-disjoint 2-connected graphs.*

4 Properties for the Ramsey graphs

In this section, we shall have some properties for the $(C_m, B_n^{(k)})$-Ramsey graph R. For any $t \geq 2$ and $k \geq 3$, let $\varepsilon = \frac{1}{2(t+k)}$, and let

$$K = 2.4 \cdot 10^7 (t+k)^5, \text{ and even } m \geq \frac{45K}{\varepsilon^4} = 1.728 \cdot 10^{10} (t+k)^9.$$

Assumption: Let $s \geq k$ be an integer. Assume that there exists a set $W \subset V(R)$ with $|W| \leq s - 1$ such that $R - W$ is a union of s vertex-disjoint 2-connected graphs induced by $V_1, \ldots, V_s$ respectively. Then $W, V_1, \ldots, V_s$ form a partition of $V(R)$. For each $1 \leq i \leq s$, let

$$W_i = \{w \in W : d_R(w, V_i) \geq 2s\}, \text{ and } U_i = V_i \cup W_i.$$

Let F be a bipartite graph with vertex set $V(F) = W \cup \{U_1, \ldots, U_s\}$ and $\{w, U_i\} \in E(F)$ if and only if $w \in U_i$.

The following lemma can be found in [1, 11]; we include a proof for completeness.

Lemma 4.1 *If $|V_i \cup V_j| \geq m$ for any $1 \leq i \neq j \leq k$, and any pair of vertices of V_i for $1 \leq i \leq k$ is connected by a path of length from two to $m/2$ using vertices from V_i, then F is a forest.*

Proof. Suppose to the contrary that $w_{i_1}, U_{j_1}, w_{i_2}, U_{j_2}, \ldots, w_{i_r}, U_{j_r}, w_{i_1}$ is a cycle of length $2r$ in F. Since for each $1 \le \ell \le r$, w_{i_ℓ} has at least two neighbors in V_{j_ℓ}, there are distinct vertices $x_\ell, y_\ell \in V_{j_\ell}$ such that x_ℓ is a neighbor of w_{i_ℓ} and y_ℓ is a neighbor of $w_{i_{\ell+1}}$. Here, $w_{i_{r+1}} = w_{i_1}$. Let P_ℓ be a path connecting x_ℓ and y_ℓ using vertices from V_{i_ℓ}. Thus $w_{i_1}, x_1, P_1, y_1, w_{i_2}, x_2, P_2, y_2, \ldots, w_{i_r}, x_r, P_r, y_r, w_{i_1}$ is a cycle in R. By the assumption, the length of P_ℓ can vary from two to $m/2$. So we can get a C_m by choosing P_ℓ properly, a contradiction. Therefore, F is a forest. $\square$

An analogous argument to [1, 11] yields the following result under a slightly relaxed condition.

Lemma 4.2 *Suppose the minimum degree $\delta(R) > m/2 + K/2 + s$, where K is the parameter derived from Lemma 3.4. We have the following properties.*

(i) $V = \cup_{i=1}^s U_i$.

(ii) $|U_i| \le m - 1$ for each $1 \le i \le s$.

(iii) $\sum_{i=1}^s |U_i| \le |V| + s - 1$.

Proof. (i) It suffices to show that each $w \in W$ is contained in some U_i. Otherwise, there exists a vertex $w \in W$ that is not in U_i for each $1 \le i \le s$. Then $d_R(w) < 2s^2 + |W| \le 2s^2 + s < m/2 + K/2 + s$ provided $m \ge 4s^2$, a contradiction.

(ii) We first show $|V_i| \le m - 1$. From the assumption, $\delta(R[V_i]) \ge \delta(R) - |W| > m/2 + K/2$. If $|V_i| \le 2\delta(R[V_i])$, then Lemma 3.2 implies that there is a C_m. Else if $|V_i| > 2\delta(R[V_i])$, then by Lemma 3.3, there is an even cycle of length at least $2\delta(R[V_i]) \ge m + K$ in $R[V_i]$. Thus C_m is guaranteed by applying Lemma 3.4 with $\varepsilon = \frac{1}{2(t+k)}$, a contradiction.

In the following, we will show $|U_i| \le m - 1$ for $1 \le i \le s$. Otherwise, suppose $|U_1| \ge m$ without loss of generality. Take a subset $U_1' \subset U_1$ with $V_1 \subset U_1'$ and $|U_1'| = m$. Let $H = R[U_1']$ and we next show H is Hamiltonian. The fact $U_1' \setminus V_1 \subset W$ implies $|U_1' \setminus V_1| < s$, and so $|V_1| > m - s$. By joining a pair of non-adjacent vertices with degree sum (in H) at least m iteratively, we get the m-closure of H, denoted by H'. Recall $\delta(R[V_1]) \ge \delta(R) - s > m/2$. It follows that V_1 forms a clique in H'. Assume $U_1' \setminus V_1 = \{w_1, \ldots, w_\ell\}$ with $\ell < s$. From the assumption $d_R(w, V_1) \ge 2s$ for each $w \in U_1' \setminus V_1$, we are able to find $x_i, y_i \in V_1$ for $1 \le i \le \ell$ such that $\{x_i, y_i\} \subset N_R(w_i, V_1)$ and they are all distinct. As V_1 forms a clique in H', we can find a C_m in H'. Therefore, H is Hamiltonian, which implies that U_1 contains C_m, a contradiction.

(iii) Note that each vertex from V_i has at least $\delta(R) - s > m/2$ neighbors in V_i. Thus $|V_i| > m/2$ and $|V_i \cup V_j| \ge m$. From (ii), $|V_i| \le |U_i| \le m - 1$, we get $\delta(R[V_i]) \ge \frac{|V_i|}{2} + 1$. It follows by Lemma 3.5 that each pair of vertices in V_i is connected by a path of length from 2 to $m/2$. Thus, by Claim 4.1, F is a forest.

In the sum $\sum_{i=1}^s |U_i|$, each vertex $w \in W$ is counted exactly $d_F(w)$ times and each vertex in $\cup_{i=1}^s V_i = V \setminus W$ is counted exactly once. Thus, $\sum_{i=1}^s |U_i| = |V| - |W| + \sum_{w \in W} d_F(w)$. Note that F is a forest, we have $\sum_{w \in W} d_F(w) \le |W| + s - 1$. Therefore, $\sum_{i=1}^s |U_i| \le |V| + s - 1$. $\square$

Lemma 4.3 *If $\delta(R) \ge 2s^2 + s$, then for any distinct $i_1, \ldots, i_k \in [s]$, $|V \setminus (\cup_{1 \le j \le k} U_{i_j})| \le n - 1$.*

Proof. For each $w \in W$ with $w \notin W_{i_1}$, it has at most $2s$ neighbors in V_{i_1}. Thus vertices from $W \setminus W_{i_1}$ have at most $2s^2$ neighbors in V_{i_1}. As $|V_{i_1}| \ge \delta(R) - |W| > 2s^2$, there is a vertex $v_{i_1} \in V_{i_1}$ such that v_{i_1} is non-adjacent to vertices in $W \setminus W_{i_1}$. Similarly, we can find $v_{i_2} \in V_{i_2}, \ldots, v_{i_k} \in V_{i_k}$ such that v_{i_j} is non-adjacent to vertices in $W \setminus W_{i_j}$ for $2 \le j \le k$. Therefore, $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are non-adjacent to all vertices in $W \setminus (\cup_{1 \le j \le k} W_{i_j})$ and so in $V \setminus (\cup_{1 \le j \le k} U_{i_j})$. As we assumed that the complement of R is $B_n^{(k)}$-free, we obtain $|V \setminus (\cup_{1 \le j \le k} U_{i_j})| \le n - 1$ as claimed. $\square$

Definition 4.1 (Duplication number) *For any $\mathscr{U} = \{U_1, U_2, \ldots, U_h\}$, we call the difference $\sum_{i=1}^{h} |U_i| - |\cup_{1\le i\le h} U_i|$ the duplication number of $\mathscr{U}$. Also, we call $\frac{1}{h}(\sum_{i=1}^{h} |U_i| - |\cup_{1\le i\le h} U_i|)$ the duplication rate of $\mathscr{U}$.*

The reader is advised to note carefully the above definitions.

Lemma 4.4 *For any h sets $U_1, U_2, \ldots, U_h$, if $|U_i \cap (\cup_{1\le j\ne i\le h} U_j)| \ge 2$ for each $1 \le i \le h$, then $\sum_{1\le i\le h} |U_i| - |\cup_{1\le i\le h} U_i| \ge h$.*

Proof. Since $|U_i \cap (\cup_{1\le j\ne i\le h} U_j)| \ge 2$ for each $1 \le i \le h$, each U_i contains at least two vertices that are also in other sets. Let x_i be the number of vertices in U_i that belong to $\cup_{1\le j\ne i\le h} U_j$. Let $x = \sum_{i=1}^{h} x_i$. Then $x \ge 2h$ because $x_i \ge 2$ for each i. Suppose there are y vertices that are in at least two sets. Then $y \le x/2$ because each such vertex is counted in at least two of the x_i's. Then $\sum_{i=1}^{h} |U_i| - |\cup_{i=1}^{h} U_i| = x - y \ge x - x/2 = x/2 \ge h$. □

The following lemma is crucial for the proof of the upper bound of Theorem 1.3(iii), which states that if any k elements of a family of sets have small duplication number, then the size of the union of these sets from the family would be large.

Lemma 4.5 *Suppose that $|U_i| = p + 1$ for each $i \in [t + k]$. For $\ell \ge 1$, suppose $k - 1 = \mu\ell + \alpha$ where $0 \le \alpha < \ell$. If $|\cup_{1\le j\le k} U_{i_j}| \ge kp + \ell + 1$ for any distinct $i_1, i_2, \ldots, i_k \in [t + k]$, then $|V| \ge \sum_{i=1}^{t+k} |U_i| - \frac{\mu-1}{\mu}(t + k - \alpha) - \alpha\mu$.*

Proof. Let $\mathscr{U} = \{U_1, U_2, \ldots, U_{t+k}\}$. As $|U_i| = p + 1$ for $i \in [t + k]$ and the union of any k elements of $\mathscr{U}$ is at least $kp + \ell + 1$, we have that the duplication number of any k elements of $\mathscr{U}$ is at most $k - \ell - 1$.

Claim 4.1 *Let b be the maximum duplication number of any $t + 1$ elements of $\mathscr{U}$ that may achieve. Then $\sum_{i=1}^{t+k} |U_i| - |V| \le k - \ell - 1 + b$.*

Proof. Suppose first that there exist $k - 1$ elements of $\mathscr{U}$, say $U_1, \ldots, U_{k-1}$, such that their duplication number is $k-\ell-1$. Then for any $i \in [k-1]$ and $j \in [k, t+k]$, $U_i \cap U_j = \varnothing$; otherwise, $U_1, \ldots, U_{k-1}$ together with some U_j for $j \in [k, t+k]$ have the duplication number at least $k - \ell$, a contradiction. So $\sum_{i=1}^{t+k} |U_i| - |V| \le (k - \ell - 1) + b$, the assertion follows for this case.

Thus, we only need to verify that the assertion holds provided that any $k - 1$ elements of $\mathscr{U}$ have duplication number at most $k - \ell - 2$. There are two subcases.

(i) There exist $k - 2$ elements of $\mathscr{U}$, say $U_1, \ldots, U_{k-2}$, such that their duplication number is $k - \ell - 2$. Let $\mathscr{U}_{k-2} = \{U_{k-1}, \ldots, U_{t+k}\}$ for convenience.

Proposition 4.1 $\sum_{i=k-1}^{t+k} |U_i| - |\cup_{k-1\le i\le t+k} U_i| \le b + 1$.

Proof. We first claim that for some U_i with $i \in [k - 1, t + k]$, $|U_i \cap (\cup_{k-1\le j\ne i\le t+k} U_j)| \le 1$. Otherwise, for each $i \in [k-1, t+k]$, $|U_i \cap (\cup_{k-1\le j\ne i\le t+k} U_j)| \ge 2$. By Lemma 4.4, the duplication number of $\mathscr{U}_{k-2}$ is at least $t + 2$.

If $t + 2 \ge k$, then we can delete from $\mathscr{U}_{k-2}$, one by one, those subsets that have the fewest common vertices with the remaining subsets. We delete a subset from $\mathscr{U}_{k-2}$ that has the fewest common elements with the remaining subsets. Then the remaining $t+1$ subsets of $\mathscr{U}_{k-2}$ have at least t duplication numbers. If the $t + 1$ subsets have exactly t duplication numbers, then there exists a subset that shares at most one vertex with the others; deleting it leaves t subsets with at least $t-1$ duplication numbers. If instead the $t+1$ subsets have at least $t+1$ duplication numbers, then we proceed similarly to the initial case to obtain t subsets with at least $t-1$ duplication numbers. We repeat this process until we have deleted subsets one by one down to a certain number. Finally, we obtain k sets $U_{i_1}, U_{i_2}, \ldots, U_{i_k}$ for $i_j \in [k-1, t+k]$ whose duplication number is at least $k-1 > k-\ell-1$, which contradicts our assumption.

So we may assume that $t+2 < k$. Note that $U_1, \ldots, U_{k-2}$ have duplication number $k-\ell-2$, so the expectation of the contribution of a set U_i for $i \in [k-2]$ to the duplication number is less than 1. Now we choose $k-t-2$ sets U_i from $\{U_1, \ldots, U_{k-2}\}$, say $U_{t+1}, U_{t+2}, \ldots, U_{k-2}$, with duplication number as large as possible. Then the duplication number of $U_{t+1}, \ldots, U_{k-2}$ is at most t less than that of $U_1, \ldots, U_{k-2}$. Recall the duplication number of $\mathscr{U}_{k-2}$ is at least $t+2$. Therefore, the duplication number of $U_{t+1}, U_{t+2}, \ldots, U_{t+k}$ is at least $(k-\ell-2)-t+(t+2) = k-\ell$. Again a contradiction. The claimed assertion follows.

Assume that $|U_{k-1} \cap (\cup_{k-1 \le j \ne i \le t+k} U_j)| \le 1$ without loss of generality. Since any $t+1$ elements of $\mathscr{U}$ have duplication number at most b, we obtain $\sum_{i=k-1}^{t+k} |U_i| - |\cup_{k-1 \le i \le t+k} U_i| \le b+1$. □

From the above proposition, we obtain that

$$\begin{aligned}
\sum_{i=1}^{t+k} |U_i| - |\cup_{1 \le i \le t+k} U_i| &= \left(\sum_{i=1}^{k-2} |U_i| - |\cup_{1 \le i \le k-2} U_i| \right) + \left(\sum_{i=k-1}^{t+k} |U_i| - |\cup_{k-1 \le i \le t+k} U_i| \right) \\
&\le (k-\ell-2) + (b+1) \\
&= k-\ell-1+b.
\end{aligned}$$

So the claim holds for this case.

(ii) Any $k-2$ elements of $\mathscr{U}$ have duplication number at most $k-\ell-3$. Similarly, there are two subcases.

We repeat the above procedure at most $k-\ell-1$ times. At each of these steps, we can inductively deduce that $\sum_{i=1}^{t+k} |U_i| - |\cup_{1 \le i \le t+k} U_i| \le k-\ell-1+b$. Indeed, suppose at the dth step, any $k-d$ elements of $\mathscr{U}$ have duplication number at most $k-\ell-d-1$. We first assume that there exist $k-d-1$ sets U_i, say $U_1, \ldots, U_{k-d-1}$, have duplication number $k-\ell-d-1$. Then by a similar argument as in Proposition 4.1, we can obtain by induction that

$$\sum_{i=k-d}^{t+k} |U_i| - \left| \bigcup_{k-d \le i \le t+k} U_i \right| \le \left(\sum_{i=k-d+1}^{t+k} |U_i| - \left| \bigcup_{k-d+1 \le i \le t+k} U_i \right| \right) + 1 \le (b+d-1)+1 = b+d$$

It follows that $\sum_{i=1}^{t+k} |U_i| - |\cup_{1 \le i \le t+k} U_i| \le (k-\ell-d-1)+(b+d) = k-\ell-1+b$. At the final step, i.e., the $(k-\ell-1)$th step, we may assume that any $\ell+1$ sets U_i have duplication number equals to 0. This implies that $\sum_{i=1}^{t+k} |U_i| - |\cup_{1 \le i \le t+k} U_i| = 0$. So the claim is proved. □

In the following, we aim to compute the maximum value of the duplication number of $\mathscr{U}$. From the proof of the above claim, we may assume that there are $k-1$ elements of $\mathscr{U}$, say $U_1, \ldots, U_{k-1}$, whose duplication number is $k-\ell-1$, and $\sum_{i=1}^{t+k} |U_i| - |V|$ achieves the maximum $k-\ell-1+b$. We will estimate the duplication number b of the remaining $t+1$ sets. Let $\mathscr{A} = \{U_1, \ldots, U_{k-1}\}$ and $\mathscr{B} = \{U_k, \ldots, U_{t+k}\}$, and let $A = \cup_{i=1}^{k-1} U_i$ and $B = V \setminus A = \cup_{i=k}^{t+k} U_i$.

Recall $U_i = V_i \cup W_i$ and note that only vertices from W_i may contribute the duplication number. Note also that the subgraph of R induced by $U_i \cup U_j$ for some distinct i and j would be connected if $W_i \cap W_j \neq \emptyset$. Since the duplication number of any k elements of $\mathscr{U}$ is at most $k-\ell-1=(k-1)-\ell$, we obtain that the subgraph induced by A in R has at least ℓ connected components.

Let A_1 be a connected component of $R[A]$. It consists of some sets from $\mathscr{A}$, and we denote this collection of sets by $\mathscr{A}_1$. Without loss of generality, we assume that $\mathscr{A}_1$ achieves the minimum duplication rate among these collections corresponding to the connected components of $R[A]$.

Proposition 4.2 *The duplication rate of the sets U_i corresponding to any connected component in B should be less than or equal to that of $\mathscr{A}_1$.*

Proof. Suppose to the contrary that there is a connected component B_1 in B that contradicts the assumption. Let $\mathscr{B}_1$ be the collection of sets U_i from B_1. Suppose that $\mathscr{A}_1$ consists of h_1 sets with duplication rate ρ_1, and $\mathscr{B}_1$ consists of h_2 sets with duplication rate ρ_2. Then $\rho_2 > \rho_1$. We claim that the duplication number of $\mathscr{A}_1$ is h_1-1. From the fact that $\rho_1 < 1$ as the duplication rate of $\mathscr{A}$ is $\frac{k-\ell-1}{k-1} < 1$, we obtain that the duplication number of $\mathscr{A}_1$ is at most h_1-1. On the other hand, since the subgraph induced by A_1 is connected, the duplication number of $\mathscr{A}_1$ is at least h_1-1. Thus the duplication number of $\mathscr{A}_1$ equals h_1-1.

If $h_1 \geq h_2$, then $\mathscr{B}_1$ has duplication number at least h_2 as $\rho_2 > \rho_1 = \frac{h_1-1}{h_1}$. Thus we can take $h_1-(h_2-1)$ sets U_i from $\mathscr{A}_1$ with maximum duplication rate, which have duplication number at least h_1-h_2 since the subgraph induced by A_1 is connected (That is, the duplication number is reduced by a maximum of h_2-1 compared to $\mathscr{A}_1$). Recall $\mathscr{A}$ has duplication number $k-\ell-1$. Thus, these $h_1-(h_2-1)$ sets U_i from $\mathscr{A}_1$, together with the sets U_i from $\mathscr{A} \setminus \mathscr{A}_1$ and the sets U_i from $\mathscr{B}_1$, yield $(k-1)-(h_2-1)+h_2=k$ elements of $\mathscr{U}$, whose duplication number is at least $(k-\ell-1)-(h_2-1)+h_2=k-\ell$, a contradiction.

If $h_1 < h_2$, then we may take h_1+1 sets U_i from $\mathscr{B}_1$ with maximum duplication rate and so they have duplication number at least h_1 as the subgraph induced by B_1 is connected. These sets, together with all sets U_i from $\mathscr{A} \setminus \mathscr{A}_1$, yield k elements of $\mathscr{U}$ with the duplication number at least $(k-\ell-1)-(h_1-1)+h_1=k-\ell$, again a contradiction. $\square$

To get the maximum value of $\sum_{i=1}^{t+k}|U_i|-|\cup_{1\leq i\leq t+k}U_i|$, it is necessary to achieve the maximum number of duplication number in $\mathscr{B}$, i.e., the duplication rate of the corresponding sets U_i of each connected component in B should reach the maximum. To achieve this, from Proposition 4.2, A should have the minimum number of connected components and the duplication rates of the corresponding sets U_i of all connected components in A are as close to the average value as possible. Since $\mathscr{A}$ has duplication number $k-\ell-1$, and $k-1=\mu\ell+\alpha$ where $0 \leq \alpha < \ell$ from the assumption, we may assume that A contains exactly ℓ connected components, among which there are α components each contains $\mu+1$ sets U_i for $i \in [t+k]$ with duplication number μ and $\ell-\alpha$ components each contains μ sets U_i for $i \in [t+k]$ with duplication number $\mu-1$. In the above construction, each of the connected components in B may reach the maximum duplication rate $(\mu-1)/\mu$ from Proposition 4.2 by noting $(\mu-1)/\mu < \mu/(\mu+1)$. Thus the $t+k$ sets U_i in $\mathscr{U}$ reaches the maximum duplication number, and so we obtain that

$$\sum_{i=1}^{t+k}|U_i| - |\cup_{1\leq i\leq t+k} U_i| \leq \alpha\mu + \tfrac{\mu-1}{\mu}(t+k-\alpha),$$

which implies that $|V| \geq \sum_{i=1}^{t+k}|U_i| - \frac{\mu-1}{\mu}(t+k-\alpha) - \alpha\mu$ as claimed. $\square$

5 Proof of Theorem 1.3

The lower bounds of Theorem 1.3 have been established in Section 2, so we only need to prove the upper bounds accordingly. In the following, for each case, we consider a red-blue edge coloring of K_N for some suitable N defined on the vertex set V, and we consistently use R and B to denote the graphs induced by all red edges and blue edges respectively. Let

$$p = \lfloor (n-1)/t \rfloor, \text{ and so } n-1-t < tp \leq n-1.$$

For any $t \geq 2$ and $k \geq 3$, let $s \geq k$, and let K and m be defined as in Section 4.

5.1 The upper bound for Theorem 1.3(i)

From Theorem 1.2, we know Theorem 1.3(i) holds for $k = 2$, we now assume that $k \geq 3$ and Theorem 1.3(i) holds for all smaller k. We will show that Theorem 1.3(i) also holds for k.

Case 1. $g_{k-1} = (t+k-2)(m-1)+1$.

For this case, $n + (k-1)p + k - 1 \leq (t+k-2)(m-1)+1$. Note that $p+1 < m-1$, so we obtain $n + kp + k < (t+k-1)(m-1)+1$. Let $N = (t+k-1)(m-1)+1$. We will show $g_k \leq N$. Consider a red-blue edge coloring of K_N on vertex set V. On the contrary, suppose that the red graph R contains no C_m and the blue graph B contains no $B_n^{(k)}$.

If there exists a vertex $v \in V$ with $d_B(v) \geq (t+k-2)(m-1)+1 = g_{k-1}$, then by the inductive hypothesis we can find either a red C_m or a blue $B_n^{(k-1)}$ in the blue neighborhood of v. In the latter case, such a $B_n^{(k-1)}$ together with v gives a blue $B_n^{(k)}$.

So we may assume $\delta(R) \geq m-1$. If R is 2-connected, then since $\delta(R) \geq m-1$, $N > 2m-2+1$, by Lemma 3.3, $ec(R) \geq 2m-2$. Thus R contains C_m by Lemma 3.4.

Now we assume that R has a cut-vertex. Note that $\delta(R) \geq m-1 \geq \frac{(t+k-1)(m-1)+1}{t+k} + (t+k)$, provided $m \geq (t+k)^2$. By Lemma 3.6 there is an $s \leq t+k-1$ and a set $W \subset V$ with $|W| \leq s-1$ such that $R - W$ is a union of vertex-disjoint 2-connected subgraphs $V_1, \ldots, V_s$. Let U_i be a subset such that $U_i = V_i \cup W_i$, where $W_i = \{w \in W : d_R(w, V_i) \geq 2s\}$. Since $\delta(R) \geq m-1 \geq m/2 + K/2 + s$, by Lemma 4.2(i)-(ii) we know that $V = \cup_{i=1}^{s} U_i$ and $|U_i| \leq m-1$. However, then $|V| = |\cup_{i=1}^{s} U_i| \leq (t+k-1)(m-1) < N$, which leads to a contradiction.

Case 2. $g_{k-1} = n + (k-1)p + (k-1)$.

We first show that $g_k \leq n + kp + k$. Let $N = n + kp + k$ and consider a red-blue edge coloring of K_N on the vertex set V. If there exists a vertex $v \in V$ with $d_B(v) \geq n + (k-1)p + (k-1) = g_{k-1}$, then we are done similarly as above. So we assume $\delta(R) \geq p+1$. If $t = 2$, by the induction hypothesis, $n + (k-1)p + (k-1) \geq k(m-1)+1$, implying $n \geq \frac{2k}{k+1}(m-1) - \frac{k-3}{k+1} \geq 3(m-1)/2$ as $k \geq 3$. Thus $2\delta(R) \geq 2p+2 \geq n \geq 3(m-1)/2$. If $t \geq 3$, then as $n-1 \geq (t-1)(m-1)$, we know $2\delta(R) \geq 2p+2 > 2(n-1)/t \geq 2(t-1)(m-1)/t \geq 4(m-1)/3$. Therefore we conclude that $2\delta(R) \geq 4(m-1)/3$.

If R is 2-connected, then by a similar argument as above, R contains C_m. Thus we assume that R has a cut-vertex. Note that $\delta(R) \geq p+1 \geq \frac{N}{t+k+1} + (t+k+1)$, provided $n \geq 2t(t+k+1)^2$. By Lemma 3.6, there is an $s \leq (t+k)$ and a set $W \subset V$ with $|W| \leq s-1$ such that $R - W$ is a union of vertex-disjoint 2-connected subgraphs $V_1, \ldots, V_s$. For each $1 \leq i \leq s$, we define U_i as before. Since $\delta(R) \geq p+1 \geq 2(m-1)/3 > m/2 + K/2 + s$, by Lemma 4.2(i)-(ii) we know that $V = \cup_{i=1}^{s} U_i$ and $|U_i| \leq m-1$.

If $s \le t+k-1$, then $|V| = |\cup_{i=1}^{s} U_i| \le (k+t-1)(m-1) < (k+t-1)(m-1)+1 \le N$, a contradiction. If $s = t+k$, as $\delta(R) \ge p+1$, then $|U_i| \ge p+2$. By Lemma 4.2(iii) and $\delta(R) \ge p+1 \ge 2(m-1)/3 > m/2 + K/2 + s$, we obtain

$$|V| \ge \sum_{i=1}^{s} |U_i| - (s-1) \ge (t+k)(p+2) - (t+k-1) \ge n + kp + k + 1 > N,$$

where the third inequality holds since $t(p+1) \ge n$, a contradiction.

Now we assume $(t+k-1)(m-1)+1 \ge n+kp+k$ and we will show $g_k \le (t+k-1)(m-1)+1$. Let $N = (t+k-1)(m-1)+1$, and consider an arbitrary red-blue edge coloring of K_N on the vertex set V. We rephrase N as $N = n+kp+k+a-1$, where $a \ge 1$. If $d_B(v) \ge n+(k-1)p+(k-1) = g_{k-1}$ for some $v \in V$, then we are done similarly as above. So we assume $\delta(R) \ge p+a$.

If R is 2-connected, then by a similar argument as above we can obtain a red C_m. So we may assume that R contains a cut-vertex. Note that $\delta(R) \ge p+a \ge \frac{N}{t+k+1} + (t+k+1)$. By Lemma 3.6, there is an $s \le (t+k)$ and a set W with $|W| \le s-1$ such that $R-W$ is a union of vertex-disjoint 2-connected subgraphs $V_1, \ldots, V_s$. For each $1 \le i \le s$, we define U_i as before. As $\delta(R) \ge p+a \ge m/2+K/2+s$, by Lemma 4.2(i)-(ii) we know that $V = \cup_{i=1}^{s} U_i$ and $|U_i| \le m-1$. If $s \le t+k-1$, then $|V| = |\cup_{i=1}^{s} U_i| \le (k+t-1)(m-1) < N$, a contradiction. If $s = t+k$, then $|U_i| \ge p+a+1$ since $\delta(R) \ge p+a$. Thus, by Lemma 4.2(iii),

$$\begin{aligned} |V| \ge \sum_{i=1}^{s} |U_i| - (s-1) &\ge (t+k)(p+a+1) - (t+k-1) \\ &\ge n + kp + ak + (a-1)t + 1 \\ &> N, \end{aligned}$$

where the third inequality holds since $t(p+1) \ge n$, a contradiction.

5.2 The upper bound for Theorem 1.3(ii)

The proof proceeds by induction on $k \ge 2$. The base case follows directly from Theorem 1.2. For the inductive step, assume $k \ge 3$ and that the assertion holds for all smaller values of k. We will show that the assertion also holds for k.

Suppose $p = m - \sigma$, where $\lceil \frac{k}{2} \rceil + 1 \le \sigma \le k$. Note that

$$n + (k-1)p = (t+k-1)p + n - tp \ge (t+k-1)(m-\sigma) > (t+k-2)(m-1)+1,$$

provided $m \ge k(t+k)$. If $p = m-k = m-(k-1)-1$, then it follows by Theorem 1.3(i) that $g_{k-1} = n+(k-1)p+k-1$. If $p = m-\sigma$, where $\lceil \frac{k}{2} \rceil + 1 \le \sigma \le k-1$, by induction, we assume that $g_{k-1} = n+(k-1)p+\sigma-1$. We will show $g_k \le n+kp+\sigma-1$ for $p = m-\sigma$, where $\lceil \frac{k}{2} \rceil + 1 \le \sigma \le k$.

Let $N = n+kp+\sigma-1$, and consider a red-blue edge coloring of K_N on the vertex set V. If there is a vertex v with $d_B(v) \ge n+(k-1)p+\sigma-1 = g_{k-1}$, then we are done similarly as above. So we may assume $\delta(R) \ge p$. If R is 2-connected, by a similar argument, R contains C_m.

Thus we assume that R contains a cut-vertex. Note that $\delta(R) \ge p \ge \frac{N}{t+k+1} + (t+k+1)$. By Lemma 3.6, there is an $s \le (t+k)$ and a set W with $|W| \le s-1$ such that $R-W$ is a union of vertex-disjoint 2-connected subgraphs $V_1, \ldots, V_s$. For each $1 \le i \le s$, we define U_i as before. Since $\delta(R) \geq m-k \geq m/2+K/2+s$, by Lemma 4.2(i)-(ii) we know that $V = \cup_{i=1}^{s} U_i$ and $|U_i| \leq m-1$. As $\delta(R) \geq p$, we have that

$$p+1 \leq |U_i| \leq p+\sigma-1.$$

If $s \leq t+k-1$, then $|V| \leq \sum_{i=1}^{s} |U_i| \leq (t+k-1)(p+\sigma-1) \leq n+kp-p+(t+k-1)(\sigma-1) < N$, where the last inequality holds provided $n \geq tk(t+k)$, which leads to a contradiction. Thus we may assume $s = t+k$.

Claim 5.1 *The size of V achieves the minimum when $|U_i| = p+1$ for all $1 \leq i \leq t+k$.*

Proof. For $|U_i| \geq p+2$, and for $|U_j| = p+y$, $1 \leq y \leq \sigma-1$, $1 \leq j \neq i \leq t+k$, set $|U_i \cap U_j| = x$. If $x \leq y-1$, then we can choose $U_j' \subseteq U_j$ of $p+y-x$ vertices that do not intersect with U_i. And if $x \geq y$, we can choose $U_j' \subseteq U_j$ of $p+1$ vertices that intersects U_i with $x+1-y \geq 1$ vertices. For each of these cases, if we substitute U_j by U_j', it will not change the size of V.

Let λ be the number of U_j''s that intersects U_i with at least one vertex. If $\lambda \geq k-1$, then there are $U_{j_1}, \ldots, U_{j_{k-1}}$ corresponding to these U_j''s such that

$$|U_i \cup U_{j_1} \cup \cdots \cup U_{j_{k-1}}| \leq (p+\sigma-1)+(k-1)p = kp+\sigma-1,$$

implying that $|V \setminus (U_i \cup U_{j_1} \cup \cdots \cup U_{j_{k-1}})| \geq n$, which contradicts Lemma 4.3.

So we assume $\lambda < k-1$. Let $\tau = |\cup_{j\neq i} (U_j' \cap U_i)|$. If $\tau \leq |U_i|-p-1$, then we can choose $U_i' \subseteq U_i$ with $|U_i|-\tau \geq p+1$ vertices which is disjoint with all other U_j''s. Thus the size of V achieves the minimum when $|U_i'| = p+1$. If $\tau > |U_i|-p-1$, then we can choose $U_i' \subseteq U_i$ of $p+1$ vertices that intersects $\tau+(p+1)-|U_i|$ vertices in total with all other U_j''s. In particular, these $\tau+(p+1)-|U_i|$ vertices are contained in $V \setminus U_i'$. So if we replace U_i by U_i', it will not effect the size of V for this case. Therefore, we conclude that the size of V achieves the minimum when $|U_i| = p+1$ for all $1 \leq i \leq t+k$. $\square$

By Lemma 4.3, the size of any k sets U_i for $1 \leq i \leq t+k$ is at least $N-(n-1) = kp+\sigma$. Now we choose k sets U_i for $1 \leq i \leq t+k$ that have maximal duplication number, which is at most $k-\sigma \leq \lfloor \frac{k}{2} \rfloor - 1$ because $\lceil \frac{k}{2} \rceil + 1 \leq \sigma \leq k$. Among these k sets U_i, there are at least two disjoint sets of them that are disjoint from all the others. Moreover, for all other U_j's, they are surely disjoint from each other and also disjoint from these k sets U_i. Otherwise, we can substitute two disjoint sets from these k sets U_i's by two intersecting sets out of them, yielding a new cluster of k sets with larger duplication number, a contradiction. Therefore,

$$|V| \geq (t+k)(p+1)-(k-\sigma) > n-1+kp+\sigma = N,$$

which leads to a contradiction.

5.3 The upper bound for Theorem 1.3(iii)

Suppose $p = m-\sigma$, where $2 \leq \sigma \leq \lceil \frac{k}{2} \rceil$, and $g_{k-1} = n+(k-1)p+\ell$ for some $\sigma-1 \leq \ell \leq \lceil \frac{k-1}{2} \rceil$, and $k-1 = \mu_\ell \ell + \alpha_\ell$ where $0 \leq \alpha_\ell < \ell$, we will show that

$$g_k \leq \begin{cases} n+kp+\ell+1 & \text{if } r_k \leq r, \\ n+kp+\ell & \text{if } r_k > r. \end{cases}$$

where $r_k = tp + t + k - \ell - n$, and $r = \frac{\mu_\ell - 1}{\mu_\ell}(t + k - \alpha_\ell) + \alpha_\ell \mu_\ell$.

Case 1. $r_k \le r$.

Let $N = n + kp + \ell + 1$, and consider a red-blue edge coloring of K_N on V. If there is a vertex v with $d_B(v) \ge n + (k-1)p + \ell = g_{k-1}$, we are done similarly. So we assume $\delta(R) \ge p+1$. If R is 2-connected, then by a similar argument as above, R contains C_m by Lemma 3.4.

Now we suppose that R contains a cut-vertex. Note that $\delta(R) \ge p + 1 \ge \frac{N}{t+k+1} + (t + k + 1)$. By Lemma 3.6, there is an $s \le (t + k)$ and a set W with $|W| \le s - 1$ such that $R - W$ is a union of vertex-disjoint 2-connected subgraphs $V_1, \ldots, V_s$. For each $1 \le i \le s$, we define U_i as before. Since $\delta(R) \ge m - k \ge m/2 + K/2 + s$, by Lemma 4.2(i)-(ii) we know that $V = \cup_{i=1}^s U_i$ and $|U_i| \le m - 1$. As $\delta(R) \ge p + 1$, then $p + 2 \le |U_i| \le p + \sigma - 1$.

If $s \le t + k - 1$, then by a similar calculation as in Theorem 1.3(ii) we obtain that

$$|V| \le \sum_{i=1}^{s} |U_i| \le (t + k - 1)(p + \sigma - 1) < N,$$

provided $n \ge tk(t + k)$, a contradiction.

If $s = t + k$, then by Lemma 4.2(iii) with $\delta(R) \ge m - k \ge m/2 + K/2 + s$, we obtain that

$$|V| \ge \sum_{i=1}^{s} |U_i| - (s - 1) \ge (t + k)(p + 2) - (t + k - 1) \ge n + kp + k + 1 > N,$$

where the third inequality holds since $t(p + 1) \ge n$, a contradiction.

Case 2. $r_k > r$.

Let $N = n + kp + \ell$, and consider a red-blue edge coloring of K_N on V. If there is a vertex v that $d_B(v) \ge n + (k - 1)p + \ell = g_{k-1}$, we are done as before. So we assume $\delta(R) \ge p$.

If R is 2-connected, then by a similar argument as above, we can obtain a C_m in R. Thus we assume that R contains a cut-vertex. Note that $\delta(R) \ge p \ge \frac{N}{t+k+1} + (t + k + 1)$. It follows by Lemma 3.6 that there is an $s \le (t + k)$ and a set W with $|W| \le s - 1$ such that $R - W$ is a union of vertex-disjoint 2-connected subgraphs $V_1, \ldots, V_s$. For each $1 \le i \le s$, we define U_i as before. Since $\delta(R) \ge m - k \ge m/2 + K/2 + s$, by Lemma 4.2(i)-(ii) we know that $V = \cup_{i=1}^s U_i$ and $|U_i| \le m - 1$. As $\delta(R) \ge p$, we have $p + 1 \le |U_i| \le p + \sigma - 1$.

If $s \le t + k - 1$, then $|V| \le \sum_{i=1}^{s} |U_i| \le (t + k - 1)(p + \sigma - 1) < N$, a contradiction. So we assume $s = t + k$. Since $\delta(R) \ge m - k \ge 2s^2 + s$ and B contains no $B_n^{(k)}$. By Lemma 4.3, for any distinct $i_1, i_2, \ldots, i_k \in [s]$, we have $|\cup_{1\le j\le k} U_{i_j}| \ge |V| - (n - 1) = kp + \ell + 1$. By a similar argument as Claim 5.1, we obtain that the size of V achieves the minimum when $|U_i| = p + 1$ for all $1 \le i \le t + k$. Therefore, by Lemma 4.5,

$$\begin{aligned}
|V| &\ge (t + k)(p + 1) - \left(\tfrac{\mu_\ell - 1}{\mu_\ell}(t + k - \alpha_\ell) + \alpha_\ell \mu_\ell\right) \\
&> (t + k)(p + 1) - r_k \\
&= n + kp + \ell \\
&= N,
\end{aligned}$$

where the first equality holds since $r_k = tp + t + k - \ell - n$. The final contradiction completes the proof. $\square$

6 Concluding remarks

• In the proof of Case 1 for the lower bound of Theorem 1.3(iii), $r_k = tp+t+k-\ell-n$ represents the duplication number of the vertices in all K_{p+1}'s in the construction of Γ_k. Meanwhile, Lemma 4.5 states that $r = \frac{\mu_\ell - 1}{\mu_\ell}(t + k - \alpha_\ell) + \alpha_\ell \mu_\ell$ is the maximum duplication number of the vertices in all K_{p+1}'s in a $(C_m, B_n^{(k)})$-Ramsey graph.

• Let us point out that Lemma 3.4 by Gould, Haxell, and Scott [9] can be slightly improved as follows [16]:

Let $c > 0$, $K = 75 \cdot 10^4/c^5$, and let G be a graph with $n \geq 18K/c^2$ vertices and minimum degree at least cn. Then G contains a cycle of length t for every even integer $t \in [4, ec(G) - K]$.

Indeed, in the proof of [9, Theorem 1]: "$\cdots$ Otherwise at least $2c^2n/9 - 2K_0$ common neighbours of z and z_0 fall onto P_i. Since P_i has length less than n, and $n \geq 90K_0c^{-4}$, there is an interval I in P_i of length at most $15c^{-2}$ that is disjoint from S and contains three common neighbours and therefore two that are an even distance apart in I, say y_1 and $y_2 \cdots$", here the condition "$n \geq 90K_0c^{-4}$" is not required and can be relaxed to "$n \geq 90K_0c^{-2}$", say. Moreover, this condition is not needed anywhere else.

From the above observation, we can thus slightly improve Theorem 1.1, Theorem 1.2, and Theorem 1.3 to obtain the exact value of $R(C_m, B_n^{(k)})$ for each fixed $k \geq 1$ and large even $m \geq \Omega(n^{7/8})$. However, these three theorems do not hold if $m \leq O(\log n/\log\log n)$ from (1) as was observed in [1]. Finding the minimal size of m for which this holds remains an interesting problem.

References

[1] P. Allen, T. Łuczak, J. Polcyn, and Y. Zhang, The Ramsey number of a long even cycle versus a star, *J. Combin. Theory Ser. B* **162** (2023), 144–153.

[2] J. A. Bondy, Pancyclic graphs I, *J. Combin. Theory Ser. B* **11** (1971), 80–84.

[3] J.A. Bondy and M. Simonovits, Cycles of even length in graphs, *J. Combin. Theory Ser. B* **16** (1974), 97–105.

[4] B. Bukh, and Z. Jiang, A bound on the number of edges in graphs without an even cycle, *Combin. Probab. Comput.* **26** (1) (2017), 1–15.

[5] M. Campos, S. Griffiths, R. Morris, and J. Sahasrabudhe, An exponential improvement for diagonal Ramsey, arXiv:2303.09521v1, 2023.

[6] G. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69–81.

[7] R. Faudree, C. Rousseau, and J. Sheehan, Cycle–book Ramsey numbers, *Ars Combin.* **31** (1991), 239–248.

[8] Z. Füredi, and M. Simonovits, The History of Degenerate (Bipartite) Extremal Graph Problems, Erdős Centennial, Bolyai Soc. Math. Stud., vol. 25, János Bolyai Math. Soc., Budapest, 2013, pp. 169–264.

[9] R. Gould, P. Haxell, and A. Scott, A note on cycle lengths in graphs, *Graphs Combin.* **18** (2002) 491–498.

[10] P. Gupta, N. Ndiaye, S. Norin, and L. Wei, Optimizing the CGMS upper bound on Ramsey numbers, arXiv:2407.19026, 2024.

[11] F. Hu, Q. Lin, T. Łuczak, B. Ning, and X. Peng, Ramsey numbers of books versus long cycles, *SIAM J. Discrete Math.* **39**(1) 2025, 550–561.

[12] B. Li and B. Ning, Eigenvalues and cycles of consecutive lengths, *J. Graph Theory* **103**(3) (2023), 486–492.

[13] Q. Lin and X. Peng, Large book-cycle Ramsey numbers, *SIAM J. Discrete Math.* **35**(1) (2021), 532–545.

[14] A. Lubotzky, R. Phillips, and P. Sarnak, Ramanujan graphs, *Combinatorica* **8** (3) (1988), 261–277.

[15] F. P. Ramsey, On a problem of formal logic, *Proc. Lond. Math. Soc.* **30** (1929), 264–286.

[16] A. Scott, personal communication.

[17] L. Shi, Ramsey numbers of long cycles versus books or wheels, *European J. Combin.* **31** (2010), 828–838.

[18] H. Voss and C. Zuluaga, Maximale gerade und ungerade Kreise in Graphen. I, *Wiss. Z. Tech. Hochsch. Ilmenau* **23**(4) (1977), 57–70 (German).

[19] J. Williamson, Panconnected graphs, II, *Period. Math. Hungar.* **8** (1977), 105–116.

[20] Y. Zhang, H. Broersma, and Y. Chen, Narrowing down the gap on cycle-star Ramsey numbers, *J. Comb.* **7**(2-3) (2016), 481–493.